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Mellin–Barnes integrals as Fourier–Mukai transforms

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Abstract

We study the generalized hypergeometric system introduced by Gelfand, Kapranov and Zelevinsky and its relationship with the toric Deligne–Mumford (DM) stacks recently studied by Borisov, Chen and Smith. We construct series solutions with values in a combinatorial version of the Chen–Ruan (orbifold) cohomology and in the K -theory of the associated DM stacks. In the spirit of the homological mirror symmetry conjecture of Kontsevich, we show that the K -theory action of the Fourier–Mukai functors associated to basic toric birational maps of DM stacks are mirrored by analytic continuation transformations of Mellin–Barnes type.

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1. Introduction

Let $\mathcal{A} = \{v_1, \dots, v_n\}$ be a collection of elements of the lattice $N \cong \mathbb{Z}^d$. We assume that the elements of \mathcal{A} generate the lattice as an abelian group, and that there exists a group homomorphism $h: N \rightarrow \mathbb{Z}$ such that $h(v) = 1$ for any element $v \in \mathcal{A}$. Let $\mathbb{L} \subset \mathbb{Z}^n$ denote the lattice of integral relations among the elements of \mathcal{A} consisting of vectors $l = (l_j) \in \mathbb{Z}^n$ such that $l_1 v_1 + \dots + l_n v_n = 0$.

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Let $\beta \in N$ be a lattice element. The *Gelfand–Kapranov–Zelevinsky hypergeometric system* (GKZ) associated to the set \mathcal{A} and parameter β is a system of differential equations on the function $\Phi(z)$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, consisting of the binomial equations

$$\left(\prod_{j, l_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{l_j} - \prod_{j, l_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-l_j} \right) \Phi = 0, \quad l \in \mathbb{L},$$

and the linear equations

$$\left(-\beta + \sum_{j=1}^n v_j z_j \frac{\partial}{\partial z_j} \right) \Phi = 0.$$

Note that it is enough to consider a finite set of binomial equations determined by a set of generators of the lattice of relations \mathbb{L} .

Gelfand, Kapranov and Zelevinsky [12] showed that this system is holonomic, so the number of solutions at a generic point is finite. They constructed explicit solutions of the system in the form of the so-called Gamma series

$$\Phi(z, \lambda) := \sum_{l \in \mathbb{L}} \prod_{j=1}^n \frac{z_j^{l_j + \lambda_j}}{\Gamma(l_j + \lambda_j + 1)},$$

where $\lambda \in \mathbb{C}^n$ is a parameter with the property that $\lambda_1 v_1 + \dots + \lambda_n v_n = \beta$. Moreover, they discovered that there is a very close connection between the regular triangulations of the polytope $\Delta = \text{Conv}(\mathcal{A})$, as described by the secondary polytope of \mathcal{A} , and the structure of the solution set.

In the context of mirror symmetry, Batyrev [3] noticed that a special case of the GKZ system is satisfied by the periods describing the variations of complex structures of Calabi–Yau hypersurfaces in toric varieties. Aspinwall, Greene and Morrison [2] used the combinatorial GKZ machinery to analyze the string theoretic phase transitions in type II string theory. The homological mirror symmetry conjecture of Kontsevich [17] provided a far reaching generalization of the earlier ideas in mirror symmetry. As further evidence for his proposal, Kontsevich [18] conjectured that the action on cohomology of the group of self-equivalences of the bounded derived category of coherent sheaves on a smooth projective Calabi–Yau variety matches the monodromy action on the cohomology of the mirror Calabi–Yau variety associated to the variations of complex structures. In the toric context, this strategy has been pursued by one of the authors in [14]. The broad goal of the current work is to offer a framework for the aforementioned ideas based on the notion of a toric Deligne–Mumford stack introduced by Borisov, Chen and Smith [8].

Sections 2 and 3 provide a geometric approach to the problem of constructing convergent Gamma series solutions to the GKZ system corresponding to a general regular triangulation of the polytope $\Delta = \text{Conv}(\mathcal{A})$. For an extensive list of works that investigate the properties of the GKZ system and the associated \mathcal{D} -module, the bibliography of the book by Saito, Sturmfels and Takayama [20] is the best resource. Our approach is closest in spirit to the methods employed by Hosono, Lian and Yau [15] and Stienstra [21] for the case of unimodular triangulations which, for some time, has been the preferred testing ground for mirror symmetry computations. However, homological mirror symmetry and the advent of D-branes in string theory have sparked renewed

interest in understanding the intricacies of the general situation. What we show is that the Chen–Ruan (orbifold) cohomology and the K -theory of the toric Deligne–Mumford stack associated to a general regular triangulation of the convex polytope Δ are the natural missing ingredients in the geometric construction of the solution set to the GKZ system.

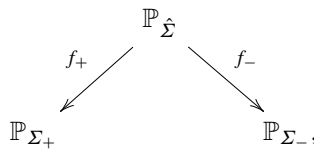
In particular, for a fan Σ supported on the cone $K = \mathbb{R}_{\geq 0}\Delta$ induced by a regular triangulation of the polytope Δ , and an arbitrary $\beta \in N$, we explicitly construct at least $\text{Vol}(\Delta)$ cohomology valued linearly independent Gamma series solutions in a certain complex domain U_Σ in \mathbb{C}^n associated to the fan Σ (Corollary 2.21(i)). When $\beta \in -K^\circ$, which is the case of mirror symmetry for Calabi–Yau complete intersections in projective toric varieties, the second part of the same corollary provides a full system of $\text{Vol}(\Delta)$ linearly independent solutions. Furthermore, Corollary 3.8 provides a mirror symmetry map defined on the dual of the Grothendieck ring of the toric DM stack \mathbb{P}_Σ

$$MS_\Sigma : (K_0(\mathbb{P}_\Sigma, \mathbb{C}))^\vee \rightarrow \text{Sol}(U_\Sigma)$$

that produces GKZ solutions which are analytic in the complex domain U_Σ in \mathbb{C}^n associated to the fan Σ .

Our K -theoretic interpretation makes essential use of results contained in the companion paper [6], where, among other things, a Stanley–Reisner type description of the Grothendieck K -theory ring of a smooth DM stack is given. It is well known that there are many subtleties involved in trying to determine the dimension of the solution set of the GKZ system (see, for example, Adolphson [1], Saito, Sturmfels and Takayama [20], Cattani, Dickenstein and Sturmfels [9], Matusevich, Miller and Walther [19]). Our methods raise the interesting issue of finding the proper K -theoretic framework for constructing GKZ solutions for general values of the parameter $\beta \in N$.

In Sections 4 and 5, we employ a combination of analytic, algebra-geometric and combinatorial methods and justify the title of the paper. We consider Σ_+ and Σ_- two fan structures induced by two regular triangulations of the polytope Δ that are joined by an edge of the secondary polytope determined by \mathcal{A} . It follows that the associated toric DM stacks \mathbb{P}_{Σ_+} and \mathbb{P}_{Σ_-} are birationally equivalent, and we have a diagram of weighted blowdowns



where $\hat{\Sigma}$ is a stacky refinement of the fans Σ_\pm . According to Bondal and Orlov [4], in the smooth fan case, and Kawamata [16], in the stacky case, the map $FM : D^b(\mathbb{P}_{\Sigma_-}) \rightarrow D^b(\mathbb{P}_{\Sigma_+})$ between the bounded derived categories of coherent sheaves of the associated DM stacks, defined by

$$FM := \mathbf{R}(f_+)_* \mathbf{L}(f_-)^*$$

is an equivalence of triangulated categories, i.e. a Fourier–Mukai functor. Our main result, Theorem 5.4, shows the commutativity of the diagram

$$\begin{array}{ccc}
 (K_0(\mathbb{P}_{\Sigma_+}, \mathbb{C}))^\vee & \xrightarrow{MS_+} & \text{Sol}(U_+) \\
 FM^\vee \downarrow & & \downarrow MB \\
 (K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}))^\vee & \xrightarrow{MS_-} & \text{Sol}(U_-).
 \end{array}$$

Not surprisingly, in accordance with general mirror symmetry principles, the left side of the diagram has to do with birational geometry, while its right side is analytic in nature. The explicit description of the analytic continuation map MB is given in Corollary 4.14 (see Definition 4.15) as an application of the Mellin–Barnes integral representation method. The use of K -theory instead of (orbifold) cohomology in this last part of the paper has many advantages. In particular, we do not need to use any type of Grothendieck–Riemann–Roch theorem for stacks.

Notation. Given a fan Σ in N , and two cones $\sigma, \tau \in \Sigma$, we write $\sigma < \tau$ or $\tau > \sigma$ to indicate that the cone σ is a face of the cone τ . A one dimensional face of a cone σ will sometimes be called a *ray* of σ . For a subset \mathcal{B} of the lattice N , we write $\mathbb{R}_{\geq 0}\mathcal{B}$ to denote the cone generated by the elements of \mathcal{B} .

2. GKZ solutions with values in toric SR cohomology

As in the previous section, assume that $\mathcal{A} = \{v_1, \dots, v_n\} \subset N \cong \mathbb{Z}^d$ generates the lattice N , and that all the elements of \mathcal{A} are located in a hyperplane $h(w) = 1$, for a linear map $h : N \rightarrow \mathbb{Z}$. In what follows, we consider regular triangulations of the polytope $\Delta = \text{Conv}(\mathcal{A})$ with all their vertices among the elements of \mathcal{A} . Every such triangulation determines a fan structure Σ supported on the cone $K = \mathbb{R}_{\geq 0}\Delta$. It is well known (see [13, Chapter 7]) that there exists a one-to-one correspondence between the regular triangulations of Δ and the maximal cones of the secondary fan determined by \mathcal{A} (or, dually, the vertices of the secondary polytope associated to \mathcal{A}).

We define the partial semigroup ring $\mathbb{C}[K, \Sigma]$ associated to the cone K and the fan Σ to be the complex vector space with a basis given by the symbols x^w for all $w \in K \cap N$ and the multiplication defined such that $x^{w_1} \cdot x^{w_2} = x^{w_1+w_2}$, whenever there exists a cone $\sigma \in \Sigma$ containing both w_1 and w_2 , and $x^{w_1} \cdot x^{w_2} = 0$, otherwise. The ideal $\mathbb{C}[K^\circ, \Sigma] \subset \mathbb{C}[K, \Sigma]$ associated to the interior of the cone K is generated by the elements x^w for all the elements $w \in K^\circ$. The ring $\mathbb{C}[K, \Sigma]$ and the ideal $\mathbb{C}[K^\circ, \Sigma]$ admit a natural positive grading induced by the hyperplane condition on \mathcal{A} .

The following result is stated in [7] (see also [5]).

Proposition 2.1. *The ring $\mathbb{C}[K, \Sigma]$ and the module $\mathbb{C}[K^\circ, \Sigma]$ (over $\mathbb{C}[K, \Sigma]$) are Cohen–Macaulay of dimension d . Moreover, for any basis (m_1, \dots, m_d) of $M = \text{Hom}(N, \mathbb{Z})$, the elements*

$$Z_i = \sum_{j, \mathbb{R}_{\geq 0}v_j \in \Sigma} \langle m_i, v_j \rangle x^{v_j}$$

form a regular sequence in $\mathbb{C}[K, \Sigma]$ (and hence in $\mathbb{C}[K^\circ, \Sigma]$).

Corollary 2.2. *The quotients*

$$\mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma] := \mathbb{C}[K, \Sigma]/(Z_1, \dots, Z_d)\mathbb{C}[K, \Sigma]$$

and

$$\mathbb{C}[K^\circ, \Sigma]/Z\mathbb{C}[K^\circ, \Sigma] := \mathbb{C}[K^\circ, \Sigma]/(Z_1, \dots, Z_d)\mathbb{C}[K^\circ, \Sigma]$$

have dimension equal to the normalized volume of Δ .

Proof. The dimensions of these vector spaces are equal to $(d - 1)!$ times the leading coefficient of the Hilbert polynomial of the graded ring and module. It is well known this leading coefficient is the quotient of the normalized volume by $(d - 1)!$, see, for example, [22, Theorem 4.16]. \square

In line with the terminology of [6, Section 3], we will call the quotient $\mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma]$ the *SR-cohomology ring*.

2.1. Γ -series with values in the completion of $\mathbb{C}[K, \Sigma]$

For any maximal dimensional cone $\sigma \in \Sigma$, we define the set $\text{Box}(\sigma)$ of elements in N to be the set

$$\left\{ v: v = \sum_{j=1}^n q_j^v v_j, 0 \leq q_j^v < 1, q_j^v = 0, \text{ if } \mathbb{R}_{\geq 0} v_j \text{ is not a ray of } \sigma \right\}.$$

Define the set $\text{Box}(\Sigma)$ of elements in N to be the union of the sets $\text{Box}(\sigma)$ for all the maximal dimensional cones $\sigma \in \Sigma$. If $v \in \text{Box}(\Sigma)$, we denote by $\sigma(v)$ the smallest cone of Σ that contains v .

The set \mathcal{A} generates the lattice N , so for each $v \in \text{Box}(\Sigma)$, we can choose a solution γ^v of the equation $\gamma_1^v v_1 + \dots + \gamma_n^v v_n = \beta$, with the property that $\gamma_j^v \equiv q_j^v \pmod{\mathbb{Z}}$. This implies that γ_j^v is integer, unless $\mathbb{R}_{\geq 0} v_j$ is a ray of $\sigma(v)$. In particular, γ_j^v is integer for $\mathbb{R}_{\geq 0} v_j \notin \Sigma$.

For a given $z = (z_1, \dots, z_n)$ in $(\mathbb{C}^*)^n$, consider the formal expression

$$x^v \prod_{j=1}^n \frac{z_j^{l_j + \gamma_j^v + D_j}}{\Gamma(l_j + \gamma_j^v + D_j + 1)},$$

where $l \in \mathbb{L}$,

$$D_j := x^{v_j} \text{ if } \mathbb{R}_{\geq 0} v_j \in \Sigma, \quad \text{and} \quad D_j := 0, \text{ otherwise,}$$

and

$$z_j^{\gamma_j^v + D_j} := e^{(\gamma_j^v + D_j)(\log |z_j| + i \arg z_j)},$$

for a choice of $(\arg z_1, \dots, \arg z_n) \in \mathbb{R}^n$.

Definition 2.3. Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be an element in \mathbb{Q}^n . For any $l = (l_1, \dots, l_n) \in \mathbb{L}$, the *support of l with respect to γ and Σ* , denoted by $\text{Supp}(l)$, consists of the elements v_j of \mathcal{A} such that $l_j + \gamma_j \notin \mathbb{Z}_{\geq 0}$. We define the set $\mathcal{S}_\Sigma(\gamma) \subset \mathbb{L}$ by the property that $l \in \mathbb{L}$ belongs to $\mathcal{S}_\Sigma(\gamma)$ if there exists a (maximal) cone σ such that all the elements of $\text{Supp}(l)$ generate rays of σ .

Note that for any $l \in \mathbb{L}$ and an arbitrary $\gamma \in \mathbb{Q}^n$, we have that

$$\mathcal{S}_\Sigma(\gamma) = (-l) + \mathcal{S}_\Sigma(\gamma - l).$$

Remark 2.4. For $v \in \text{Box}(\Sigma)$ and $\gamma^v \in \mathbb{Q}^n$ a corresponding solution to $\gamma_1^v v_1 + \dots + \gamma_n^v v_n = \beta$, any $l \in \mathcal{S}_\Sigma(\gamma^v)$ has the property that the cone $\mathbb{R}_{\geq 0} \text{Supp}(l)$ belongs to the fan Σ , and $\sigma(v)$ is a subcone of the cone $\mathbb{R}_{\geq 0} \text{Supp}(l)$.

The motivation behind Definition 2.3 is the following result.

Proposition 2.5. *The expression*

$$x^v \prod_{j=1}^n \frac{z_j^{l_j + \gamma_j^v + D_j}}{\Gamma(l_j + \gamma_j^v + D_j + 1)}$$

vanishes in the completion of the ring $\mathbb{C}[K, \Sigma]$, unless $l \in \mathcal{S}_\Sigma(\gamma^v)$.

Proof. Suppose that the expression is non-zero and let $\sigma(v)$ be the smallest cone of Σ that contains v .

Notice that j th factor in the product is divisible by D_j for all negative integer $l_j + \gamma_j^v$. This in particular implies that $l_j + \gamma_j^v \geq 0$ for $\mathbb{R}_{\geq 0} v_j \notin \sigma$, since these $l_j + \gamma_j^v$ are integer. Because of the factor x^v in the expression, the set of all v_j such that $l_j + \gamma_j^v$ is a negative integer must lie in a cone of Σ that contains $\sigma(v)$. Then the rays of this cone contain all v_j for which $l_j + \gamma_j^v$ is either negative or non-integer. \square

To any maximal cone σ of the fan Σ we associate the set $\mathcal{C}_\sigma \subset \mathbb{L} \otimes \mathbb{R} \subset \mathbb{R}^n$ defined by the property that $x = (x_1, \dots, x_n) \in \mathbb{L} \otimes \mathbb{R}$ belongs to \mathcal{C}_σ , if $x_j \geq 0$ whenever $\mathbb{R}_{\geq 0} v_j$ is not a ray σ . It follows that \mathcal{C}_σ is a cone in \mathbb{R}^n generated by elements of \mathbb{L} . Moreover, the Minkowski sum

$$\mathcal{C}_\Sigma := \sum_{\sigma \in \Sigma(d)} \mathcal{C}_\sigma$$

is a cone in \mathbb{R}^n whose dual cone $\mathcal{C}_\Sigma^\vee \subset (\mathbb{R}^n)^\vee$ has non-empty interior (see, for example, [13, p. 219]). The cone \mathcal{C}_Σ^\vee is the maximal cone associated to the chosen regular triangulation in the non-pointed secondary fan determined in $(\mathbb{R}^n)^\vee$ by the set \mathcal{A} .

Lemma 2.6. *Let $\gamma = (\gamma_1, \dots, \gamma_n)$ be an element in \mathbb{Q}^n . There exists $b = (b_1, \dots, b_n) \in \mathbb{L}$ such that $\mathcal{S}_\Sigma(-b + \gamma) \subset \mathcal{C}_\Sigma$, or, equivalently, such that $\mathcal{S}_\Sigma(\gamma) \subset (-b) + \mathcal{C}_\Sigma$.*

Proof. Given $\gamma \in \mathbb{Q}^n$, for each maximal cone $\sigma \in \Sigma$, there exists a unique element $\gamma^\sigma = (\gamma_1^\sigma, \dots, \gamma_n^\sigma) \in \mathbb{L} \otimes \mathbb{Q}$ such that $\gamma_j^\sigma = \gamma_j$, when $\mathbb{R}_{\geq 0} v_j \notin \sigma$. Note that the set of all $l =$

$(l_1, \dots, l_n) \in \mathbb{L}$ such that $l_j + \gamma_j \geq 0$ when $\mathbb{R}_{\geq 0} v_j \notin \sigma$, is contained in the translated cone $(-\gamma^\sigma) + \mathcal{C}_\sigma \subset \mathbb{L} \otimes \mathbb{R}$, where \mathcal{C}_σ is the cone described in the first part of the previous lemma. We have that $(-\gamma^\sigma) + \mathcal{C}_\sigma \subset (-\gamma^\sigma) + \mathcal{C}_\Sigma$.

We claim that there exists an element $b \in \mathbb{L}$ such that $(-\gamma^\sigma) + \mathcal{C}_\Sigma \subset (-b) + \mathcal{C}_\Sigma$, for all the maximal simplices $\sigma \in \Sigma$. Indeed, it is enough to find $b \in \mathbb{L}$ such that $b - \gamma^\sigma \in \mathcal{C}_\Sigma$ for all the maximal simplices $\sigma \in \Sigma$. This can be achieved by choosing a finite number of generators in \mathbb{L} for the cone \mathcal{C}_Σ , and choosing $b \in \mathbb{L}$ to be an appropriate positive integral linear combination of the generators.

It follows that

$$\bigcup_{\sigma \in \Sigma(d)} (-\gamma^\sigma) + \mathcal{C}_\sigma \subset (-b) + \mathcal{C}_\Sigma,$$

where the union is taken over all the maximal cones $\sigma \in \Sigma$. Hence

$$\mathcal{S}_\Sigma(-b + \gamma) \subset \bigcup_{\sigma \in \Sigma(d)} ((b - \gamma^\sigma) + \mathcal{C}_\sigma) \subset \mathcal{C}_\Sigma.$$

This also means that

$$\mathcal{S}_\Sigma(\gamma) = (-b) + \mathcal{S}_\Sigma(-b + \gamma) \subset (-b) + \mathcal{C}_\Sigma. \quad \square$$

Corollary 2.7. For any $v \in \text{Box}(\Sigma)$, with $v = \sum_{j=1}^n q_j^v v_j$, with $0 \leq q_j < 1$, and $q_j = 0$ if $\mathbb{R}_{\geq 0} v_j \not\subset \sigma(v)$, there exists a choice of $\gamma^v \in \mathbb{Q}^n$ such that

$$\gamma_1^v v_1 + \dots + \gamma_n^v v_n = \beta, \quad \gamma_j^v \equiv q_j^v \pmod{\mathbb{Z}},$$

with the property that $\mathcal{S}_\Sigma(\gamma^v) \subset \mathcal{C}_\Sigma$.

Proposition 2.8. For each maximal cone σ of Σ , there exists $c_\sigma \in \mathcal{C}_\sigma^\vee$, such that the series

$$\sum_{l \in \mathbb{L} \cap \mathcal{C}_\sigma} \prod_{j=1}^n \frac{z_j^{l_j + \lambda_j}}{\Gamma(l_j + \lambda_j + 1)}$$

is absolutely convergent for $(z, \lambda) \in U_\sigma \times \mathbb{C}^n$, and defines an analytic function in $U_\sigma \times \mathbb{C}^n$. Here, $z_j^{\lambda_j} = e^{\lambda_j(\log |z_j| + i \arg z_j)}$, and the open set U_σ is defined by

$$U_\sigma := \{(z_1, \dots, z_n) \in \mathbb{C}^n : (-\log |z_1|, \dots, -\log |z_n|) \in \mathcal{C}_\sigma^\vee + c_\sigma, \\ (\arg z_1, \dots, \arg z_n) \in (-\pi, \pi) \times \dots \times (-\pi, \pi)\}.$$

Proof. The argument restriction on the z variables stems from the presence of the terms $z_j^{\lambda_j}$. Hence, it is enough to show that the series

$$\sum_{l \in \mathbb{L} \cap \mathcal{C}_\sigma} \prod_{j=1}^n \frac{z_j^{l_j}}{\Gamma(l_j + \lambda_j + 1)}$$

converges absolutely for $(z, \lambda) \in \mathbb{C}^n \times \mathbb{C}^n$ with

$$\{z = (z_1, \dots, z_n) \in \mathbb{C}^n: (-\log |z_1|, \dots, -\log |z_n|) \in \mathcal{C}_\sigma^\vee + c_\sigma\},$$

for some $c_\sigma \in \mathcal{C}_\sigma^\vee$, to be determined below.

In order to be able to apply the Weierstrass convergence theorem for sequences of holomorphic functions, we have to investigate the uniform convergence of the sequence of partial sums of this series for $\|\lambda\| \leq \delta$.

For any $l = (l_1, \dots, l_n) \in \mathbb{L}$, we have $\sum_{j=1}^n l_j = 0$. Hence, we see that

$$\left| \sum_{j=1}^n \Re(l_j + \lambda_j + 1) \right| \leq \delta + n,$$

and

$$\sum_{j=1}^n |\Im(l_j + \lambda_j + 1)| \leq \delta + n,$$

so we can apply Lemma A.4 of Appendix A. There exists a positive constant $A > 0$, such that

$$\left| \prod_{j=1}^n \frac{z_j^{l_j}}{\Gamma(l_j + \lambda_j + 1)} \right| \leq A \cdot (4n)^{\|\lambda\|} e^{\sum l_j \log |z_j|} = A \cdot e^{\|\lambda\| \log(4n) + \sum l_j \log |z_j|}.$$

For some $\epsilon > 0$, the absolute and uniform convergence of the series is guaranteed for those $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ such that

$$\|\lambda\| \log(4n) + \sum_{j=1}^n l_j \log |z_j| \leq -\epsilon \|\lambda\|,$$

for any $l \in \mathbb{L} \cap \mathcal{C}_\sigma$.

Choose $\tilde{l}_1, \dots, \tilde{l}_p$ to be a set of generators of the cone \mathcal{C}_σ . We can choose c_σ deep enough in the interior of the cone \mathcal{C}_σ^\vee , such that, for any $u \in \mathcal{C}_\sigma^\vee + c_\sigma$,

$$\langle u, \tilde{l}_i \rangle \geq (\epsilon + \log(4n)) \log \|\tilde{l}_i\|$$

for any $i, 1 \leq i \leq p$. This implies that, for any $l \in \mathbb{L} \cap \mathcal{C}_\sigma$ and any $u \in \mathcal{C}_\sigma^\vee + c_\sigma$, we have that

$$\langle u, l \rangle \geq (\epsilon + \log(4n)) \log \|l\|.$$

It follows that the series converges absolutely in the region U_σ introduced in the statement of the proposition. The region contains an open set of \mathbb{C}^n , since $\mathcal{C}_\Sigma^\vee \subset \mathcal{C}_\sigma^\vee$ and the cone \mathcal{C}_Σ^\vee has non-empty interior [13, p. 219]. This ends the proof of the proposition. \square

Corollary 2.9. *Let J be a subset of the set of the maximal cones of the fan Σ , and $\mathcal{C}_J := \sum_{\sigma \in J} \mathcal{C}_\sigma$. Then there exists $c_J \in \mathcal{C}_J^\vee$ such that the series*

$$\sum_{l \in \mathbb{L} \cap \mathcal{C}_J} \prod_{j=1}^n \frac{z_j^{l_j + \lambda_j}}{\Gamma(l_j + \lambda_j + 1)}$$

is absolutely convergent for $(z, \lambda) \in U_J \times \mathbb{C}^n$, and defines an analytic function in $U_J \times \mathbb{C}^n$, where

$$U_J := \{(z_1, \dots, z_n) \in \mathbb{C}^n : (-\log |z_1|, \dots, -\log |z_n|) \in \mathcal{C}_J^\vee + c_J, \\ (\arg z_1, \dots, \arg z_n) \in (-\pi, \pi) \times \dots \times (-\pi, \pi)\}.$$

In the case $J = \Sigma(d)$, we will use the notations c_Σ and U_Σ to denote $c_{\Sigma(d)}$ and $U_{\Sigma(d)}$, respectively.

Proof. We have that $\mathcal{C}_\Sigma^\vee \subset \mathcal{C}_J^\vee = \bigcap_{\sigma \in J} \mathcal{C}_\sigma^\vee$, therefore the cone \mathcal{C}_J^\vee has non-empty interior. It is then enough to apply the previous proposition, and to note that, it is possible to choose $c_J \in \mathcal{C}_J^\vee$ such that $c_J - c_\sigma \in \mathcal{C}_J^\vee$, for all $\sigma \in J$. The corollary follows after we see that

$$\mathcal{C}_J^\vee + c_J = (\mathcal{C}_J^\vee + (c_J - c_\sigma)) + c_\sigma \subset \mathcal{C}_J^\vee + c_\sigma \subset \mathcal{C}_\sigma^\vee + c_\sigma,$$

for all $\sigma \in J$. \square

Corollary 2.10. *For any $v \in \text{Box}(\Sigma)$, the series*

$$\sum_{l \in \mathcal{S}_\Sigma(\gamma^v)} \sum_{l \in \mathcal{S}_\Sigma(\gamma^v)} \prod_{j=1}^n \frac{z_j^{l_j + \gamma_j^v + \lambda_j}}{\Gamma(l_j + \gamma_j^v + \lambda_j + 1)}$$

defines an analytic function in the domain $(z, \lambda) \in U_\Sigma \times \mathbb{C}^n$.

Proof. By Lemma 2.6, we can find an element $b \in \mathbb{L}$ such that $\mathcal{S}_\sigma(\gamma^v) \subset (-b) + \mathcal{C}_\Sigma$. Hence, the above series is bounded in absolute value by the series

$$\sum_{l \in \mathbb{L} \cap \mathcal{C}_\Sigma} \left| \prod_{j=1}^n \frac{z_j^{l_j - b_j + \gamma_j^v + \lambda_j}}{\Gamma(l_j - b_j + \gamma_j^v + \lambda_j + 1)} \right|.$$

The change of variable $\lambda_j \rightarrow \lambda_j - b_j + \gamma_j^v$, implies that, by Corollary 2.9 with $J = \Sigma(d)$, the series is absolute convergent for $(z, \lambda) \in U_\Sigma \times \mathbb{C}^n$. \square

Definition 2.11. Let $z = (z_1, \dots, z_n)$ be a point in the open set $U_\Sigma \subset \mathbb{C}^n$ introduced in Corollary 2.9. The Γ -series with values in the completion of $\mathbb{C}[K, \Sigma]$ is defined as

$$\Phi_\Sigma(z_1, \dots, z_n) := \sum_{v \in \text{Box}(\Sigma)} x^v \sum_{l \in \mathbb{L}} \prod_{j=1}^n \frac{z_j^{l_j + \gamma_j^v + D_j}}{\Gamma(l_j + \gamma_j^v + D_j + 1)},$$

where, as before,

$$D_j := x^{v_j} \quad \text{if } \mathbb{R}_{\geq 0} v_j \in \Sigma \quad \text{and} \quad D_j := 0, \quad \text{otherwise,}$$

and

$$z_j^{D_j} := e^{(\nu_j^v + D_j)(\log |z_j| + i \arg z_j)}$$

for a choice of $(\arg z_1, \dots, \arg z_n) \in \mathbb{R}^n$.

Proposition 2.12. *The series $\Phi_\Sigma(z_1, \dots, z_n)$ defines a map from the region $U_\Sigma \subset \mathbb{C}^n$ (defined in Corollary 2.9) to the completion of the graded ring $\mathbb{C}[K, \Sigma]$.*

Proof. According to Proposition 2.5, for each $v \in \text{Box}(\Sigma)$, the non-zero terms of the series come from $l \in S_\Sigma^v$. We then apply Corollary 2.9 by setting $\lambda_j = D_j$, with $D_j = x^{v_j}$ if $\mathbb{R}_{\geq 0} v_j \in \Sigma$, and $D_j = 0$, otherwise. The result follows. \square

2.2. GKZ solutions with values in SR cohomology

The equality

$$\Phi_\Sigma(z_1, \dots, z_n) = \sum_{w \in K} \Phi_w(z_1, \dots, z_n) x^w$$

holds in the completion of the ring $\mathbb{C}[K, \Sigma]$. Let R be the subring of $\mathbb{C}[K, \Sigma]$ generated by the elements x^{v_j} for $\mathbb{R}_{\geq 0} v_j \in \Sigma$. We can view $\mathbb{C}[K, \Sigma]$ as an R -module.

Definition 2.13. The leading term module $M(\beta) \subset \mathbb{C}[K, \Sigma]$ associated to the vector β and the fan Σ is the R -submodule of $\mathbb{C}[K, \Sigma]$ generated by the elements

$$x^v \cdot \prod_{r_j < 0, \mathbb{R}_{\geq 0} v_j \in \Sigma, \mathbb{R}_{\geq 0} v_j \not\prec \sigma(v)} x^{v_j},$$

for all relations $v + \sum r_j v_j = \beta$, with $v \in \text{Box}(\Sigma)$, $r \in \mathbb{Z}^n$, such that $r_j \geq 0$ if $\mathbb{R}_{\geq 0} v_j \notin \Sigma$, and $\sigma(v)$ is the smallest cone that contains v .

For a better understanding of the modules $M(\beta)$, let us choose $P > 0$ to be the least common multiple of all the indexes of sublattices in \mathbb{Z}^d generated by all the possible simplices with vertices among the vectors of \mathcal{A} . In particular, for any simplicial fan supported on the cone K whose rays are generated by elements of \mathcal{A} , Pv will be in the semigroup generated by the elements of \mathcal{A} , for any v in the twisted sector of that fan. We fix a fan Σ .

Proposition 2.14. *A lattice element $w \in N \cap K$ has the property that $x^w \in M(\beta)$ if and only if there exists some integer $k > 0$ such that*

$$(\beta - w) + kPw$$

is in the semigroup generated by all the elements of \mathcal{A} .

Proof. An element w has the property that $x^w \in M(\beta)$ if and only if

$$w = v + \sum_{r_j < 0, \mathbb{R}_{\geq 0} v_j \not\prec \sigma(v)} v_j + \sum n_j v_j,$$

for some v in the twisted sector of the fan, $n_j \in \mathbb{Z}_{\geq 0}$ and we only use v_j generating rays of some maximal cone σ of the fan that contains as a subcone the cone $\sigma(v)$,

$$v = \sum_{j=1}^n q_j v_j, \quad 0 \leq q_j < 1, \quad q_j = 0 \text{ if } v_j \text{ is not a ray of } \sigma(v).$$

Here r is a solution to $v + \sum_j r_j v_j = \beta$, $r_j \in \mathbb{Z}$, which corresponds to the twisted sector v , where the vectors v_j corresponding to the negative r_j are rays of the maximal cone σ . In particular, r_j are non-negative for $\mathbb{R}_{\geq 0} v_j$ is not a cone of the fan.

Choose the positive integer k such that $r_j + kPq_j > 0$, for $\mathbb{R}_{\geq 0} v_j \prec \sigma(v)$, and $r_j + kP - 1 > 0$ for $\mathbb{R}_{\geq 0} v_j \not\prec \sigma(v)$. Then

$$\begin{aligned} (\beta - w) + kPw &= \sum_{\mathbb{R}_{\geq 0} v_j \prec \sigma(v)} (r_j + kPq_j)v_j + \sum_{r_j < 0, \mathbb{R}_{\geq 0} v_j \not\prec \sigma(v)} (r_j + kP - 1)v_j \\ &+ \sum_{r_j \geq 0} r_j v_j + \sum (kP - 1)n_j v_j, \end{aligned}$$

which proves the only if part of the lemma.

To show the if part assume that for some integer $k > 0$ we have

$$(\beta - w) + kPw = \sum l_j v_j$$

where $l_j \in \mathbb{Z}_{\geq 0}$ and the sum is taken over all the vectors v_j in \mathcal{A} . There exists a maximal cone σ and a corresponding twisted sector v with the associated minimal cone $\sigma(v)$ such that

$$w = v + \sum_{\mathbb{R}_{\geq 0} v_j \prec \sigma} c_j v_j, \quad c_j \in \mathbb{Z}_{\geq 0}.$$

This allows us to write

$$\beta = (1 - kP)v - \sum_{v_j \in \sigma} (kP - 1)c_j v_j + \sum l_j v_j.$$

We can see that in this presentation of β , for those v_j with $\mathbb{R}_{\geq 0} v_j \not\prec \sigma(v)$ that have negative coefficients, i.e. $-(kP - 1)c_j + l_j < 0$, we must have $c_j > 0$. Then we can see that w is in fact written as required in the beginning of the proof. \square

Note that the condition that $(\beta - w) + kPw$ belongs to the semigroup generated by the elements of \mathcal{A} does not depend on the choice of the simplicial fan Σ .

Recall that $\mathbb{C}[K^\circ, \Sigma]$ is the ideal of the ring $\mathbb{C}[K, \Sigma]$ generated by the elements x^w for $w \in K^\circ$.

Corollary 2.15.

- (i) For any $\beta \in N$, we have that $\mathbb{C}[K^\circ, \Sigma] \subset M(\beta) \subset \mathbb{C}[K, \Sigma]$.
- (ii) If $\beta \in -K^\circ$, then $M(\beta) = \mathbb{C}[K^\circ, \Sigma]$.

Proof. (i) The second inclusion is true by the definition of the module $M(\beta)$. To show the first inclusion observe that, for any $w \in K^\circ$, there exists a small rational number $\epsilon > 0$ such that $w - \epsilon \sum v_j \in K$. Hence, w can be written as a *positive* linear combination of all the elements v_j with rational coefficients. We use this representation of w and any representation of $\beta - w$ as an integer linear combination of v_j to see that for a sufficiently large and sufficiently divisible k , the element $\beta - w + kPw$ is a positive linear combination of v_j . Proposition 2.14 then completes the argument.

(ii) It is enough to note that, when $\beta \in -K^\circ$, the condition that $-\beta + (kP - 1)w$ belongs to the semigroup generated by the elements of \mathcal{A} implies that $w \in K^\circ$. \square

Recall that, for any basis (m_1, \dots, m_d) of $\text{Hom}(N, \mathbb{Z})$, the elements

$$Z_i = \sum_{j, \mathbb{R}_{\geq 0} v_j \in \Sigma} \langle m_i, v_j \rangle x^{v_j}$$

form a regular sequence in $\mathbb{C}[K, \Sigma]$. There is only a finite number of possible elements of the form $\prod_{r_j < 0} x^{v_j}$, so the R -module $M(\beta)$ is finitely generated. As a direct summand of the finite dimensional vector space $\mathbb{C}[K, \Sigma]/(Z_1, \dots, Z_d)\mathbb{C}[K, \Sigma]$, the vector space $R/(Z_1, \dots, Z_d)\mathbb{C}[K, \Sigma]$ is also finite dimensional. Hence, the following holds:

Proposition 2.16. *The vector space*

$$M(\beta)/ZM(\beta) := M(\beta)/(Z_1, \dots, Z_d)M(\beta)$$

is finite dimensional.

As a consequence of Corollary 2.15 and Proposition 2.12, the $\mathbb{C}[K, \Sigma]$ -valued Γ -series Φ_Σ induces a map from the region in $U_\Sigma \subset \mathbb{C}^n$ to the finite dimensional vector space $M(\beta)/ZM(\beta)$,

$$(z_1, \dots, z_n) \mapsto \Psi_\Sigma(z_1, \dots, z_n) := \sum_{v \in \text{Box}(\Sigma)} \sum_{l \in \mathbb{L}} R_{\gamma^v + l}(z_1, \dots, z_n),$$

where each term of the Γ -series is interpreted mod $(Z_1, \dots, Z_d)M(\beta)$, or modulo $(Z_1, \dots, Z_d)\mathbb{C}[K, \Sigma]$, respectively. By a slight abuse of notation, we will also denote by $\Psi_\Sigma(z_1, \dots, z_n)$ the same map with values in the finite dimensional vector space $\mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma]$.

Proposition 2.17. *For any linear map $h : M(\beta)/ZM(\beta) \rightarrow \mathbb{C}$, (or, $h : \mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma] \rightarrow \mathbb{C}$) the function $h \cdot \Psi_\Sigma(z_1, \dots, z_n)$ satisfies the GKZ hypergeometric equations corresponding to the set \mathcal{A} .*

Proof. The binomial GKZ equations

$$\left(\prod_{l_j > 0} \left(\frac{\partial}{\partial z_j} \right)^{l_j} - \prod_{l_j < 0} \left(\frac{\partial}{\partial z_j} \right)^{-l_j} \right) \Psi_\Sigma = 0$$

are satisfied because of the gamma identity.

For the linear GKZ equations, note first that, our choice that $D_j = x^{v_j} = 0$ if $\mathbb{R}_{\geq 0} v_j \notin \Sigma$, shows that

$$\sum_{j=1}^n \langle m, v_j \rangle D_j = \sum_{v_j \in \Sigma} \langle m, v_j \rangle D_j,$$

for any $m \in M = \text{Hom}(N, \mathbb{Z})$. Hence

$$\left(-\beta + \sum_{j=1}^n v_j z_j \frac{\partial}{\partial z_j} \right) \Psi_\Sigma = \left(\sum_{j=1}^n v_j D_j \right) \Psi_\Sigma.$$

It remains to observe that $\sum_{j=1}^n v_j D_j$ is a linear combination of the Z_j 's and that, by definition, the $\Psi_\Sigma(z_1, \dots, z_n)$ takes values in $M(\beta) \subset \mathbb{C}[K, \Sigma]$. \square

Definition 2.18. We call the induced maps

$$\Psi_\Sigma : U_\Sigma \rightarrow M(\beta)/ZM(\beta)$$

and

$$\Psi_\Sigma : U_\Sigma \rightarrow \mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma]$$

the *GKZ solution map* and the *SR cohomology valued GKZ solution map*, respectively.

The next result deals with the linear independence of the solutions obtained above.

Proposition 2.19. *If $h : M(\beta)/ZM(\beta) \rightarrow \mathbb{C}$ is a linear map such that $h \cdot \Psi_\Sigma = 0$, then $h = 0$.*

Proof. It is clear that solutions induced by elements of $M(\beta)$ corresponding to different elements in $\text{Box}(\Sigma)$ are linearly independent, since the different fractional powers induce different monodromy behaviors.

Hence, we can restrict our attention to one element of $\text{Box}(\Sigma)$ at a time. Consider $v \in \text{Box}(\Sigma)$ with

$$v = \sum_{j=1}^n q_j v_j, \quad 0 \leq q_j < 1, \quad q_j = 0 \text{ if } \mathbb{R}_{\geq 0} v_j \notin \sigma(v).$$

Assume that there exists an element $x \in M(\beta)/ZM(\beta)$ corresponding to $v \in \text{Box}(\Sigma)$ such that $h(x) \neq 0$. Let L be the largest degree of such an element. Furthermore, we can assume that x lifts to a monomial, i.e. $x = x^w \pmod{ZM(\beta)}$. Here

$$w = v + \sum_{r_j < 0, \mathbb{R}_{\geq 0} v_j \not\subseteq \sigma(v)} v_j + \sum n_j v_j,$$

where $\sum r_j v_j = \beta$, $r_j \equiv q_j \pmod{\mathbb{Z}}$, $n_j \in \mathbb{Z}_{\geq 0}$, and we only use v_j generating rays of some maximal cone σ of the fan that contains the cone $\sigma(v)$ as a subcone. The non-negative integers n_j are zero unless $v_j \in \Sigma$.

Let $\epsilon > 0$ be some small positive number. For each j such that $v_j \in \Sigma$, consider the loop of the form $z_j(t) = \epsilon e^{2\pi i t}$, $z_i(t) = \epsilon$, $i \neq j$, $0 \leq t \leq 1$. The action of the induced monodromy operator T_j on the Γ -series Ψ_Σ with values in $M(\beta)/ZM(\beta)$ is given by $\exp(D_j)$. As a result, there is a polynomial $g(T_j)$ such that $g(T_j)\Psi_\Sigma = D_j\Psi_\Sigma$, for every j such that $\mathbb{R}_{\geq 0} v_j \in \Sigma$. As a result, we have

$$\prod_j g(T_j)^{n_j} h(\Psi_\Sigma)(z) = h\left(\prod_j D_j^{n_j} \Psi_\Sigma(z)\right).$$

We now claim that the resulting function is non-zero.

The definition of the Γ -series $\Psi_\Sigma(z_1, \dots, z_n)$ and the fact that $D_j = x^{v_j}$ are nilpotent in $M(\beta)/ZM(\beta)$ shows that any induced solution of the GKZ system can be written as the product of a monomial in the variables z_j and an element of $\mathbb{C}[u_k^{-1}, \log u_k][[u_k]]$ where u_k , $1 \leq k \leq n-d$, are torus invariant variables. As a consequence of this fact, it is enough to show that the ‘‘Fourier coefficient’’ of z^r , for some r , is non-zero. We choose the element $r = (r_1, \dots, r_n)$ introduced above, with the property that $\sum r_j v_j = \beta$.

The ‘‘Fourier coefficient’’ of z^r in the expansion of $h(\prod_j D_j^{n_j} \Psi_\Sigma(z))$ is given by

$$h\left(\prod D_j^{n_j} \cdot x^v \cdot \prod \frac{1}{\Gamma(r_j + D_j + 1)}\right).$$

Notice that the terms that occur in the expansion of the expression in the argument of h have degree at least L . Moreover, x^w is the only monomial of that degree that occurs, and it has a non-zero coefficient. Since $h(x^w) \neq 0$, the maximal property of L implies that z^r has a non-zero coefficient times $h(x^w)$, so it is non-zero. This ends the proof of the linear independence result. \square

The following result is proved by different methods in [20, Section 3.5].

Proposition 2.20. *For any $\beta \in N$,*

$$\dim M(\beta)/ZM(\beta) \geq \text{Vol}(\Delta),$$

where $\text{Vol}(\Delta)$ is the normalized volume of the polytope Δ .

Proof. We view $M(\beta)$ as a graded $\mathbb{C}[y_1, \dots, y_d]$ module with $y_i = Z_i$. Since R is finitely generated as a module over this ring, and since $M(\beta)$ is finitely generated over R , $M(\beta)$ is finitely

generated. By the graded Nakayama lemma, a basis of $M(\beta)/ZM(\beta)$ can be lifted to a set of generators of $M(\beta)$ as a module over $\mathbb{C}[y_1, \dots, y_d]$. By Corollary 2.15(i), the dimension of the k th graded component of $M(\beta)$ grows as a polynomial of the form

$$\text{Vol}(\Delta) \frac{k^d}{(d-1)!} + \text{lower degree terms.}$$

Since the dimension of the degree k component of $\mathbb{C}[y_1, \dots, y_d]$ grows like $k^d/(d-1)! + \dots$, the number of generators of $M(\beta)$ is at least $\text{Vol}(\Delta)$. \square

Corollary 2.21.

(i) For any $\beta \in N$, the map

$$\begin{aligned} (M(\beta)/ZM(\beta))^\vee &\rightarrow \text{Sol}(U_\Sigma), \\ f &\rightarrow f \cdot \Psi_\Sigma, \end{aligned}$$

produces at least $\text{Vol}(\Delta)$ linearly independent GKZ solutions which are analytic in U_Σ .

(ii) If $\beta \in -K^\circ$, the above map produces exactly $\text{Vol}(\Delta)$ linearly independent GKZ solutions which are analytic in U_Σ .

3. GKZ solutions with values in K -theory

We first recall some results about reduced toric Deligne–Mumford stacks and their K -theory. According to [8], a stacky fan Σ in the abelian group N is defined by a usual simplicial fan in $N \otimes \mathbb{R}$ and a finite set of vectors v_j ($1 \leq j \leq n$) in N generating the rays of the fan. In the context of this work, N is always a lattice.

According to [6], the Grothendieck group $K_0(\mathbb{P}_\Sigma)$ is generated by the classes R_j of the invertible sheaves \mathcal{L}_j corresponding to the one dimensional cones of the fan Σ . Moreover, the ring $K_0(\mathbb{P}_\Sigma)$ is isomorphic to the quotient of the Laurent polynomial ring $\mathbb{Z}[R_1^\pm, \dots, R_n^\pm]$ by the ideal generated by the relations:

- $\prod_{j=1}^n R_j^{(m, v_j)} = 1$, for all $m \in M = \text{Hom}(N, \mathbb{Z})$,
- $\prod_{j \in J} (1 - R_j) = 0$, for any set $J \subset \{1, \dots, n\}$, such that $\sum_{j \in J} \mathbb{R}_{\geq 0} v_j$ is not a cone of the fan Σ .

The next proposition is a restatement of the main results of [6, Section 5].

Proposition 3.1.

(i) The ring $K_0(\mathbb{P}_\Sigma, \mathbb{C})$ is Artinian. Its maximum ideals are in one-to-one correspondence with elements of $\text{Box}(\Sigma)$ as follows. An element $v = \sum_{j=1}^n q_j v_j$, with $0 \leq q_j < 1$, and $q_j = 0$ if $\mathbb{R}_{\geq 0} v_j \not\subset \sigma(v)$, corresponds to the maximum ideal determined by the n -tuple of roots of unity $(y_1^v, \dots, y_n^v) \in \mathbb{C}^n$ with $y_j^v = e^{2\pi i q_j}$.

- (ii) The $K_0(\mathbb{P}_\Sigma, \mathbb{C})$ is a direct sum of Artinian local rings obtained by localizing at maximal ideal corresponding to all elements $v \in \text{Box}(\Sigma)$,

$$K_0(\mathbb{P}_\Sigma, \mathbb{C}) = \bigoplus_{v \in \text{Box}(\Sigma)} (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v.$$

- (iii) There is a natural vector space isomorphism between $K_0(\mathbb{P}_\Sigma, \mathbb{C})$ and the SR-cohomology ring $\mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma]$. It is induced by isomorphisms of \mathbb{C} -algebras

$$\begin{aligned} (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v &\cong x^v \cdot \mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma] \\ &\cong H_{SR}(\mathbb{P}_{\Sigma/\sigma(v)}, \mathbb{C}), \end{aligned}$$

for each element $v \in \text{Box}(\Sigma)$. Here $H_{SR}(\mathbb{P}_{\Sigma/\sigma(v)}, \mathbb{C})$ denotes the SR-cohomology ring of the toric smooth Deligne–Mumford stack induced by the quotient fan $\Sigma/\sigma(v)$.

- (iv) For any element $v \in \text{Box}(\Sigma)$, with $v = \sum_{j=1}^n q_j^v v_j$ with $0 \leq q_j^v < 1$, the \mathbb{C} -algebra isomorphism

$$(K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v \cong H_{SR}(\mathbb{P}_{\Sigma/\sigma(v)}, \mathbb{C})$$

is given by

$$R_j = y_j^v e^{D_j}, \quad 1 \leq j \leq n,$$

where $y_j^v = e^{2\pi i q_j^v}$, and R_j and D_j are the generators of the two rings corresponding to the vectors v_j .

A few facts and some notation used in the spectral theory of linear operators on finite dimensional vector spaces are collected in Appendix B. We will use them to construct GKZ solutions associated to the set \mathcal{A} .

Assume that the stacky fan Σ is supported on the cone K generated by the elements of the set \mathcal{A} , and that $v_j \in \mathcal{A}$. In particular, consistent with the second set relations that hold in $K_0(\mathbb{P}_\Sigma)$, we can assume $R_j = 1$ whenever $\mathbb{R}_{\geq 0} v_j$ is not a cone in Σ . The linear operators $\mathcal{R}_j : K_0(\mathbb{P}_\Sigma, \mathbb{C}) \rightarrow K_0(\mathbb{P}_\Sigma, \mathbb{C})$, $1 \leq j \leq n$, defined by $\mathcal{R}_j(x) := R_j x$, are mutually commuting with spectra $s(\mathcal{R}_j) = \{y_j^v : v \in \text{Box}(\Sigma)\}$ corresponding to the direct sum decomposition of $K_0(\mathbb{P}_\Sigma, \mathbb{C})$. We cover each root of unity y_j^v ($v \in \text{Box}(\Sigma)$, $1 \leq j \leq n$) with an open disc B_j^v of radius $\epsilon > 0$, such that any two discs centered at different points have disjoint closures, and the origin of the complex plane is not inside any of them. The multidiscs $B^v := B_1^v \times \cdots \times B_n^v \subset \mathbb{C}^n$ have the property that $\bar{B}^v \cap \bar{B}^{v'} = \emptyset$, for any two disjoint v, v' in $\text{Box}(\Sigma)$.

For an element $v \in \text{Box}(\Sigma)$, and $U \subset \mathbb{C}^n$ an open simply connected domain, consider the analytic function $\varphi : U \times B^v \rightarrow \mathbb{C}$,

$$\varphi(z, r) := \prod_{j=1}^n \frac{z_j^{\frac{1}{2\pi i} \log_j r_j}}{\Gamma(\frac{1}{2\pi i} \log_j r_j + 1)},$$

where \log_j are arbitrary log branches. The spectrum of the restriction of operator \mathcal{R}_j to $(K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v$ consists of the unique value $y_j^v \in B_j^v$. Since the function $\varphi(z, r)$ is analytic for $z \in U$ and $r = (r_1, \dots, r_n) \in B_1^v \times \dots \times B_n^v$, the linear operator

$$\varphi(z, \mathcal{R}) : (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v \rightarrow (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v$$

is well defined.

Proposition 3.2. *The linear operator*

$$\varphi(z, \mathcal{R}) : (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v \rightarrow (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v$$

is zero, unless there exists a cone $\sigma \in \Sigma$ with $\sigma(v) \subset \sigma$ whose rays contain all $v_j \in \mathcal{A}$ for which $\frac{1}{2\pi i} \log_j y_j^v$ is not a non-negative integer.

Proof. The result is just a rephrasing of Proposition 2.5. For $r_j \in B_j^v$, we can write that

$$\frac{1}{2\pi i} \log_j r_j = \frac{1}{2\pi i} \log_j y_j^v + (r_j - y_j^v) \psi(r_j),$$

where ψ is analytic in B_j^v . It follows that, in the domain B^v , the function $\varphi(z, r)$ is the product of an analytic function and the analytic function $\prod_{j \in J} (r_j - 1)$, where $J \subset \{1, \dots, n\}$ consists of all j such that $\frac{1}{2\pi i} \log_j y_j^v$ is a negative integer.

Theorem VII.1.5(b) in [10] implies that the linear operator $\varphi(z, \mathcal{R})$ is then the product of a linear operator and the linear operator

$$\prod_{j \in J} (\mathcal{R}_j - I).$$

According to Theorem 3.1(iv), the action of this operator on the space $(K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v$ is a multiple of the element $\prod_{j \in J} D_j$ viewed as an element in the SR-cohomology ring $H_{SR}(\mathbb{P}_{\Sigma/\sigma(v)}, \mathbb{C})$. The combinatorial definition of this ring provides the final step of the proof. \square

For some $v \in \text{Box}(\Sigma)$, we make a choice of log branches, such that

$$\gamma_1^v v_1 + \dots + \gamma_n^v v_n = \beta, \quad \text{where } \gamma_j^v := \frac{1}{2\pi i} \log_j y_j^v.$$

Corollary 2.10 shows that the function $\mathcal{E}_\Sigma^v(z, r) : U_\Sigma \times B^v \rightarrow \mathbb{C}$, defined by

$$\mathcal{E}_\Sigma^v(z, r) := \sum_{l \in \mathcal{S}_\Sigma(\gamma^v)} \prod_{j=1}^n \frac{z_j^{l_j + \frac{1}{2\pi i} \log_j r_j}}{\Gamma(l_j + \frac{1}{2\pi i} \log_j r_j + 1)} \tag{1}$$

is analytic.

Corollary 3.3. For some $v \in \text{Box}(\Sigma)$, and $z \in U_\Sigma$, the linear operator

$$\mathcal{E}_\Sigma^v(z, \mathcal{R}) : (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v \rightarrow (K_0(\mathbb{P}_\Sigma, \mathbb{C}))_v$$

is given by

$$\mathcal{E}_\Sigma^v(z, \mathcal{R}) = \sum \prod_{j=1}^n \frac{z_j^{\frac{1}{2\pi i} \log_j \mathcal{R}_j}}{\Gamma(\frac{1}{2\pi i} \log_j \mathcal{R}_j + 1)},$$

where the summation is taken over all the possible log branches such that

$$(\log_1 y_1^v)v_1 + \dots + (\log_n y_n^v)v_n = (2\pi i)\beta.$$

Proof. For a fixed choice of log branches as above, and $\gamma_j^v = \frac{1}{2\pi i} \log_j y_j^v$, Proposition 3.2 implies that the non-zero terms in the summation

$$\sum_{l \in \mathbb{L}} \prod_{j=1}^n \frac{z_j^{l_j + \frac{1}{2\pi i} \log_j \mathcal{R}_j}}{\Gamma(l_j + \frac{1}{2\pi i} \log_j \mathcal{R}_j + 1)},$$

correspond to those $l \in \mathbb{L}$ with $l \in \mathcal{S}_\Sigma(\gamma^v)$. \square

Definition 3.4. The Γ -series with values in K -theory is the map $\mathcal{E}_\Sigma(z, \mathcal{R})$ from the region $U_\Sigma \subset \mathbb{C}^n$ to the space of linear operators on $K_0(\mathbb{P}_\Sigma, \mathbb{C})$. For any $z \in U_\Sigma$, the linear operator $\mathcal{E}_\Sigma(z, \mathcal{R}) : K_0(\mathbb{P}_\Sigma, \mathbb{C}) \rightarrow K_0(\mathbb{P}_\Sigma, \mathbb{C})$ is associated to the analytic function

$$\mathcal{E}_\Sigma(z, r) : U_\Sigma \times B \rightarrow \mathbb{C}$$

with the property that $\mathcal{E}_\Sigma(z, r) = \mathcal{E}_\Sigma^v(z, r)$, for all $v \in \text{Box}(\Sigma)$, and $(z, r) \in U_\Sigma \times B^v$, where B is a domain in \mathbb{C}^n such that

$$\bigcup_{v \in \text{Box}(\Sigma)} B^v \subset B.$$

Remark 3.5. The definition allows for some ambiguity in choosing the domain B and the function $\mathcal{E}_\Sigma(z, r)$. However, the operator $\mathcal{E}_\Sigma(z, \mathcal{R})$ is independent of these choices.

Remark 3.6. If we regard the K -theory ring $A = K_0(\mathbb{P}_\Sigma, \mathbb{C})$ as a module over itself, we see that the linear operators $\mathcal{R}_j : A \rightarrow A$, given by $\mathcal{R}_j(x) = R_j x$ are A -module endomorphisms. For an arbitrary commutative ring A , the map $e : \text{End}_A(A) \rightarrow A$, defined by $e(\Phi) := \Phi(1)$ is a ring isomorphism, so we can regard the K -theory endomorphisms as K -theory elements.

Proposition 3.7. For any $\beta \in N$, the Γ -series with values in SR cohomology (cf. Definition 2.18)

$$\Psi_\Sigma(z) : U_\Sigma \rightarrow \mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma]$$

can be written as

$$\Psi_{\Sigma}(z) = Ch(\mathcal{E}_{\Sigma}(z, \mathcal{R})(1)),$$

where $Ch: K_0(\mathbb{P}_{\Sigma}, \mathbb{C}) \rightarrow \mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma]$ is the \mathbb{C} -algebra isomorphism between the K -theory and the SR -cohomology ring of Proposition 3.1(iii), and $\mathcal{E}_{\Sigma}(z, \mathcal{R})(1): U_{\Sigma} \rightarrow K_0(\mathbb{P}_{\Sigma}, \mathbb{C})$ is the K -theory valued Γ -series.

Proof. It is clear that it is enough to check the statement for each twisted sector $v \in \text{Box}(\Sigma)$. We have to prove that

$$\Psi_{\Sigma}(z) = Ch(\mathcal{E}_{\Sigma}^v(z, \mathcal{R})(1))$$

in $(K_0(\mathbb{P}_{\Sigma}, \mathbb{C}))_v \cong H_{SR}(\mathbb{P}_{\Sigma/\sigma(v)}, \mathbb{C})$.

We have that $\mathcal{E}_{\Sigma}^v(z, \mathcal{R})(1)$ is equal in $(K_0(\mathbb{P}_{\Sigma}, \mathbb{C}))_v$ to

$$\sum_{0 \leq j_1, \dots, j_n < v} \partial_1^{j_1} \dots \partial_n^{j_n} \mathcal{E}_{\Sigma}^v(y_1^v, \dots, y_n^v) \frac{(R_1 - y_1^v)^{j_1}}{j_1!} \dots \frac{(R_n - y_n^v)^{j_n}}{j_n!}$$

for a sufficiently large positive integer v . It follows that, as an element of $H_{SR}(\mathbb{P}_{\Sigma/\sigma(v)}, \mathbb{C})$, $Ch(\mathcal{E}_{\Sigma}^v(z, \mathcal{R})(1))$ is equal to

$$\sum_{0 \leq j_1, \dots, j_n < v} \partial_1^{j_1} \dots \partial_n^{j_n} \mathcal{E}_{\Sigma}^v(y_1^v, \dots, y_n^v) \frac{(y_1^v e^{D_1} - y_1^v)^{j_1}}{j_1!} \dots \frac{(y_n^v e^{D_n} - y_n^v)^{j_n}}{j_n!}.$$

This is the Taylor polynomial approximation of the analytic function $\mathcal{E}_{\Sigma}^v(z, (y_1^v e^{D_1}, \dots, y_n^v e^{D_n}))$ around the origin $D_1 = \dots = D_n = 0$. Note that

$$\mathcal{E}_{\Sigma}^v(z, (y_1^v e^{D_1}, \dots, y_n^v e^{D_n})) = \sum_{l \in \mathcal{S}_{\Sigma}(\gamma^v)} \prod_{j=1}^n \frac{z^{l_j + \gamma_j^v + D_j}}{\Gamma(l_j + \gamma_j^v + D_j + 1)},$$

as analytic functions with $z \in U_{\Sigma}$ and (D_1, \dots, D_n) in a neighborhood of the origin in \mathbb{C}^n . Hence, the above Taylor polynomial is equal to $\Psi_{\Sigma}(z)$ when viewed as elements of $H_{SR}(\mathbb{P}_{\Sigma/\sigma(v)}, \mathbb{C})$. \square

Corollary 3.8. For any $\beta \in N$, the mirror symmetry map $MS_{\Sigma}: (K_0(\mathbb{P}_{\Sigma}, \mathbb{C}))^{\vee} \rightarrow \text{Sol}(U_{\Sigma})$ given by

$$MS_{\Sigma}(f) := f(\mathcal{E}_{\Sigma}(z, \mathcal{R})(1))$$

produces GKZ solutions which are analytic in U_{Σ} .

Remark 3.9. The dimension of the space of GKZ solutions with values in K -theory or SR cohomology is generally not easy to calculate or even estimate. Moreover, the K -theoretic meaning of the leading term module $M(\beta)$, for general β , is unclear but it is perhaps worth investigating.

4. Analytic continuation of hypergeometric series

Consider an *oriented* edge of the secondary polytope starting at the vertex of the secondary polytope corresponding to some regular triangulation \mathcal{T}_+ and ending at the vertex corresponding to another regular triangulation \mathcal{T}_- , and let Σ_+ and Σ_- be the induced fans supported on the cone K . In order to ease up some of the heavy notation, in this section and the next one, we will usually replace the subscript Σ_{\pm} by \pm . For example, we will write $S_{\pm}(\gamma)$ instead of $S_{\Sigma_{\pm}}(\gamma)$.

Theorem 2.10 in [13] shows that there exists a circuit I (i.e. a minimal linearly dependent set of elements) in \mathcal{A} determining an integral relation of the form

$$h_1 v_1 + \cdots + h_n v_n = 0, \quad h = (h_1, \dots, h_n) \in \mathbb{L},$$

with $I = I_+ \cup I_-$, where

$$I_+ := \{v_j: h_j > 0\}, \quad I_- := \{v_j: h_j < 0\},$$

such that the triangulations are both supported on the circuit I and are obtained by a modification based on the circuit I (see pp. 231–233 in the book [13] for detailed explanations). Moreover, each of the fans Σ_{\pm} has the property that, for every subset $\mathcal{F} \subset \mathcal{A} \setminus I$, if $\mathcal{F} \cup (I \setminus v_0)$ are the rays of a maximal cone in Σ_{\pm} for *some* $v_0 \in I_{\pm}$, then the elements of $\mathcal{F} \cup (I \setminus v)$ are the rays of a maximal cone in Σ_{\pm} for *any* $v \in I_{\pm}$. Such a subset \mathcal{F} is said to be a *separating* set for the fans Σ_{\pm} . Furthermore, the modification is obtained by replacing the set of all maximal cones generated by sets of the form $\mathcal{F} \cup (I \setminus v)$ ($v \in I_+$) of Σ_+ with the set of all maximal cones generated by $\mathcal{F} \cup (I \setminus v)$ ($v \in I_-$), with \mathcal{F} a separating set.

Definition 4.1. We say that a maximal cone of the fans Σ_{\pm} generated by a set of the form $\mathcal{F} \cup (I \setminus v)$, with \mathcal{F} a separating set and $v \in I_{\pm}$, is *essential*. We denote by $\Sigma_{\pm}^{\text{es}}(d)$ the sets of essential maximal cones of the fans Σ_{\pm} .

We assume that the element $h = (h_1, \dots, h_n) \in \mathbb{L}$ is *primitive* in \mathbb{L} , i.e. it is not a non-trivial integral multiple of any other element in the lattice. This means that $\mathbb{L}/\langle h \rangle$ is itself a lattice. We introduce the notation

$$\mathbb{L}' := \mathbb{L}/\langle h \rangle,$$

and let $p: \mathbb{L} \rightarrow \mathbb{L}'$ denote the canonical projection.

Definition 4.2. For any $r = (r_1, \dots, r_n) \in (\mathbb{C}^*)^n$, we denote by $\mathcal{I}(r) \subset \mathbb{C}^*$ the finite set of complex numbers t such that $r_j t^{h_j} = 1$ for some j with $v_j \in I_-$.

Remark 4.3. For a given $r = (r_1, \dots, r_n) \in (\mathbb{C}^*)^n$, note that any two values $t, s \in \mathcal{I}(r)$ such that $(r_1 t^{h_1}, \dots, r_n t^{h_n}) = (r_1 s^{h_1}, \dots, r_n s^{h_n})$ are in fact equal. Indeed, since r_j are non-zero, we have that $t^{h_j} = s^{h_j}$ for all j . But $(h_1, \dots, h_n) \in \mathbb{L}$ is primitive, so we can find integers d_1, \dots, d_n such that $d_1 h_1 + \cdots + d_n h_n = 1$. Then $t = t^{d_1 h_1 + \cdots + d_n h_n} = s^{d_1 h_1 + \cdots + d_n h_n} = s$.

Let $\text{Box}(\Sigma_{\pm}^{\text{es}}) \subset \text{Box}(\Sigma_{\pm})$ be the subsets consisting of those elements $v \in \text{Box}(\Sigma_{\pm})$ with the property that the minimal cones containing v in Σ_{\pm} are subcones of one of the maximal cones

in $\Sigma_{\pm}^{es}(d)$, respectively. We now describe the effect that a modification has on the corresponding twisted sectors. We use the notation and the results of Proposition 3.1(i).

Proposition 4.4.

(i)
$$\text{Box}(\Sigma_+) \setminus \text{Box}(\Sigma_+^{es}) = \text{Box}(\Sigma_-) \setminus \text{Box}(\Sigma_-^{es}).$$

Moreover, for any element $v \in N$ belonging to the two sets above, with

$$v = \sum_{j=1}^n q_j v_j, \quad 0 \leq q_j < 1, \quad q_j = 0 \text{ if } \mathbb{R}_{\geq 0} v_j \not\subset \sigma(v),$$

the minimal cone $\sigma(v)$ containing v is unchanged under the modification. Moreover, the corresponding sets of n -tuples of roots of unity (y_1^v, \dots, y_n^v) , $y_j^v = e^{2\pi i q_j}$, are also unchanged.

(ii) For any $v \in \text{Box}(\Sigma_+^{es})$, with

$$v = \sum_{j=1}^n q_j v_j, \quad 0 \leq q_j < 1, \quad q_j = 0 \text{ if } \mathbb{R}_{\geq 0} v_j \not\subset \sigma(v) \in \Sigma_+,$$

and $y_j^v = e^{2\pi i q_j}$, the n -tuples $(y_1^v t^{h_1}, \dots, y_n^v t^{h_n}) \in (\mathbb{C}^*)^n$ with $t \in \mathcal{I}(y^v)$, determine maximum ideals of $K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C})$ corresponding to elements of $\text{Box}(\Sigma_-^{es})$. Moreover, any n -tuple of roots of unity induced by some element of $\text{Box}(\Sigma_-^{es})$ is obtained in this way, for appropriate $v \in \text{Box}(\Sigma_+^{es})$ and $t \in \mathcal{I}(y^v)$.

Proof. (i) For any $v \in \text{Box}(\Sigma_+) \setminus \text{Box}(\Sigma_+^{es})$, we have that $\sigma(v)$ is a subcone of a maximal cone of Σ_+ that survives the modification. Hence $\sigma(v)$ is also a cone of Σ_- . We still have to show that $v \notin \text{Box}(\Sigma_-^{es})$. For, assume that $\sigma(v)$ is a subcone of maximal cone $\sigma \in \Sigma_-^{es}(d)$ that changes under the modification, and the rays of σ consist of the elements of $\mathcal{F} \cup (I \setminus v_-)$ for some separating set \mathcal{F} and $v_- \in I_-$. Since $\sigma(v)$ is a cone of Σ_+ and the elements of I_+ do not generate a cone in this fan, there exists some vector $v_+ \in I_+$ which does not generate a ray of $\sigma(v)$. Hence $\mathcal{F} \cup (I \setminus v_+)$ is a maximal cone in $\Sigma_+^{es}(d)$ containing $\sigma(v)$ as a subcone. But this is a contradiction with our starting assumption that $v \notin \text{Box}(\Sigma_+^{es})$. The roles of Σ_+ and Σ_- can be reversed, and the statement follows.

(ii) First, consider some n -tuple (y_1^v, \dots, y_n^v) corresponding to an element $v \in \text{Box}(\Sigma_+^{es})$. We show that the n -tuple $(y_1^v t^{h_1}, \dots, y_n^v t^{h_n})$ with $t \in \mathbb{C}$ a root of unity such that $y_i^v t^{h_i} = 1$, for some i with $v_i \in I_-$, corresponds to an element of $\text{Box}(\Sigma_-^{es})$. For, note that there exists a maximal cone $\sigma \in \Sigma_+^{es}(d)$ generated by $\mathcal{F} \cup (I \setminus w)$ for some separating set \mathcal{F} and $w \in I_+$, such that

$$v = \sum_{j=1}^n q_j v_j, \quad 0 \leq q_j < 1, \quad q_j = 0 \text{ if } \mathbb{R}_{\geq 0} v_j \not\subset \sigma.$$

Choose a rational number $q \in \mathbb{Q}$ such that $q_i + qh_i \in \mathbb{Z}$, for some i with $v_i \in I_-$. By adding the linear relation $qh_1v_1 + \dots + qh_nv_n = 0$ to the above expression of v , we can write that

$$v = \sum_{j=1}^n (q_j + qh_j)v_j,$$

where $q_j + qh_j \in \mathbb{Z}$, whenever $v_j \notin \mathcal{F} \cup (I \setminus v_i)$. Since the elements of $\mathcal{F} \cup (I \setminus v_i)$ generate the rays of a maximal cone in $\Sigma_-^s(d)$, we conclude that $(y_1^v t^{h_1}, \dots, y_n^v t^{h_n})$, with $y_j^v = e^{2\pi i q_j v}$, $t = e^{2\pi i q}$, corresponds indeed to an element in $\text{Box}(\Sigma_-^{\text{es}})$ which differs from v by an integral linear combination of the vectors $v_j \in \mathcal{A}$. This shows that, for any $v \in \text{Box}(\Sigma_+^{\text{es}})$, the procedure described in the second part of the proposition produces a subset of $\text{Box}(\Sigma_-^{\text{es}})$. We still have to show that the union of all these subsets, for all $v \in \text{Box}(\Sigma_+^{\text{es}})$, is equal to $\text{Box}(\Sigma_-^{\text{es}})$.

It is enough to show that, after starting with an n -tuple of roots of unity (y_1^v, \dots, y_n^v) corresponding to some element $v \in \text{Box}(\Sigma_+^{\text{es}})$ and applying the above procedure from $\text{Box}(\Sigma_+^{\text{es}})$ to $\text{Box}(\Sigma_-^{\text{es}})$ and back, the n -tuple (y_1^v, \dots, y_n^v) is recovered. Note that for such an n -tuple there exists an element $v_i \in I_+$ such that $y_i^v = 1$. It follows that, if $(y_1^v t^{h_1}, \dots, y_n^v t^{h_n})$ (for the appropriate $t \in \mathbb{C}$) corresponds to an element in $\text{Box}(\Sigma_-^{\text{es}})$, then $(y_1^v t^{h_1} (t^{-1})^{h_1}, \dots, y_n^v t^{h_n} (t^{-1})^{h_n})$ in $\text{Box}(\Sigma_+^{\text{es}})$ is an allowed choice under the procedure, since $y_i^v t^{h_i} (t^{-1})^{h_i} = y_i^v = 1$, with $v_i \in I_+$, as noted above. This ends the proof of the proposition. \square

For an element $v \in \text{Box}(\Sigma_+)$, consider the associated $y^v \in (\mathbb{C}^*)^n$. Choose n log branches, all of which are denoted \log_+ by a slight abuse of notation, such that $\gamma_1^v v_1 + \dots + \gamma_n^v v_n = \beta$, where

$$\gamma_j^v := \frac{1}{2\pi i} \log_+ y_j^v.$$

We now choose a branch of $\log t$, and for any $t \in \mathcal{I}(y^v)$, we set

$$\gamma_j^v(t) := \gamma_j^v + h_j \log t.$$

We still have that $\gamma_1^v(t)v_1 + \dots + \gamma_n^v(t)v_n = \beta$. For each $t \in \mathcal{I}(y^v)$, it is possible to find log branches that will be denoted \log_- such that

$$\gamma_j^v(t) = \frac{1}{2\pi i} \log_-(y_j^v t^{h_j}).$$

We now exhibit some special subsets of the sets $\mathcal{S}_\pm(\gamma) \subset \mathbb{L}$ introduced in Definition 2.3.

Definition 4.5. For an arbitrary $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Q}^n$, we define the set $\mathcal{S}_\pm^{\text{es}}(\gamma^v) \subset \mathbb{L}$ by the property that $l \in \mathbb{L}$ belongs to $\mathcal{S}_\pm^{\text{es}}(\gamma)$ if there exists a maximal cone $\sigma \in \Sigma_\pm^{\text{es}}(d)$ such that all the elements of $\text{Supp}(l)$ with respect to Σ_\pm and γ generate rays of σ .

When there is no danger of confusion, we will simply talk about $\text{Supp}(l)$ with no mention of the fan and the element γ .

Proposition 4.6. For any $v \in \text{Box}(\Sigma_+)$, with $y^v, \gamma^v(t)$ ($t \in \mathcal{I}(y^v)$) chosen as above, the following are true:

(i) If $1 \notin \mathcal{I}(y^v)$, then $\mathcal{S}_+^{\text{es}}(\gamma^v) = \mathcal{S}_+(\gamma^v)$. If $1 \in \mathcal{I}(y^v)$, then $\gamma^v(1) = \gamma^v$, and

$$\mathcal{S}_+(\gamma^v) \setminus \mathcal{S}_+^{\text{es}}(\gamma^v) = \mathcal{S}_-(\gamma^v) \setminus \mathcal{S}_-^{\text{es}}(\gamma^v).$$

(ii) Under the natural projection $p: \mathbb{L} \rightarrow \mathbb{L}'$, the images of the sets $\mathcal{S}_+^{\text{es}}(\gamma^v)$ and $\mathcal{S}_-^{\text{es}}(\gamma^v(t))$ coincide, for any $t \in \mathcal{I}(y^v)$.

Proof. (i) Assume that there exists $l \in \mathcal{S}_+(\gamma^v) \setminus \mathcal{S}_+^{\text{es}}(\gamma^v)$. This means that there exists a maximal simplex $\sigma \in \Sigma_+$ such that all the vectors in $\text{Supp}(l)$ generate rays in σ . Since any such simplex σ is unchanged under the modification, and the rays of $\sigma(v)$ are in the support of l , we see that $v \in \text{Box}(\Sigma_-)$ since $\sigma \in \Sigma_-(d)$. Hence $l \in \mathcal{S}_-(\gamma^v)$ and $1 \in \mathcal{I}(y^v)$ with $\gamma^v(1) = \gamma^v$.

We still have to show that, if $l \in \mathcal{S}_+(\gamma^v) \setminus \mathcal{S}_+^{\text{es}}(\gamma^v)$, then $l \notin \mathcal{S}_-^{\text{es}}(\gamma^v)$. For, assume that there exists a maximal cone $\sigma' \in \Sigma_-^{\text{es}}(d)$, σ' whose rays are generated by the elements of $\mathcal{F} \cup (I \setminus v)$, $v \in I_-$, such that the elements of the support of l generate rays in σ . Note that, since $l \in \mathcal{S}_+(\gamma^v)$ and the elements of I_+ do not generate a cone in Σ_+ , there must be some i with $v_i \in I_+$ such that $l_i + \gamma_i^v \in \mathbb{Z}_{\geq 0}$. This means that the elements of the support of l generate rays of the cone of $\Sigma_+^{\text{es}}(d)$ generated by $\mathcal{F} \cup (I \setminus v_i)$. However, this contradicts the assumption that $l \notin \mathcal{S}_+^{\text{es}}(\gamma^v)$. With this, we have proved that $\mathcal{S}_+(\gamma^v) \setminus \mathcal{S}_+^{\text{es}}(\gamma^v) \subset \mathcal{S}_-(\gamma^v) \setminus \mathcal{S}_-^{\text{es}}(\gamma^v)$. The inverse inclusion is obtained by employing the completely analogous argument with the roles of Σ_{\pm} reversed.

(ii) Given $l \in \mathbb{L}$, the elements of the fiber $p^{-1}(p(l))$ consist of elements of the form $l + mh$ with $m \in \mathbb{Z}$.

Let l be an element of $\mathcal{S}_+^{\text{es}}(\gamma^v)$. This means that the vectors v_j such that $l_j + \gamma_j^v \notin \mathbb{Z}_{\geq 0}$ are among the elements of a set of the form $\mathcal{F} \cup (I \setminus v_i)$ with \mathcal{F} a separating set and $v_i \in I_+$. In particular, we see that the minimal cone of Σ_+ that contains v is itself contained in a maximal cone of Σ_+ that gets replaced under the modification. Hence $v \in \text{Box}(\Sigma_+^{\text{es}})$. According to Proposition 4.4(ii), given $t \in \mathcal{I}(y^v)$, the n -tuple $(y_1^v t^{h_1}, \dots, y_n^v t^{h_n})$ corresponds to an element in $\text{Box}(\Sigma_-^{\text{es}})$. Therefore, there exists some $v_k \in I_-$ such that $y_k^v t^{h_k} = 1$, so $\gamma_k^v(t) = \gamma_k^v + h_k(\frac{1}{2\pi i} \log t) \in \mathbb{Z}$.

We can choose an integer $m \ll 0$ such that $(l_k + mh_k) + \gamma_k^v(t) \in \mathbb{Z}_{\geq 0}$. As a consequence, the set $\text{Supp}(l + mh)$ is a subset of $\mathcal{F} \cup (I \setminus v_k)$. But the elements of the latter set generate a maximal cone in $\Sigma_-^{\text{es}}(d)$, so $l + mh \in \mathcal{S}_-^{\text{es}}(\gamma^v(t))$. We have shown that $\mathcal{S}_+^{\text{es}}(\gamma^v) \subset \mathcal{S}_-^{\text{es}}(\gamma^v(t))$.

The inverse inclusion is obtained by reversing the roles of Σ_{\pm} and by replacing t with t^{-1} . The result follows. \square

For any $l \in \mathbb{L}$, we define the integers $m_+(l), m_{-,t}(l), t \in \mathcal{I}(y^v)$, to be

$$\begin{aligned} m_+(l) &:= \min\{m \in \mathbb{Z}, mh_j + l_j + \gamma_j^v \in \mathbb{Z}_{\geq 0}, \text{ for some } j, v_j \in I_+\}, \\ m_{-,t}(l) &:= \max\{m \in \mathbb{Z}, mh_j + l'_j + \gamma_j^v(t) \in \mathbb{Z}_{\geq 0}, \text{ for some } j, v_j \in I_-\}. \end{aligned} \tag{2}$$

For any $v \in \text{Box}(\Sigma_+)$ and $t \in \mathcal{I}(y^v)$, there exists some j', j'' with $v_{j'} \in I_+, v_{j''} \in I_-$ such that $\gamma_{j'}^v, \gamma_{j''}^v(t) \in \mathbb{Z}$. Hence, the functions $m_+, m_{-,t}: \mathbb{L} \rightarrow \mathbb{Z}$ are well defined. Moreover, they are piecewise linear with a finite number of linear restrictions.

Proposition 4.7. Let l be an element of the lattice \mathbb{L} such that $p(l)$ belongs to the image of the sets $\mathcal{S}_+^{\text{es}}(\gamma^v), \mathcal{S}_-^{\text{es}}(\gamma^v(t))$ under the projection $p: \mathbb{L} \rightarrow \mathbb{L}'$.

The element $mh + l \in \mathbb{L}$ belongs to $\mathcal{S}_+^{\text{es}}(\gamma^v)$ if and only if $m \geq m_+(l)$, and to $\mathcal{S}_-^{\text{es}}(\gamma^v(t))$, $t \in \mathcal{I}(\gamma^v)$, if and only if $m \leq m_{-,t}(l)$.

Proof. If $m < m_+(l)$, then $mh_j + l_j + \gamma_j^v \notin \mathbb{Z}_{\geq 0}$, for all $v_j \in I_+$, so I_+ contains all the elements of $\text{Supp}(mh + l)$. But the elements of I_+ do not generate the rays of a cone in Σ_+ , hence $mh + l \notin \mathcal{S}_+^{\text{es}}(\gamma^v)$. A similar argument shows that, for $t \in \mathcal{I}(\gamma^v)$, we have that $mh + l \notin \mathcal{S}_-^{\text{es}}(\gamma^v(t))$ if $m < m_{-,t}(l)$.

If $m \geq m_+(l)$, let i be such that $mh_i + l_i + \gamma_i^v \in \mathbb{Z}_{\geq 0}$. Since $p(l)$ is in image of $\mathcal{S}_+^{\text{es}}(\gamma^v)$ under the projection $p: \mathbb{L} \rightarrow \mathbb{L}'$, we conclude that the elements $v_j \in \mathcal{A} \setminus I$ such that $l_j + \gamma_j^v \notin \mathbb{Z}_{\geq 0}$ form a separating set \mathcal{F} . Hence $\mathcal{F} \cup (I \setminus \{v_i\})$ are the rays of a maximal simplex in $\Sigma_+^{\text{es}}(d)$, so $mh + l \in \mathcal{S}_+^{\text{es}}(\gamma^v)$. A similar argument shows that, for $t \in \mathcal{I}(\gamma^v)$, if $m \leq m_{-,t}(l)$, then $mh + l \in \mathcal{S}_-^{\text{es}}(\gamma^v(t))$. This ends the proof of the proposition. \square

Remark 4.8. If $v \in \text{Box}(\Sigma_+) \setminus \text{Box}(\Sigma_+^{\text{es}})$, any maximal cone of Σ_+ containing $\sigma(v)$ is left unchanged by the modification. Since for any $l \in \mathcal{S}_+^{\text{es}}(\gamma^v)$ we would have that $\sigma(v) < \mathbb{R}_{\geq 0} \text{Supp}(l) < \sigma$ with $\sigma \in \Sigma_+^{\text{es}}(d)$, we conclude that $\mathcal{S}_+^{\text{es}}(\gamma^v) = \emptyset$. Moreover, by Proposition 4.4(i), we also have that $v \in \text{Box}(\Sigma_-) \setminus \text{Box}(\Sigma_-^{\text{es}})$ and $1 \in \mathcal{I}(\gamma^v)$ with $\gamma^v(1) = \gamma^v$. Hence

$$\mathcal{S}_+(\gamma^v) = \mathcal{S}_-(\gamma^v) \quad \text{and} \quad \mathcal{S}_+^{\text{es}}(\gamma^v) = \mathcal{S}_-^{\text{es}}(\gamma^v) = \emptyset.$$

We are now ready to describe the analytic continuation of the analytic function $\mathcal{E}_+^v(z, r): U_+ \times B^v \rightarrow \mathbb{C}$ given by (see (1))

$$\mathcal{E}_+^v(z, r) = \sum_{l \in \mathcal{S}_+(\gamma^v)} \prod_{j=1}^n \frac{z_j^{l_j + \frac{1}{2\pi i} \log_+ r_j}}{\Gamma(l_j + \frac{1}{2\pi i} \log_+ r_j + 1)}.$$

As mentioned above, in order to simplify our notation, we choose to denote the n possibly distinct log branches with the same symbol \log_+ .

Remark 4.9. For the given $v \in \text{Box}(\Sigma_+)$, the open domain $U_+ \times B^v$ in $\mathbb{C}^n \times \mathbb{C}^n$ has been defined in Section 3, and we choose the branches \log_+ such that

$$\mathcal{S}_+(\gamma^v) \subset \mathcal{C}_+.$$

The existence of such a choice follows from Corollary 2.7.

It will also be useful to consider the analytic function $(\mathcal{E}_+^v)^{\text{es}}(z, r): U_+ \times B^v \rightarrow \mathbb{C}$ given by

$$(\mathcal{E}_+^v)^{\text{es}}(z, r) = \sum_{l \in \mathcal{S}_+^{\text{es}}(\gamma^v)} \prod_{j=1}^n \frac{z_j^{l_j + \frac{1}{2\pi i} \log_+ r_j}}{\Gamma(l_j + \frac{1}{2\pi i} \log_+ r_j + 1)}. \tag{3}$$

As above, for each $t \in \mathcal{I}(\gamma^v)$, we can choose n log branches such that

$$\frac{1}{2\pi i} \log_-(y_j^v t^{h_j}) = \gamma_j^v(t) = \gamma_j^v + h_j \log t,$$

where $\log t$ is a fixed choice of branch.

The analytic functions $\mathcal{E}_-^{v(t)}(z, r), (\mathcal{E}_-^{v(t)})^{\text{es}}(z, r) : U_- \times B^{v(t)} \rightarrow \mathbb{C}$ are then defined analogously, for $t \in \mathcal{I}(y^v)$ such that there exists a corresponding twisted sector $v(t) \in \text{Box}(\Sigma_-)$ associated to $(y_1^v t^{h_1}, \dots, y_n^v t^{h_n})$. Proposition 4.4 shows that such $v(t)$ exists for $t = 1 \in \mathcal{I}(y^v)$ when $v \in \text{Box}(\Sigma_+) \setminus \text{Box}(\Sigma_+^{\text{es}})$, and for all $t \in \mathcal{I}(y^v)$ when $v \in \text{Box}(\Sigma_+^{\text{es}})$.

The analytic continuation will be performed along a path of the form $(z(u), y^v)$ starting at a point (z_+, y^v) in $U_+ \times B^v$ and ending at a point (z_-, y^v) in $U_- \times B^v$. The path $z(u) = (z_1(u), \dots, z_n(u))$ is defined so that, for all $u, 0 \leq u \leq 1$,

$$\begin{aligned} \arg z_j(u) &= \arg(z_+)_j = \arg(z_-)_j, \\ \log |(z_j(u))| &= (1 - u) \log |(z_+)_j| + u \log |(z_-)_j| \end{aligned}$$

for all $j, 1 \leq j \leq n$.

The points $z_{\pm} \in U_{\pm}$ are chosen such that the conditions (A1)–(A3) are satisfied.

(A1) According to Corollary 2.9, the domains $U_{\pm} \subset \mathbb{C}^n$ have the form

$$\begin{aligned} \{(z_1, \dots, z_n) \in \mathbb{C}^n : (-\log |z_1|, \dots, -\log |z_n|) \in \mathcal{C}_{\pm}^{\vee} + c_{\pm}, \\ (\arg z_1, \dots, \arg z_n) \in (-\pi, \pi) \times \dots \times (-\pi, \pi)\}, \end{aligned}$$

for some appropriate $c_{\pm} \in \mathcal{C}_{\pm}^{\vee}$. Since the cones \mathcal{C}_+^{\vee} and \mathcal{C}_-^{\vee} are adjacent in the secondary fan, and the common facet is included in a hyperplane determined by the element $h = (h_1, \dots, h_n) \in \mathbb{L}$, we can choose the points $z_+ = ((z_+)_1, \dots, (z_+)_n) \in U_+$ and $z_- = ((z_-)_1, \dots, (z_-)_n) \in U_-$, such that

$$\arg(z_+)_j = \arg(z_-)_j, -\log |(z_+)_j| + \log |(z_-)_j| = Ah_j, \quad A > 0,$$

for all $j, 1 \leq j \leq n$.

- (A2) The points z_{\pm} such that $(-\log |(z_{\pm})_1|, \dots, -\log |(z_{\pm})_n|) \in \mathcal{C}_J^{\vee} + c_J$, where J is the set of common maximal simplices of Σ_{\pm} , and $\mathcal{C}_J = \sum_{\sigma \in J} \mathcal{C}_{\sigma}$. The element c_J , which is located deep inside the cone \mathcal{C}_J^{\vee} , and the associated open domain $U_J \subset \mathbb{C}^n$, are provided by the results of Corollary 2.9. By convexity, for any point $z(u)$ on the analytic continuation path, we see that $(-\log |z_1(u)|, \dots, -\log |z_n(u)|) \in \mathcal{C}_J^{\vee} + c_J$.
- (A3) The analytic continuation path is chosen so that, along the path $z(u)$, we have that $-2\pi < \arg y(u) < 0$, where the complex number $y(u)$ is given by

$$y(u) := e^{i\pi \sum_{j, v_j \in I_-} h_j} \prod_{j=1}^n (z_j(u))^{h_j}.$$

Note that, at least for $-\pi < \arg z_j(u) = \arg(z_+)_j = \arg(z_-)_j < 0$, we have that

$$\arg y(u) = \sum_{j, v_j \in I_+} h_j \arg z_j(u) + \sum_{j, v_j \in I_-} h_j (\pi + \arg z_j(u)) < 0,$$

which shows that it is possible to choose the points z_{\pm} in U_{\pm} such that the argument of $y(u)$ is between -2π and 0 for all $u, 0 \leq u \leq 1$.

Theorem 4.10.

- (i) For any $v \in \text{Box}(\Sigma_+)$, the function $\mathcal{E}_+^v(z, r) - (\mathcal{E}_+^v)^{\text{es}}(z, r)$ is analytic in the open domain $U_J \times B^v$, and the open domain U_J contains the sets U_{\pm} and the path $z(u)$.
 If $1 \notin \mathcal{I}(y^v)$, the function is identically zero in the domain $U_J \times B^v$.
 If $1 \in \mathcal{I}(y^v)$, then the analytic functions $\mathcal{E}_+^v(z, r) - (\mathcal{E}_+^v)^{\text{es}}(z, r)$ and $\mathcal{E}_-^{v(1)}(z, r) - (\mathcal{E}_-^{v(1)})^{\text{es}}(z, r)$ are equal for all $(z, r) \in U_J \times B^v$.
- (ii) For any $v \in \text{Box}(\Sigma_+^{\text{es}})$, the analytic continuation along the path $(z(u), y^v)$ of the germ of the analytic function $(\mathcal{E}_+^v)^{\text{es}}(z, r)$ at $(z_+, y^v) \in U_+ \times B^v$ is given by the germ at $(z_-, y^v) \in U_- \times B^v$ of an analytic function defined as follows.
 If $1 \notin \mathcal{I}(y^v)$, the function can be written as

$$- \sum_{t \in \mathcal{I}(y^v)} \int_{C_t} T(r, \hat{t}) (\mathcal{E}_-^{v(t)})^{\text{es}}(z, r \hat{t}^h) d\hat{t} + \prod_{j, v_j \in I_+} (1 - r_j^{-1}) \phi(z, r).$$

If $1 \in \mathcal{I}(y^v)$, the function can be written as

$$(\mathcal{E}_-^{v(1)})^{\text{es}}(z, r) - \sum_{t \in \mathcal{I}(y^v)} \int_{C_t} T(r, \hat{t}) (\mathcal{E}_-^{v(t)})^{\text{es}}(z, r \hat{t}^h) d\hat{t} + \prod_{j, v_j \in I_+} (1 - r_j^{-1}) \varphi(z, r).$$

Here, the integration kernel $T(r, \hat{t})$ is defined by

$$T(r, \hat{t}) := \frac{1}{2\pi i (\hat{t} - 1)} \prod_{j, v_j \in I_-} \frac{1 - r_j^{-1}}{1 - r_j^{-1} \hat{t}^{-h_j}},$$

$\phi(z, r)$, $\varphi(z, r)$ are analytic functions on $U_- \times B^v$, and the contours C_t are disjoint circles in the \hat{t} -plane centered at the points $t \in \mathcal{I}(y^v)$, counterclockwise oriented, such that, for $r \in B^v$, all the poles of $T(r, \hat{t})$ are contained inside the discs bounded by the contours C_t .

Proof. (i) The series representation of the function $\mathcal{E}_+^v(z, r) - (\mathcal{E}_+^v)^{\text{es}}(z, r)$ is

$$\sum_{l \in \mathcal{S}_+(\gamma^v) \setminus \mathcal{S}_+^{\text{es}}(\gamma^v)} \prod_{j=1}^n \frac{z_j^{l_j + \frac{1}{2\pi i} \log_+ r_j}}{\Gamma(l_j + \frac{1}{2\pi i} \log_+ r_j + 1)}.$$

If $1 \notin \mathcal{I}(y^v)$, then by Proposition 4.6(i) we have that $\mathcal{S}_+(\gamma^v) \setminus \mathcal{S}_+^{\text{es}}(\gamma^v) = \emptyset$, so the function is identically zero indeed.

According to the same result, if $1 \in \mathcal{I}(y^v)$, then $\mathcal{S}_+(\gamma^v) \setminus \mathcal{S}_+^{\text{es}}(\gamma^v) = \mathcal{S}_-(\gamma^v) \setminus \mathcal{S}_-^{\text{es}}(\gamma^v)$. For any l in these sets, there exists a maximal cone σ belonging to both Σ_+ and Σ_- . We can apply Corollary 2.9 for both fans Σ_{\pm} and J their common subset of maximal cones. Condition (A2) imposed on the analytic continuation path $z(u)$ ensures that, for $0 \leq u \leq 1$, the point $(-\log |z_1(u)|, \dots, -\log |z_n(u)|) \in C_J^v + c_J$. Note that the branches $\log_{\pm} r_j$ are identical for $(r_1, \dots, r_n) \in B^v$. It follows that the functions $\mathcal{E}_{\pm}^v(z, r) - (\mathcal{E}_{\pm}^v)^{\text{es}}(z, r)$ coincide on $U_J \times B^v$, with U_J the open domain in \mathbb{C}^n provided by Corollary 2.9.

(ii) To simplify notation, we set

$$\lambda_j := \frac{1}{2\pi i} \log_+ r_j.$$

For any $l \in \mathbb{L}$, $m_+(l)$ was defined as the minimum integer m such that $mh_j + l_j + \gamma_j^v \in \mathbb{Z}_{\geq 0}$, for some j with $v_j \in I_+$ (cf. formula (2)). This shows that $m_+(l + mh) = m_+(l) + m$ for any $l \in \mathbb{L}$ and $m \in \mathbb{Z}$. In particular $m_+(l - m_+(l)h) = 0$, and $m_+(l - mh) \neq 0$ for any other integer m such that $m \neq m_+(l)$.

Hence, we can define the piecewise linear injection $\iota : \mathbb{L}' \rightarrow \mathbb{L}$ by

$$\iota(p(l)) := l - m_+(l)h. \tag{4}$$

Note that $p \circ \iota = \text{Id}_{\mathbb{L}'}$. We have that $m_+(\iota(p(l))) = 0$, and $\iota(p(l))$ is the unique element of the fiber $p^{-1}(p(l))$ with this property. For any $l \in \mathbb{L}$, it will be convenient to introduce the notations

$$l' := \iota(p(l)),$$

and

$$\mathcal{S}' := \iota(p(\mathcal{S}_+^{\text{es}}(\gamma^v))) = \iota(p(\mathcal{S}_-^{\text{es}}(\gamma^v(t)))) , \quad t \in \mathcal{I}(y^v).$$

We see that

$$m_+(l') = 0, \quad \text{for any } l' \in \mathcal{S}'.$$

It is worth noting that, for any $l \in \mathcal{S}_+^{\text{es}}(\gamma^v)$, we have that $l' = \iota(p(l))$ belongs to $\mathcal{S}_+^{\text{es}}(\gamma^v)$. This is an immediate consequence of Proposition 4.7, since $m_+(l') = 0$. Moreover, the choice of the branches \log_+ according to Remark 4.9 shows that

$$\mathcal{S}' \subset \mathcal{S}_+^{\text{es}}(\gamma^v) \subset \mathcal{C}_+.$$

For any $l = mh + l' \in \mathbb{L}$ and any $\lambda \in \mathbb{C}^k$, we can write that

$$\begin{aligned} & \prod_{j=1}^n \frac{z_j^{l_j + \lambda_j}}{\Gamma(l_j + \lambda_j + 1)} \\ &= \prod_{j=1}^n z_j^{l'_j + \lambda_j} \cdot (-1)^{-\sum_{j,v_j \in I_-} l'_j} \cdot \frac{\prod_{j,v_j \in I_-} (\sin(-\pi \lambda_j) / \pi) \Gamma(-mh_j - l'_j - \lambda_j)}{\prod_{j,v_j \notin I_-} \Gamma(mh_j + l'_j + \lambda_j + 1)} \\ & \cdot \left((-1)^{\sum_{j,v_j \in I_-} h_j} \prod_{j=1}^n z_j^{h_j} \right)^m. \end{aligned} \tag{5}$$

Since $r \in B^v$, we see that the values of the parameters λ_j are localized around γ_j^v such that the only possible integer value for each λ_j is γ_j^v . For $\|\lambda - \gamma^v\| < \epsilon$, consider the Mellin–Barnes integral (of the type analyzed in Lemma A.6)

$$I := \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} I(s) ds,$$

with the integrand $I(s)$ given by

$$\frac{\prod_{j,v_j \in I_-} (\sin(-\pi \lambda_j)/\pi) \Gamma(-sh_j - l'_j - \lambda_j) \Gamma(-s) \Gamma(1+s)}{\prod_{j,v_j \notin I_-} \Gamma(sh_j + l'_j + \lambda_j + 1)} (e^{i\pi} y)^s,$$

where

$$y := e^{i\pi \sum_{j,v_j \in I_-} h_j} \prod_{j=1}^n z_j^{h_j},$$

the path of integration is parallel to the imaginary axis, and a is a strictly negative real number such that $\epsilon < |a| < 1$. In particular, the contour avoids any poles of the integrand. The hypotheses of Lemma A.6 are satisfied, with $H = 2$ and $\beta = 0$. The integral is absolutely convergent and defines an analytic function of y for $-\pi < \arg y < 0$, and is equal to the sum of the residues at poles on the right of the contour for $|y| < \rho$, and to the negative of the sum of the residues at poles on the left of the contour for $|y| > \rho$, where in this case $\rho = \prod_{j,v_j \in I} |h_j|^{h_j}$.

This special form of the Mellin–Barnes integral has been chosen such that the residue at any pole $s = m \in \mathbb{Z}$ is exactly the last line of formula (5). The other poles of the integrand $I(s)$ are the poles of the product $\prod_{j,v_j \in I_-} \Gamma(-sh_j - l'_j - \lambda_j)$. Set $s = m + \theta$, $m \in \mathbb{Z}$, with

$$\theta := \frac{1}{2\pi i} \log \hat{t},$$

where the branch $\log \hat{t}$ is the one chosen above, and we have that

$$\frac{1}{2\pi i} \log_{-}(y_j^v t^{h_j}) = \gamma_j^v(t) = \gamma_j^v + h_j \log t,$$

for $t \in \mathcal{I}(y^v)$. We see that each pole of the integrand is the sum of some $m \in \mathbb{Z}$ and a complex number θ such that $\lambda_j + h_j \theta \in \mathbb{Z}$, for some j with $v_j \in I_-$. This is equivalent to the condition that

$$\frac{1}{2\pi i} (\log_{+} r_j + h_j \log \hat{t}) = \frac{1}{2\pi i} \log_{-}(r_j \hat{t}^{h_j}) \in \mathbb{Z},$$

for some j with $v_j \in I_-$. Let $\mathcal{I}(r)$ be the set of such values \hat{t} .

Recall that the set $\mathcal{I}(y^v)$ consists of those roots of unity t that satisfy $\frac{1}{2\pi i} \log_{-}(y_j^v t^{h_j}) \in \mathbb{Z}$, for some j with $v_j \in I_-$. Note that for values $r \in (\mathbb{C}^*)^n$ in an open infinitesimal neighborhood of $y^v \in (\mathbb{C}^*)^n$, the elements of the set $\mathcal{I}(r)$ are clustered around the elements of $\mathcal{I}(y^v)$. More

precisely, for each $t \in \mathcal{I}(y^v)$, we can choose mutually disjoint discs centered at t , such that any element of the set $\mathcal{I}(r)$ is contained in one of these discs. For each $t \in \mathcal{I}(y^v)$, the contour C_t is the boundary of the corresponding disc. If $1 \in \mathcal{I}(y^v)$, we choose the circle C_1 centered at 1 which contains only the pole $\hat{t} = 1$ inside.

The Mellin–Barnes integral $(2\pi i)I$ introduced above is then equal to

$$\begin{aligned}
 & - \sum_{t \in \{1\} \cup \mathcal{I}(y^v)} \sum_{m \geq 0} \int_{C_t} \frac{\prod_{j, v_j \in I_-} (\sin(-\pi \lambda_j) / \pi) \Gamma(-h_j(m + \theta) - l'_j - \lambda_j)}{\prod_{j, v_j \notin I_-} \Gamma(h_j(m + \theta) + l'_j + \lambda_j + 1)} \\
 & \cdot \Gamma(-m - \theta) \Gamma(1 + m + \theta) (e^{i\pi} y)^{m + \theta} \frac{d\hat{t}}{2\pi i \hat{t}},
 \end{aligned}$$

when $|y| < \rho$, and to

$$\begin{aligned}
 & \sum_{t \in \{1\} \cup \mathcal{I}(y^v)} \sum_{m < 0} \int_{C_t} \frac{\prod_{j, v_j \in I_-} (\sin(-\pi \lambda_j) / \pi) \Gamma(-h_j(m + \theta) - l'_j - \lambda_j)}{\prod_{j, v_j \notin I_-} \Gamma(h_j(m + \theta) + l'_j + \lambda_j + 1)} \\
 & \cdot \Gamma(-m - \theta) \Gamma(1 + m + \theta) (e^{i\pi} y)^{m + \theta} \frac{d\hat{t}}{2\pi i \hat{t}},
 \end{aligned}$$

when $|y| > \rho$. Of course, the closed contours C_t avoid all the poles of the integrand when λ is localized near γ^v .

In both cases, a direct application of the Γ -identity, shows that, for each $m \in \mathbb{Z}$, the above integrand in \hat{t} is equal to

$$\begin{aligned}
 & \frac{\pi e^{-i\pi\theta}}{2\pi i \sin(-\pi\theta)} \prod_{j, v_j \in I_-} \frac{(-1)^{-h_j m - l'_j} \sin(-\pi \lambda_j)}{\sin(-\pi(\lambda_j + h_j \theta))} \prod_{j=1}^n \frac{1}{\Gamma(h_j(m + \theta) + l'_j + \lambda_j + 1)} y^{m + \theta} \\
 & = -2\pi i e^{-\pi i \sum_{j, v_j \in I_-} h_j \theta} (-1)^{\sum_{j, v_j \in I_-} (-h_j m - l'_j)} T(r, \hat{t}) \prod_{j=1}^n \frac{1}{\Gamma(h_j(m + \theta) + l'_j + \lambda_j + 1)} y^{m + \theta} \\
 & = -2\pi i (-1)^{-\sum_{j, v_j \in I_-} l'_j} T(r, \hat{t}) \prod_{j=1}^n \frac{z_j^{h_j(m + \theta)}}{\Gamma(h_j(m + \theta) + l'_j + \lambda_j + 1)},
 \end{aligned}$$

where, by the statement of the theorem, the function $2\pi i T(r, \hat{t})$ is equal to

$$\frac{1}{e^{2\pi i \theta} - 1} \prod_{j, v_j \in I_-} \frac{1 - e^{-2\pi i \lambda_j}}{1 - e^{-2\pi i(\lambda_j + h_j \theta)}} = \frac{1}{\hat{t} - 1} \prod_{j, v_j \in I_-} \frac{1 - r_j^{-1}}{1 - r_j^{-1} \hat{t}^{-h_j}}.$$

Therefore, the Mellin–Barnes integral I introduced above is equal to

$$(-1)^{-\sum_{j, v_j \in I_-} l'_j} \sum_{t \in \{1\} \cup \mathcal{I}(y^v)} \sum_{m \geq 0} \int_{C_t} T(r, \hat{t}) \prod_{j=1}^n \frac{z_j^{h_j(m + \theta)}}{\Gamma(h_j(m + \theta) + l'_j + \lambda_j + 1)} d\hat{t},$$

when $|y| < \rho$, and to

$$-(-1)^{-\sum_{j,v_j \in I_-} l'_j} \sum_{t \in \{1\} \cup \mathcal{I}(y^v)} \sum_{m < 0} \int_{C_t} T(r, \hat{t}) \prod_{j=1}^n \frac{z_j^{h_j(m+\theta)}}{\Gamma(h_j(m+\theta) + l'_j + \lambda_j + 1)} d\hat{t},$$

when $|y| > \rho$. It follows that the analytic continuation of the sum of the former series along the path $z(u)$ is the sum of the latter series.

Recall now that the cones \mathcal{C}_-^\vee and \mathcal{C}_+^\vee have a common facet (maximal dimensional face) which we denote by $\tilde{\mathcal{C}}$. It is the unique facet of the cones \mathcal{C}_\pm^\vee orthogonal to the element $h \in \mathbb{L}$. In order to proceed with the analytic continuation, we need two lemmas.

Lemma 4.11. *For any real constants $k, A > 0$, there exists an element \tilde{c} deep in the interior of the cone $\tilde{\mathcal{C}}$ such that, for any $l' \in \mathcal{S}'$, we have that*

$$\langle u, l' \rangle \geq k \|l'\|,$$

for any $u \in \tilde{\mathcal{C}} + \tilde{c} + a$, and any $a \in \mathbb{L} \otimes \mathbb{R}$ with $\|a\| \leq A$.

Proof. Note first that

$$\langle a, l' \rangle \geq -A \|l'\|,$$

since $\|a\| \leq A$.

The dual of projection $p: \mathbb{L} \rightarrow \mathbb{L}'$ is a lattice embedding $(\mathbb{L}')^\vee \hookrightarrow \mathbb{L}^\vee$ such that the image of $(\mathbb{L}')^\vee \otimes \mathbb{R}$ is the hyperplane in $\mathbb{L}^\vee \otimes \mathbb{R}$ generated by the cone $\tilde{\mathcal{C}}$. Hence, with a slight abuse of notation, for any $x \in \tilde{\mathcal{C}}$, we can write that

$$\langle x, l' \rangle = \langle x, p(l') \rangle,$$

where we use identical bracket notations for the pairings between \mathbb{L}^\vee and \mathbb{L} , and $(\mathbb{L}')^\vee$ and \mathbb{L}' , respectively.

As noted above, the choice of the branches \log_+ implies that $\mathcal{S}' \subset \mathcal{C}_+$, which shows that

$$p(\mathcal{S}') \subset p(\mathcal{C}_+) \subset \tilde{\mathcal{C}}^\vee \subset \mathbb{L}' \otimes \mathbb{R}.$$

For any constant $M > 0$, we can then choose an element \tilde{c} deep enough in the interior of the cone $\tilde{\mathcal{C}}$ (see the end of the proof of Proposition 2.8) such that, for any $x \in \tilde{\mathcal{C}} + \tilde{c}$ we have that

$$\langle x, p(l') \rangle \geq M \|p(l')\|,$$

for any $l' \in \mathcal{S}'$. But $l' = \iota(p(l'))$ where the piecewise linear injection $\iota: \mathbb{L}' \rightarrow \mathbb{L}$ has been defined by formula (4). Hence, there exists a constant $K > 0$ such that

$$\|p(l')\| \geq K \|l'\|.$$

We conclude that for an appropriate choice of the element $\tilde{c} \in \tilde{\mathcal{C}}$ we have that

$$\langle x, l' \rangle \geq (k + A) \|l'\|,$$

for any $x \in \tilde{\mathcal{C}} + \tilde{c}$ and $l' \in S'$. The lemma follows. \square

Lemma 4.12. *There exists a value $A > 0$, and an element $\tilde{c} \in \tilde{\mathcal{C}}$, such that the set*

$$V_A := \{\tilde{\mathcal{C}} + \tilde{c} + a, \|a\| < A\},$$

intersects the sets $\mathcal{C}_\pm^\vee + c_\pm$, and, such that the integral

$$\int_{a+i\infty}^{a-i\infty} \sum_{l' \in S'} \prod_{j=1}^n z_j^{l'_j + \lambda_j} (-1)^{-\sum_{j, v_j \in l_-} l'_j} I(s) ds$$

is absolutely convergent, and defines an analytic function of (z, r) in an open domain containing the region $U \times B^v$ defined by the restrictions

$$U := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n: (-\log |z_1|, \dots, -\log |z_n|) \in V_A, -2\pi < \arg y < 0, \\ (\arg z_1, \dots, \arg z_n) \in (-\pi, \pi) \times \dots \times (-\pi, \pi)\},$$

where, as before,

$$y = e^{i\pi \sum_{j, v_j \in l_-} h_j} \prod_{j=1}^n z_j^{h_j}.$$

Proof. The proof of the lemma is very similar to that of Proposition 2.8 and its corollaries. The definition of V_A , as well as the restriction (A3) imposed on the path $z(u)$ show that U is an open set in \mathbb{C}^n . Moreover, it is clear that by choosing a large enough $A > 0$ we can ensure that the intersection of V_A with the sets $\mathcal{C}_\pm^\vee + c_\pm$ is non-empty.

The hypotheses of Lemma A.5 are satisfied. Hence, on the line $s = a + it, t \in \mathbb{R}$, the absolute value of the integrand is bounded above by a positive constant multiple of

$$|y|^a e^{-(\pi + \arg y)t} (|t| + 1)^{R+n/2} e^{-\pi|t|} \sum_{l' \in S'} (4ek)^{\|l'\|} e^{\sum l'_j \log |z_j|},$$

where $R > 0$ is a positive constant determined by (r_1, \dots, r_n) .

As in the proof of Proposition 2.8, the above estimate ignores the factors $z_j^{\lambda_j}$. Lemma 4.11 allows us to choose \tilde{c} deep inside the cone $\tilde{\mathcal{C}}$ such that there exists some $\epsilon > 0$, for which

$$(4ek)^{\|l'\|} e^{\sum l'_j \log |z_j|} \leq e^{-\epsilon \|l'\|},$$

for any $l' \in S'$ and $z \in U$. The argument restriction on y insures the absolute convergence of the integral. \square

Note that the region $U \subset \mathbb{C}^n$ defined in the previous lemma imposes no restrictions on $|y|$, i.e. U contains points $z = (z_1, \dots, z_n)$ whose associated coordinate y has an absolute value that is arbitrarily small or large, as needed. This observation allows the analytic continuation procedure between the regions $|y| < \rho$ and $|y| > \rho$ to be performed along the path $z(u)$.

Hence, Lemma 4.12 implies that the analytic continuation along the path $z(u)$ from the subdomain $|y| < \rho$ of $U \times B^v$ of the series

$$\frac{1}{2\pi i} \sum_{t \in \{1\} \cup \mathcal{I}(y^v)} \sum_{l' \in \mathcal{S}'} \sum_{m \geq 0} \int_{C_t} T(r, \hat{t}) \prod_{j=1}^n \frac{z_j^{h_j(m+\theta)+l'_j+\lambda_j}}{\Gamma(h_j(m+\theta)+l'_j+\lambda_j+1)} d\hat{t}, \tag{6}$$

to the subdomain $|y| > \rho$ of $U \times B^v$, is the series

$$-\frac{1}{2\pi i} \sum_{t \in \{1\} \cup \mathcal{I}(y^v)} \sum_{l' \in \mathcal{S}'} \sum_{m < 0} \int_{C_t} T(r, \hat{t}) \prod_{j=1}^n \frac{z_j^{h_j(m+\theta)+l'_j+\lambda_j}}{\Gamma(h_j(m+\theta)+l'_j+\lambda_j+1)} d\hat{t}. \tag{7}$$

For convenience, we introduce the notation

$$P_t(l', m) := \frac{1}{2\pi i} \int_{C_t} T(r, \hat{t}) \prod_{j=1}^n \frac{z_j^{h_j(m+\theta)+l'_j+\lambda_j}}{\Gamma(h_j(m+\theta)+l'_j+\lambda_j+1)} d\hat{t}.$$

Lemma 4.13. *For any $t \in \mathcal{I}(y^v)$, the series*

$$S_{-,t} := \sum_{l' \in \mathcal{S}'} \sum_{0 \leq m \leq \max(0, m_{-,t}(l'))} P_t(l', m)$$

is absolutely convergent for $(z, r) \in U \times B^v$, and defines an analytic function in the domain $U \times B^v$. The integers $m_{-,t}$ have been defined by formulae (2).

Proof. For any $t \in \mathcal{I}(y^v)$, the series $S_{-,t}$ is absolutely convergent in the subdomain of $U \times B^v$ characterized by $|y| < \rho$, since it is a subseries of the series (7) which is absolutely convergent in that region. Moreover, by Proposition 4.6(ii) we have that $p(\mathcal{S}_+^{\text{es}}(y^v)) = p(\mathcal{S}_-^{\text{es}}(y^v(t)))$, so, if $l' \in \mathcal{S}'$ and $m \leq m_{-,t}(l')$, Proposition 4.7 shows that $l + mh \in \mathcal{S}_+^{\text{es}}(y^v(t))$. Hence, the series $S_{-,t}$ is a subseries of the absolutely convergent series in the subdomain $|y| < \rho$ obtained by integrating over C_t the series defining the function $\Xi_-^{v(t)}(z, r\hat{t}^h)$. In conclusion, the absolute convergence of $S_{-,t}$ is independent of the magnitude of y , therefore it holds everywhere in $U \times B^v$. \square

As a direct consequence of Lemma 4.13 we see that, for any $t \in \mathcal{I}(y^v)$, the analytic continuation along the path $z(u)$ from the subdomain $|y| < \rho$ of $U \times B^v$ of the series

$$\sum_{l' \in \mathcal{S}'} \sum_{m \geq 0} P_1(l', m) + \sum_{t \in \mathcal{I}(y^v) \setminus \{1\}} \sum_{l' \in \mathcal{S}'} \sum_{m > \max(0, m_{-,t}(l'))} P_t(l', m), \tag{8}$$

to the subdomain $|y| > \rho$ of $U \times B^v$, is the series

$$- \sum_{l' \in \mathcal{S}'} \sum_{m < 0} P_1(l', m) - \sum_{t \in \mathcal{I}(y^v) \setminus \{1\}} \sum_{l' \in \mathcal{S}'} \sum_{m \leq \max(0, m_{-,t}(l'))} P_t(l', m). \tag{9}$$

It is important to remember that, for any $l' \in \mathcal{S}'$, we have that $m_+(l') = 0$.

Note that, if $t \in \mathcal{I}(y^v) \setminus \{1\}$ and $m > m_{-,t}(l')$, then the function in \hat{t}

$$T(r, \hat{t}) \prod_{j=1}^n \frac{z_j^{h_j(m+\theta)+l'_j+\lambda_j}}{\Gamma(h_j(m+\theta)+l'_j+\lambda_j+1)} \tag{10}$$

has no poles inside the contour C_t . Hence, the second term in formula (8) is always zero. Furthermore, the second term in formula (9) is equal to

$$- \sum_{t \in \mathcal{I}(y^v) \setminus \{1\}} \int_{C_t} T(r, \hat{t}) (\mathcal{E}_-^{v(t)})^{\text{es}}(z, r\hat{t}^h) d\hat{t}.$$

If $1 \notin \mathcal{I}(y^v)$, the function (10) has only the simple pole $\tilde{t} = 1$ inside the contour C_1 , for all integers m . Hence, in this case, the series (8) is equal to $(\mathcal{E}_+^v)^{\text{es}}(z, r)$.

Moreover, if $1 \notin \mathcal{I}(y^v)$, and $m < 0 = m_+(l')$, the residue of the function (10) at $\tilde{t} = 1$ can be written as a product of an analytic function on $\mathbb{C}^n \times B^v$ and the product $\prod_{j, v_j \in I_+} (1 - r_j^{-1})$. Hence, in this case, the first term in formula (9) is of the form

$$\prod_{j, v_j \in I_+} (1 - r_j^{-1}) \phi(z, r),$$

with $\phi(z, r)$ an analytic function on $U_- \times B^v$. This proves part (ii) of the theorem in the case $1 \notin \mathcal{I}(y^v)$.

Let us now assume that $1 \in \mathcal{I}(y^v)$. We only have to analyze the first terms in the series (8) and (9). Note that, if $m > m_{-,1}(l')$, then

$$P_1(l', m) = \prod_{j=1}^n \frac{z_j^{mh_j+l'_j+\lambda_j}}{\Gamma(mh_j+l'_j+\lambda_j+1)}.$$

It is convenient to introduce the notation

$$R(l', m) := \prod_{j=1}^n \frac{z_j^{mh_j+l'_j+\lambda_j}}{\Gamma(mh_j+l'_j+\lambda_j+1)}.$$

We proceed with a case by case analysis according to signs of the integers $m_{-,1}(l')$. Namely, we write the set \mathcal{S}' as a disjoint union of the subsets \mathcal{S}_1 and \mathcal{S}_2 , where

$$\begin{aligned} \mathcal{S}_1 &:= \{l' \in \mathcal{S}' : m_{-,1}(l') \geq 0\}, \\ \mathcal{S}_2 &:= \{l' \in \mathcal{S}' : m_{-,1}(l') < 0\}. \end{aligned}$$

The terms of the series $\sum_{l' \in \mathcal{S}'} \sum_{m \geq 0} P_1(l', m)$ (the first half of the series (8)) coming from $l' \in \mathcal{S}_1$ add up to

$$\sum_{l' \in \mathcal{S}_1} \left(\sum_{m \geq 0} R(l', m) - \sum_{0 \leq m \leq m_{-1}(l')} R(l', m) + \sum_{0 \leq m \leq m_{-1}(l')} P_1(l', m) \right).$$

The terms of the series $\sum_{l' \in \mathcal{S}'} \sum_{m \geq 0} P_1(l', m)$ coming from $l' \in \mathcal{S}_2$ add up to

$$\sum_{l' \in \mathcal{S}_2} \sum_{m \geq 0} R(l', m).$$

The terms of the series $-\sum_{l' \in \mathcal{S}'} \sum_{m < 0} P_1(l', m)$ (the first half of the series (9)) coming from $l' \in \mathcal{S}_1$ add up to

$$-\sum_{l' \in \mathcal{S}_1} \sum_{m < 0} P_1(l', m).$$

The terms of the series $-\sum_{l' \in \mathcal{S}'} \sum_{m < 0} P_1(l', m)$ coming from $l' \in \mathcal{S}_2$ add up to

$$-\sum_{l' \in \mathcal{S}_2} \left(\sum_{m \leq m_{-1}(l')} P_1(l', m) + \sum_{m_{-1}(l') < m < 0} R(l', m) \right).$$

Note that, since $m_+(l') = 0$ for all $l' \in \mathcal{S}'$, the second part of this series is of the form

$$\prod_{j, v_j \in I_+} (1 - r_j^{-1}) \varphi_1(z, r),$$

where $\varphi_1(z, r)$ is an analytic function in the domain $U_- \times B^v$.

When we put everything together, the series (8) is equal to

$$\sum_{l' \in \mathcal{S}_1 \cup \mathcal{S}_2} \sum_{m \geq 0} R(l', m) - \sum_{l' \in \mathcal{S}_1} \sum_{0 \leq m \leq m_{-1}(l')} R(l', m) + \sum_{l' \in \mathcal{S}_1} \sum_{0 \leq m \leq m_{-1}(l')} P_1(l', m).$$

It is important to note that Lemma 4.13 implies that the last terms of the previous formula define analytic functions in $U \times B^v$, so they do not change under analytic continuation.

The series (9) is equal to

$$-\sum_{l' \in \mathcal{S}_1} \sum_{m < 0} P_1(l', m) - \sum_{l' \in \mathcal{S}_2} \sum_{m \leq m_{-1}(l')} P_1(l', m) + \prod_{j, v_j \in I_+} (1 - r_j^{-1}) \varphi_1(z, r).$$

It follows that the analytic continuation along the path $z(u)$ from the subdomain $|y| < \rho$ of $U \times B^v$ of the series

$$(\mathcal{E}_+^v)^{\text{es}}(z, r) = \sum_{l' \in \mathcal{S}_1 \cup \mathcal{S}_2} \sum_{m \geq 0} R(l', m)$$

to the subdomain $|y| > \rho$ of $U \times B^v$, is the series

$$\sum_{l' \in \mathcal{S}_1} \sum_{0 \leq m \leq m_{-1}(l')} R(l', m) - \sum_{l' \in \mathcal{S}_1 \cup \mathcal{S}_2} \sum_{m \leq m_{-1}(l')} P_1(l', m) + \prod_{j, v_j \in I_+} (1 - r_j^{-1}) \varphi_1(z, r) - \frac{1}{2\pi i} \sum_{t \in \mathcal{I}(y^v) \setminus \{1\}} \int_{C_t} T(r, \hat{t}) (\mathcal{E}_-^{v(t)})^{\text{es}}(z, r \hat{t}^h) d\hat{t}.$$

Moreover, for $(z, r) \in U_- \times B^v$, there exists an analytic function $\varphi_2(z, r)$ in $U_- \times B^v$, such that

$$\sum_{l' \in \mathcal{S}_1} \sum_{0 \leq m \leq m_{-1}(l')} R(l', m) = (\mathcal{E}_-^{v(t)})^{\text{es}}(z, r) + \prod_{j, v_j \in I_+} (1 - r_j^{-1}) \varphi_2(z, r),$$

where

$$\left(\sum_{l' \in \mathcal{S}_1} \sum_{m < 0} + \sum_{l' \in \mathcal{S}_2} \sum_{m \leq m_{-1}(l')} \right) R(l', m) = - \prod_{j, v_j \in I_+} (1 - r_j^{-1}) \varphi_2(z, r).$$

The proof of the theorem is then finished if we note that

$$\sum_{l' \in \mathcal{S}_1 \cup \mathcal{S}_2} \sum_{m \leq m_{-1}(l')} P_1(l', m) = \int_{C_1} T(r, \hat{t}) (\mathcal{E}_-^{v(1)})^{\text{es}}(z, r \hat{t}^h) d\hat{t}. \quad \square$$

Let

$$B := \bigcup_{v \in \text{Box}(\Sigma_+)} B^v.$$

Consistent with Definition 3.4, we define the functions

$$\mathcal{E}_+, (\mathcal{E}_+)^{\text{es}} : U_+ \times B \rightarrow \mathbb{C},$$

such that $\mathcal{E}_+(z, r) = \mathcal{E}_+^v(z, r)$, $(\mathcal{E}_+)^{\text{es}}(z, r) = (\mathcal{E}_+^v)^{\text{es}}(z, r)$, for all $v \in \text{Box}(\Sigma_+)$ and $(z, r) \in U_+ \times B^v$. We also define

$$\mathcal{E}_-, (\mathcal{E}_-)^{\text{es}} : U_- \times \bigcup_{v \in \text{Box}(\Sigma_{\pm})} B^v \rightarrow \mathbb{C},$$

such that $\mathcal{E}_-(z, r) = \mathcal{E}_-^v(z, r)$, $(\mathcal{E}_-)^{\text{es}}(z, r) = (\mathcal{E}_-^v)^{\text{es}}(z, r)$, for all $v \in \text{Box}(\Sigma_{\pm})$ and $(z, r) \in U_- \times B^v$. For $v \in \text{Box}(\Sigma_+) \setminus \text{Box}(\Sigma_-)$, we set $\mathcal{E}_-^v = (\mathcal{E}_-^v)^{\text{es}} = 0$. This choice is consistent with the freedom in choosing the function \mathcal{E}_- discussed in Remark 3.5.

Let $\mathcal{I} \subset \mathbb{C}^*$ be the set of roots of unity of a large enough order such that $\mathcal{I}(y^v) \subset \mathcal{I}$ for all $v \in \text{Box}(\Sigma_+)$. When $t \notin \mathcal{I}(y^v)$, we see that $\mathcal{E}_-^{v(t)} = (\mathcal{E}_-^{v(t)})^{\text{es}} = 0$. With these conventions, the results of the theorem can be expressed in a more convenient way as follows.

Corollary 4.14.

- (i) The function $\mathcal{E}_+(z, r) - (\mathcal{E}_+)^{\text{es}}(z, r)$ is analytic in the open domain $U_J \times B$, and the open domain U_J contains the sets U_{\pm} and the path $z(u)$. The analytic functions $\mathcal{E}_+(z, r) - (\mathcal{E}_+)^{\text{es}}(z, r)$ and $\mathcal{E}_-(z, r) - (\mathcal{E}_-)^{\text{es}}(z, r)$ are equal for all $(z, r) \in U_J \times B$.
- (ii) The analytic continuation along the path $(z(u), y)$ of the germ of the analytic function $(\mathcal{E}_+)^{\text{es}}(z, r)$ at $(z_+, y) \in U_+ \times B$ is given by the germ at $(z_-, y) \in U_- \times B$ of the analytic function

$$(\mathcal{E}_-)^{\text{es}}(z, r) - \sum_{I \in \mathcal{I}} \int_{C_I} T(r, \hat{t})(\mathcal{E}_-)^{\text{es}}(z, r\hat{t}^h) d\hat{t} + \prod_{j, v_j \in I_+} (1 - r_j^{-1})\varphi(z, r).$$

Here, $\varphi(z, r)$ is an analytic function on $U_- \times B$, while the integration kernel $T(r, \hat{t})$ and the contours C_I are defined as in the statement of Theorem 4.10.

We finish this section with a definition and notation which will be useful in the next section.

Definition 4.15. The analytic continuation operator along the path $z(u)$ from the domain $U_+ \times B$ to the domain $U_- \times B$ is called the *Mellin–Barnes operator*. For any analytic function $\phi(z, r)$ in $U_+ \times B$, $MB(\phi)(z, r)$ denotes its analytic continuation along the path $z(u)$ to the domain $U_- \times B$.

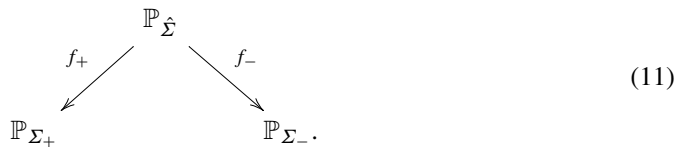
5. Toric birational maps vs. analytic continuation

As in the previous section, we consider a modification associated with the integral relation $h_1 v_1 + \dots + h_n v_n = 0$ and the corresponding circuit $I = \{v_j, h_j \neq 0\} \subset \mathcal{A}$ that determines a change of the fan Σ_+ into the fan Σ_- . Let $\hat{v} \in N$ be the vector

$$\hat{v} := \sum_{j, v_j \in I_+} h_j v_j = \sum_{j, v_j \in I_-} (-h_j) v_j.$$

Let $\hat{\Sigma}$ be a stacky fan refining the fans Σ_{\pm} obtained by replacing the cones generated in the fans Σ_{\pm} by sets of type $I \setminus v_{\pm}$, with $v_{\pm} \in I_{\pm}$ respectively, with cones generated by sets of the type $\hat{v} \cup I \setminus \{v_-, v_+\}$ for $v_{\pm} \in I_{\pm}$. It is important to note that the possibly non-primitive vector \hat{v} is part of the information defining the stacky fan $\hat{\Sigma}$. This definition makes sense even if one of the sets I_- or I_+ has only one element. In that case, the new vector \hat{v} replaces the corresponding generator of a one dimensional cone in Σ_- or Σ_+ .

We have the following diagram of weighted blowdowns (possibly in codimension one, if either $|I_-|$, or $|I_+|$ is equal to one):



We will study the properties of the “Fourier–Mukai” map

$$FM: K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}) \rightarrow K_0(\mathbb{P}_{\Sigma_+}, \mathbb{C})$$

defined by

$$FM := (f_+)_*(f_-)^*.$$

We will use the same notation $R_j, 1 \leq j \leq n$, for the K -theory classes induced by the vectors v_j in any of the toric DM stacks $\mathbb{P}_{\Sigma_{\pm}}, \mathbb{P}_{\hat{\Sigma}}$. For details about how the correspondence between vectors and K -classes works see [6, Section 4]. In particular, we may happen that $R_j = 1$ when the vector v_j does not generate a cone in the corresponding fan. We denote by \hat{R} the K -theory class in $\mathbb{P}_{\hat{\Sigma}}$ induced by the vector \hat{v} .

Let σ be a cone of the fan Σ_- generated by the vectors v_j with $v_j \in J \subset \mathcal{A}$. Assume that σ is not a subcone of any essential maximal cone (see Definition 4.1).

In other words, any maximal cone of Σ_- containing σ as a subcone is also a cone of Σ_+ and $\hat{\Sigma}$. It follows that the quotient fans $\Sigma_{\pm}/\sigma, \hat{\Sigma}/\sigma$ are all unchanged, and by Proposition 4.2 of [8], they define closed toric substacks in the ambient toric stacks. Note that the toric substack induced by the cone σ and the exceptional toric substack induced by the vector \hat{v} in $\mathbb{P}_{\hat{\Sigma}}$ have empty intersection. Hence, the restrictions of the maps f_{\pm} to the closed substack $\mathbb{P}_{\hat{\Sigma}/\sigma}$ of $\mathbb{P}_{\hat{\Sigma}}$ are isomorphisms onto their images in $\mathbb{P}_{\Sigma_{\pm}}$, the closed substacks $\mathbb{P}_{\Sigma_{\pm}/\sigma}$. As a direct consequence, the following proposition holds:

Proposition 5.1. *For any polynomial $\phi(r_1, \dots, r_n) \in \mathbb{C}[r_1, \dots, r_n]$, we have that*

$$FM\left(\prod_{j, v_j \in J} (1 - R_j)\phi(R)\right) = \prod_{j, v_j \in J} (1 - R_j)\phi(R).$$

Proof. It is enough to check the statement for ϕ an arbitrary monomial $\prod_{i=1}^n r_i^{m_i}$. We can also assume that J contains no elements of I . If it did, the elements of $J \setminus (J \cap I)$ would generate a cone that is not a subcone of any essential maximal cone of Σ_- , and the statement for $J \setminus (J \cap I)$ would imply the one for J . Note that, since the elements of J generate a cone in Σ_- , it is not possible for I to be a subset of J .

According to Theorem 9.1 (the general case) and Corollary 9.4 (the blowdown of codimension one case) in [6], we have that $(f_+)_*(1) = 1$. By the projection formula, it is then enough to show that the pull-backs of the classes $\prod_{j, v_j \in J} (1 - R_j) \prod_{i=1}^n R_i^{m_i}$ from $K_0(\mathbb{P}_{\Sigma_{\pm}}, \mathbb{C})$ coincide in $K_0(\mathbb{P}_{\hat{\Sigma}}, \mathbb{C})$. Proposition 8.1 in [6] implies that these pull-backs are written, with a slight abuse of notation, as

$$\prod_{j, v_j \in J} (1 - R_j) \prod_{i=1}^n R_i^{m_i} \hat{R}^{m_{\pm}}$$

for some integers m_{\pm} .

However, our assumption on the set J implies that the vector \hat{v} and the elements of J do not generate a cone in $\hat{\Sigma}$, so [6, Theorem 4.10] shows that

$$\prod_{j, v_j \in J} (1 - R_j)(1 - \hat{R}) = 0$$

in $K_0(\mathbb{P}_{\hat{\Sigma}}, \mathbb{C})$. We conclude that

$$\prod_{j, v_j \in J} (1 - R_j) \hat{R}^{m_{\pm}} = \prod_{j, v_j \in J} (1 - R_j)$$

for any integers m_{\pm} . This ends the proof of the proposition. \square

Proposition 5.2. *For any polynomial $\phi(r_1, \dots, r_n) \in \mathbb{C}[r_1, \dots, r_n]$, we have that $FM(\phi(R)) = (FM(\phi)(\mathcal{R}))(1)$, where the function $FM(\phi)$ is defined by*

$$FM(\phi)(r) := \phi(r) - \sum_{t \in \mathcal{I}} \int_{C_t} T(r, \hat{t}) \phi(r \hat{t}^h) d\hat{t}.$$

The integration kernel $T(r, \hat{t})$, the set of roots of unity $\mathcal{I} \subset \mathbb{C}^*$, as well as the contours C_t , are those used in the statement of Corollary 4.14.

Proof. According to Theorem 9.1 (the general case) and Corollary 9.4 (the blowdown of codimension one case) in [6], if \hat{R} denotes the K -theory class determined by the vector \hat{v} in $K_0(\mathbb{P}_{\hat{\Sigma}}, \mathbb{C})$, we have the following equality of formal power series in \hat{t} :

$$(f_+)_* \left(\frac{1}{1 - \hat{R}^{-1} \hat{t}} \right) = \frac{1}{1 - \hat{t}} - \frac{\hat{t}}{1 - \hat{t}} \cdot \prod_{j, v_j \in I_-} \frac{1 - R_j^{-1}}{1 - R_j^{-1} \hat{t}^{-h_j}}.$$

At this point, it is more convenient to work with linear operators on K -theory, rather than the K -theory itself. Let

$$\hat{\mathcal{R}}: K_0(\mathbb{P}_{\hat{\Sigma}}, \mathbb{C}) \rightarrow K_0(\mathbb{P}_{\hat{\Sigma}}, \mathbb{C})$$

be the linear map (and ring endomorphism) given by multiplication with the class \hat{R} . In order to understand the spectrum of $\hat{\mathcal{R}}$, note first that Proposition 4.4 can be applied for the relation

$$\hat{v} + \sum_{j, v_j \in I_-} h_j v_j = 0$$

inducing the modification from the fan Σ_+ to the fan $\hat{\Sigma}$. Proposition 4.4 implies then that the maximum ideals of $K_0(\mathbb{P}_{\hat{\Sigma}}, \mathbb{C})$ are among the $(n + 1)$ -tuples of roots of unity of the form $(t, y_1^v t^{p_1}, \dots, y_n^v t^{p_n})$ with $v \in \text{Box}(\Sigma_+)$, $t \in \mathcal{I}(y^v)$, and p_j is equal to h_j , for j with $v_j \in I_-$, and zero, otherwise. We conclude that the spectrum of $\hat{\mathcal{R}}$ consists of roots of unity contained in \mathcal{I} .

It follows that, for any integer k , the operator $\hat{\mathcal{R}}^k$ admits the Cauchy integral representation (see Appendix B)

$$\hat{\mathcal{R}}^k = -\frac{1}{2\pi i} \sum_{t \in \mathcal{I}} \int_{C_t} \hat{t}^{k-1} (I - \hat{t} \hat{\mathcal{R}}^{-1})^{-1} d\hat{t}.$$

Thus, for any polynomial in one variable $\psi(r) \in \mathbb{C}[r]$, we have that

$$\psi(\hat{R}) = -\frac{1}{2\pi i} \sum_{t \in \mathcal{I}} \int_{C_t} \psi(t) \hat{t}^{-1} (I - \hat{t} \hat{\mathcal{R}}^{-1})^{-1} (1) d\hat{t}.$$

The push down formula implies then that

$$\begin{aligned} (f_+)_*(\psi(\hat{R})) &= -\frac{1}{2\pi i} \sum_{t \in \mathcal{I}} \int_{C_t} \frac{\psi(\hat{t}) \hat{t}^{-1}}{1 - \hat{t}} d\hat{t} \\ &+ \frac{1}{2\pi i} \sum_{t \in \mathcal{I}} \int_{C_t} \frac{\psi(\hat{t})}{1 - \hat{t}} \prod_{j, v_j \in I_-} (I - \mathcal{R}_j^{-1})(I - \hat{t}^{-h_j} \mathcal{R}_j^{-1})^{-1} (1) d\hat{t}. \end{aligned}$$

The important fact to note is that, in the second line of the formula above, the contours $C_t, t \in \mathcal{I}$, enclose all the values \hat{t} where the operators $I - \hat{t}^{-h_j} \mathcal{R}_j^{-1}$ with j such that $v_j \in I_-$, are not invertible on $K_0(\mathbb{P}_{\Sigma_+}, \mathbb{C})$.

We now analyze the behavior under the pull-back $(f_-)^*$ of a monomial class $\prod R_j^{m_j}$ in $K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C})$, with m_j positive integers. Proposition 8.1 in [6] implies that

$$(f_-)^* \left(\prod_{j=1}^n R_j^{m_j} \right) = \prod_{j, v_j \notin I_+} R_j^{m_j} \prod_{j, v_j \in I_+} (R_j \hat{R}^{h_j})^{m_j} = \hat{R}^{\sum_{j, v_j \in I_+} h_j m_j} (f_+)^* \left(\prod_{j=1}^n R_j^{m_j} \right),$$

where we have used the observation that the modifications from Σ_- and Σ_+ to $\hat{\Sigma}$ are induced by the relations $\hat{v} = \sum_{j, v_j \in I_-} (-h_j) v_j$ and $\hat{v} = \sum_{j, v_j \in I_+} h_j v_j$, respectively. Hence, by the projection formula, we obtain that

$$\begin{aligned} (f_+)_*(f_-)^* \left(\prod_{j=1}^n R_j^{m_j} \right) &= \frac{1}{2\pi i} \sum_{t \in \mathcal{I}} \prod_{j=1}^n R_j^{m_j} \cdot \int_{C_t} \frac{\hat{t}^{\sum_{j, v_j \in I_+} h_j m_j}}{1 - \hat{t}} \left(-\hat{t}^{-1} + \prod_{j, v_j \in I_-} (I - \mathcal{R}_j^{-1})(I - \hat{t}^{-h_j} \mathcal{R}_j^{-1})^{-1} (1) \right) d\hat{t}. \end{aligned}$$

Since $h_j = 0$ for those j with $v_j \notin I$, we see that

$$\prod_{j=1}^n R_j^{m_j} \hat{t}^{\sum_{j, v_j \in I_+} h_j m_j} = \prod_{j=1}^n (R_j \hat{t}^{h_j})^{m_j}.$$

We conclude that, for any polynomial $\phi(r_1, \dots, r_n)$, we have that

$$\begin{aligned} FM(\phi(R_1, \dots, R_n)) &= (f_+)_*(f_-)^*(\phi(R_1, \dots, R_n)) \\ &= \frac{1}{2\pi i} \sum_{t \in \mathcal{I}} \int_{C_t} \phi(R_1 \hat{t}^{h_1}, \dots, R_n \hat{t}^{h_n}) \left(\frac{\hat{t}^{-1}}{\hat{t} - 1} - 2\pi i T(\mathcal{R}, \hat{t})(1) \right) d\hat{t} \\ &= \phi(R_1, \dots, R_n) - \sum_{t \in \mathcal{I}} \int_{C_t} T(\mathcal{R}, \hat{t}) \phi(\hat{t}^{h_1} \hat{\mathcal{R}}_1, \dots, \hat{t}^{h_n} \hat{\mathcal{R}}_n)(1) d\hat{t}. \end{aligned}$$

This ends the proof of the proposition. \square

Remark 5.3. The statements of the previous two propositions describe the Fourier–Mukai action on polynomial classes $\phi(R_1, \dots, R_n)$ in $K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C})$. Both results can be easily extended to the case when $\phi(r_1, \dots, r_n)$ is an analytic function in the domain $\bigcup_{v \in \text{Box}(\Sigma_-)} B^v$, where $B^v \subset \mathbb{C}^n$ are disjoint open sets around the n -tuples of roots of unity y^v (see Section 3 for more details on the choice of the sets B^v). It is then enough to choose a polynomial $\psi(r_1, \dots, r_n)$ (cf. Appendix B) such that the linear operators $\phi(\mathcal{R}_1, \dots, \mathcal{R}_n)$ and $\psi(\mathcal{R}_1, \dots, \mathcal{R}_n)$ coincide on $K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C})$, and to apply the previous two results for the polynomial ψ .

Theorem 5.4. *The following diagram is commutative:*

$$\begin{array}{ccc} (K_0(\mathbb{P}_{\Sigma_+}, \mathbb{C}))^\vee & \xrightarrow{MS_+} & \text{Sol}(U_+) \\ FM^\vee \downarrow & & \downarrow MB \\ (K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}))^\vee & \xrightarrow{MS_-} & \text{Sol}(U_-). \end{array}$$

Proof. For an arbitrary linear function $f : K_0(\mathbb{P}_{\Sigma_+}, \mathbb{C}) \rightarrow \mathbb{C}$, we have that

$$MB(MS_+(f)) = f(MB(\mathcal{E}_+)(z, \mathcal{R})(1)),$$

and that

$$MS_-(FM^\vee(f)) = f(FM(\mathcal{E}_-)(z, \mathcal{R})(1)).$$

We first write $\mathcal{E}_\pm(z, r) = (\mathcal{E}_\pm - (\mathcal{E}_\pm)^{\text{es}})(z, r) + (\mathcal{E}_\pm)^{\text{es}}(z, r)$. In the notation of Section 4, for any $v \in \text{Box}(\Sigma_-)$, the analytic function $(\mathcal{E}_-^v - (\mathcal{E}_-^v)^{\text{es}})(z, r)$ on $U_- \times B^v$ is the sum of a series made out of terms of the type

$$\varphi(z, r) = \prod_{j=1}^n \frac{z_j^{l_j + \frac{1}{2\pi i} \log_- r_j}}{\Gamma(l_j + \frac{1}{2\pi i} \log_- r_j + 1)}$$

with $l \in \mathcal{S}_-(\gamma^v) \setminus \mathcal{S}_-^{\text{es}}(\gamma^v)$. By Proposition 3.1(ii), $K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C})$ is a direct sum of Artinian local rings obtained by localizing at the maximal ideals $(R_1 - y_1^v, \dots, R_n - y_n^v)$ corresponding to all elements $v \in \text{Box}(\Sigma_-)$,

$$K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}) = \bigoplus_{v \in \text{Box}(\Sigma_-)} (K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}))_v.$$

Since $\varphi(z, r)$ vanishes for values of r outside of B^v , we see that

$$\varphi(z, \mathcal{R}) = \varphi(z, \mathcal{R})E_v,$$

where $E_v : K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}) \rightarrow K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C})$ is the projection onto the subspace $(K_0(\mathbb{P}_{\Sigma_-}, \mathbb{C}))_v$ corresponding to v .

Since $l \in \mathcal{S}_-(\gamma^v) \setminus \mathcal{S}_-^{\text{es}}(\gamma^v)$, the elements of $\text{Supp}(l)$ generate a cone in Σ_- that is not a subcone of any essential maximal cone. In particular, this shows that v is also in $\text{Box}(\Sigma_+)$ with the same associated $y^v \in (\mathbb{C}^*)^n$.

There exists an analytic function $\psi(z, r)$ in $U_- \times B^v$ such that

$$\varphi(z, r) = \prod_{j, v_j \in J} (1 - r_j)\psi(z, r),$$

where J consists of those v_j in $\text{Supp}(l)$ such that $l_j + 1/(2\pi i) \log_- y_j^v$ is a negative integer. In particular, $y_j^v = 1$ for such j . We introduce the analytic function $\tilde{\varphi}(z, r)$ on $\mathbb{C}^n \times B^v$ given by

$$\tilde{\varphi}(z, r) := \prod_{j, v_j \in \sigma(v)} (1 - r_j)^{-1}\psi(z, r).$$

Proposition 5.1 and Remark 5.3 imply that

$$FM\left(\prod_{j, v_j \in \text{Supp}(l)} (I - \mathcal{R}_j)\tilde{\varphi}(z, \mathcal{R})(1)\right) = \prod_{j, v_j \in \text{Supp}(l)} (I - \mathcal{R}_j)\tilde{\varphi}(z, \mathcal{R})(1),$$

which means that

$$FM(\varphi(z, \mathcal{R})(1)) = \varphi(z, \mathcal{R})(1).$$

We conclude that

$$FM((\mathcal{E}_-^v - (\mathcal{E}_-^v)^{\text{es}})(z, \mathcal{R})(1)) = (\mathcal{E}_-^v - (\mathcal{E}_-^v)^{\text{es}})(z, \mathcal{R})(1),$$

for all $v \in \text{Box}(\Sigma_-)$. Hence

$$FM((\mathcal{E}_- - (\mathcal{E}_-)^{\text{es}})(z, \mathcal{R})(1)) = (\mathcal{E}_- - (\mathcal{E}_-)^{\text{es}})(z, \mathcal{R})(1).$$

Corollary 4.14(i) shows then that

$$FM((\mathcal{E}_- - (\mathcal{E}_-)^{\text{es}})(z, \mathcal{R})(1)) = MB((\mathcal{E}_+ - (\mathcal{E}_+)^{\text{es}})(z, \mathcal{R})(1)).$$

Moreover, Corollary 4.14(ii) and Proposition 5.2 (combined with Remark 5.3) show that

$$FM((\mathcal{E}_-)^{\text{es}}(z, \mathcal{R})(1)) = MB((\mathcal{E}_+)^{\text{es}}(z, \mathcal{R})(1)),$$

which ends the proof of the theorem. \square

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Appendix A. Auxiliary analytic results and estimates

In what follows, we define the norm of $r = (r_1, \dots, r_n) \in \mathbb{C}^n$ to be the positive real number $\|r\| := |r_1| + \dots + |r_n|$. As a convention, the arguments of all the complex numbers used here are chosen in $[-\pi, \pi]$.

Lemma A.1. *Let a_1, \dots, a_p be strictly positive real numbers. Then*

$$\prod_{j=1}^p a_j^{-a_j} \leq \left(\frac{\|a\|}{p}\right)^{-\|a\|}.$$

Proof. Note that the function $f(x) = -x \log x$ is concave down on $(0, +\infty)$. This means that

$$\frac{\sum_{j=1}^n -x_j \log x_j}{n} \leq -\frac{\|x\|}{n} \log \frac{\|x\|}{n}.$$

After applying the exponential to the two sides of the above inequality, the desired result is obtained. \square

Lemma A.2. *There exists a positive constant $M > 0$ such that*

$$\left| \frac{1}{\Gamma(z)} \right| \leq M \cdot (|x| + |y|)^{-x+1/2} e^{x+y\theta},$$

for any complex number $z = x + iy = Re^{i\theta}$.

Proof. According to Stirling’s formula, we have that, for a fixed complex number u and for any $\delta > 0$,

$$\Gamma(z + u) = (2\pi)^{1/2} z^{z+u-1/2} e^{-z} O(1) \tag{A.1}$$

where, as $|z| \rightarrow \infty$, $O(1)$ goes to 1 uniformly in

$$|\arg z| < \pi - \delta.$$

As a direct consequence, we have that, if we write $z = x + iy = Re^{i\theta}$, $|\theta| < \pi - \delta$, then ($u = 0$)

$$\left| \frac{1}{\Gamma(z)} \right| = (2\pi)^{-1/2} R^{-x+1/2} e^{x+y\theta} O(1), \quad \text{when } R \rightarrow \infty.$$

If $-\pi \leq \theta \leq -\pi + \delta$, or $\pi - \delta \leq \theta \leq \pi$, we use the Gamma identity $\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z)$, $z \notin \mathbb{Z}_{\leq 0}$, and again Stirling’s formula for $\Gamma(1 - z)$ with $u = 1$ to write that

$$\left| \frac{1}{\Gamma(z)} \right| = e^{\pi|y|} |e^{-2\pi(|y|\pm ix)} - 1| (2\pi)^{-1/2} R^{-x+1/2} e^{x+y\theta^*} O(1),$$

where $\theta^* = \arg(-z)$. Note, however, that, for any $z = x + iy = Re^{i\theta}$,

$$\pi|y| + y\theta^* = y\theta.$$

Clearly, there exist a positive constant $a > 0$, such that

$$|e^{-2\pi(|y|\pm ix)} - 1| < b,$$

for any complex number $z = x + iy$.

Since

$$\frac{1}{\sqrt{2}}(|x| + |y|) \leq R = \sqrt{x^2 + y^2} \leq |x| + |y|,$$

we can replace R by $|x| + |y|$ above, and conclude that there exist a positive constant $M > 0$, such that

$$\left| \frac{1}{\Gamma(z)} \right| \leq M \cdot (|x| + |y|)^{-x+1/2} e^{x+y\theta},$$

for any complex number $z = x + iy = Re^{i\theta}$. This ends the proof of the lemma. \square

Lemma A.3. For any $\delta > 0$, there exists a positive constant $B > 0$, such that

$$\prod_{j=1}^n (|x_j| + |y_j|)^{-x_j+1/2} \leq B \cdot (4n)^{\|x\|},$$

for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ in \mathbb{C}^n such that

$$|x_1 + \dots + x_n| \leq \delta, \quad \|y\| \leq \delta.$$

Proof. Note that

$$\prod_{j=1}^n (|x_j| + |y_j|)^{-x_j+1/2} = \prod_{j=1}^n \left(\frac{|x_j| + |y_j|}{\|x\| + \|y\|} \right)^{-x_j+1/2} (\|x\| + \|y\|)^{\sum(-x_j+1/2)}. \quad (\text{A.2})$$

We see that

$$\prod_{j=1}^n \left(\frac{|x_j| + |y_j|}{\|x\| + \|y\|} \right)^{1/2} \leq 1.$$

If $x_j \leq 0$, then

$$\left(\frac{|x_j| + |y_j|}{\|x\| + \|y\|} \right)^{-x_j} \leq 1,$$

while if $x_j > 0$, then

$$\left(\frac{|x_j| + |y_j|}{\|x\| + \|y\|} \right)^{-x_j} \leq \left(\frac{x_j}{\|x\| + \delta} \right)^{-x_j}.$$

Let p be the number of strictly positive x_j , and assume that $p > 0$. We apply Lemma A.1 to the positive real numbers

$$\frac{x_j}{\|x\| + \delta}, \quad x_j > 0,$$

and get that

$$\prod_{x_j > 0} \left(\frac{x_j}{\|x\| + \delta} \right)^{-x_j} \leq \left(\frac{\sum_{x_j > 0} x_j}{p(\|x\| + \delta)} \right)^{-\sum_{x_j > 0} x_j}.$$

Since

$$\sum_{x_j > 0} x_j + \sum_{x_j \leq 0} x_j > -\delta, \quad \text{and} \quad \|x\| = \sum_{x_j > 0} x_j - \sum_{x_j \leq 0} x_j,$$

we see that

$$\|x\| < 2 \sum_{x_j > 0} x_j + \delta.$$

Hence

$$\begin{aligned} \left(\frac{\sum_{x_j > 0} x_j}{p(\|x\| + \delta)} \right)^{-\sum_{x_j > 0} x_j} &\leq \left(\frac{\sum_{x_j > 0} x_j}{2p(\sum_{x_j > 0} x_j + \delta)} \right)^{-\sum_{x_j > 0} x_j} \\ &\leq (2p)^{\sum_{x_j > 0} x_j} \left(1 + \frac{\delta}{\sum_{x_j > 0} x_j} \right)^{\sum_{x_j > 0} x_j} \leq K \cdot (2n)^{\|x\|}, \end{aligned}$$

for some positive constant $K > 0$.

We conclude that

$$\prod_{j=1}^n \left(\frac{|x_j| + |y_j|}{\|x\| + \|y\|} \right)^{-x_j+1/2} \leq K \cdot (2n)^{\|x\|}, \tag{A.3}$$

and the inequality obviously holds also in the case when there are no positive x_j 's, i.e. when $p = 0$.

Moreover

$$(\|x\| + \|y\|)^{\sum(-x_j+1/2)} \leq (\|x\| + \|y\|)^{\pm\delta+k/2}.$$

But $\|y\| \leq \delta$, so there exists a positive constant $B > 0$ (depending on δ) such that

$$(\|x\| + \|y\|)^{\sum(-x_j+1/2)} \leq B \cdot 2^{\|x\|}, \tag{A.4}$$

for any value of $\|x\| \geq 0$.

By combining the formulae (A.2), (A.3) and (A.4), we can write

$$\prod_{j=1}^n (|x_j| + |y_j|)^{-x_j+1/2} \leq K \cdot B \cdot (4n)^{\|x\|},$$

which proves the lemma. \square

Lemma A.4. For any $\delta > 0$, there exists a positive constant $A > 0$, such that

$$\left| \prod_{j=1}^n \frac{1}{\Gamma(x_j + iy_j)} \right| \leq A \cdot (4n)^{\|x\|},$$

for any $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ in \mathbb{C}^n such that

$$|x_1 + \dots + x_n| \leq \delta, \quad \|y\| \leq \delta.$$

Proof. Clearly, we can safely assume that $\|x\| + \|y\| > 0$. According to Lemma A.2, we have that

$$\left| \prod_{j=1}^n \frac{1}{\Gamma(x_j + iy_j)} \right| \leq M^n e^{\sum x_j + \sum y_j \theta_j} \prod_{j=1}^k (|x_j| + |y_j|)^{-x_j+1/2}.$$

Note first that

$$\sum x_j + \sum y_j \theta_j \leq \delta + \pi \delta.$$

According to the previous lemma, there exists a constant $B > 0$ such that

$$\prod_{j=1}^n (|x_j| + |y_j|)^{-x_j+1/2} \leq B \cdot (4n)^{\|x\|}.$$

Hence

$$\left| \prod_{j=1}^n \frac{1}{\Gamma(x_j + iy_j)} \right| \leq M^n e^{\delta + \pi \delta} B \cdot (4n)^{\|x\|},$$

which ends the proof of the lemma. \square

Lemma A.5. Let $h = (h_1, \dots, h_n)$ be a fixed element in \mathbb{R}^n , and I_-, I_+ two subsets that determine a partition of $\{1, \dots, n\}$. Define

$$H := \sum_{j \in I_-} |h_j| - \sum_{j \in I_+} |h_j|.$$

For any $\epsilon > 0$ and $\delta > 0$, there exists a positive constant $A > 0$, such that

$$\left| \frac{\prod_{j \in I_-} \Gamma(1 - x_j - iy_j - i(h_j t))}{\prod_{j \in I_+} \Gamma(x_j + iy_j + i(h_j t))} \right| \leq A \cdot (|t| + 1)^{\delta + n/2} (4en)^{\|x\|} e^{-\pi H |t|/2},$$

for any $t \in \mathbb{R}$, and any $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in \mathbb{C}^n such that

$$|x_1 + \dots + x_n| \leq \delta, \quad \|y\| \leq \delta,$$

and such that, for all $j \in I_-$, the complex numbers $1 - x_j - iy_j - i(h_j t)$ are located at a distance greater than ϵ from any integer.

Proof. For $j \in I_-$, we use the Gamma identity to write

$$\left| \Gamma(1 - x_j - iy_j - i(h_j t)) \right| = \frac{2\pi e^{-\pi |y_j + h_j t|}}{|e^{-2\pi i(y_j + h_j t \pm ix_j)} - 1|} \left| \frac{1}{\Gamma(x_j + iy_j + i(h_j t))} \right|.$$

The hypothesis that $1 - x_j - iy_j - i(h_j t)$ are located at a distance greater than ϵ from any integer, for all $j \in I_-$, guarantees that

$$1/|e^{-2\pi i(y_j + h_j t \pm ix_j)} - 1|$$

is bounded from above for all $j \in I_-$.

If we restrict the real number t to a bounded range $|t| \leq \Lambda$, then Lemma A.4 provides the required result with a constant A that depends on Λ . It remains to understand what happens for $|t| > \Lambda$, for some fixed $\Lambda > 1$.

Note that

$$-|y_j + h_j t| \leq |y_j| - |h_j t|.$$

As a result of the above considerations, the quotient of products of Gamma functions invoked in the statement of the lemma is bounded above by a constant multiplied by

$$e^{-\pi \sum_{j \in I_-} |h_j t|} \prod_{j=1}^n \left| \frac{1}{\Gamma(x_j + iy_j + i(h_j t))} \right|.$$

According to Lemma A.2, this expression is bounded by a constant multiplied by

$$e^{-\pi \sum_{j \in I_-} |h_j t| + \sum_{j=1}^n (x_j + (y_j + h_j t)\theta_j)} \prod_{j=1}^n (|x_j| + |y_j| + |h_j t|)^{-x_j + 1/2},$$

with $x_j + iy_j + ih_j t = R_j e^{i\theta_j}$. Since it is the case that $(y_j + h_j t)\theta_j = |y_j + h_j t||\theta_j| \leq (|y_j| + |h_j t|)|\theta_j| \leq \delta\pi + |h_j t||\theta_j|$, and $\sum_{j=1}^n x_j < \delta$, we infer that the above expression is also bounded above by a constant multiplied by

$$e^{(-\pi \sum_{j \in I_-} |h_j| + \sum_{j=1}^n |h_j \theta_j|)|t|} \prod_{j=1}^n (|x_j| + |y_j| + |h_j t|)^{-x_j + 1/2}. \tag{A.5}$$

We first study the sum $\sum_{j=1}^n |h_j \theta_j| |t|$. Note that for any x, s with $s \neq 0$ one has

$$|s| \arctan(|x|/|s|) \leq |x|,$$

which shows that

$$(|\arg(x + is)| - \pi/2)|s| \leq |x|$$

for all x and s . When we apply this to $x = x_j$ and $s = y_j + h_j t$, we get

$$|\theta_j| |y_j + h_j t| \leq \pi/2 |y_j + h_j t| + |x_j|$$

which implies

$$|\theta_j| |h_j| |t| \leq \pi/2 |h_j| |t| + |x_j| + C,$$

where the constant C depends on δ only. We sum this over all j and exponentiate to get that the factor

$$e^{(-\pi \sum_{j \in I_-} |h_j| + \sum_{j=1}^n |h_j \theta_j|)|t|}$$

in formula (A.5) is bounded above by a constant (independent of the x_j 's) multiplied by

$$e^{-\pi H |t|/2} e^{\|x\|},$$

where $H = \sum_{j \in I_-} |h_j| - \sum_{j \in I_+} |h_j|$.

Let us now analyze the other factor of the formula (A.5) for $|t| > \Lambda$. We see that

$$\prod_{j=1}^n (|x_j| + |y_j| + |h_j t|)^{-x_j + 1/2} = |t|^{\sum (-x_j + 1/2)} \prod_{j=1}^n (|x_j/t| + |y_j/t| + |h_j|)^{-x_j + 1/2}.$$

Since $|t| > \Lambda > 1$, we see immediately that

$$|t|^{\sum (-x_j + 1/2)} \leq (|t| + 1)^{\delta + n/2}.$$

According to Lemma A.3, we have that

$$\prod_{j=1}^n (|x_j/t| + |y_j/t| + |h_j|)^{-x_j/t+1/2} \leq B \cdot (4n)^{\|x/t\|} \leq B \cdot (4n)^{\|x\|/A}.$$

Hence the factor $\prod_{j=1}^n (|x_j| + |y_j| + |h_j t|)^{-x_j+1/2}$ is bounded above by a constant multiplied by

$$(|t| + 1)^{\delta+n/2} (4n)^{\|x\|},$$

which ends the proof of the lemma. \square

The following property is essentially stated in [11, p. 49] and [23, §14.5]. We include a proof for completeness.

Lemma A.6. Consider the integral

$$\int_{\gamma+i\infty}^{\gamma-i\infty} \frac{\prod \Gamma(A_j s + a_j) \prod \Gamma(-C_j s + c_j)}{\prod \Gamma(B_j s + b_j) \prod \Gamma(-D_j s + d_j)} y^s ds,$$

with γ real, and A_j, B_j, C, D_j all real and strictly positive. The path of integration is parallel to the imaginary axis for large $|s|$, but it can be curved elsewhere so that it avoids the poles of the integrand. Introduce the following notations:

$$H := \sum A_j + \sum C_j - \sum B_j - \sum D_j,$$

$$\beta := \sum A_j - \sum C_j - \sum B_j + \sum D_j,$$

$$\eta := \Re \left(\sum \left(a_j - \frac{1}{2} \right) + \sum \left(c_j - \frac{1}{2} \right) - \sum \left(b_j - \frac{1}{2} \right) - \sum \left(d_j - \frac{1}{2} \right) \right),$$

$$\rho := \left(\prod A_j^{-A_j} \right) \left(\prod C_j^{C_j} \right) \left(\prod B_j^{B_j} \right) \left(\prod D_j^{-D_j} \right).$$

(i) For

$$s = \gamma + it, \quad y = R e^{i\theta},$$

the absolute value of the integrand has the asymptotic form

$$e^{-\frac{1}{2} H \pi |t|} |t|^{\beta \gamma + \eta} R^\gamma e^{-\theta t} \rho^{-\gamma}$$

when $|t|$ is large. Therefore, if

$$H > 0,$$

the integral is absolutely convergent (and defines an analytic function of y) in any domain contained in

$$|\arg y| < \min\left(\pi, \frac{H\pi}{2}\right).$$

(ii) Moreover, if $\beta = 0$, the integral is equal to the sum of the residues on the right of the contour for $|y| < \rho$, and to the negative of the sum of the residues on the left of the contour for $|y| > \rho$ (these facts are obtained by closing the contour to the right, respectively to the left, with a semicircle of radius $r \rightarrow \infty$).

Proof. Part (i) is a direct consequence of Stirling’s formula (A.1).

For part (ii), assume that $|y| < \rho$. It is enough to show that the integral over a semicircle C of radius $M > 0$ on the right of the imaginary axis and centered at the origin goes to zero when M goes to $+\infty$. It is possible to choose a sequence $M_n \rightarrow +\infty$ while making sure that the expressions $-C_j s + c_j$ and $-D_j s + d_j$ for $s = M_n e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$, are at a distance greater than some $\epsilon > 0$ from all negative integers.

The integrand can be written as

$$I(s) := \frac{\prod \Gamma(A_j s + a_j) \prod \Gamma(D_j s + 1 - d_j) \prod \sin(\pi(-D_j s + d_j))}{\prod \Gamma(B_j s + b_j) \prod \Gamma(C_j s + 1 - c_j) \prod \sin(\pi(-C_j s + c_j))} y^s.$$

Note that we are working under the assumptions that $H > 0$ and $\beta = 0$. It follows that

$$\sum C_j - \sum D_j = \sum A_j - \sum B_j = H/2 > 0.$$

Stirling’s formula (A.1) implies that, for $H = 0$ and $s = M_n e^{i\theta}$

$$O(M_n^\eta) \frac{\prod \sin(\pi(-D_j s + d_j))}{\prod \sin(\pi(-C_j s + c_j))} (y/\rho)^s,$$

where the symbol O is independent of $\theta = \arg s$ when s is on the semicircle.

Note that we write $a_n = O(b_n)$ for two sequences $(a_n), (b_n)$, if

$$|a_n/b_n| < K, \quad n \gg 0,$$

with K independent of n .

We have that

$$4|\sin(x + iy)|^2 = e^{2y} + e^{-2y} - 2\cos(2x), \quad x, y \in \mathbb{R}.$$

The choice explained above of the semicircles $s = M_n e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$, guarantees that

$$\left| \frac{1}{\prod \sin(\pi(-C_j s + c_j))} \right| < K,$$

with K independent of n .

It follows that

$$\frac{\prod \sin(\pi(-D_j s + d_j))}{\prod \sin(\pi(-C_j s + c_j))} = O\left(\exp\left(\left(\sum D_j - \sum C_j\right) M_n \pi |\sin \theta|\right)\right),$$

and, for $|y/\rho| < 1$,

$$(y/\rho)^s = O\left(\exp\left(M_n \cos \theta \log |y/\rho| - M_n \sin \theta \arg y\right)\right).$$

But $\sum D_j - \sum C_j = -H/2$, so

$$\begin{aligned} & \frac{\prod \Gamma(A_j s + a_j) \prod \Gamma(D_j s + 1 - d_j) \prod \sin(\pi(-D_j s + d_j))}{\prod \Gamma(B_j s + b_j) \prod \Gamma(C_j s + 1 - c_j) \prod \sin(\pi(-C_j s + c_j))} y^s \\ & = O\left(M_n^\eta \exp\left(M_n \left(-\frac{1}{2} H \pi |\sin \theta| - \arg y \sin \theta\right) + M_n \cos \theta \log |y/\rho|\right)\right). \end{aligned}$$

Choose $\delta > 0$ such that $\delta < \frac{1}{2} H \pi \pm \arg y$, and we see that the above expression is in fact

$$O\left(M_n^\eta \exp\left(-\delta M_n |\sin \theta| + M_n \cos \theta \log |y/\rho|\right)\right).$$

Hence, for $|y/\rho| < 1$, and $-\pi/2 \leq \theta \leq \pi/2$, the integrand tends to zero sufficiently rapidly (when $n \rightarrow \infty$) to ensure that the integral along the semicircle tends to zero. \square

Appendix B. Functions of linear operators

We briefly recall some facts from the spectral theory of linear operators on finite dimensional vector spaces. The details can be found in [10, Section VII.1], where the case of one linear operator is treated. In our discussion, $\mathcal{R}_1, \dots, \mathcal{R}_n$ are mutually commuting linear operators on a finite dimensional complex vector space V , i.e. $\mathcal{R}_i \mathcal{R}_j = \mathcal{R}_j \mathcal{R}_i$ for all $i, j, 1 \leq i, j \leq n$. If $P(x_1, \dots, x_n)$ is a polynomial with complex coefficients, then $P(\mathcal{R}_1, \dots, \mathcal{R}_n)$ is a well defined linear operator on V . For a linear operator \mathcal{R} on V , its *spectrum* $s(\mathcal{R})$ is the set of complex numbers λ such that $\mathcal{R} - \lambda I$ is not one-to-one. The *index* $\nu(\lambda)$ of a complex number λ is the smallest non-negative integer ν such that

$$\{x: (\mathcal{R} - \lambda I)^\nu x = 0\} = \{x: (\mathcal{R} - \lambda I)^{\nu+1} x = 0\}.$$

Given two complex polynomials P, Q , in n variables then

$$P(\mathcal{R}_1, \dots, \mathcal{R}_n) = Q(\mathcal{R}_1, \dots, \mathcal{R}_n)$$

if and only if $P - Q$ is divisible by $(x_1 - \lambda_1)^{\nu(\lambda_1)} \dots (x_n - \lambda_n)^{\nu(\lambda_n)}$ for any $(\lambda_1, \dots, \lambda_n) \in s(\mathcal{R}_1) \times \dots \times s(\mathcal{R}_n) \subset \mathbb{C}^n$. This property provides the definition of the linear operator $f(\mathcal{R}_1, \dots, \mathcal{R}_n)$ for every function $f: \mathbb{C}^n \rightarrow \mathbb{C}$ which is analytic in an open domain (not necessarily connected!) that contains $s(\mathcal{R}_1) \times \dots \times s(\mathcal{R}_n) \subset \mathbb{C}^n$. Indeed, it is enough to consider a polynomial $P(x_1, \dots, x_n)$ such that

$$\partial_1^{m_1} \dots \partial_n^{m_n} P(\lambda_1, \dots, \lambda_n) = \partial_1^{m_1} \dots \partial_n^{m_n} f(\lambda_1, \dots, \lambda_n),$$

for all $\lambda_j \in s(\mathcal{R}_j)$ and $0 \leq m_j \leq v(\lambda_j) - 1$, and set

$$f(\mathcal{R}_1, \dots, \mathcal{R}_n) := P(\mathcal{R}_1, \dots, \mathcal{R}_n).$$

It follows that, for such a function f , the linear operator $f(\mathcal{R}_1, \dots, \mathcal{R}_n)$ can be expressed as

$$\sum_{\lambda_1 \in s(\mathcal{R}_1), \dots, \lambda_n \in s(\mathcal{R}_n)} \sum_{0 \leq j_1 < v(\lambda_1), \dots, 0 \leq j_n < v(\lambda_n)} \frac{(\mathcal{R}_1 - \lambda_1 I)^{j_1} E_1(\lambda_1)}{j_1!} \dots \frac{(\mathcal{R}_n - \lambda_n I)^{j_n} E_n(\lambda_n)}{j_n!} \partial_1^{j_1} \dots \partial_n^{j_n} f(\lambda_1, \dots, \lambda_n),$$

where the operators $E_j(\lambda)$ are the usual projections onto the kernels of the operators $(\mathcal{R}_j - \lambda I)^{v(\lambda)}$ for some eigenvalue $\lambda \in s(\mathcal{R}_j)$ (compare to [10, Theorem VII.1.8]). As a direct consequence, there exists a Cauchy type integral representation of the operator $f(\mathcal{R}_1, \dots, \mathcal{R}_n)$. Namely, assume that f is analytic in an open domain in \mathbb{C}^n containing $U_1 \times \dots \times U_n$, where, for all j , $1 \leq j \leq n$, U_j contains $s(\mathcal{R}_j)$, and the (positive oriented) boundary B_j of U_j consists of a finite union of closed rectifiable Jordan curves. Then $f(\mathcal{R}_1, \dots, \mathcal{R}_n)$ is given by

$$\frac{1}{(2\pi i)^n} \int_{B_1 \times \dots \times B_n} f(\lambda_1, \dots, \lambda_n) (\lambda_1 I - \mathcal{R}_1)^{-1} \dots (\lambda_n I - \mathcal{R}_n)^{-1} d\lambda_1 \dots d\lambda_n$$

(compare to [10, Theorem VII.1.10]).

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