Total restrained domination in graphs with minimum degree two

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Abstract

In this paper, we continue the study of total restrained domination in graphs, a concept introduced by Telle and Proskurowksi (Algorithms for vertex partitioning problems on partial k-trees, SIAM J. Discrete Math. 10 (1997) 529–550) as a vertex partitioning problem. A set $S$ of vertices in a graph $G = (V, E)$ is a total restrained dominating set of $G$ if every vertex is adjacent to a vertex in $S$ and every vertex of $V \setminus S$ is adjacent to a vertex in $V \setminus S$. The minimum cardinality of a total restrained dominating set of $G$ is the total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$. Let $G$ be a connected graph of order $n$ with minimum degree at least 2 and with maximum degree $\Delta$ where $\Delta \leq n - 2$. We prove that if $n \geq 4$, then $\gamma_{tr}(G) \leq n - \frac{2}{3} \Delta - \frac{2}{9} \sqrt{3 \Delta - 8} - \frac{7}{9}$ and this bound is sharp. If we restrict $G$ to a bipartite graph with $\Delta \geq 3$, then we improve this bound by showing that $\gamma_{tr}(G) \leq n - \frac{2}{3} \Delta - \frac{2}{3} \Delta - \frac{7}{3} \Delta - 8 - \frac{7}{9}$ and that this bound is sharp.

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1. Introduction

The concept of domination in graphs, with its many variations, is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [13,14]. In this paper, we focus on two variations on the domination theme that are well studied in graph theory called total domination and restrained domination.

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$, and let $S \subseteq V$. The set $S$ is a dominating set (DS) if every vertex in $V \setminus S$ is adjacent to a vertex of $S$. The set $S$ is a total dominating set (TDS) if every vertex in $V$ is adjacent to a vertex of $S$, while $S$ is a restrained dominating set (RDS) if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V \setminus S$. If $S$ is simultaneously a TDS and a RDS, then $S$ is a total restrained dominating set (TRDS) of $G$. The minimum cardinality of a TRDS of $G$ is the total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$. We denote the domination, total domination, and restrained domination numbers of $G$ by $\gamma(G)$, $\gamma_t(G)$, and $\gamma_r(G)$, respectively. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [3] and further studied, for example, in [1,8,9,16], while restrained domination was introduced by Telle and Proskurowski [18], albeit indirectly, as a vertex partitioning problem and further studied, for example, in [5–7,11,15]. The concept of total restrained domination in

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graphs was also introduced in [18], albeit indirectly, as a vertex partitioning problem and has been studied, for example, in [12,17,19].

A TRDS can be interpreted as a red–blue coloring of the vertices, with the red vertices forming the TRDS. We call a red–blue coloring of vertices such that every blue vertex has both a red and a blue neighbor and every red vertex has a red neighbor a tr-coloring (total restrained coloring) of $G$. The total restrained domination number $\gamma_{tr}(G)$ of $G$ is the minimum number of red vertices of $G$ in a tr-coloring of $G$. We call a tr-coloring of $G$ that colors $\gamma_{tr}(G)$ vertices red a $\gamma_{tr}$-coloring of $G$.

For notation and graph theory terminology we in general follow [13]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = \{v\} \cup N(v)$. For a set $S$ of vertices, the open neighborhood of $S$ is defined by $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ by $N[S] = N(S) \cup S$. If $X, Y \subseteq V$, then the set $X$ is said to dominate the set $Y$ if $Y \subseteq N[S]$, while $X$ totally dominates the set $Y$ if $Y \subseteq N(S)$. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. We denote the degree of $v$ in $G$ by $d_G(v)$, or simply by $d(v)$ if the graph $G$ is clear from context. The minimum degree (respectively, maximum degree) among the vertices of $G$ is denoted by $\delta(G)$ (respectively, $\Delta(G)$). A vertex of degree $k$ we call a degree-$k$ vertex. The set $S$ is a packing if the vertices in $S$ are pairwise at distance at least 3 apart in $G$, i.e., if $u, v \in S$, then $d_G(u, v) \geq 3$. Equivalently, $S$ is a packing if the closed neighborhoods of vertices in $S$ are pairwise disjoint. We denote a path on $n$ vertices by $P_n$.

2. Main results

Let $G$ be a connected graph of order $n$ and maximum degree $\Delta$. Berge [2] was the first to observe that $\gamma(G) \leq n - \Delta$, and graphs achieving this bound were characterized in [10]. Cockayne, Dawes and Hedetniemi [3] observed that if $n \geq 3$ and $\Delta \leq n - 2$, then $\gamma_1(G) \leq n - \Delta$. Recently it was shown in [4] that if $\delta(G) \geq 2$, then $\gamma_{tr}(G) \leq n - \Delta$. Hence, if $\delta(G) \geq 2$, then both the total domination and the restrained domination numbers are bounded above by $n - \Delta$. Our aim in this paper is to investigate a bound on the total restrained domination number in terms of the order and maximum degree of the graph. We shall show:

**Theorem 1.** If $G$ is a connected graph of order $n \geq 4$, maximum degree $\Delta$ where $\Delta \leq n - 2$, and minimum degree at least 2, then

$$\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1,$$

and this bound is sharp.

If we restrict our attention to bipartite graphs, then we show that the bound of Theorem 1 can be improved.

**Theorem 2.** If $G$ is a connected bipartite graph of order $n \geq 5$, maximum degree $\Delta$ where $3 \leq \Delta \leq n - 2$, and minimum degree at least 2, then

$$\gamma_{tr}(G) \leq n - \frac{3}{2} \Delta - \frac{2}{5} \sqrt{3\Delta - 8} - \frac{7}{9},$$

and this bound is sharp.

3. Notation

Before proceeding with proofs of our main results, namely Theorems 1 and 2, we introduce some additional notation. By a proper subgraph of a graph $G$ we mean a subgraph of $G$ different from $G$. We define a vertex as small if it has degree 2, and large if it has degree more than 2. We define a ray as a path (not necessarily induced) of length 3 the two internal vertices of which are small vertices. Let $G$ be a graph with minimum degree at least 2, and let $L'$ be the set of all large vertices of $G$. Suppose $|L'| \geq 1$ and let $C$ be any component of $G - L'$; it is a path. If $C$ has only one vertex, or has at least two vertices but the two ends of $C$ are adjacent in $G$ to different large vertices, then we say that $C$ is a
2-path. Otherwise we say that $C$ is a 2-handle. For notational convenience, we let

$$\phi(n, A) = n - \frac{A}{2} - 1 \quad \text{and} \quad \psi(n, A) = n - \frac{2}{3}A - \frac{2}{9}\sqrt{3A} - \frac{8}{9}.$$ 

4. Proof of Theorem 1

We proceed by induction on $\ell = n + m$, where $m$ denotes the size of $G$. We wish to show that $\gamma_{tr}(G) \leq \phi(n, A)$. Note that $n \geq 4$ and $m \geq 4$, and so $\ell \geq 8$. When $\ell = 8$, the graph $G$ is a 4-cycle, and so $\gamma_{tr}(G) = 2 = \phi(4, 2) = \phi(n, A)$. This establishes the base case. Let $\ell \geq 9$ and let $n' \geq 4$, $m' \geq 3$ and $\ell' \geq 3$ be integers with $n' + m' < \ell$ and $\ell' \leq n' - 2$. For the inductive hypothesis, assume that all connected graphs $G'$ of order $n'$ and size $m'$ with maximum degree $\Delta'$ and minimum degree at least 2 satisfy $\gamma_{tr}(G') \leq \phi(n', A')$. Let $G = (V, E)$ be a connected graph of order $n$ and size $m$ with $\ell = n + m$, maximum degree $\Delta$ where $\Delta \leq n - 2$ and minimum degree at least 2. We begin with the following claim.

Claim A. If a connected proper subgraph $G'$ of $G$ of order $n'$ has maximum degree $\Delta$ where $\Delta \leq n' - 2$ and minimum degree at least 2, and if the subgraph $G - V(G')$ contains no isolated vertices, then $\gamma_{tr}(G) \leq \phi(n, A)$.

Proof. Let $G'$ have size $m'$. Then, $n' + m' < \ell$, and so $G'$ satisfies the inductive hypothesis. Let $n'' = n - k$ where $k \geq 0$. Then by the inductive hypothesis, $\gamma_{tr}(G') \leq \phi(n'', A) = \phi(n - k, A) = \phi(n, A) - k$. Any $\gamma_{tr}$-coloring of $G'$ can be extended to a $tr$-coloring of $G$ by coloring every vertex in $V(G) \setminus V(G')$ with the color red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + k \leq \phi(n, A)$, as desired. □

Let $v$ be a vertex of maximum degree $\Delta$ in $G$, and let $\mathcal{L}$ be the set of all large vertices of $G$.

Claim B. We may assume that the set $\mathcal{L} \setminus \{v\}$ is an independent set in $G$.

Proof. Suppose $e = uv$ is an edge of $G$ joining two vertices $u$ and $w$ of $\mathcal{L} \setminus \{v\}$. If $G - e$ is a connected, then $G$ satisfies the statement of Claim A, and so $\gamma_{tr}(G) \leq \phi(n, A)$. Hence we may assume that $e$ is a bridge of $G$. Let $G_u$ be the component of $G - e$ containing $u$, and $G_w$ the component containing $w$. We may assume that $v \in V(G_u)$. Then, $G_{uv}$ is a connected subgraph of $G$ of order $n'$ with maximum degree $\Delta$ and minimum degree at least 2. If $\Delta \leq n' - 2$, then $G_u$ satisfies the statement of Claim A, and so $\gamma_{tr}(G) \leq \phi(n, A)$. Hence we may assume that $v$ dominates $V(G_{uv})$. Let $x \in N(u) \setminus \{v, w\}$. Then, $x \in V(G_{uv})$ and, since $G[\mathcal{L}]$ contains no cycles, $x$ is a small vertex. Coloring the vertices in $(V(G_{uv}) \setminus \{v\}) \cup \{v, x\}$ red and coloring all other vertices blue produces a $tr$-coloring of $G$, and so $\gamma_{tr}(G) \leq n - \Delta < \phi(n, A)$, as desired. □

By Claim B, the only edges in $G[\mathcal{L}]$, if any, are incident with $v$.

Claim C. We may assume that $G$ contains no ray.

Proof. Suppose that $G$ contains a ray $P: u, u_1, u_2, w$. Thus both $u_1$ and $u_2$ are small vertices of $G$. If $\Delta = n - 2$, then $u$ or $w$, say $u$, is a vertex of maximum degree $\Delta$ in $G$. Coloring $u$ and $u_1$ red and every other vertex blue produces a $tr$-coloring of $G$, and so $\gamma_{tr}(G) = 2 = \phi(n, A)$. Hence we may assume that $\Delta \leq n - 3$. Let $G'$ be the graph obtained from $G$ by removing the vertex $u_1$ and adding the edge $uw$. Then, $G'$ is a connected graph of order $n' = n - 1$ and size $m' = m - 1$, with maximum degree $\Delta$ where $\Delta \leq n' - 2$, and minimum degree at least 2. Applying the inductive hypothesis to $G'$, we have that $\gamma_{tr}(G') \leq \phi(n', A) = \phi(n - 1, A) = \phi(n, A) - 1$. Any $\gamma_{tr}$-coloring of $G'$ can be extended to a $tr$-coloring of $G$ by coloring the vertex $u_1$ red, unless $u$ and $u_2$ are both colored blue, in which case we recolor $u_2$ red and color $u_1$ blue. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 1 \leq \phi(n, A)$, as desired. □

By Claim C, every 2-path in $G$ has order 1, while every 2-handle of $G$ has order 2. Thus every large vertex in $G$ is either adjacent to $v$ or at distance 2 from some large vertex.

Claim D. We may assume that every two vertices in $\mathcal{L} \setminus \{v\}$ have at most one common small neighbor.
Thus we may assume $G = d(u)$ and $d(u) < \omega$. Suppose that $w$ is a small vertex of order 2 (with both ends adjacent to $v$ and $v$ is not a common small neighborhood of $w$). Let $G'$ be the graph obtained from $G$ by deleting $u$ and adding the edge $w$. Then $G'$ has order $n' = n - 1$, size $m' = m - 1$, maximum degree $A' \geq A$ and minimum degree at least 2. By the inductive hypothesis, $\gamma_t(G') \leq \phi(n', A) = \phi(n - 1, A) = \phi(n, A) - 1$. Any $\gamma_t$-coloring of $G'$ colors $u$ or $w$ red, and can therefore be extended to a tr-coloring of $G$ by coloring $w$ red. Hence, $\gamma_t(G) \leq \gamma_t(G') + 1 = \phi(n, A)$, as desired. \hfill \square

Before proceeding further, we introduce some additional notation. For each $u \in L$, let $H_u$ denote the graph obtained from $G$ by deleting $u$ and all 2-paths and 2-handles that have an end adjacent with $u$, and let $n_u = |V(H_u)|$.

**Claim E.** If $\gamma_t(H_u) \leq \phi(n_u, A) + 1$ for some $u \in L \setminus \{v\}$, then $\gamma_t(G) \leq \phi(n, A)$.

**Proof.** Let $u \in L \setminus \{v\}$ and suppose that $\gamma_t(H_u) \leq \phi(n_u, A) + 1$. By Claim B, every neighbor of $u$ is either a small vertex or the vertex $v$. By Claims C and D, every small neighbor of $u$ is either on a 2-path of order 1 or on a 2-handle of order 2 (with both ends adjacent to $u$).

Suppose first that $u$ is adjacent to the ends of a 2-handle $x, y$. If $d(u) = 3$, let $w$ be a neighbor of $u$ different from $x$ and $y$ (possibly, $w = v$). Let $G'$ be the graph obtained from $G$ by deleting $u$ and joining $x$ and $y$ to $w$. Then $G'$ has order $n' = n - 1$, size $m' = m - 1$, maximum degree $A' \geq A$ and minimum degree at least 2. By the inductive hypothesis, $\gamma_t(G') \leq \phi(n', A) \leq \phi(n - 1, A) = \phi(n, A) - 1$. Any $\gamma_t$-coloring of $G'$ can be extended to a tr-coloring of $G$ by coloring $w$ red, and so $\gamma_t(G) \leq \gamma_t(G') + 1 \leq \phi(n, A)$. Hence we may assume that $d(u) \geq 4$. Then, $G - \{x, y\}$ satisfies the statement of Claim A, and so $\gamma_t(G) \leq \phi(n_u, A) + n - n_u - 1 \leq (\phi(n_u, A) + 1) + n - n_u - 1 = \phi(n, A)$, as desired. \hfill \square

**Claim F.** We may assume that $G$ has no 2-handle.

**Proof.** Suppose that $G$ has a 2-handle $x, y$. Let $u$ be the large vertex adjacent to $x$ and $y$. If $d(u) = 3$ or if $d(u) \geq 4$ and $d(u) < A$, then by using a similar argument as in the proof of Claim E, the result follows. Hence we may assume that $d(u) = A \geq 4$. Renaming vertices, if necessary, we may assume that $u = v$. We consider two cases.

**Case 1:** There is a neighbor $w$ of $v$ that has no common neighbor with $v$.

Suppose that $w$ is a small vertex. Let $z$ be the (large) neighbor of $w$ different from $v$. Then, $vz$ is not an edge of $G$. Suppose $v$ and $z$ have at least two common small neighbors. Let $G'$ be the graph obtained from $G$ by deleting $w$ and adding the edge $vz$. Then, $G'$ has order $n' = n - 1$, size $m' = m - 1$, maximum degree $A' = A - 2$ and minimum degree at least 2. If $A' = n' - 1$, then coloring $v$ and $w$ red and all remaining uncolored vertices of $G$ blue produces a tr-coloring of $G$, and so $\gamma_t(G) = 2 \leq n - A \leq \phi(n, A)$. Thus we may assume $A \leq n' - 2$. By the inductive hypothesis, $\gamma_t(G') \leq \phi(n', A') \leq \phi(n, A) - 1$. Any $\gamma_t$-coloring of $G'$ colors $v$ red, and can therefore be extended to a tr-coloring of $G$ by coloring $w$ red. Hence, $\gamma_t(G) \leq \gamma_t(G') + 1 \leq \phi(n, A)$. Thus we may assume $v$ and $z$ have exactly one common neighbor, namely $w$. Then, $H_z$ has order $n_z = n - 4$, size $m_z < m$, maximum degree $A' = A - 1$ and minimum degree at least 2. If $A' = n_z - 1$, then coloring $v$, $w$, and $z$ red and all remaining uncolored vertices of $G$ blue produces a tr-coloring of $G$. Hence, $\gamma_t(G) = 3 < n - A \leq \phi(n, A)$. Thus we may assume $A' \\leq n_z - 2$. By the inductive hypothesis, $\gamma_t(H_z) \leq \phi(n_z, A') \leq \phi(n, A) + 1$. Hence, $H_z$ satisfies the statement of Claim E, and so $\gamma_t(G) \leq \phi(n, A)$.

Thus we may assume that $w$ is a large vertex. Then, $H_u$ has order $n_u \leq n - 3$, size $m_u < m$, maximum degree $A' = A - 1$ and minimum degree at least 2. If $A' = n_u - 1$, then coloring $v$ and $w$ red and all remaining uncolored vertices of $G$ blue produces a tr-coloring of $G$. Hence, $\gamma_t(H_u) = 2 \leq n - A \leq \phi(n, A)$. Thus we may assume $A' \leq n_u - 2$. By the inductive hypothesis, $\gamma_t(H_u) \leq \phi(n_u, A') = \phi(n, A) + 1$. Hence, $H_w$ satisfies the statement of Claim E, and so $\gamma_t(G) \leq \phi(n, A)$.

**Case 2:** Every neighbor of $v$ lies in a common triangle with $v$. Since $A \leq n - 2$, at least one vertex of $G$ is not a neighbor of $v$. Let $S$ denote the set of all those vertices that are isolated in the subgraph induced by $V(G) \setminus N[v]$. Let
$H$ be the subgraph of $G$ induced by $N[v] \cup S$. Suppose first that $S = \emptyset$. Then $V(H) = N[v]$. Hence every vertex in $V(G) \setminus N[v]$ has degree at least 1 in $G - V(H)$. By Claim B, there exist a vertex $w$ of degree 1 in $G - V(H)$. But then $w$ is adjacent to a vertex $v_1 \in N(v)$ and $v_1$ lies in a common triangle with $v$ and a small vertex $v'$. We obtain a tr-coloring of $G$ that colors $n - \Delta + 1$ vertices red by coloring $v$, $x$ and $y$ red, coloring every vertex in $V(G) \setminus (N[v] \cup \{w\})$ red and coloring all remaining uncolored vertices of $G$ blue. Thus, since $\Delta \geq 4$, $\gamma_{tr}(G) \leq n - \Delta + 1 \leq \phi(n, \Delta)$. Hence we may assume that $S \neq \emptyset$, and so $H$ satisfies the statement of Claim A. This implies that $H = G$, and therefore $S = V(G) \setminus N[v]$.

By Claim B, every vertex of $S$ is a small vertex of $G$ and $N(S) \subseteq \mathcal{L}(v)$. Let $\mathcal{L}_S = \mathcal{L} \cap N(S)$. Then, $\mathcal{L}_S$ is an independent set and every vertex of $\mathcal{L}$ lies in a common triangle with $v$. By Claim D, every two vertices in $\mathcal{L}_S$ have at most one common neighbor.

Suppose every vertex in $\mathcal{L}_S$ has at least two common neighbors with $v$. Observe that every vertex in $S$ is adjacent to exactly two vertices in $\mathcal{L}_S$ and every vertex in $\mathcal{L}_S$ lies in a common triangle with $v$. Let $v_1$ and $v_2$ be two vertices in $\mathcal{L}_S$. Then $v_1$ (respectively, $v_2$) has at least two common neighbors with $v$. Let $w_1$ be a common neighbor of $v_1$ and $v$ and let $w_2$ be a common neighbor of $v_2$ and $v$. Let $G'$ be the graph obtained from $G$ by deleting the vertices $x$, $y$, $w_1$ and $w_2$. Then, $G'$ has order $n' = n - 4$, size $m' < m$, maximum degree $\Delta'$ where $\Delta' = \Delta - 4 \leq n' - 2$ and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \phi(n', \Delta') = \phi(n - 4, \Delta - 4) = \phi(n, \Delta) - 2$. If a $\gamma_{tr}$-coloring of $G'$ colors $v$ red, then we color $v_1$ and $w_2$ red and color $x$ and $y$ blue. Otherwise, we color $w_1$ and $w_2$ blue and color $x$ and $y$ red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 2 \leq \phi(n, \Delta)$. Thus we may assume there is a vertex in $\mathcal{L}_S$, say $z$, that has exactly one common neighbor with $v$.

Then, $H_z$ has order $n_z \leq n - 3$, size $m_z < m$, common degree $\Delta' = \Delta - 2$ and minimum degree at least 2. If $\Delta' = n_z - 1$, then coloring $v$ red, coloring $z$ and its common neighbor with $v$ red and coloring the remaining uncolored vertices of $G$ blue, produces a tr-coloring of $G$. Therefore, since $\Delta > 4$, $\gamma_{tr}(G) = 3 < \phi(n, \Delta)$. Thus we may assume that $\Delta' \leq n_z - 2$. Then, $H_z$ satisfies the inductive hypothesis, and so $\gamma_{tr}(H_z) \leq \phi(n_z, \Delta') = \phi(n_z, \Delta) + 1$. Hence, $H_z$ satisfies the statement of Claim E, and so $\gamma_{tr}(G) \leq \phi(n, \Delta)$, as desired.

Claim G. We may assume that every vertex in $\mathcal{L}(v)$ has a neighbor that is not a neighbor of $v$.

Proof. Suppose $\mathcal{L}(v)$ contains a vertex $w$ such that every neighbor of $w$ is a neighbor of $v$. Then $v$ and $w$ contain at least two common neighbors. Let $x$ be a common neighbor of $v$ and $w$ and let $G' = G - x$. Then $G'$ has order $n' = n - 1$, size $m' = m - 2$, maximum degree $\Delta' = \Delta - 1 \leq n' - 2$ and minimum degree at least 2. Hence, $G'$ satisfies the inductive hypothesis, and so $\gamma_{tr}(G') \leq \phi(n', \Delta') = \phi(n - 1, \Delta - 1) = \phi(n, \Delta) - \frac{1}{2}$. Every $\gamma_{tr}$-coloring of $G'$ colors $v$ or $w$ red. If $G'$ color both $v$ and $w$ red, then we recolor $w$ blue and color $x$ blue; otherwise, we color $x$ blue. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') < \phi(n, \Delta)$, as desired.

Claim H. We may assume that every vertex in $\mathcal{L}(v)$ is adjacent to $v$.

Proof. Suppose $\mathcal{L}(v)$ contains a vertex $w$ that is not adjacent to $v$. Suppose first that $v$ and $w$ have no common neighbors. Then, $H_w$ satisfies the statement of Claim E, and so $\gamma_{tr}(G) < \phi(n, \Delta)$. Hence we may assume that $v$ and $w$ have at least one common neighbor. By Claim B, every common neighbor of $v$ and $w$ is a small vertex. Let $x$ be a common neighbor of $v$ and $w$.

Suppose $v$ and $w$ have at least two common neighbors. Let $G'$ be the graph obtained from $G$ by deleting $x$ and adding the edge $vw$. Then, $G'$ has order $n' = n - 1$, size $m' < m$, maximum degree $\Delta$ and minimum degree at least 2. If $\Delta = n' - 1$, then coloring $u$ and $x$ red and all remaining uncolored vertices of $G$ blue produces a tr-coloring of $G$. Hence $\gamma_{tr}(G) = 2 \leq n - \Delta = \phi(n, \Delta)$. Thus we may assume $\Delta \leq n' - 2$. By the inductive hypothesis, $\gamma_{tr}(G') \leq \phi(n', \Delta') \leq \phi(n, \Delta) - 1$. Any $\gamma_{tr}$-coloring of $G'$ colors $v$ or $w$ red, and can therefore be extended to a tr-coloring of $G$ by coloring $x$ red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 1 \leq \phi(n, \Delta)$. Thus we may assume that $v$ and $w$ have at most one common neighbor. But then $H_w$ satisfies the statement of Claim E, and so $\gamma_{tr}(G) < \phi(n, \Delta)$, as desired.

Claim I. We may assume that every vertex in $\mathcal{L}(v)$ has a common neighbor with $v$.

Proof. Suppose $\mathcal{L}(v)$ contains a vertex $w$ that has no common neighbor with $v$. By Claim H, $v$ and $w$ are adjacent in $G$. Then, $H_w$ satisfies the statement of Claim E, and so $\gamma_{tr}(G) \leq \phi(n, \Delta)$, as desired.
With our earlier assumptions, we have that \( L \setminus \{v\} \) is an independent set and that any two vertices in \( L \setminus \{v\} \) have at most one common neighbor. Furthermore, each vertex in \( L \setminus \{v\} \) is adjacent to \( v \), has at least one common neighbor with \( v \) and has at least one neighbor that is a small vertex not adjacent to \( v \). Let \( |L \setminus \{v\}| = k \). Then \( k \leq n/2 \). We now color \( v \) and every neighbor of \( v \) that is a small vertex blue and color the remaining uncolored vertices of \( G \) red. Hence, \( \gamma_{tr}(G) \leq n - (\Delta - k + 1) = n - \Delta + k - 1 \leq n - \Delta + \Delta/2 - 1 = \phi(n, \Delta) \). This establishes the upper bound of the theorem.

It remains for us to show that this upper bound is sharp. Let \( G \) be the graph obtained from a complete graph on \( t \geq 4 \) vertices in which every edge is subdivided exactly once and identifying one vertex \( v \) that is a large vertex and joining \( v \) to every other large vertex of the resulting graph. Then, \( n = t + \left(\frac{t}{2}\right) = 1 + 2(t - 1) + \left(\frac{t - 1}{2}\right) \) and \( \Delta = 2(t - 1) \). Every tr-coloring of \( G \) colors at least \( t - 1 \) large vertices red (in order to totally dominate all the small vertices) and therefore colors at least \( \left(\frac{t - 1}{2}\right) \) small vertices red. Thus, \( \gamma_{tr}(G) \geq (t - 1) + \left(\frac{t - 1}{2}\right) = n - (t - 1) - 1 = n - \Delta/2 - 1 = \phi(n, \Delta) \). Since the graph \( G \) satisfies the conditions of the theorem, we have already established that \( \gamma_{tr}(G) \leq \phi(n, \Delta) \). Consequently, \( \gamma_{tr}(G) = \phi(n, \Delta) \). This concludes the proof of Theorem 1.

5. Proof of Theorem 2

Before presenting a proof of Theorem 2, we first prove a key lemma that will be very useful in proving our main result. By a weak partition of a set we mean a partition of the set in which some of the subsets may be empty.

**Lemma 3.** Let \( G \) be a connected bipartite graph of order \( n \geq 5, \) maximum degree \( \Delta \geq 3 \) and minimum degree at least \( 2. \) If \( G \) has a vertex of maximum degree \( \Delta \) that is adjacent only to degree-2 vertices, then \( \gamma_{tr}(G) \leq \psi(n, \Delta) \) and this bound is sharp.

**Proof.** Let \( v \) be a vertex of maximum degree \( \Delta \) in \( G \). By assumption, every vertex adjacent to \( v \) has degree 2. Define \( A \) such that \( u \in A \) if and only if \( N(u) \subseteq N(v) \). Note that all the vertices in \( A \) are at distance two from \( v \). Let \( B \) be the set of all vertices at distance two from \( v \) which do not lie in \( A \). Hence every vertex in \( B \) has at least one neighbor that is not a neighbor of \( v \). Since \( G \) is a bipartite graph, the set \( B \) is an independent set in \( G \). Define \( C \) such that \( u \in C \) if and only if \( N(u) \subseteq B \). Let \( D = V \setminus (N[v] \cup A \cup B \cup C) \). Thus if \( D \neq \emptyset \), then \( G[D] \) contains no isolated vertex. Note that \( V = (N[v], A, B, C, D) \) is a weak partition of the set \( V \).

Let \( |A| = a \) and \( |B| = b \). If \( a \geq 1 \), let \( A = \{v_1, \ldots, v_a\} \) and if \( b \geq 1 \), let \( B = \{w_1, \ldots, w_b\} \). If \( a \geq 1 \), then for \( i = 1, \ldots, a \), let \( d(v_i) = |N(v_i)| = \ell_i \) and let

\[
\ell = \sum_{i=1}^{a} \ell_i.
\]

By assumption \( \delta(G) \geq 2 \), and so \( \ell_i \geq 2 \) for \( i = 1, \ldots, a \). Thus, \( \ell \geq 2a \). If \( b \geq 1 \), then for \( i = 1, \ldots, b \), let \( N_i = N(v) \cap N(w_i) \) and let \( |N_i| = r_i \). Let

\[
r = \sum_{i=1}^{b} r_i.
\]

Then, \( \ell + r = \Delta \geq 3 \). Let

\[
\alpha = n - \frac{3}{2}(\ell + r) - \frac{3}{2}\sqrt{3(\ell + r) - 8} - \frac{7}{9}.
\]

Then, \( \alpha = \psi(n, \Delta) \). We wish to show that \( \gamma_{tr}(G) \leq \alpha \).

**Claim J.** We may assume that \( \ell \geq 1 \).

**Proof.** Suppose that \( \ell = 0 \). Then, \( \Delta = r \). We now produce a tr-coloring of \( G \) as follows. Color the vertex \( v \) blue and color all but one neighbor of \( v \) blue. Color all remaining uncolored vertices red. This produces a tr-coloring of \( G \) that
colors $\Delta$ vertices blue, and so $\gamma_{t_2}(G) \leq n - \Delta$. However,
\[
\begin{align*}
n - \Delta &\leq x, \\
\iff n - \Delta &\leq n - \frac{2}{3}\Delta - \frac{2}{9}\sqrt{3\Delta^2 - 8} - \frac{7}{9}, \\
\iff 0 &\leq (\Delta - 3)^2.
\end{align*}
\]
By assumption $\Delta \geq 3$, implying that $n - \Delta \leq x$, whence $\gamma_{t_2}(G) \leq x$, as desired. \hfill \Box

**Claim K.** We may assume that $r \geq 1$.

**Proof.** Suppose that $r = 0$. Then, $\Delta = \ell$. We now produce a $t_2$-coloring of $G$ as follows. Color each vertex in $A$ and all but one neighbor of each vertex of $A$ blue. Color all remaining uncolored vertices red. This produces a $t_2$-coloring of $G$ that colors $\Delta$ vertices blue, and so $\gamma_{t_2}(G) \leq n - \Delta \leq x$, whence $\gamma_{t_2}(G) \leq x$, as desired. \hfill \Box

By Claim J, $\ell \geq 1$ and by Claim K, $r \geq 1$. Renaming vertices if necessary, we may assume that amongst all the sets, $N_i$, $1 \leq i \leq b$, the set $N_1$ is one of maximum cardinality. That is, among all vertices in $B$, the vertex $w_1$ has the maximum number of common neighbors with $v$. By the Pigeonhole Principle, $|N_1| = r_1 \geq r/b$.

We next define a weak partition $B = (B_1, B_2, B_3)$ of the independent set $B$ and a weak partition $C = (C_1, C_2)$ of the set $C$ as follows. Let $B_1$ be a maximum subset of vertices in $B$ that contain the vertex $w_1$ and such that $B_1$ is a packing in $G$. By the maximality of $B_1$, each vertex in $B \setminus B_1$ is at distance 2 from some vertex of $B_1$. Recall that $C = \{u \in V \mid N(u) \subseteq B\}$. Since $B_1$ is a packing, each vertex of $C$ is adjacent to at most one vertex of $B_1$ and therefore to at least one vertex of $B \setminus B_1$. Let $B_2$ be a minimum subset of vertices in $B \setminus B_1$ that dominates $C$. Let $C_1$ be the set of vertices in $C$ that are adjacent only to vertices of $B_2$, and let $C_2 = C \setminus C_1$. Thus, $C_1$ is defined such that $u \in C$ if and only if $N(u) \subseteq B_2$. By the minimality of the set $B_2$, we note that each vertex of $B_2$ is adjacent to at least one vertex of $C_2$ that is not adjacent to any other vertex of $B_2$. Thus, $|C_2| \geq |B_2|$. Let $B_3 = B \setminus (B_1 \cup B_2)$.

We now consider two $t_2$-colorings of $G$, one in which $v$ is colored red and the other in which $v$ is colored blue.

**Claim L.** There exists a $t_2$-coloring of $G$ that colors the vertex $v$ red and colors at least $\ell + 2\sqrt{r} - 1$ vertices blue.

**Proof.** We begin by coloring each vertex in $A$ and all but one neighbor of each vertex in $A$ blue. We then color each vertex in the set $B_1 \cup B_2 \cup C_2$ blue. For each vertex of $B_1$, color all but one of its common neighbors with $v$ blue. For each vertex in $B_3$ that is dominated by $D$, color all of its common neighbors with $v$ blue. For each vertex in $B_3$ that is not dominated by $D$ (and is therefore dominated by $C_2$), color all but one of its common neighbors with $v$ blue. Color all remaining uncolored vertices red. In particular, note that $r_1 - 1$ common neighbors of $v$ and $w_1$ are colored blue. Further note that $v$ is colored red and each vertex in the set $B_2 \cup C_1 \cup D$ is colored red. In this way we produce a $t_2$-coloring of $G$ that colors at least $\ell + (r_1 - 1) + |B_1| + |C_2| + |B_3| \geq \ell + (r/b - 1) + |B_1| + |B_2| + |B_3| = \ell + r/b + b - 1$ vertices blue. Since the function $r/b + b$ (for $r$ fixed) is minimized when $b = \sqrt{r}$, it follows that our $t_2$-coloring of $G$ colors at least $\ell + 2\sqrt{r} - 1$ vertices blue, as desired. \hfill \Box

**Claim M.** There exists a $t_2$-coloring of $G$ that colors $v$ blue and colors at least $\ell/2 + r + 1$ vertices blue.

**Proof.** We begin by coloring $v$ blue. We color each vertex in $A$ red and we color exactly one neighbor of each vertex in $A$ red. We then color all remaining uncolored vertices in $N(v)$ blue. Thereafter, we color all remaining uncolored vertices in $G$ red. In this way we produce a $t_2$-coloring of $G$ that colors $v$ blue and colors all but a neighbors of $v$ blue. Since $\ell \geq 2a$, this $t_2$-coloring of $G$ colors $\ell + r - a + 1 \geq \ell/2 + r + 1$ vertices blue, as desired. \hfill \Box

Let $\mathcal{C}$ be a $\gamma_{t_2}$-coloring of $G$. Hence among all $t_2$-colorings of $G$, the coloring $\mathcal{C}$ maximizes the number of vertices that can be colored blue. If $\mathcal{C}$ colors $v$ red, then, by Claim L, $\mathcal{C}$ colors at least $\ell + 2\sqrt{r} - 1$ vertices blue. On the other hand, if $\mathcal{C}$ colors $v$ blue, then, by Claim M, $\mathcal{C}$ colors at least $\ell/2 + r + 1$ vertices blue. Hence letting
\[
\begin{align*}
x_1 &= n - \frac{\ell}{2} - r - 1 \\
x_2 &= n - \ell - 2\sqrt{r} + 1,
\end{align*}
\]
we have that
\[ \gamma_{tr}(G) \leq \min\{x_1, x_2\}. \]

We consider two possibilities.

**Case 1:** \( \ell \leq 2(r - 2\sqrt{r} + 2) \). Then, \( x_1 \leq x_2 \), and so \( \gamma_{tr}(G) \leq x_1 \). Hence it suffices for us to show that \( x_1 \leq x \). Note that in this case since \( r \geq 1 \), we have \( \ell \leq 2r \). In particular, note that \(-3\ell + 6r + 4 > 0\). Now,

\[ x_1 \leq x, \]
\[ \iff n - \frac{\ell}{2} - r - 1 \leq n - \frac{2}{9}(\ell + r) - \frac{2}{9}\sqrt{3(\ell + r)} - 8 - \frac{7}{9}, \]
\[ \iff 4\sqrt{3(\ell + r)} - 8 \leq -3\ell + 6r + 4, \]
\[ \iff 0 \leq 9\ell^2 - 36(r + 2)\ell + (36r^2 + 144), \]
\[ \iff \ell \leq 2(r - 2\sqrt{r} + 2) \text{ or } \ell \geq 2(r + 2\sqrt{r} + 2). \]

By assumption, \( \ell \leq 2(r - 2\sqrt{r} + 2) \), implying that \( x_1 \leq x \), whence \( \gamma_{tr}(G) \leq x \), as desired.

**Case 2:** \( \ell \geq 2(r - 2\sqrt{r} + 2) \). Then, \( x_2 \leq x_1 \), and so \( \gamma_{tr}(G) \leq x_2 \). Hence it suffices for us to show that \( x_2 \leq x \). Note that in this case since \( r \geq 1 \), we have \( 3\ell - 6r + 18\sqrt{r} - 16 > 6\sqrt{r} - 4 > 0 \). Now,

\[ x_2 \leq x, \]
\[ \iff n - \ell - 2\sqrt{r} + 1 \leq n - \frac{2}{9}(\ell + r) - \frac{2}{9}\sqrt{3(\ell + r)} - 8 - \frac{7}{9}, \]
\[ \iff 2\sqrt{3(\ell + r)} - 8 \leq 3\ell - 6r + 18\sqrt{r} - 16, \]
\[ \iff 0 \leq 9\ell^2 - (36r - 108\sqrt{1} + 108)\ell + (36r^2 - 216\sqrt{r}^3/2 + 504r - 576\sqrt{r} + 288), \]
\[ \iff \ell \leq 2(r - 4\sqrt{r} + 2) \text{ or } \ell \geq 2(r - 2\sqrt{r} + 2). \]

By assumption, \( \ell \geq 2(r - 2\sqrt{r} + 2) \), implying that \( x_2 \leq x \), whence \( \gamma_{tr}(G) \leq x \), as desired.

In both cases, the desired upper bound follows. It remains for us to establish that the upper bound is sharp. Let \( t \geq 2 \) be an integer, and let \( a = t^2 - 2t + 2, b = t, \ell = 2(t^2 - 2t + 2) \) and \( r = t^2 \). Let \( H = aP_3 \cup bK_1,t \). Let \( A \) be the set of \( a \) central vertices of the paths \( P_3 \), and let \( B \) be the set of \( b \) central vertices of the stars \( K_1,t \). Let \( G \) be the graph obtained from \( H \) by forming a clique on the set \( B \), subdividing each edge of the resulting complete graph on these \( b \) vertices exactly once, and adding a new vertex \( v \) and joining it to every vertex of degree 1 in \( H \). Then, \( v \) has maximum degree in \( G \), namely \( 2a + bt = \ell + r \). By construction, \( G \) is a connected bipartite graph of order \( n \), maximum degree \( A \) and minimum degree at least 2, where \( A = \ell + r \) and \( n = 1 + a + b + \left(\frac{b}{2}\right) + \ell + r \). Thus,

\[ A = 3t^2 - 4t + 4 \quad \text{and} \quad \]
\[ n = \frac{9}{2}t^2 - \frac{11}{2}t + 7. \]

Furthermore, the vertex \( v \) is a vertex of maximum degree \( A \) in \( G \) that is only adjacent to degree-2 vertices. Thus the conditions of the lemma are satisfied. We show that the graph \( G \) achieves the upper bound of the lemma. Let \( \% \) be a \( \gamma_{tr} \)-coloring of \( G \). We consider two possibilities.

Suppose \( \% \) colors the vertex \( v \) blue. Then every vertex of \( B \) is red (since each common neighbor of \( v \) and a vertex of \( B \) must have a red neighbor), whence every degree-2 vertex joining two vertices of \( B \) is red. Furthermore, each vertex of \( A \) is colored red (since each common neighbor of \( v \) and a vertex of \( A \) must have a red neighbor). Since each vertex of \( A \) must have a red neighbor, one neighbor of each vertex of \( A \) is colored red. Thus at least \( n - (r + \ell/2 + 1) = n - 2t^2 + 2t - 3 \) vertices are colored red.
On the other hand, suppose 6 colors the vertex $v$ red. If two vertices of $B$ are colored blue, then the common neighbor of these two vertices has no red neighbor, a contradiction. Hence at least $b - 1$ vertices of $B$ are colored red. Let $B' \subseteq B - \{u\}$ be a subset of $b - 1$ vertices of $B$ that are colored red. Every degree-2 vertex joining two vertices of $B'$ is red. Every degree-2 vertex joining $v$ and a vertex of $B'$ is red. At least one neighbor of the vertex in $B' \setminus B'$ is colored red. Since each vertex of $A$ must have a red neighbor, one neighbor of each vertex of $A$ is colored red. Thus at least 

$$n - (\ell + 2t - 1) = n - 2t^2 + 2t - 3 \text{ vertices are colored red.}$$

In both cases, the $\gamma_{tr}$-coloring of $G$ colors at least $n - 2t^2 + 2t - 3$ vertices red. Hence, $\gamma_{tr}(G) \geq n - 2t^2 + 2t - 3 = \psi(n, A)$. Since the upper bound of the lemma has been established, we know that $\gamma_{tr}(G) \leq \psi(n, A)$. Consequently, $\gamma_{tr}(G) = \psi(n, A)$.

We are now ready to prove the main result of this section. Recall Theorem 2.

**Theorem 2.** If $G$ is a connected bipartite graph of order $n \geq 5$, maximum degree $A$ where $3 \leq A \leq n - 2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq \psi(n, A)$, and this bound is sharp.

**Proof.** We proceed by induction on $\ell = n + m$, where $m$ denotes the size of $G$. Note that $n \geq 5$ and $m \geq 6$, and so $\ell \geq 11$. When $\ell = 11$, $G = K_{2,3}$ and $\gamma_{tr}(G) = 2 = \psi(5, 3) = \psi(n, A)$. This establishes the base case. For the inductive hypothesis, let $\ell \geq 12$ and assume that for all connected bipartite graphs $G$ of order $n' \geq 5$ and size $m'$ with $n' + m' < \ell$ that have maximum degree $A'$ where $3 \leq A' \leq n' - 2$ and minimum degree at least 2 that $\gamma_{tr}(G') \leq \psi(n, A')$. Let $G$ be a connected bipartite graph of order $n \geq 5$ and size $m$ with $\ell = n + m$, maximum degree $A$ where $3 \leq A \leq n - 2$ and minimum degree at least 2.

The proof of the following claim is almost identical to the proof of Claim A, and is therefore omitted.

**Claim N.** If a connected proper subgraph $G'$ of $G$ of order $n'$ has maximum degree $A$ where $3 \leq A \leq n' - 2$ and minimum degree at least 2, and if the subgraph $G - V(G')$ contains no isolated vertices, then $\gamma_{tr}(G) \leq \psi(n, A)$.

Let $v$ be a vertex of maximum degree $A$ in $G$. Recall that $L$ is the set of all large vertices of $G$.

**Claim O.** We may assume that the set $L \setminus \{v\}$ is an independent set in $G$.

**Proof.** Suppose $e = uvw$ is an edge of $G$ joining two vertices $u$ and $w$ of $L \setminus \{v\}$. If $G - e$ is a connected, then $G$ that satisfies the statement of Claim N, and so $\gamma_{tr}(G) \leq \psi(n, A)$. Hence, we may assume that $e$ is a bridge of $G$. Let $G_u$ be the component of $G - e$ containing $u$, and $G_w$ be the component containing $w$. We may assume that $v \in V(G_u)$. Hence, $G_u$ is a connected proper subgraph of $G$ of order $n'$ with maximum degree $A$ where $A \geq 3$ and minimum degree at least 2. If $v$ dominates $G_u$, then $u$ and $v$ have a common neighbor. But this contradicts our assumption that $G$ is bipartite. Hence, $A \leq n' - 2$. Thus, $G_u$ satisfies the statement of Claim N, and so $\gamma_{tr}(G_u) \leq \psi(n, A)$, as desired.

By Claim O, the only edges in $G[L']$, if any, are incident with $v$.

**Claim P.** We may assume that every two vertices in $L' \setminus \{v\}$ have at most one common neighbor different from $v$.

**Proof.** Suppose $L' \setminus \{v\}$ contains two vertices $u$ and $w$ that have at least two common neighbors different from $v$. Then both these common neighbors are small. Let $x$ be a small vertex that is a common neighbor of $u$ and $w$. Then, $G' = G - x$ is a connected bipartite graph of order $n' = n - 1$, size $m' = m - 2$, maximum degree $A$ where $3 \leq A \leq n' - 2$ and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \psi(n', A) = \psi(n - 1, A) = \psi(n, A) - 1$. Any $\gamma_{tr}$-coloring of $G'$ colors $u$ or $w$ red, and can therefore be extended to a $tr$-coloring of $G$ by coloring $x$ red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 1 = \psi(n, A)$, as desired.

**Claim Q.** We may assume that there is no 2-handle whose ends are adjacent with a vertex in $L \setminus \{v\}$ that is adjacent with $v$.

**Proof.** Suppose that there is a 2-handle $C$ whose ends are adjacent with a vertex $u \in L \setminus \{v\}$ that is adjacent with $v$. Since $G$ is bipartite, the $V(C)$ consists of an odd number of vertices. Let $C$ be the 2-handle $u_1, u_2, \ldots, u_k$, for some $k \geq 3$. 
Suppose that $k \geq 5$. Let $G'$ be the graph obtained from $G$ by deleting the vertices $u_1$ and $u_2$ and adding the edge $uu_3$. Then, $G'$ is a connected bipartite graph of order $n' = n - 2$, size $m' < m$, maximum degree $\Delta$ where $3 \leq \Delta \leq n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \psi(n', \Delta) = \psi(n, \Delta) - 2$. If a $\gamma_{tr}$-coloring of $G'$ colors $u$ or $u_3$ red, then we color $u_1$ and $u_2$ red. Otherwise, we recolor $u_3$ red, color $u_2$ red and color $u_1$ blue. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 2 \leq \psi(n, \Delta)$. Thus we may assume that $k = 3$.

If $d(u) \geq 4$, then $G - V(C)$ satisfies the statement of Claim N, and so $\gamma_{tr}(G) \leq \psi(n, \Delta)$. Hence we may assume $d(u) = 3$. Let $G'$ be the graph obtained from $G$ by deleting $u$ and adding the edges $uu_1$ and $uu_3$. Then, $G'$ is a connected bipartite graph of order $n' = n - 1$, size $m' = m - 1$, maximum degree $\Delta' = \Delta + 1$ where $3 \leq \Delta' \leq n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \psi(n', \Delta') = \psi(n - 1, \Delta + 1) = \psi(n, \Delta + 1) - 1$. For $\Delta \geq 3$, $\psi(n, \Delta) - \psi(n, \Delta + 1) = 2(\sqrt{3\Delta - 5} - \sqrt{3\Delta - 8})/9$, and so $\gamma_{tr}(G) \leq \psi(n, \Delta) - \psi(n, \Delta + 1) \leq 8/9$. Thus, $\gamma_{tr}(G') \leq \psi(n, \Delta + 1) - 1 < \psi(n, \Delta) - 1$.

If a $\gamma_{tr}$-coloring of $G'$ colors $v$ red, then we color $u$ red. If a $\gamma_{tr}$-coloring of $G'$ colors $v$ blue and colors a neighbor of $v$ different from $u_1$ and $u_3$ red, then we color $u$ blue. If a $\gamma_{tr}$-coloring of $G'$ colors $v$ blue and colors every neighbor of $v$ different from $u_1$ and $u_2$, then exactly one of $u_1$ and $u_2$ is colored red, and we recolor $u_3$ blue and color $u$ red. In this way, we produce a $tr$-coloring of $G$ from a $\gamma_{tr}$-coloring of $G'$ that colors at most $\gamma_{tr}(G') + 1$ vertices in $G$ red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 1 < \psi(n, \Delta) - 2/9 < \psi(n, \Delta)$, as desired. □

Claim R. If $v$ has a large neighbor $u$, then we may assume that every vertex at distance 2 from $u$ in $G - v$ is a large vertex in $G$.

Proof. Suppose there is a vertex at distance 2 from $u$ in $G - v$ that is a small vertex in $G$. We consider two cases.

Case 1: There is a small vertex at distance 2 from $u$ in $G - v$ that is not adjacent to $v$ in $G$. Let $y$ be a small vertex at distance 2 from $u$ in $G - v$ that is not adjacent to $v$ in $G$, and let $x$ be the common (small) neighbor of $u$ and $y$. Let $w$ be the neighbor of $y$ different from $x$. Since $G$ is bipartite, $u$, $w$, and $y$ are not adjacent. If $w$ is large, then $G' = G - \{x, y\}$ satisfies the statement of Claim N, and so $\gamma_{tr}(G) \leq \psi(n, \Delta)$. Hence we may assume $w$ is a small vertex. By Claim Q, $u$, $w$, and $y$ are not adjacent vertices. Let $N(w) = \{y, z\}$.

Let $G'$ be the graph obtained from $G$ by deleting $x$ and $y$ and adding the edge $uw$. Then, $G'$ is a connected bipartite graph of order $n' = n - 2$, size $m' < m$, maximum degree $\Delta$ where $3 \leq \Delta \leq n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \psi(n', \Delta) \leq \psi(n, \Delta) - 2$. If a $\gamma_{tr}$-coloring of $G'$ colors $u$ or $w$ red, then we color $x$ and $y$ red. Otherwise, if a $\gamma_{tr}$-coloring of $G'$ colors both $u$ and $w$ blue, then it colors $z$ red and we can therefore recolor $u$ red, color $y$ red and color $x$ blue. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 2 \leq \psi(n, \Delta)$.

Case 2: Every small vertex at distance 2 from $u$ in $G - v$ is adjacent to $v$ in $G$. Let $y$ be a small vertex at distance 2 from $u$ in $G - v$ and let $x$ be the common (small) neighbor of $u$ and $y$. Then, $v$ and $y$ are adjacent vertices in $G$. Let $S$ be the set of vertices that belong to a 2-path where one end is adjacent to $u$ and the other end is adjacent to a large vertex different from $v$ (possibly, $S = \emptyset$). Then, every vertex of $S$ is the common small neighbor of $u$ and a large vertex different from $v$. Furthermore, by Claim P, every two vertices in $S$ have only the vertex $u$ as a common neighbor. Let $T$ denote the set of vertices that lie on a 2-path with one end adjacent to $u$ and the other end adjacent to $v$. Then, $T_u$ and $T_v$ be the vertices in $T$ adjacent with $u$ and $v$, respectively. Then, $x \in T_u$ and $y \in T_v$, and $N(u) = T_u \cup \{v\}$.

Case 2.1: $S \neq \emptyset$. Let $G'$ be the component of $G - S$ that contains $v$ (possibly, $G' = G - S$). Then, $G'$ is a connected bipartite graph of order $n' = n - k$ where $k \geq |S|$, size $m' < m$, maximum degree $\Delta$ where $3 \leq \Delta \leq n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \psi(n', \Delta) = \psi(n, \Delta) - k$. Note that in $G'$, $N(u) = T_u \cup \{v\}$.

Consider a $\gamma_{tr}$-coloring of $G'$. If $S'$ colors both $u$ and $v$ blue, then it colors every vertex in $T_r$ red, and so we recolor $u$ red and every vertex of $T_v$ blue. If $S'$ colors $u$ blue and $v$ red, then it colors every vertex in $T_u$ red and every vertex in $T_v$ blue, and so we recolor $u$ red and recolor every vertex in $T_v$ blue. In both cases, we must have that $T = \{x, y\}$, for otherwise, we produce a new $tr$-coloring of $G'$ that colors fewer vertices red than does $S'$, which is impossible. Hence, in both cases, we produce a new $\gamma_{tr}$-coloring of $G'$ that colors $u$ red. Therefore, we may assume that $S'$ colors $u$ red. But then we can extend $S'$ to a $tr$-coloring of $G$ by coloring all remaining $k$ uncolored vertices red. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + k \leq \psi(n, \Delta)$.

Case 2.2: $S = \emptyset$. Then, $N(u) = T_u \cup \{v\}$. Since $d(u) \geq 3$, $|T_u| \geq 2$. Let $G' = G - \{x, y\}$. Then, $G'$ is a connected bipartite graph of order $n' = n - 2$, size $m' < m$, maximum degree $\Delta'$ where $\Delta - 1 \leq \Delta' \leq \Delta$ and $\Delta' \leq n' - 2$, and minimum degree at least 2. By the inductive hypothesis, $\gamma_{tr}(G') \leq \psi(n', \Delta') = \psi(n, \Delta) - 2\Delta$. Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') \leq \gamma_{tr}(G') + 2 = \psi(n, \Delta) - 2\Delta$. Therefore, $\gamma_{tr}(G) \leq \psi(n, \Delta)$, as desired.
degree at least 2. If $A' = A$, then $G'$ satisfies the statement of Claim N, and the desired result follows. Hence, we may assume that $A' = A - 1$.

If $A' = 2$, then $G' = C_4$ and $G$ is obtained from a 6-cycle by adding an edge between two vertices at distance 3 apart on the cycle. Coloring $u$ and $v$ red and coloring every other vertex of $G$ blue produces a tr-coloring of $G$, and so $\gamma_{tr}(G) = 2 < 3 = \psi(6, 3) = \psi(n, A)$. Hence we may assume $A' \geq 3$ (and so, $A \geq 4$). Applying the inductive hypothesis to $G'$, $\gamma_{tr}(G') \leq \psi(n', A') = \psi(n - 2, A - 1) = \psi(n, A - 1) - 2$. For $A \geq 4$, $\psi(n, A - 1) - \psi(n, A) \leq \frac{8}{9}$, and so $\gamma_{tr}(G) \leq \psi(n, A) - \frac{10}{9}$.

Consider a $\gamma_{tr}$-coloring $\mathcal{C}'$ of $G'$. If $\mathcal{C}'$ colors both $u$ and $v$ blue, then it colors every vertex in $T \setminus \{x, y\}$ red, and so we recolor $u$ red and color every vertex in $T_v$ blue to produce a new $\gamma_{tr}$-coloring of $G'$. Such a $\gamma_{tr}$-coloring of $G'$ can be extended to a tr-coloring of $G$ by coloring $x$ red and $y$ blue. If $\mathcal{C}'$ colors both $u$ and $v$ red, then it can be extended to a tr-coloring of $G$ by coloring both $x$ and $y$ blue. If $\mathcal{C}'$ colors $u$ red and $v$ blue (respectively, $u$ blue and $v$ red), then it can be extended to a tr-coloring of $G$ by coloring $x$ red and $y$ blue (respectively, $x$ blue and $y$ red). Hence, $\gamma_{tr}(G) \leq \gamma_{tr}(G') + 1 \leq \gamma_{tr}(G') - 1/9 < \psi(n, A)$, as desired. □

Claim S. We may assume that $v$ has no large neighbor.

Proof. Suppose that $v$ has a large neighbor $u$. Since $G$ is bipartite, $u$ and $v$ have no common neighbors. By Claim O, every neighbor of $u$ different from $v$ is small. By Claim P, every two small neighbors of $u$ have only the vertex $u$ as a common neighbor. By Claim R, every vertex at distance 2 from $u$ in $G - v$ is large in $G$. Let $U = N[u]\{v\}$ and let $|U| = k$. We now consider the graph $G' = G - U$. Then, $\delta(G') \geq 2$.

Suppose $G'$ is disconnected. Let $F$ be a component of $G'$ that does not contain the vertex $v$. Then the component of $G - V(F)$ that contains $v$ has the vertices in $U$ and satisfies the statement of Claim N, and the desired result follows. Hence, we may assume that $G'$ is connected. Let $G'$ have maximum degree $A'$.

Suppose $A' = 2$. Then $G$ can be obtained from a 4-cycle $v = v_1, v_2, v_3, v_4, v$ by adding a new vertex $u$, joining it to each of $v, v_2$ and $v_4$, and then subdividing the edges $uv_2$ and $uv_4$ exactly once. Thus, $\gamma_{tr}(G) \leq 4 = \psi(7, 3) = \psi(n, A)$. Hence, we may assume that $A' \geq 3$ (and so, $A \geq 4$).

Thus, $G'$ is a connected graph of order $n' = n - k$, size $m' < m$, maximum degree $A'$ where $A - 1 \leq A' \leq A$ and $A' \leq n' - 2$, and minimum degree at least 2. If $A' = A$, then the desired result follows readily from Claim N. Hence we may assume that $A' = A - 1$. Applying the inductive hypothesis to $G'$, $\gamma_{tr}(G') \leq \psi(n', A') = \psi(n - k, A - 1) = \psi(n, A - 1) - k$. For $A \geq 4$, $\psi(n, A - 1) = \psi(n, A) - k + \frac{8}{9}$, and so $\gamma_{tr}(G') \leq \psi(n, A) - k + \frac{8}{9}$.

Let $\mathcal{C}'$ be a $\gamma_{tr}$-coloring of $G'$. If $\mathcal{C}'$ colors a vertex in $G'$ that has a common small neighbor with $u$ blue, then we extend $\mathcal{C}'$ to a tr-coloring of $G$ by coloring this common small neighbor blue and coloring all remaining uncolored vertices of $G$ red. Otherwise, we extend $\mathcal{C}'$ by coloring $u$ and a small neighbor of $u$ blue and coloring all remaining uncolored vertices of $G$ red. In this way, we extend $\mathcal{C}'$ to a tr-coloring of $G$ by coloring at most $k - 1$ additional vertices red, and so $\gamma_{tr}(G) \leq \gamma_{tr}(G') + k - 1 \leq \psi(n, A) - 1/9 < \psi(n, A)$. □

By Claim S, the vertex $v$ of maximum degree in $G$ is adjacent only to degree-2 vertices. Hence by Lemma 3, $\gamma_{tr}(G) \leq \psi(n, A)$ and this bound is sharp. This completes the proof of the Theorem 2. □

6. Closing remark

As remarked by one the referees, for any integer $k \geq 2$, there exists an infinite family of graphs $G$ with minimum degree $k$ such that $\gamma_{tr}(G)/|V(G)|$ tends to one when $|V(G)|$ goes to infinity. One such family contains graphs $G_k, G_{k+1}, G_{k+2}, \ldots$ where $G_r$ is constructed as follows. For integers $r \geq k+1$, let $B_r$ be the bipartite graph formed by taking as one partite set a set of $r$ elements, and as the other partite set $B$ all the $k$-element subsets of $A$, and joining each element of $A$ to those subsets it is a member of. Note that $|B| = \binom{r}{k}$. Let $G_r$ be obtained from $B_r$ by forming a clique on the set $A$, and so the set $A$ induces a complete graph on $r$ vertices in $G_r$. Then, $G_r$ has minimum degree $k$ and $|V(G)| = r + \binom{r}{k}$.

In every tr-coloring of $G_r$, at least $r - k + 1$ vertices in $A$ need to be colored red (to totally dominate the set $B$) and therefore at least $\binom{r-k+1}{k}$ vertices in $B$ need to be colored red. Thus, $\gamma_{tr}(G)/|V(G)| \geq |r - k + 1 + \binom{r-k+1}{k} + \binom{r-k+1}{k} = \binom{r-k+1}{k}$, which tends to one when $r$ tends to infinity. However, this is not the case for the total domination number $\gamma_t(G)$ since if $G$ is a graph with minimum degree $k \geq 2$, then $\gamma_t(G)/|V(G)| \leq (1 + \ln k)/k$. 


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