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Higher order approximate periodic solutions for nonlinear oscillators with the Hamiltonian approach

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ABSTRACT

In this work, the Hamiltonian approach is applied to obtain the natural frequency of the Duffing oscillator, the nonlinear oscillator with discontinuity and the quintic nonlinear oscillator. The Hamiltonian approach is then extended to the second and third orders to find more precise results. The accuracy of the results obtained is examined through time histories and error analyses for different values for the initial conditions. Excellent agreement of the approximate frequencies and the exact solution is demonstrated. It is shown that this method is powerful and accurate for solving nonlinear conservative oscillatory systems.

1. Introduction

The study of nonlinear equations in mechanics and physics is of interest to many researchers. There are a large variety of approximate approaches for solving nonlinear equations, such as the energy balance method [1–8], the frequency amplitude formulation [9–12], the variational approach [13–15] and other methods [16–20]. Professor He introduced the Hamiltonian approach [21–23] and applied it for solving nonlinear equations. Further, many other researchers have used this method for solving nonlinear equations [24–31]. Khan et al. [26] employed it for analyzing the nonlinear vibration of a rigid rod on a circular surface. Nonlinear oscillators with rational and irrational terms were studied by Yildirim et al. [27] by means of this approach. Xu [28,29] has employed the Hamiltonian approach for solving plasma physics equations and vibration analysis of a simple pendulum. Nonlinear oscillators with fractional power and nonlinear oscillations of a point charge in the electric field of a charged ring were investigated by Cveticanin et al. [30] and Yildirim et al. [31] respectively. Table 1 shows recent developments and applications of the Hamiltonian approach.

In this work, we apply and modify this method and also obtain the natural frequency of the Duffing equation, the nonlinear oscillator with discontinuity and the nonlinear oscillator with a quintic term with high accuracy. The solution procedure of this work demonstrates that this method is very simple and accurate for solving nonlinear equations.

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Table 1
Recent developments and applications of the Hamiltonian approach.

Nonlinear equations	Frequency response from the Hamiltonian approach
$\ddot{x}+\frac{c}{x}=0$	$\omega=\frac{\sqrt{2c}}{A}$
$\ddot{x}+\frac{x}{x^2+1}=0$	$\omega=\sqrt{\frac{\int_0^{\frac{\pi}{2}}\left\{\frac{\cos^2 t}{1+A^2\cos^2 t}\right\}}{\int_0^{\frac{\pi}{2}}\sin^2 t dt}}$
$\ddot{x}+\frac{x^3}{x^2+1}=0$	$\omega=\sqrt{\frac{\int_0^{\frac{\pi}{2}}\left\{\cos^2 t-\gamma\frac{\cos^2 t}{1+A^2\cos^2 t}\right\}}{\int_0^{\frac{\pi}{2}}\sin^2 t dt}}$
$\ddot{x}+x-\gamma\frac{x}{\sqrt{x^2+1}}=0$	$\omega=\sqrt{\frac{\int_0^{\frac{\pi}{2}}\left\{\cos^2 t-\gamma\frac{\cos^2 t}{\sqrt{1+A^2\cos^2 t}}\right\}}{\int_0^{\frac{\pi}{2}}\sin^2 t dt}}$
$\ddot{x}+\sum_i c_i^2 x\|x\|^{\alpha_i-1}=0$	$\omega_{\mathrm{HA}}=\sqrt{\frac{\sum_i c_i a_i A^{\alpha_i-1}}{a_0}}a_i=\int_0^{\frac{\pi}{2}}\cos^{\alpha_i+1}\theta d\theta=\frac{\pi}{2^{\alpha_i+1}(\alpha_i+1)B\left(\frac{\alpha_i+3}{2},\frac{\alpha_i+3}{2}\right)}$
$\ddot{x}+\frac{x}{(x^2+1)^{\frac{3}{2}}}=0$	$\omega_{\mathrm{HA}}=\sqrt{-\frac{1}{A}\frac{\partial}{\partial A}\left(\frac{1}{\sqrt{1+A^2}}\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{A^2}{1+A^2}\right)+\cdots\right\}\right)}$
$\ddot{x}+\frac{3}{4}x^2\ddot{x}+\frac{3}{4}x\dot{x}^2+\frac{3gx}{l}\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)=0$	$\omega=\sqrt{\frac{3g(192-72A^2+5A^4)}{l(192+72A^2)}}$
$\ddot{x}+\mathrm{sgn}(x)=0$	$\omega=1.128379/\sqrt{A}$
$\ddot{x}+\omega_0^2\sin x=0$	$\omega=\omega_0\sqrt{\frac{4\int_0^{\frac{\pi}{2}}\sin(A\cos\Omega)\cos\Omega d\Omega}{\pi A}}$

2. The solution procedure

2.1. Example 1

Consider the Duffing equation

$$\ddot{x}+x^3=0 \qquad x(0)=A, \qquad \dot{x}(0)=0. \tag{1}$$

Professor He applied the first-order Hamiltonian approach for solving Eq. (1) and obtained

$$\omega_{\mathrm{FAH}}=\sqrt{\frac{3}{4}A^2}. \tag{2}$$

We aim to obtain the natural frequency of Eq. (1) by using second-order and third-order Hamiltonian approaches.

2.1.1. The second-order Hamiltonian approach

Assume that the solution can be expressed as

$$x=a\cos(\omega t)+b\cos(3\omega t). \tag{3}$$

According to the initial condition,

$$A=a+b. \tag{4}$$

Its Hamiltonian can be easily obtained for Eq. (1); it reads

$$H=\frac{1}{2}\dot{x}^2+\frac{1}{4}x^4. \tag{5}$$

Integrating Eq. (5) with respect to time from 0 to $T/4$, we have

$$\tilde{H}(u)=\int_0^{\frac{T}{4}}\left(\frac{1}{2}\dot{x}^2+\frac{1}{4}x^4\right)dt. \tag{6}$$

Substituting Eqs. (3)–(6), we obtain

$$\begin{aligned}\tilde{H}(x)&=\int_0^{\frac{T}{4}}\left[\frac{1}{2}(a\omega\sin\omega t+3b\omega\sin 3\omega t)^2+\frac{1}{4}(a\cos\omega t+b\cos 3\omega t)^4\right]dt\\&=\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}\omega(a\sin t+3b\sin 3t)^2+\frac{1}{4\omega}(a\cos t+b\cos 3t)^4\right]dt\\&=\frac{\pi}{8}\omega\left(a^2+9b^2\right)+\frac{\pi}{64\omega}\left(3a^4+4a^3b+12a^2b^2+3b^4\right).\end{aligned} \tag{7}$$

Set

$$\frac{\partial}{\partial a}\left(\frac{\partial \tilde{H}}{\partial\left(1/\omega\right)}\right) = -\frac{\pi}{8}2a\omega^2 + \frac{\pi}{64}\left(12a^3 + 12a^2b + 24ab^2\right) = 0 \tag{8}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial \tilde{H}}{\partial\left(1/\omega\right)}\right) = -\frac{\pi}{8}18b\omega^2 + \frac{\pi}{64}\left(4a^3 + 24a^2b + 12b^3\right) = 0. \tag{9}$$

After some mathematical simplification and using MATLAB software, we achieve

$$a = 0.9571460091530A \tag{10}$$

$$b = 0.0428953990847A. \tag{11}$$

We have obtained the following frequency–amplitude relationship for the Duffing equation:

$$\omega_{\mathrm{SAH}} = \sqrt{0.720588473580A^2}. \tag{12}$$

The second approximate Hamiltonian approach provides accurate approximations to the exact frequency ω_{ex} for very large values of the oscillation amplitude and the relative error for ω_{SOH} is lower than 0.1961%.

2.1.2. *The third-order Hamiltonian approach*

Here we assume that the solution can be written as

$$x = a\cos\left(\omega t\right) + b\cos\left(3\omega t\right) + c\cos\left(5\omega t\right). \tag{13}$$

According to the initial condition,

$$A = a + b + c. \tag{14}$$

Substituting Eq. (13) to Eq. (6), we obtain

$$\begin{aligned}\tilde{H}\left(x\right) &= \int_0^{\frac{T}{4}}\left[\frac{1}{2}\omega\left(a\sin\omega t + 3b\sin 3\omega t + 5c\sin 5\omega t\right)^2 + \frac{1}{4\omega}\left(a\cos\omega t + b\cos 3\omega t + c\cos 5\omega t\right)^4\right]\mathrm{d}t\\ &= \int_0^{\frac{\pi}{2}}\left[\frac{1}{2}\omega\left(a\sin t + 3b\sin 3t + 5c\sin 5t\right)^2 + \frac{1}{4\omega}\left(a\cos t + b\cos 3t + c\cos 5t\right)^4\right]\mathrm{d}t\\ &= \frac{\pi}{8}\omega\left(a^2 + 9b^2 + 25c^2\right) + \frac{\pi}{64\omega}\left(3(a^4 + b^4 + c^4) + 12(b^2c^2 + ab^2c + a^2bc + a^2c^2 + a^2b^2) + 4a^3b\right).\end{aligned} \tag{15}$$

Set

$$\frac{\partial}{\partial a}\left(\frac{\partial \tilde{H}}{\partial\left(1/\omega\right)}\right) = -a\omega^2 + \frac{1}{4}\left(3a^3 + 3b^2c + 6abc + 3a^2b + 6ac^2 + 6ab^2\right) = 0 \tag{16}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial \tilde{H}}{\partial\left(1/\omega\right)}\right) = -18b\omega^2 + \frac{1}{2}\left(3b^3 + 6bc^2 + 6abc + 3ca^2 + a^3 + 6ba^2\right) = 0 \tag{17}$$

$$\frac{\partial}{\partial c}\left(\frac{\partial \tilde{H}}{\partial\left(1/\omega\right)}\right) = -50c\omega^2 + \frac{1}{2}\left(6cb^2 + 3ab^2 + 3ba^2 + 6ca^2 + 3c^3\right) = 0. \tag{18}$$

After some mathematical simplification and using MATLAB software, we obtain

$$a = 0.955091126192990848A \tag{19}$$

$$b = 0.043051911447186058A \tag{20}$$

$$c = 0.00185696235982A. \tag{21}$$

We have obtained the following frequency–amplitude relationship:

$$\omega_{\mathrm{TAH}} = \sqrt{0.71789616293340A^2}. \tag{22}$$

The third approximate Hamiltonian approach provides more accurate approximations to the exact frequency ω_{ex} for very large values of the oscillation amplitude, and the relative error for ω_{TOH} is lower than 0.0088%. The accuracies of the solutions for each of the approximations are shown in Figs. 1–3 for small and large values of the initial amplitude.

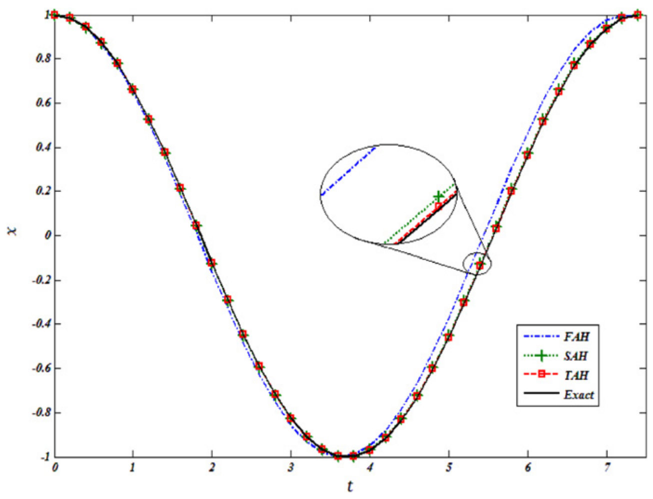

Fig. 1. Comparison between the solutions obtained and the exact one for $A = 0.1$.

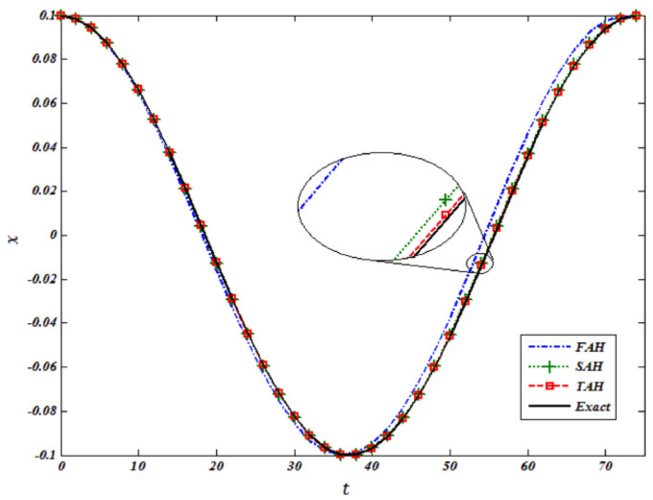

Fig. 2. Comparison between the solutions obtained and the exact one for $A = 1$.

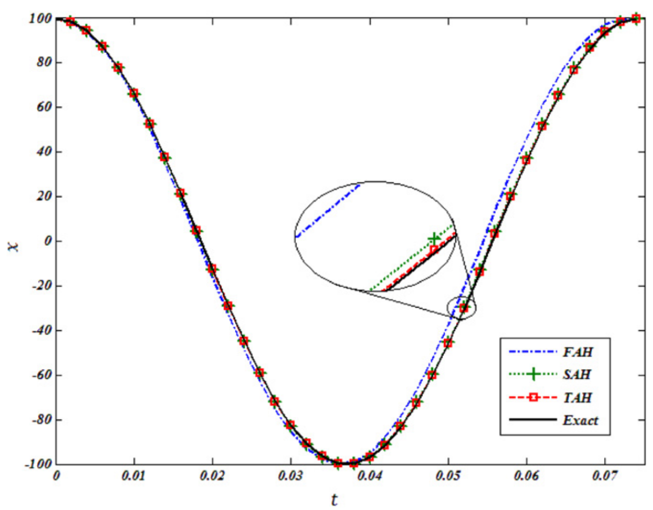

Fig. 3. Comparison between the solutions obtained and the exact one for $A = 100$.

2.2. Example 2

In this section the following nonlinear oscillator with discontinuity is analyzed [32]:

$$\ddot{x} + x\,|x| = 0, \qquad x\,(0) = A \quad \text{and} \quad \dot{x}\,(0) = 0. \tag{23}$$

2.2.1. The first-order Hamiltonian approach

The Hamiltonian of this equation is constructed as

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{3}x^2\,|x|\,. \tag{24}$$

Assume the first approximate solution of Eq. (23) as

$$x = A\cos\omega t. \tag{25}$$

Then

$$\tilde{H}(x) = \int_0^{\frac{T}{4}} \left(\frac{1}{2}\dot{x}^2 + \frac{1}{3}x^3\right) \mathrm{d}t + \int_{\frac{T}{4}}^{\frac{T}{2}} \left(\frac{1}{2}\dot{x}^2 - \frac{1}{3}x^3\right) \mathrm{d}t. \tag{26}$$

Putting the solution assumed into Eq. (26), we have

$$\begin{aligned}\tilde{H}\,(x) &= \int_0^{\frac{T}{4}} \left(\frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{1}{3}A^3\cos^3\omega t\right)\mathrm{d}t + \int_{\frac{T}{4}}^{\frac{T}{2}} \left(\frac{1}{2}A^2\omega^2\sin^2\omega t - \frac{1}{3}A^3\cos^3\omega t\right)\mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}A^2\omega\sin^2 t + \frac{1}{3\omega}A^3\cos^3 t\right)\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} \left(\frac{1}{2}A^2\omega\sin^2 t - \frac{1}{3\omega}A^3\cos^3 t\right)\mathrm{d}t \\ &= \frac{\pi}{4}\omega A^2 + \frac{4}{9\omega}A^3. \end{aligned} \tag{27}$$

Set

$$\frac{\partial}{\partial A}\left(\frac{\partial\tilde{H}}{\partial\left(\frac{1}{\omega}\right)}\right) = -\frac{\pi}{2}A\omega^2 + \frac{4}{3}A^2. \tag{28}$$

So, the first approximate frequency can be obtained as

$$\omega_{\mathrm{FAH}} = \sqrt{\frac{8}{3\pi}A}. \tag{29}$$

Here, the relative error of the solution for large amplitudes is about 0.7255%.

2.2.2. The second-order Hamiltonian approach

For the second-order Hamiltonian approach we consider the following equation as the response of the system:

$$x = a\cos\omega t + b\cos 3\omega t, \quad a + b = A. \tag{30}$$

Substituting the above equation into Eq. (26), we have

$$\begin{aligned}\tilde{H}\,(u) &= \int_0^{\frac{T}{4}} \left(\frac{1}{2}\omega^2\left(a\sin\omega t + 3b\sin 3\omega t\right)^2 + frac13\left(a\cos\omega t + b\cos 3\omega t\right)^3\right)\mathrm{d}t \\ &\quad + \int_{\frac{T}{4}}^{\frac{T}{2}} \left(\frac{1}{2}\omega^2\left(a\sin\omega t + 3b\sin 3\omega t\right)^2 - \frac{1}{3}\left(a\cos\omega t + b\cos 3\omega t\right)^3\right)\mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\omega\left(a\sin t + 3b3\sin 3t\right)^2 + \frac{1}{3\omega}\left(a\cos t + b\cos 3t\right)^3\right)\mathrm{d}t \\ &\quad + \int_{\frac{\pi}{2}}^{\pi} \left(\frac{1}{2}\omega\left(a\sin t + 3b\sin 3t\right)^2 - \frac{1}{3\omega}\left(a\cos t + b\cos 3t\right)^3\right)\mathrm{d}t \\ &= \frac{\pi}{4}\omega\left(a^2 + 9b^2\right) + \frac{1}{3\omega}\left(\frac{4}{3}a^3 + \frac{4}{5}a^2 b + \frac{108}{35}ab^2 - \frac{4}{9}b^3\right). \end{aligned} \tag{31}$$

Set

$$\frac{\partial}{\partial a}\left(\frac{\partial \tilde{H}}{\partial\left(\frac{1}{\omega}\right)}\right) = -\frac{\pi}{2}a\omega^2 + \frac{4}{3}a^2 + \frac{8}{15}ab + \frac{36}{35}b^2 = 0 \tag{32}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial \tilde{H}}{\partial\left(\frac{1}{\omega}\right)}\right) = -\frac{9\pi}{2}b\omega^2 + \frac{4}{15}a^2 + \frac{75}{35}ab - \frac{4}{9}b^2 = 0. \tag{33}$$

After some mathematical simplification, the frequency amplitude relationship and the constant values of a and b can be eventually obtained as

$$a = 0.97424661242708A \tag{34}$$

$$b = 0.002575338757292A \tag{35}$$

$$\omega_{\mathrm{SAH}} = 0.91441568459242\sqrt{A}. \tag{36}$$

For very large values of the oscillation amplitude the relative error for ω_{SAH} is lower than 0.02905%.

2.2.3. The third-order Hamiltonian approach

For the third-order Hamiltonian approach we assume Eq. (13) as the trial function. Substituting Eq. (13) into Eq. (26) leads to

$$\begin{aligned}\tilde{H}(x) = & \int_0^{\frac{T}{4}}\left\{\frac{1}{2}\omega^2(a\sin\omega t + 3b\sin 3\omega t + 5c\sin 5\omega t)^2 + \frac{1}{3}(a\cos\omega t + b\cos 3\omega t + c\cos 5\omega t)^3\right\}\mathrm{d}t \\ & + \int_{\frac{T}{4}}^{\frac{T}{2}}\left\{\frac{1}{2}\omega^2(a\sin\omega t + 3b\sin 3\omega t + 5c\sin 5\omega t)^2 - \frac{1}{3}(a\cos\omega t + b\cos 3\omega t + c\cos 5\omega t)^3\right\}\mathrm{d}t \\ = & \int_0^{\frac{\pi}{2}}\left\{\frac{1}{2}\omega(a\sin t + 3b\sin 3t + 5c\sin 5t)^2 + \frac{1}{3\omega}(a\cos t + b\cos 3t + c\cos 5t)^3\right\}\mathrm{d}t \\ & + \int_{\frac{\pi}{2}}^{\pi}\left\{\frac{1}{2}\omega(a\sin t + 3b\sin 3t + 5c\sin 5t)^2 - \frac{1}{3}(a\cos t + b\cos 3t + c\cos 5t)^3\right\}\mathrm{d}t \\ = & \frac{\pi}{4}\omega\left(a^2 + 9b^2 + 25c^2\right) + \frac{2}{3\omega}\left\{\frac{2}{3}a^3 + \frac{2}{5}a^2b + \frac{54}{35}ab^2 - \frac{2}{9}b^3 - \frac{2}{35}a^2c\right. \\ & \left. + \frac{20}{21}abc + \frac{50}{33}ac^2 + \frac{54}{55}cb^2 - \frac{50}{91}bc^2 + \frac{2}{15}c^3\right\}.\end{aligned} \tag{37}$$

Set

$$\frac{\partial}{\partial a}\left(\frac{\partial \tilde{H}}{\partial\left(\frac{1}{\omega}\right)}\right) = -\frac{\pi}{4}a\omega^2 + \frac{2}{3}a^2 + \frac{4}{15}ab + \frac{18}{35}b^2 - \frac{4}{105}ac + \frac{20}{63}bc + \frac{50}{99}c^2 = 0 \tag{38}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial \tilde{H}}{\partial\left(\frac{1}{\omega}\right)}\right) = -\frac{9\pi}{4}b\omega^2 + \frac{2}{15}a^2 + \frac{36}{35}ab - \frac{2}{9}b^2 + \frac{36}{55}bc + \frac{20}{63}ac - \frac{50}{273}c^2 = 0 \tag{39}$$

$$\frac{\partial}{\partial c}\left(\frac{\partial \tilde{H}}{\partial\left(\frac{1}{\omega}\right)}\right) = -\frac{25\pi}{4}c\omega^2 - \frac{2}{105}a^2 + \frac{20}{63}ab + \frac{18}{55}b^2 - \frac{100}{273}bc + \frac{100}{99}ac + \frac{2}{15}c^2 = 0. \tag{40}$$

Consequently we can obtain

$$a = 0.97491406508A \tag{41}$$

$$b = 0.0257281322963A \tag{42}$$

$$c = -0.0006421973766A. \tag{43}$$

And the natural frequency obtained by the third approximation is

$$\omega_{\mathrm{TAH}} = 0.9147335203935\sqrt{A}. \tag{44}$$

For very large values of the oscillation amplitude the relative error for ω_{TAH} is lower than 0.00571%. So it is obvious that higher orders reach better results. The accuracies of the solutions for each of the approximations are shown in Figs. 4–6 for small and large values of the initial amplitude.

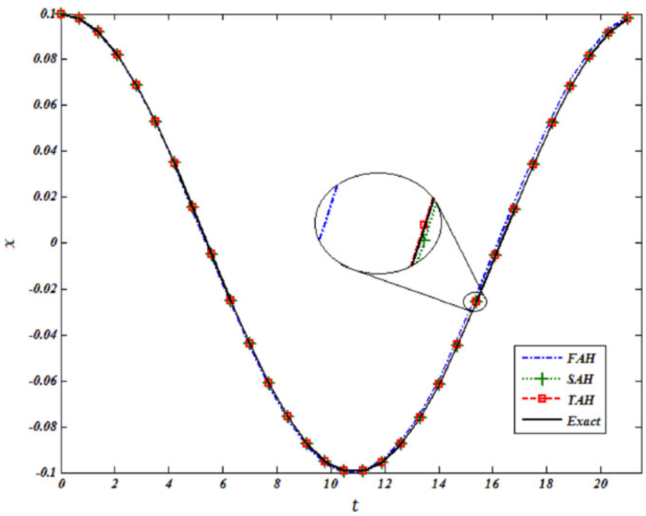

Fig. 4. Comparison between the solutions obtained and the exact one for $A = 0.1$.

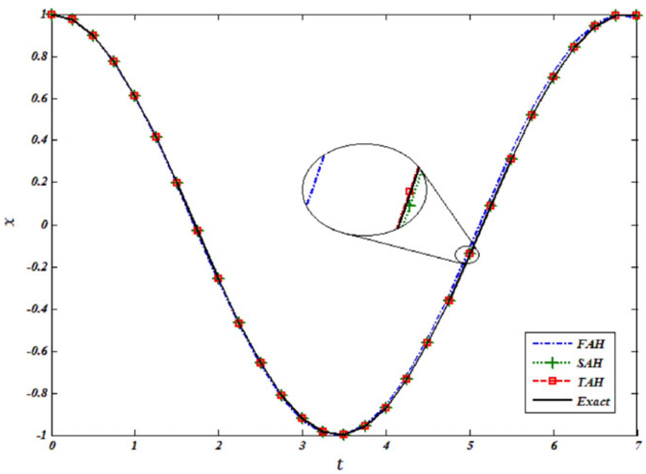

Fig. 5. Comparison between the solutions obtained and the exact one for $A = 1$.

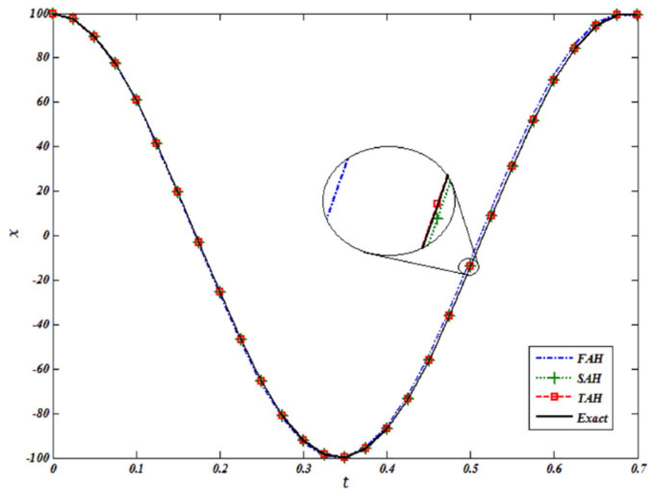

Fig. 6. Comparison between the solutions obtained and the exact one for $A = 100$.

2.3. Example 3

Consider the nonlinear oscillator with a quintic term [33]

$$\ddot{x} + x + x^5 = 0 \qquad x(0) = A, \qquad \dot{x}(0) = 0. \tag{45}$$

2.3.1. The first-order Hamiltonian approach

For the first approximate of the Hamiltonian approach, assume Eq. (25) as a trial function. Its Hamiltonian can be easily obtained for Eq. (45); it reads

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{6}x^6. \tag{46}$$

Integrating Eq. (46) with respect to time from 0 to $T/4$, we have

$$\tilde{H}(x) = \int_0^{\frac{T}{4}} \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{6}x^6\right) \mathrm{d}t. \tag{47}$$

Substituting Eqs. (25)–(47), we obtain

$$\tilde{H}(x) = \int_0^{\frac{T}{4}} \left[\frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{1}{2}A^2 \cos^2 \omega t + \frac{1}{6}A^6 \cos^6 \omega t\right] \mathrm{d}t \tag{48}$$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}A^2\omega \sin^2 t + \frac{1}{2\omega}A^2 \cos^2 t + \frac{1}{6\omega}A^6 \cos^6 t\right] \mathrm{d}t \\ &= \frac{\pi}{8}A^2\omega + \frac{\pi}{8\omega}A^2 + \frac{5\pi}{192\omega}A^6. \end{aligned} \tag{49}$$

Set

$$\frac{\partial}{\partial A}\left(\frac{\partial \tilde{H}}{\partial (1/\omega)}\right) = -\frac{\pi}{4}A\omega^2 + \frac{\pi}{4}A + \frac{5\pi}{32}A^5 = 0. \tag{50}$$

Finally

$$\omega_{\mathrm{FAH}} = \sqrt{1 + \frac{5}{8}A^4}. \tag{51}$$

2.3.2. The second-order Hamiltonian approach

In this case, by substituting Eqs. (3)–(47), we obtain

$$\begin{aligned} \tilde{H}(x) &= \int_0^{\frac{T}{4}} \left[\frac{1}{2}(a\omega \sin \omega t + 3b\omega \sin 3\omega t)^2 + \frac{1}{2}(a\cos \omega t + b\cos 3\omega t)^2 + \frac{1}{6}(a\cos \omega t + b\cos 3\omega t)^6\right] \mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}\omega (a\sin t + 3b\sin 3t)^2 + \frac{1}{2\omega}(a\cos t + b\cos 3t)^2 + \frac{1}{6\omega}(a\cos t + b\cos 3t)^6\right] \mathrm{d}t \\ &= \frac{\pi}{8}\omega\left(a^2 + 9b^2\right) + \frac{\pi}{8\omega}\left(a^2 + b^2\right) + \frac{5\pi}{192\omega}\left(a^6 + 3a^5b + 9a^4b^2 + 6a^3b^3 + 9a^2b^4 + b^6\right). \end{aligned} \tag{52}$$

Set

$$\frac{\partial}{\partial a}\left(\frac{\partial \tilde{H}}{\partial (1/\omega)}\right) = -\frac{\pi}{8}2a\omega^2 + \frac{\pi}{8\omega}2a + \frac{5\pi}{192\omega}\left(6a^5 + 15a^4b + 36a^3b^2 + 18a^2b^3 + 18ab^4\right) = 0 \tag{53}$$

$$\frac{\partial}{\partial b}\left(\frac{\partial \tilde{H}}{\partial (1/\omega)}\right) = -\frac{\pi}{8}18b\omega^2 + \frac{\pi}{8\omega}2b + \frac{5\pi}{192\omega}\left(3a^5 + 18a^4b + 18a^3b^2 + 36a^2b^3 + 6b^5\right) = 0. \tag{54}$$

Thus, the value of the natural frequency can be obtained by some mathematical simplifications. Numerical values of frequencies obtained by the first-order and second-order Hamiltonian approaches and the relative errors are shown in Table 2 for a range of initial amplitudes. It is seen that the second-order results are much closer to the exact ones than the first-order results.

Table 2
Comparison between the frequency obtained and the exact one for some values of A.

A	ω_{FAH} (Relative error, %)	ω_{SAH} (Relative error, %)	ω_{ex}
0.1	1.0000312495 (0.000336431791976)	1.0000288337 (0.000094856293736)	1.0000278851
0.5	1.0193441519 (0.202847005609007)	1.0178618081 (0.057130698240214)	1.0172806286
1	1.2747548784 (2.135393404306345)	1.2556602111 (0.605498213683614)	1.2481029699
5	19.7895174272 (5.839820494782060)	19.0112231014 (1.677286868708726)	18.6976105351
10	79.0632658066 (5.855088549884582)	75.9462023930 (1.681759503938878)	74.6900946281
100	7905.6942136665 (5.856109159419318)	7593.9619158418 (1.682058499832310)	7468.3400669493
500	197642.3537630536 (5.856109261345213)	189849.0463026337 (1.682058556472065)	186708.500002678

3. Conclusion

The higher orders of the Hamiltonian approach for obtaining better approximate solutions for the Duffing equation, the nonlinear oscillator with discontinuity and the nonlinear oscillator with a quintic term were introduced. The accuracy and validity of the solutions obtained have been examined by comparing against the exact ones as regards time histories and in a table. We demonstrated that this approach is very accurate and simple for solving nonlinear equations. Finally, we conclude that the Hamiltonian approach is a method with good capabilities for solving nonlinear conservative oscillatory systems.

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