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Tangent vectors to a zero set at abnormal points ☆

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Abstract

We are concerned with sharp characterization of the contingent cone to the set defined by a finite number of equality constraints in the absence of classical regularity (constrained qualification).

Keywords: Tangent vector; Contingent cone; Nonlinear mapping; Constrained optimization; Second-order optimality conditions

1. Introduction

Let S be a given set in a Banach space X. The *contingent cone* (Bouligand tangent) $T_S(\bar{x})$ to S at a point $\bar{x} \in S$ is comprised by all vectors $d \in X$ possessing the following property: there exist a sequence $\{d^k\} \subset X$ convergent to d, and a sequence of positive numbers $\{t_k\}$ convergent to zero, such that $\bar{x} + t_k d^k \in S$, $\forall k$. The contingent cone to any set at any point is closed. We refer to the elements of the contingent cone as *tangent vectors*.

In addition to being of independent interest, the significance of this particular notion of tangency has to do with the fact that it turns out to be the most natural and convenient in the context of optimality conditions for constrained optimization problems.

Namely, consider the optimization problem

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$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in S, \end{aligned} \tag{1}$$

where $f : X \to \mathbf{R}$ is a smooth function. Then the following first-order optimality conditions are standard. If $\bar{x} \in S$ is a local solution of problem (1), then

$$\langle f'(\bar{x}), d \rangle \geqslant 0, \quad \forall d \in T_S(\bar{x}). \tag{2}$$

This primal necessary condition can be presented in the form

$$-f'(\bar{x}) \in \big(T_S(\bar{x})\big)^\circ,$$

where $(T_S(\bar{x}))^\circ = \{x^* \in X^* \mid \langle x^*, d \rangle \leqslant 0, \ \forall d \in T_S(\bar{x})\}$ is the polar cone of $T_S(\bar{x})$; X^* stands for the (topological) dual space of X. Hence, a specific characterization of $(T_S(\bar{x}))^\circ$ for specific S leads to the primal–dual form of necessary optimality condition.

Necessary condition (2) is quite sharp, as it is close to sufficient, in the sense that the same cone is involved in both. Specifically, if X is finite-dimensional, and for $\bar{x} \in S$ it holds that

$$\langle f'(\bar{x}), d \rangle > 0, \quad \forall d \in T_S(\bar{x}) \setminus \{0\},$$

then $\bar{x}$ is a strict local solution of (1).

Another important area of application of the notion of tangency in question is in the bifurcation theory. Specifically, let

$$S = \big\{(\sigma, u) \in \Sigma \times U \mid F(\sigma, u) = 0\big\},$$

where Σ, U and Y are Banach spaces, and $F : \Sigma \times U \to Y$ is a smooth mapping. Suppose that $\bar{u} \in U$ is a trivial solution, that is

$$\Sigma \times \{\bar{u}\} \subset S. \tag{3}$$

In order to establish the existence of bifurcation at $(\bar{\sigma}, \bar{u})$ for some $\bar{\sigma} \in \Sigma$, it is sufficient to show that there exists a vector $(v, v) \in T_S(\bar{\sigma}, \bar{u})$ such that $v \neq 0$. This observation is the essence of numerous bifurcation theorems [1,9–12,16–20].

Summarizing, sharp characterization of the contingent cone to a set given by more specific constraints is certainly of interest. In this paper, we derive the sharpest known characterization of this kind for the set given by a finite number of equality constraints:

$$S = \big\{x \in X \mid F(x) = 0\big\}, \tag{4}$$

where $F : X \to \mathbf{R}^l$ is a sufficiently smooth mapping.

By necessity, $T_S(\bar{x}) \subset \ker F'(\bar{x})$, and the classical Lyusternik theorem says that the converse inclusion is true provided the point $\bar{x}$ is *normal*, that is

$$\operatorname{rank} F'(\bar{x}) = l. \tag{5}$$

That is, the normality condition (5) guarantees the equality $T_S(\bar{x}) = \ker F'(\bar{x})$. However, the inclusion $T_S(\bar{x}) \supset \ker F'(\bar{x})$ is not necessarily true when (5) is violated.

The first tangent cone results valid without the normality condition were obtained in [21] (under the assumption that $F'(\bar{x}) = 0$) and [7] (in the general setting; see also [3,8,15,

16]). Let P be the orthogonal projector onto $(\operatorname{im} F'(\bar{x}))^{\perp}$ in $\mathbf{R}^l$ (note that (5) is violated if and only if $\operatorname{im} F'(\bar{x})$ is a proper subspace in $\mathbf{R}^l$). Define the mapping

$$\Phi(\bar{x},\cdot): X \to \mathbf{R}^l, \quad \Phi(\bar{x},d) = F'(\bar{x})d + \frac{1}{2}PF''(\bar{x})[d,d], \tag{6}$$

and the cone

$$\begin{aligned} T(\bar{x}) &= \{d \in X \mid \Phi(\bar{x},d) = 0\} \\ &= \{d \in \ker F'(\bar{x}) \mid F''(\bar{x})[d,d] \in \operatorname{im} F'(\bar{x})\}. \end{aligned}$$

Then by necessity $T_S(\bar{x}) \subset T(\bar{x})$. Moreover, if F is 2-*regular at* $\bar{x}$ *with respect to a* given *direction* $d \in T(\bar{x})$, that is

$$\operatorname{rank} \frac{\partial \Phi}{\partial d}(\bar{x},d) = l, \tag{7}$$

then $d \in T_S(\bar{x})$. In particular, if F is 2-regular at $\bar{x}$ with respect to all $d \in T(\bar{x}) \setminus \{0\}$ (or at least with respect to all d comprising a dense subset of $T(\bar{x})$), then $T_S(\bar{x}) = T(\bar{x})$. Under the relaxed smoothness assumptions, these results were obtained in [13,14].

Observe that the partial derivative in 2-regularity condition (7) can be evaluated explicitly: for $\xi \in X$,

$$\frac{\partial \Phi}{\partial d}(\bar{x},d)\xi = F'(\bar{x})\xi + PF''(\bar{x})[d,\xi].$$

It follows that (7) can be equivalently rewritten in the form

$$\operatorname{im} F'(\bar{x}) + F''(\bar{x})\big[d, \ker F'(\bar{x})\big] = \mathbf{R}^l. \tag{8}$$

Indeed, (7) is obviously equivalent to

$$PF''(\bar{x})\big[d, \ker F'(\bar{x})\big] = \big(\operatorname{im} F'(\bar{x})\big)^{\perp}. \tag{9}$$

Suppose that (8) holds. Then for any $y \in (\operatorname{im} F'(\bar{x}))^{\perp}$, there exist $x^1 \in X$ and $x^2 \in \ker F'(\bar{x})$ such that

$$F'(\bar{x})x^1 + F''(\bar{x})[d,x^2] = y.$$

Applying P to both sides of the last equality, we obtain

$$PF''(\bar{x})[d,x^2] = y,$$

and hence, (9) is established. Suppose now that (9) holds. Then for any $y = y^1 + y^2 \in \mathbf{R}^l$, $y^1 \in \operatorname{im} F'(\bar{x})$, $y^2 \in (\operatorname{im} F'(\bar{x}))^{\perp}$, there exists $x^2 \in \ker F'(\bar{x})$ such that

$$PF''(\bar{x})[d,x^2] = y^2.$$

Define $\tilde{y} = F''(\bar{x})[d,x^2]$, then $\tilde{y} = \tilde{y}^1 + y^2$, $\tilde{y}^1 \in \operatorname{im} F'(\bar{x})$, and $y^1 - \tilde{y}^1 \in \operatorname{im} F'(\bar{x})$, that is, there exists $x^1 \in X$ such that $F'(\bar{x})x^1 = y^1 - \tilde{y}^1$. This finally leads to

$$F'(\bar{x})x^1 + F''(\bar{x})[d,x^2] = y^1 - \tilde{y}^1 + \tilde{y}^1 + y^2 = y,$$

and hence, (8) holds.

It is now evident that normality condition (5) implies 2-regularity with respect to every direction $d \in X$, but not vice versa. 2-regularity (e.g., with respect to every direction

$d \in T(\bar{x}) \setminus \{0\}$, which is relevant in the context of tangency) is a substantially weaker assumption than normality. At the same time, 2-regularity is also violated in many important cases. The following simple example is sort of model in these considerations.

Example 1.1. Let $X = \mathbf{R}^2$, $l = 1$, $F(x) = x_1^2 - x_2^3$, $x = (x_1, x_2)$, $\bar{x} = 0$. It is easy to see that $T(\bar{x})$ is the linear subspace spanned by $(0, 1)$, and F is not 2-regular at $\bar{x}$ with respect to any $d \in T(\bar{x})$. Geometric considerations suggest that $T_S(\bar{x})$ is the ray spanned by $(0, 1)$, and, e.g., $(0, 1) \in T_S(\bar{x})$, while $(0, -1) \notin T_S(\bar{x})$.

Hence, some further development of the results of [7,21] is needed, in order to obtain tools sharp enough to separate the two cases in Example 1.1. In Section 3, we provide an improvement which does this job.

Our approach is based on the theory of second-order optimality conditions developed in [3,4]. This theory was previously used in order to obtain the implicit function theorems relevant for abnormal points [5], as well as bifurcation theorems [6]. We briefly review the necessary results from [3,4] in Section 2.

This work was initiated by [2], where the higher-order (higher than 2) optimality conditions were used for characterization of tangent directions.

A few words about our notation which is fairly standard. Above, we have already used $\operatorname{im} A$ for the image space of a linear operator A, and $\ker A$ for its null space, respectively. All finite-dimensional spaces are supposed to be equipped with the Euclid inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|\cdot|$. The orthogonal complement of a subspace M in a finite-dimensional space is denoted by $M^\perp$. The symbol $\langle \cdot, \cdot \rangle$ will be used for the duality pairing as well.

2. Second-order necessary conditions

In this section, we briefly review the second-order necessary optimality conditions developed in [3,4]. We emphasize that these conditions are meaningful even in the context of abnormal solutions, while for customary necessary conditions this is not the case.

Let Z be a Banach space, and consider the optimization problem

$$\begin{aligned} &\text{minimize} && f(z) \\ &\text{subject to} && F(z) = 0, \end{aligned} \tag{10}$$

where a cost function $f : Z \to \mathbf{R}$ and a constraint mapping $F : Z \to \mathbf{R}^l$ are assumed to be twice continuously differentiable in a neighborhood of a point $\bar{z} \in Z$.

Define the Lagrangian function of problem (10): for $z \in Z$, $\lambda_0 \in \mathbf{R}$, and $\lambda \in \mathbf{R}^l$,

$$L(z, \lambda_0, \lambda, \mu) = \lambda_0 f(z) + \big\langle \lambda, F(z) \big\rangle.$$

Define the cone $\Lambda(\bar{z})$ of Lagrange multipliers associated with $\bar{z}$. This cone is comprised by all pairs $(\lambda_0, \lambda) \in (\mathbf{R}_+ \times \mathbf{R}^l) \setminus \{0\}$ such that

$$\frac{\partial L}{\partial z}(\bar{z}, \lambda_0, \lambda) = 0.$$

By the Lagrange principle, if $\bar{z}$ is a local solution of (10), then $\Lambda(\bar{z}) \neq \emptyset$.

The central role in the optimality conditions presented below is played by the cone $\Lambda_l(\bar{z})$ which we define next, and which is in general smaller than the cone of all Lagrange multipliers. The cone $\Lambda_l(\bar{z})$ is comprised by all pairs $(\lambda_0, \lambda) \in \Lambda(\bar{z})$ possessing the following property: there exists a linear subspace $\Pi = \Pi(\lambda_0, \lambda)$ in Z such that

$$\Pi \subset \ker F'(\bar{z}), \quad \operatorname{codim} \Pi \leqslant l,$$
$$\left\langle \frac{\partial^2 L}{\partial z^2}(\bar{z}, \lambda_0, \lambda)\zeta, \zeta \right\rangle \geqslant 0, \quad \forall \zeta \in \Pi.$$

Theorem 2.1 [3,4]. *If $\bar{z}$ is a local solution of problem* (10)*, then $\Lambda_l \neq \emptyset$, and, moreover,*

$$\max_{\substack{(\lambda_0,\lambda)\in\Lambda_l \\ |\lambda_0|+|\lambda|=1}} \left\langle \frac{\partial^2 L}{\partial z^2}(\bar{z}, \lambda_0, \lambda)\zeta, \zeta \right\rangle \geqslant 0, \quad \forall \zeta \in \ker F'(\bar{z}). \tag{11}$$

3. Tangent vectors

We are now in a position to apply Theorem 2.1 in order to characterize the contingent cone to the set S given by (4). The key observation is the following: it is easy to see that $d \in T_S(\bar{x})$ if and only if $\bar{z} = (0, 0)$ is not a local solution of the optimization problem

$$\begin{aligned} &\text{minimize} \quad -t \\ &\text{subject to} \quad F\big(\bar{x} + t(d + x)\big) = 0 \end{aligned} \tag{12}$$

in unknown $z = (x, t) \in Z = X \times \mathbf{R}$.

Throughout the rest of this paper, we assume that S is defined in (4), where $F : X \to \mathbf{R}^l$ is four times continuously differentiable in a neighborhood of a point $\bar{x} \in S$.

Let R be the orthogonal projector onto $\operatorname{im} F'(\bar{x})$, while P be the orthogonal projector onto $(\operatorname{im} F'(\bar{x}))^\perp$ in $\mathbf{R}^l$. Recall that the mapping $\Phi(\bar{x}, \cdot)$ is defined in (6). For a given $d \in X$, define the cone $\mathcal{Y}_l(\bar{x}; d)$ comprised by all elements $\lambda \in (\operatorname{im}(\partial\Phi/\partial d)(\bar{x}, d))^\perp \setminus \{0\}$ possessing the properties

$$\big\langle \lambda, \eta(d) \big\rangle \geqslant 0, \tag{13}$$

and there exists a linear subspace $\Pi = \Pi(\lambda)$ in Z such that

$$\frac{\partial \Phi}{\partial d}(\bar{x}, d)\xi + \tau\eta(d) = 0, \quad \forall (\xi, \tau) \in \Pi, \ \operatorname{codim} \Pi \leqslant l, \tag{14}$$

$$\begin{aligned} \Big\langle \lambda, \frac{1}{2}PF''(\bar{x})[\xi, \xi] + \tau RF''(\bar{x})[d, \xi] + \frac{\tau}{2}PF'''(\bar{x})[d, d, \xi] \\ + \frac{\tau^2}{3!}RF'''(\bar{x})[d, d, d] + \frac{\tau^2}{4!}PF^{(4)}(\bar{x})[d, d, d, d] \Big\rangle \geqslant 0, \quad \forall (\xi, \tau) \in \Pi, \end{aligned} \tag{15}$$

where

$$\eta(d) = \frac{1}{2}RF''(\bar{x})[d, d] + \frac{1}{3!}PF'''(\bar{x})[d, d, d].$$

Theorem 3.1. *For a given vector $d \in T(\bar{x})$, the following condition is sufficient for the inclusion $d \in T_S(\bar{x})$: either $\mathcal{Y}_l(\bar{x}; d) = \emptyset$, or there exist $\xi \in X$ and a real number τ such that*

$$\frac{\partial \Phi}{\partial d}(\bar{x}, d)\xi + \tau\eta(d) = 0, \tag{16}$$

$$\left\langle \lambda, \frac{1}{2}PF''(\bar{x})[\xi,\xi] + \tau RF''(\bar{x})[d,\xi] + \frac{\tau}{2}PF'''(\bar{x})[d,d,\xi] + \frac{\tau^2}{3!}RF'''(\bar{x})[d,d,d] + \frac{\tau^2}{4!}PF^{(4)}(\bar{x})[d,d,d,d] \right\rangle < 0, \quad \forall \lambda \in \mathcal{Y}_l(\bar{x}; d). \tag{17}$$

Proof. According to the Hadamard lemma, for every $\xi \in X$ sufficiently close to zero, there exists a symmetric bilinear mapping $Q(\xi) : X \times X \to \mathbf{R}^l$ such that

$$F(\bar{x} + \xi) = F'(\bar{x})\xi + \frac{1}{2}F''(\bar{x})[\xi,\xi] + Q(\xi)[\xi,\xi], \tag{18}$$

$Q(0) = 0$, $Q(\cdot)$ is twice continuously differentiable near zero, and $\forall \xi \in X$,

$$\big(Q'(0)\xi\big)[\xi,\xi] = \frac{1}{3!}F'''(\bar{x})[\xi,\xi,\xi], \tag{19}$$

$$\big(Q''(0)[\xi,\xi]\big)[\xi,\xi] = \frac{1}{4!}F^{(4)}(\bar{x})[\xi,\xi,\xi,\xi]. \tag{20}$$

Using (18), the equality constraints of problem (12) can be equivalently rewritten in the form

$$\begin{aligned} 0 &= RF\big(\bar{x} + t(d+x)\big) \\ &= t\left(F'(\bar{x})x + \frac{t}{2}RF''(\bar{x})[d+x,d+x] + tRQ\big(t(d+x)\big)[d+x,d+x]\right), \end{aligned} \tag{21}$$

$$\begin{aligned} 0 &= PF\big(\bar{x} + t(d+x)\big) \\ &= t^2\left(PF''(\bar{x})[d,x] + \frac{1}{2}PF''(\bar{x})[x,x] + PQ\big(t(d+x)\big)[d+x,d+x]\right), \end{aligned} \tag{22}$$

where the definition of R and P and the inclusion $d \in T(\bar{x})$ are taken into account.

It is clear from (21) and (22) that the feasible set of problem (12) consists of two pieces. One piece is defined by the equality $t = 0$, and the cost function of problem (12) takes only the zero value on this piece. Another piece is defined by $t \neq 0$ and the equalities

$$F'(\bar{x})x + \frac{t}{2}RF''(\bar{x})[d+x,d+x] + tRQ\big(t(d+x)\big)[d+x,d+x] = 0, \tag{23}$$

$$PF''(\bar{x})[d,x] + \frac{1}{2}PF''(\bar{x})[x,x] + PQ\big(t(d+x)\big)[d+x,d+x] = 0. \tag{24}$$

In order to prove that $d \in T_S(\bar{x})$, it suffice to show that under our assumptions, the necessary optimality conditions given by Theorem 2.1 are violated at $\bar{z} = (0,0)$ for the following optimization problem:

$$\begin{aligned} &\text{minimize} && -t \\ &\text{subject to} && (23) \text{ and } (24). \end{aligned} \tag{25}$$

Note that the constraints of this problem are twice continuously differentiable near $\bar{z}$.

Define the Lagrangian function of problem (25): for $z = (x,\ t) \in Z$, $\lambda_0 \in \mathbf{R}$, and $\lambda \in \mathbf{R}^l$,

$$\begin{aligned} L(z, \lambda_0, \lambda) = -\lambda_0 t + \Big\langle \lambda, F'(\bar{x})x + \frac{t}{2} RF''(\bar{x})[d+x, d+x] \\ + tRQ\big(t(d+x)\big)[d+x, d+x] + PF''(\bar{x})[d, x] \\ + \frac{1}{2} PF''(\bar{x})[x, x] + PQ\big(t(d+x)\big)[d+x, d+x] \Big\rangle. \end{aligned}$$

Employing (19) and (20), by direct computations we obtain, for $\zeta = (\xi, \tau) \in Z$,

$$\begin{aligned} \left\langle \frac{\partial L}{\partial z}(\bar{z}, \lambda_0, \lambda), \zeta \right\rangle &= -\lambda_0 \tau - \left\langle \lambda, F'(\bar{x})\xi + PF''(\bar{x})[d, \xi] \right. \\ &\quad \left. + \frac{\tau}{2} RF''(\bar{x})[d, d] + \frac{\tau}{3!} PF'''(\bar{x})[d, d, d] \right\rangle \\ &= \left\langle \lambda, \frac{\partial \Phi}{\partial d}(\bar{x}, d)\xi \right\rangle + \left(\left\langle \lambda, \frac{1}{2} RF''(\bar{x})[d, d] \right. \right. \\ &\quad \left. \left. + \frac{1}{3!} PF'''(\bar{x})[d, d, d] \right\rangle - \lambda_0 \right) \tau, \end{aligned} \tag{26}$$

$$\begin{aligned} \left\langle \frac{\partial^2 L}{\partial z^2}(\bar{z}, \lambda_0, \lambda)\zeta, \zeta \right\rangle &= \left\langle \lambda, \frac{1}{2} PF''(\bar{x})[\xi, \xi] + \tau RF''(\bar{x})[d, \xi] \right. \\ &\quad + \frac{\tau}{2} PF'''(\bar{x})[d, d, \xi] + \frac{\tau^2}{3!} RF'''(\bar{x})[d, d, d] \\ &\quad \left. + \frac{\tau^2}{4!} PF^{(4)}(\bar{x})[d, d, d, d] \right\rangle. \end{aligned} \tag{27}$$

By (26), we see that the cone $\Lambda(\bar{z})$ is comprised by all pairs $(\lambda_0, \lambda) \in (\mathbf{R} \times \mathbf{R}^l) \setminus \{0\}$ such that $\lambda \in (\operatorname{im}(\partial\Phi/\partial d)(\bar{x}, d))^\perp$ and

$$\left\langle \lambda, \frac{1}{2} RF''(\bar{x})[d, d] + \frac{1}{3!} PF'''(\bar{x})[d, d, d] \right\rangle = \lambda_0 \geqslant 0.$$

Moreover, from (19), (23), (24) and (27) and the definition of $\mathcal{Y}_l(\bar{x}; d)$ it follows that the cone $\Lambda_l(\bar{z})$ is comprised by pairs $(\langle \lambda, \eta(d) \rangle, \lambda)$, $\lambda \in \mathcal{Y}_l(\bar{x}; d)$. It is now evident that the necessary conditions of Theorem 2.1 are violated at $\bar{z}$ if and only if $\mathcal{Y}_l(\bar{x}; d) = \emptyset$, or there exists $(\xi, \tau) \in Z$ satisfying (16) and (17). □

We now consider the important particular case when

$$\eta(d) \notin \operatorname{im} \frac{\partial \Phi}{\partial d}(\bar{x}, d). \tag{28}$$

In this case, the sufficient condition for tangency given by Theorem 3.1 takes a considerably simpler form.

Let the cone $\tilde{\mathcal{Y}}_{l-1}(\bar{x}; d)$ be comprised by elements $\lambda \in (\operatorname{im}(\partial\Phi/\partial d)(\bar{x}, d))^{\perp} \setminus \{0\}$ satisfying (13) and possessing the following property: there exists a linear subspace $L = L(\lambda)$ in X such that

$$L \subset \ker \frac{\partial \Phi}{\partial d}(\bar{x}, d), \quad \operatorname{codim} L \leqslant l - 1, \tag{29}$$

$$\left\langle \lambda, PF''(\bar{x})[\xi, \xi] \right\rangle \geqslant 0, \quad \forall \xi \in L. \tag{30}$$

Note that (28) implies $\operatorname{codim}\ker(\partial\Phi/\partial d)(\bar{x}, d) = \operatorname{rank}(\partial\Phi/\partial d)(\bar{x}, d) \leqslant l - 1$.

Corollary 3.1. *For a given vector $d \in T(\bar{x})$ satisfying* (28)*, the following condition is sufficient for the inclusion $d \in T_S(\bar{x})$: either $\tilde{\mathcal{Y}}_{l-1}(\bar{x}; d) = \emptyset$, or there exists*

$$\xi \in \ker \frac{\partial \Phi}{\partial d}(\bar{x}, d) \tag{31}$$

such that

$$\left\langle \lambda, PF''(\bar{x})[\xi, \xi] \right\rangle < 0, \quad \forall \lambda \in \tilde{\mathcal{Y}}_{l-1}(\bar{x}; d). \tag{32}$$

Proof. Under (28), for a given $\lambda \in \mathbf{R}^l$, a linear subspace Π in Z satisfies (14) and (15) if and only if $\Pi = L \times \{0\}$, where the linear subspace L in X satisfies (29) and (30). This follows from the fact that the equality in (14) can hold for some $\xi \in X$ with $\tau = 0$ only, and the obvious relation between $\operatorname{codim}\Pi$ and $\operatorname{codim} L$. It is evident now that $\mathcal{Y}_l(\bar{x}; d) = \tilde{\mathcal{Y}}_{l-1}(\bar{x}; d)$, and (16) and (17) take place for some $\xi \in X$ and a real number τ if and only if ξ satisfies (31) and (32), and $\tau = 0$. Application of Theorem 3.1 completes the proof. □

Example 3.1 (*Compare with Example 1.1*). Let $X = \mathbf{R}^2$, $l = 1$, $F(x) = x_1^2 - x_2^3 + \omega(x)$, $x = (x_1, x_2)$, where $\omega : \mathbf{R}^2 \to \mathbf{R}$ is an arbitrary function four times continuously differentiable near 0 and such that its first three derivatives vanish at 0. Let $\bar{x} = 0$.

We have $R = 0$, $P = 1$, and for $d = (d_1, d_2) \in T(\bar{x})$,

$$\frac{\partial \Phi}{\partial d}(\bar{x}, d) = 0, \quad d_1 = 0.$$

In particular,

$$\eta(d) = -d_2^3 \neq 0$$

provided $d \neq 0$. Furthermore, the cone $\tilde{\mathcal{Y}}_{l-1}(\bar{x}; d)$ is comprised by all numbers $\lambda \neq 0$ such that

$$\lambda\eta(d) = -\lambda d_2^3 \geqslant 0,$$
$$\lambda F''(\bar{x})[\xi, \xi] = 2\lambda\xi_1^2 \geqslant 0, \quad \forall \xi = (\xi_1, \xi_2) \in \mathbf{R}^2,$$

and in particular, $\lambda \geqslant 0$. If $d_2 > 0$, then obviously $\tilde{\mathcal{Y}}_{l-1}(\bar{x}; d) = \emptyset$, and we conclude that $d \in T_S(\bar{x})$ according to Corollary 3.1. At the same time, if $d_2 < 0$, then it is easy to see that the sufficient condition for tangency given by Corollary 3.1 is not satisfied.

Example 3.2. Let $X = \mathbf{R}^3$, $l = 2$, $F(x) = (x_1^2 + x_2^2 - x_3^3, x_1(x_1 - x_3 + x_3^3)) + \omega(x)$, $x = (x_1, x_2, x_3)$, where $\omega : \mathbf{R}^3 \to \mathbf{R}^2$ is an arbitrary mapping four times continuously differentiable near 0 and such that its first three derivatives vanish at 0. Let $\bar{x} = 0$.

Here, $\tilde{\mathcal{Y}}_{l-1}(\bar{x}; d) = \emptyset$ for $d = (\delta, 0, \delta)$ with $\delta > 0$, and hence, $d \in T_S(\bar{x})$ according to Corollary 3.1.

Finally, the sufficient condition for tangency given by Theorem 3.1 can be simplified even without the additional assumption (28), though the resulting condition is somewhat less subtle. Let $\tilde{\mathcal{Y}}_l$ be the cone defined similar to $\tilde{\mathcal{Y}}_{l-1}$, but with $l - 1$ in the right-hand side of the inequality in (29) replaced by l.

Corollary 3.2. *For a given vector $d \in T(\bar{x})$, the following condition is sufficient for the inclusion $d \in T_S(\bar{x})$: either $\tilde{\mathcal{Y}}_l(\bar{x}; d) = \emptyset$, or there exists $\xi \in X$ satisfying* (31)*, and such that*

$$\langle \lambda, PF''(\bar{x})[\xi, \xi] \rangle < 0, \quad \forall \lambda \in \tilde{\mathcal{Y}}_l(\bar{x}; d). \tag{33}$$

Proof. Let $\lambda \in \mathcal{Y}_l(\bar{x}; d)$; then $\lambda \in (\operatorname{im}(\partial\Phi/\partial d)(\bar{x}, d))^\perp \setminus \{0\}$, (13) holds, and there exists a linear subspace Π in Z satisfying (14) and (15). Set $L = \{\xi \in X \mid (\xi, 0) \in \Pi\}$. Obviously, the inclusion in (29) holds, $\operatorname{codim} L \leqslant \operatorname{codim} \Pi \leqslant l$, and (30) is satisfied. Hence, $\lambda \in \tilde{\mathcal{Y}}_l(\bar{x}; d)$, and this proves the inclusion $\mathcal{Y}_l(\bar{x}; d) \subset \tilde{\mathcal{Y}}_l(\bar{x}; d)$. Now, if $\xi \in X$ satisfies (31) and (33), then (16) and (17) also hold with this ξ and $\tau = 0$. In order to complete the proof, it is sufficient to refer to Theorem 3.1. □

Unlike Corollary 3.1, Corollary 3.2 does not make the job in Example 3.1. We next modify this example in order to demonstrate the situation when both Corollaries 3.1 and 3.2 are applicable.

Example 3.3. Let $X = \mathbf{R}^3$, $l = 1$, $F(x) = x_1^2 + x_2^2 - x_3^3 + \omega(x)$, $x = (x_1, x_2, x_3)$, where $\omega : \mathbf{R}^3 \to \mathbf{R}$ is an arbitrary function four times continuously differentiable near 0 and such that its first three derivatives vanish at 0. Let $\bar{x} = 0$.

Here, $\tilde{\mathcal{Y}}_l(\bar{x}; d) = \emptyset$ for $d = (0, 0, d_2)$ with $d_2 > 0$, and both Corollaries 3.1 and 3.2 are appropriate in order to show that $d \in T_S(\bar{x})$.

Moreover, Corollary 3.2 can be applicable when Corollary 3.1 is not. Of course, this can happen only when (28) is violated, and in particular, condition (13) can be dropped in the definition of $\tilde{\mathcal{Y}}_l(\bar{x}; d)$.

Example 3.4. Let $X = \mathbf{R}^5$, $l = 1$, $F(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2 + \omega(x)$, $x = (x_1, x_2, x_3, x_4, x_5)$, where $\omega : \mathbf{R}^5 \to \mathbf{R}$ is an arbitrary function four times continuously differentiable near 0 and such that its first three derivatives vanish at 0. Let $\bar{x} = 0$.

We have $R = 0$, $P = 1$, and for $d = (0, 0, 0, 0, d_5) \in T(\bar{x})$,

$$\frac{\partial \Phi}{\partial d}(\bar{x}, d) = 0, \quad \eta(d) = 0.$$

The cone $\tilde{\mathcal{Y}}_l(\bar{x};d)$ is comprised by all numbers $\lambda \neq 0$ such that

$$\lambda F''(\bar{x})[\xi,\xi] = 2\lambda\big(\xi_1^2+\xi_2^2-\xi_3^2-\xi_4^2\big) \geqslant 0, \quad \forall \xi = (\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) \in L,$$

for some linear subspace L in X such that $\operatorname{codim} L \leqslant 1$. Such λ does not exists, since the index of the quadratic form $\xi \to \lambda F''(\bar{x})[\xi,\xi] : X \to \mathbf{R}$ should be less or equal to 1, but this is obviously not the case for any $\lambda \neq 0$. Hence, $\tilde{\mathcal{Y}}_l(\bar{x};d) = \emptyset$, and by Corollary 3.2 we conclude that $d \in T_S(\bar{x})$.

Recall that, in the 2-regular case, $(\operatorname{im}(\partial\Phi/\partial d)(\bar{x},d))^{\perp} = \{0\}$, and hence, all the cones $\mathcal{Y}_l(\bar{x};d)$, $\tilde{\mathcal{Y}}_{l-1}(\bar{x};d)$, and $\tilde{\mathcal{Y}}_l(\bar{x};d)$, are necessarily empty. In particular, Theorem 3.1 and each of Corollaries 3.1 and 3.2 are applicable in the 2-regular case.

Finally, we emphasize that Theorem 3.1 and each of Corollaries 3.1 and 3.2 can be applicable only provided the corresponding cone $\mathcal{Y}_l(\bar{x};d)$, $\tilde{\mathcal{Y}}_{l-1}(\bar{x};d)$, or $\tilde{\mathcal{Y}}_l(\bar{x};d)$, is pointed (the empty cone is pointed, by definition).

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