ELSEVIER

Discrete Mathematics 161 (1996) 323–327

DISCRETE MATHEMATICS

Note

Characterization of self-complementary symmetric digraphs

Hong Zhang

Department of Mathematical Sciences, Indiana University-Purdue University at Fort Wayne, Fort Wayne, IN 46805, USA

Received 21 October 1992; revised 28 May 1995

Abstract

The class of self-complementary symmetric digraphs is characterized and it is shown that the number of vertices of such a digraph is an odd prime power.

1. Introduction

The digraphs we consider here are finite and simple. A digraph D is said to be *vertex transitive* if its automorphism group G acts transitively on the vertex set of the digraph. The automorphism group also induces a permutation group on the set of directed edges. A digraph is called *edge transitive* if for any two directed edges (a, b) and (c, d) there exists an automorphism α such that $\alpha a = c$, $\alpha b = d$. A digraph is *symmetric* if it is both vertex transitive and edge transitive. A digraph D is called *self-complementary* if it is isomorphic to its complement D^c. Similarly, an undirected graph is said to be symmetric if its automorphism group acts transitively on the vertex set the edge set and it is said to be self-complementary if it is isomorphic to its complement.

In this paper we consider the class of digraphs which are both self-complementary and symmetric. In [4], an algebraic characterization of the self-complementary symmetric graphs was given. In particular, it is shown that there exists a self-complementary symmetric graph on v vertices if and only if $v = p^n \equiv 1 \pmod 4$, where p is a prime. We shall extend the results to self-complementary symmetric digraphs and show that there exists such a digraph on v vertices if and only if v is a power of an odd prime.

We first establish some notation.

Let G be a group and H be a subset of G. The *Cayley digraph* $D(G, H)$ is defined as the digraph with vertex set $V = G$ and edge set $E = \{(a, b)\colon a^{-1}b \in H\}$.

0012-365X/96/$15.00
SSDI 0012-365X(95)00236-7

The in-degree and out-degree of a regular digraph is denoted by δ^- and δ^+, respectively.

There is a natural bijection between the class of undirected graphs and the class of directed graphs whose edges always occur in pairs oriented in opposite directions. To each undirected graph $X = (V, E)$ is associated a directed graph $X' = (V, E')$ with $E' = \{(a, b), (b, a): ab \in E\}$. This mapping is clearly invertible.

Let G be a permutation group on a set D and a be an element of D. The *stabilizer* of G at a is defined as $G_a = \{\theta \in G: \theta a = a\}$.

A permutation group G acting on a set D is said to be *doubly transitive* if for any two ordered pairs (a, b) and (c, d) in D, where $a \neq b$ and $c \neq d$, there exists an $\alpha \in G$ such that $\alpha a = c$, $\alpha b = d$. A permutation group G is said to be *doubly homogeneous*, if for any two unordered pairs $\{a, b\}$ and $\{c, d\}$ in D, where $a \neq b$ and $c \neq d$, there exists an $\alpha \in G$ such that $\{\alpha a, \alpha b\} = \{c, d\}$.

Let V be an n dimensional vector space over a finite field F with q elements. The group of all invertible linear transformations on V is called the *general linear group* and is denoted by $GL(V)$ or $GL(n, q)$.

Let p be a prime and F the finite field with p^n elements. A *semilinear transformation* on F is a mapping of the form $x \to ax^\sigma + b$, where a and b are elements of F, $a \neq 0$, and σ is a field automorphism of F. The set of all semilinear transformations on the field F of p^n elements is denoted by $T(p^n)$.

2. Construction

A family of self-complementary symmetric digraphs can be constructed based on finite fields. The construction is similar to those in [2, 4].

Let p be an odd prime, n a positive integer, and let F be the finite field with p^n elements. It is well-known that the multiplicative group F^* is cyclic. Let H be the subgroup of F^* of index 2 and let F_+ be the additive group of group of F. Then Cayley digraphs $D(F_+, H)$ is called the *Paley digraph* and is denoted by $P(p^n)$.

Theorem 1. *The Paley digraph $P(p^n)$ is self-complementary and symmetric.*

Proof. Since $P(p^n)$ is a Cayley graph, it is vertex transitive.

Let $\alpha \in H$. Multiplication by α of the elements of F induces an automorphism of $P(p^n)$. Hence, H is a subgroup of the automorphism group of $P(p^n)$. Let (a, b) be any edge of $P(p^n)$. Then $b - a \in H$. The addition of $-a$ maps the edge to $(0, b - a)$ which is mapped to $(0, 1)$ by the multiplication by $(b - a)^{-1}$. Hence, D is edge transitive.

Since $[F^* : H] = 2$, $F^* = H \cup \theta H$. For any edge (a, b) of $P(p^n)$, $b - a \in H$ and $\theta(b - a) \notin H$. Hence, $(\theta a, \theta b)$ is an edge of $P(p^n)^c$. Therefore, θ is an isomorphism from $P(p^n)$ to $P(p^n)^c$. Hence, $P(p^n)$ is self-complementary and symmetric. $\square$

The number of vertices of the graph we constructed above is an odd prime power p^n. We shall show that the condition is necessary for self-complementary symmetric digraphs.

3. Automorphism group

The family of self-complementary symmetric digraphs can be divided into two classes as shown in the following lemma.

Lemma 1. *Let D be a self-complementary symmetric digraph with v vertices. Then*

(1) *v is odd.*

(2) *If $v \equiv 1 \pmod 4$, then there is an undirected self-complementary symmetric graph X such that D is the corresponding digraph of X under the bijection defined in Section* 1, i.e., $X' = D$.

(3) *If $v \equiv 3 \pmod 4$, then D is a tournament.*

Proof. Since D is vertex transitive it is regular. The digraph D being self-complementary implies that $\delta^- = \delta^+ = (v-1)/2$. Hence, v must be odd. By edge transitivity, either all adjacent vertices are joined by edges in both directions or none are. Hence, D is either a tournament or corresponds to an undirected graph, which must also be self-complementary and symmetric.

If $v \equiv 1 \pmod 4$, then δ^+ is even. Since D is edge transitive the order of its automorphism group G is even. The digraph cannot be a tournament since the automorphism group of a tournament has an odd order. Hence, D corresponds to an undirected self-complementary symmetric graph.

If $v \equiv 3 \pmod 4$, the the number of edges $v(v-1)/2$ is odd. Hence, D cannot correspond to an undirected graph and it must be a tournament. □

The symmetry requirements of the self-complementary symmetric digraphs imply certain transitivity conditions on their automorphism groups. Obviously the automorphism group of such a digraph cannot be doubly transitive. However, we can show that the automorphism group is a normal subgroup of a doubly transitive group.

Theorem 2. *Let G be the group of automorphisms of a self-complementary symmetric digraph D. Then there is a doubly transitive permutation group $\bar{G}$ acting on the vertex set of D such that G is a normal subgroup of $\bar{G}$ with index 2. Furthermore, if $v \equiv 3 \pmod 4$, then G is doubly homogeneous.*

Proof. Since D is self-complementary, there is an isomorphism θ from D to D^c. Let $\bar{G} = \langle G, \theta \rangle$. Clearly $\theta^2 \in G$ and $\theta^{-1} G \theta = G$. Hence, $\bar{G} = G \cup \theta G$ and G is a normal subgroup of $\bar{G}$ with index 2.

Let (a, b) and (c, d) be any two ordered pairs of distinct vertices of D. If they are both edges of D or both edges of D^c then since D and D^c are symmetric, there exists an $\alpha \in G$ such that $(\alpha a, \alpha b) = (c, d)$. If one of them is an edge of D and the other an edge of D^c, then $(\theta a, \theta b)$ and (c, d) will both be edges of either D or D^c. Hence there is an $\alpha \in G$ such that $(\alpha\theta a, \alpha\theta b) = (c, d)$. Since, $\alpha\theta \in \bar{G}$, $\bar{G}$ is a doubly transitive permutation group on $V(D)$.

If $v \equiv 3 \pmod 4$, then by Lemma 1, D is tournament. The edge transitivity of D clearly implies G is doubly homogeneous. □

A result of Burnside [1] shows that a doubly transitive group has a unique minimal normal subgroup which is either elementary abelian or simple. We can show that in our case, the doubly transitive group $\bar{G}$ has an elementary abelian group as its minimal normal subgroup and $\bar{G}_0$ can be considered as a subgroup of $GL(n, p)$.

Theorem 3. *Let $D = (V, E)$ be a self-complementary symmetric digraph with v vertices and G be the automorphism group. Let $\bar{G}$ be the doubly transitive group on V defined in Theorem 2. Then the minimal normal subgroup of $\bar{G}$ is elementary abelian. The vertex set V of D can be identified with an n-dimensional vector space over a field of p elements for some odd prime p such that the minimal normal subgroup of $\bar{G}$ is the group of all translations on the vector space and $\bar{G}_0$ is a subgroup of $GL(n, p)$. If $v \equiv 3 \pmod 4$, then G is a subgroup of $T(v)$.*

Proof. If $v \equiv 1 \pmod 4$, D corresponds to an undirected self-complementary symmetric graph by Lemma 1. The statement follows from Theorem 3 in [4].

If $v \equiv 3 \pmod 4$, then G is doubly homogeneous by Theorem 2. In [3], Kantor proved that a doubly homogeneous group which is not doubly transitive must be a subgroup of $T(v)$. □

4. Characterization

Based on the information of the automorphism groups, we give an algebraic characterization of the self-complementary symmetric digraphs.

Theorem 4. *A digraph is self-complementary symmetric if and only if it is isomorphic to a Cayley digraph $D(V_+, O_H)$, where*

(1) V_+ is the additive group of the vector space V of dimension n over the finite field with p elements, where p is an odd prime.

(2) O_H is an orbit of a group H, $H \subset \bar{H} \subset GL(V)$, $[\bar{H} : H] = 2$, $\bar{H}$ is transitive on $V \backslash \{0\}$, and H is not transitive on $V \backslash \{0\}$.

Furthermore, if the number of vertices of D is v and $v \equiv 3 \pmod 4$, then D is isomorphic to the Paley digraph $P(v)$.

Proof. Let D be a self-complementary symmetric digraph and G be its group of automorphisms. By Theorem 3, the vertex set $V(D)$ can be identified with a vector space V. Denote the group of translations on the vector space by R. Then R is the minimal normal subgroup of $\bar{G}$ and $\bar{G}_0 \subset GL(V)$. Clearly, R is regular and $\bar{G} = R\bar{G}_0$. Since G is transitive on V, R is also a subgroup of G and $G = RG_0$. Let $H = G_0$ and $\bar{H} = \bar{G}_0$. Then $H \subset \bar{H} \subset GL(V)$ and $[\bar{H}:H] = 2$. Let $O_H = \{x\colon (0, x)$ is an edge of $D\}$. Since G_0 is transitive on O_H, it is easy to see that O_H is an orbit of H and D is the Cayley digraph $D(V_+, O_H)$.

Conversely, it is clear that the automorphism group G of the digraph $D(V_+, O_H)$ contains all translations of V and the group H. Since $[\bar{H}:H] = 2$, $\bar{H} = H \cup \theta H$ for some $\theta \in GL(V)$. Let a be a fixed element of O_H. Since $\bar{H}$ is transitive on $V \backslash \{0\}$, $V \backslash \{0\} = \bar{H}(a) = H(a) \cup \theta H(a)$. It follows that H has two orbits $H(a)$, $\theta H(a)$ on $V \backslash \{0\}$, and $O_H = H(a)$. Hence $D(V_+, O_H)$ is both vertex transitive and edge transitive. Clearly, the mapping θ induces an isomorphism from $D(V_+, O_H)$ to its complement. Hence, $D(V_+, O_H)$ is self-complementary and symmetric.

If $v \equiv 3 \pmod 4$, then G is a subgroup of $T(v)$ by Theorem 3. An automorphism $\alpha \in G_0$ has the form $\alpha(x) = ax^{\sigma}$. We claim that a is a square. If a is a nonsquare, an odd power of α has the form $\alpha^k(x) = a'x^{\sigma'}$, where a' is again a nonsquare. Hence, the order of α is even. This is a contradiction since D is a tournament by Lemma 1 and the order of G must be odd. Now it is clear that two orbits of G_0 on $D \backslash \{0\}$ are the set of squares and the set of nonsquares. Hence, D is isomorphic to the Paley graph $P(v)$. □

Corollary. *There exists a self-complementary symmetric digraph on v vertices if and only if $v = p^n$, where p is an odd prime.*

Proof. By Theorem 4, the number of vertices v of a self-complementary symmetric graph must be p^n. Conversely for every number v satisfying the condition, by the construction in Section 2, there exists a self-complementary symmetric digraph on v vertices. □

Acknowledgements

The author wishes to thank the referees for their helpful suggestions.

References

[1] W. Burnside, Theory of Groups of Finite Order (Cambridge Univ. Press, Cambridge, 1911).
[2] C.Y. Chao, On the classification of symmetric graphs with a prime number of vertices, Trans. Amer. Math. Soc. 158 (1971) 247–256.
[3] W.M. Kantor, Automorphism groups of designs, Math. Z. 109 (1969) 246–252.
[4] H. Zhang, Self-complementary symmetric graphs, J. Graph Theory 16-1 (1992) 1–5.