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Explicit min–max polynomials on the disc

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Abstract

Denote by $\Pi^2_{n+m-1} := \{\sum_{0\le i+j\le n+m-1} c_{i,j}x^i y^j : c_{i,j} \in \mathbb{R}\}$ the space of polynomials of two variables with real coefficients of total degree less than or equal to $n+m-1$. Let $b_0, b_1, \ldots, b_l \in \mathbb{R}$ be given. For $n, m \in \mathbb{N}, n \ge l+1$ we look for the polynomial $b_0x^ny^m + b_1x^{n-1}y^{m+1} + \cdots + b_lx^{n-l}y^{m+l} + q(x,y), q(x,y) \in \Pi^2_{n+m-1}$, which has least maximum norm on the disc and call such a polynomial a min–max polynomial. First we introduce the polynomial $2P_{n,m}(x,y) = xG_{n-1,m}(x,y) + yG_{n,m-1}(x,y) = 2x^ny^m + q(x,y)$ and $q(x,y) \in \Pi^2_{n+m-1}$, where $G_{n,m}(x,y) := 1/2^{n+m}(U_n(x)U_m(y) + U_{n-2}(x)U_{m-2}(y))$, and show that it is a min–max polynomial on the disc. Then we give a sufficient condition on the coefficients $b_j, j = 0, \ldots, l, l$ fixed, such that for every $n, m \in \mathbb{N}, n \ge l+1$, the linear combination $\sum_{\nu=0}^{l} b_\nu P_{n-\nu,m+\nu}(x,y)$ is a min–max polynomial. In fact the more general case, when the coefficients b_j and l are allowed to depend on n and m, is considered. So far, up to very special cases, min–max polynomials are known only for $x^ny^m, n, m \in \mathbb{N}_0$.

1. Introduction

We study here the following uniform approximation problem: let $\mathcal{D} := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ be the bivariate unit disc, let $\mathcal{P}$ be a homogeneous polynomial of two variables of degree $n+m, n,\ m \in \mathbb{N}_0,\ n+m \ge 1$, i.e.,

$$\mathcal{P}(x,y) := \sum_{i=0}^{n+m} a_i x^{n+m-i} y^i, \quad a_i \in \mathbb{R},\ i = 0, \ldots, n+m, \tag{1}$$

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and let $\Pi^2_{n+m-1} := \{\sum_{0\le i+j\le n+m-1} c_{i,j}x^i y^j \ : \ c_{i,j} \in \mathbb{R}\}$ denote the space of polynomials of two variables with real coefficients of total degree less than or equal to $n+m-1$. For each $n, m \in \mathbb{N}_0,\ n+m \ge 1$, we would like to know a polynomial $p^* \in \Pi^2_{n+m-1}$ such that $\|\mathcal{P} - p^*\|$ has the least max norm on $\mathcal{D}$, i.e.,

$$\begin{aligned}\|\mathcal{P} - p^*\| &:= \max_{(x,y)\in\mathcal{D}} |\mathcal{P}(x,y) - p^*(x,y)| \\ &= \min_{p\in\Pi^2_{n+m-1}} \max_{(x,y)\in\mathcal{D}} |\mathcal{P}(x,y) - p(x,y)|.\end{aligned}$$

$\mathcal{P} - p^*$ is called min–max polynomial on $\mathcal{D}$. For the bivariate monomial case $\mathcal{P}(x,y) = x^n y^m$, Gearhart [3] discovered in his fundamental paper that for every $n, m \in \mathbb{N}_0,\ n+m \ge 1$, the polynomials

$$\begin{aligned}G_{n,m}(x,y) &:= \frac{1}{2^{n+m}}(U_n(x)U_m(y) + U_{n-2}(x)U_{m-2}(y)) \\ &= x^n y^m + q(x,y), \quad q \in \Pi^2_{n+m-1}\end{aligned} \tag{2}$$

are min–max polynomials on the disc, where $U_n(x) := \sin(n+1)\arccos x/\sin\arccos x,\ -1 \le x \le 1,\ n \in \mathbb{N}_0$, and $U_n(x) := -U_{-n-2}(x)$ when $n \in \mathbb{Z},\ n < 0$. Another best approximation to $x^n y^m$ for each $n, m \in \mathbb{N}$ has been found by Reimer, the so-called Reimer polynomials; see [8]. For monomials of special degree, other best approximations have been found by Bojanov et al. [1, p. 491], Braß[2, p. 59] and Newman and Xu [7]; see [1, p. 494]. A sufficient condition for a polynomial to be a min–max polynomial is given in [12, Theorem 2], but most likely it holds very rarely, if at all; for a more detailed discussion see Remark 3.9 at the end of the paper.

The key property of the Gearhart polynomials $G_{n,m}$ is that they become on the boundary $\partial\mathcal{D}$ of the disc

$$G_{n,m}(\cos\varphi, \sin\varphi) = \begin{cases} \dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\cos(n+m)\varphi, & \text{if } m \text{ even}, \\ \dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\sin(n+m)\varphi, & \text{if } m \text{ odd}. \end{cases} \tag{3}$$

Since $|G_{n,m}(x,y)|$ attains its maximum on $\mathcal{D}$ at the points $(\cos\varphi_j, \sin\varphi_j),\ j = 1, \ldots, 2(n+m)$, where the φ_j's are the zeros of $\sin(n+m)\varphi$ when m is even and the zeros of $\cos(n+m)\varphi$ when m is odd, it follows by the results of Shapiro [11] that the function σ defined on the plane by

$$\sigma(x,y) := \begin{cases} (-1)^j, & \text{if } (x,y) = (\cos\varphi_j, \sin\varphi_j), \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

is an extremal signature with respect to Π^2_{n+m-1} (for details on extremal signatures see [4,5,10]); hence $G_{n,m}$ is a min–max polynomial on $\mathcal{D}$. In the same way one obtains immediately that if the coefficients a_i of the polynomial $\mathcal{P}(x,y)$ from (1) satisfy one of the following four conditions:

$$\begin{aligned} &a_{2j+1} = 0 \quad \text{and} \quad \pm a_{4j}, \mp a_{4j+2} \in \mathbb{R}_0^+ \quad \text{for } j = 0, 1, 2, \ldots, \\ &a_{2j} = 0 \quad \text{and} \quad \pm a_{4j+1}, \mp a_{4j+3} \in \mathbb{R}_0^+ \quad \text{for } j = 0, 1, 2, \ldots, \end{aligned} \tag{5}$$

(that is, all the coefficients of odd index are zero and the coefficients of even index alternate in sign, or the converse situation), then $\sum_{i=0}^{n+m} a_i G_{n+m-i,i}$ is a min–max polynomial on $\mathcal{D}$; see [3, Theorem 3.1]. In fact, a more general statement with any min–max polynomial for monomials instead of $G_{n+m-i,i}$ was proven. Though it was repeatedly pointed out in the literature that it would be of high interest to find min–max polynomials with given coefficients a_i there has been no progress on this question since the seventies, that is, since Gearhart [3].

In this paper we attack the problem in the following way. First we introduce new bivariate min–max polynomials, defined as follows, for $n, m \in \mathbb{N}$:

$$\begin{aligned} P_{n,m}(x,y) &:= \frac{1}{2} x G_{n-1,m}(x,y) + \frac{1}{2} y G_{n,m-1}(x,y) \\ &= x^n y^m + q(x,y), \quad q \in \Pi^2_{n+m-1}. \end{aligned} \tag{6}$$

Taking into consideration that for $n = 0$ or $m = 0$ the min–max polynomial for the monomial case is unique (see [3]), we define also, for $n, m \in \mathbb{N}$,

$$P_{n,0}(x,y) := G_{n,0}(x,y) \quad \text{and} P_{0,m}(x,y) := G_{0,m}(x,y), \quad \text{respectively.} \tag{7}$$

Representing the given polynomial $\mathcal{P}(x,y)$ from (1) in terms of $P_{n,m}(x,y)$, i.e.,

$$\mathcal{P}(x,y) = \sum_{i=0}^{n+m} a_i P_{n+m-i,i}(x,y) + q(x,y), \quad q \in \Pi^2_{n+m-1}$$

we derive a sufficient condition on the a_i's, which depends on the length of the linear combination (that is, on the difference of the indices of the last and first non-zero coefficient) but not on n, m, such that $\mathcal{P}(x,y) - q(x,y)$ is a min–max polynomial.

The paper is organized as follows. In the next section we present and discuss the result just announced. The proofs are given in Section 3.

2. The main result

As mentioned above, the polynomials $P_{n,m}$ introduced in (6) will be crucial in what follows.

Proposition 2.1. *The polynomial $P_{n,m}(x,y)$ given by (6) is a min–max polynomial on $\mathcal{D}$.*

Loosely speaking, in contrast to the Gearhart polynomials from (2), the $P_{n,m}$'s have the advantage of being stable with respect to perturbation, since the global extremal values are attained at the boundary of $\mathcal{D}$ only.

Notation 2.2. Let for $n, m \in \mathbb{N}_0,\ i \in \mathbb{N}$,

$$Q_{n,m}(x,y) := U_n(x)U_{m-2}(y) + U_{n-2}(x)U_m(y), \tag{8}$$

$$\begin{aligned} \Sigma_i(x,y) &:= \frac{1}{2}[Q_{i,i}(x,y) + 4xyU_{i-1}(x)U_{i-1}(y)] \\ &\quad + \frac{2[(-1-(-1)^i)/2]^{i/2-1} - Q_{i,i}(x,y)}{2(1-x^2-y^2)}, \end{aligned} \tag{9}$$

and

$$\sigma_i(x,y) := U_i(x)U_{i-2}(y) + \frac{2[(-1-(-1)^i)/2]^{i/2-1} - Q_{i,i}(x,y)}{2(1-x^2-y^2)}. \tag{10}$$

It will be proved in Proposition 3.8 that $\Sigma_i(x,y)$ and $\sigma_i(x,y)$ are polynomials which satisfy the following inequalities on $\mathcal{D}$:

$$|\Sigma_i(x,y)| \le M_i \quad \text{and} \quad |\sigma_i(x,y)| \le M_i, \tag{11}$$

where

$$M_i := i^3/3 + i^2. \tag{12}$$

In addition $0 \leq (-1)^{i/2-1}\Sigma_i(x, y)$ and $0 \leq (-1)^{i/2-1}\sigma_i(x, y)$ if i is even.

Our main result is now the following sufficient condition for a polynomial to be a min–max polynomial on $\mathcal{D}$.

Theorem 2.3. *Let $n, m, k \in \mathbb{N}_0$ be such that $n + m \geq 3$ and $1 \leq k \leq n + m - 2$, and $a_i \in \mathbb{R}$, $i = 0, \ldots, k$, be given. If for all $(x, y) \in \mathcal{D}$*

$$\sum_{i=1}^{k} a_i^2 - 2\sum_{i=1}^{k-1}\left(\sum_{j=1}^{k-i} a_j a_{j+i}\right)\Sigma_i(x, y) - 2a_0\sum_{i=1}^{k} a_i\sigma_i(x, y) \geq 0, \tag{13}$$

then

$$\sum_{i=0}^{k} a_i P_{n+m-i,i}(x, y) = \sum_{i=0}^{k} a_i x^{n+m-i} y^i + q(x, y),$$

for $q \in \Pi^2_{n+m-1}$, is a min–max polynomial on $\mathcal{D}$.

Furthermore, the minimum deviation of the min–max polynomial is given by

$$\left\|\sum_{i=0}^{k} a_i P_{n+m-i,i}(x, y)\right\|^2 = \frac{\left(\sum_{i=0}^{[k/2]}(-1)^i a_{2i}\right)^2 + \left(\sum_{i=0}^{[(k-1)/2]}(-1)^i a_{2i+1}\right)^2}{2^{2(n+m-1)}}. \tag{14}$$

Note that as a_0 appears in the third expression only, it plays a special role. As an immediate consequence of Theorem 2.3 we obtain:

Corollary 2.4. *Let $\tilde{n}, \tilde{m}, l \in \mathbb{N}$ be such that $l \leq \tilde{n} - 2$ and suppose that $b_0, b_1, \ldots, b_l \in \mathbb{R}$ satisfy*

$$\sum_{i=0}^{l} b_i^2 - 2\sum_{i=1}^{l}\left(\sum_{j=0}^{l-i} b_j b_{j+i}\right)\Sigma_i(x, y) \geq 0 \tag{15}$$

(which implies that $\sum_{\nu=0}^{l} b_\nu P_{\tilde{n}-\nu,\tilde{m}+\nu} = b_0 x^{\tilde{n}} y^{\tilde{m}} + \cdots + b_l x^{\tilde{n}-l} y^{\tilde{m}+l} + \tilde{q}(x, y)$, $\tilde{q} \in \Pi^2_{\tilde{n}+\tilde{m}-1}$ is a min–max polynomial on $\mathcal{D}$ with respect to $\Pi^2_{\tilde{n}+\tilde{m}-1}$). Then for every $n, m \in \mathbb{N}$ satisfying $n \geq \tilde{n}$, $m \geq \tilde{m}$ and $n - m = \tilde{n} - \tilde{m}$, the polynomial $\sum_{\nu=0}^{l} b_\nu P_{n-\nu,m+\nu}(x, y) = b_0 x^n y^m + \cdots + b_l x^{n-l} y^{m+l} + q(x, y)$, $q \in \Pi^2_{n+m-1}$, is a min–max polynomial on $\mathcal{D}$.

Next let us consider some special cases. When we fix two consecutive coefficients we obtain:

Corollary 2.5. *Let us have $n, m, i \in \mathbb{N}_0$, $n + m \geq 3$, $i \leq n + m - 3$, and $a, b \in \mathbb{R}$. Then $aP_{n+m-i,i}(x, y) + bP_{n+m-(i+1),i+1}(x, y)$ is a min–max polynomial on $\mathcal{D}$.*

The very special case $ax^{n+m} + bx^{n+m-1}y$ appears in [3, p. 201], proved there in a completely different way.

Checking condition (13) with the help of Mathematica when three consecutive coefficients are given yields:

Corollary 2.6. *Let us have* $n, m \in \mathbb{N}_0,\ n+m \geq 5,\ 1 \leq i \leq n+m-4$, *and let* $b, c \in \mathbb{R}$ *be such that*

$$c \geq 5 + |b| + 2\sqrt{6+3|b|} \quad \text{or} \quad c \leq 5 + |b| - 2\sqrt{6+3|b|}.$$

Then $P_{n+m-i,i}(x,y) + bP_{n+m-(i+1),i+1}(x,y) + cP_{n+m-(i+2),i+2}(x,y)$ *is a min–max polynomial on* $\mathcal{D}$.

The case $b = 0$ and $c \leq 0$ is a special case of (5) and is already known [3, Theorem 3.1].

In the following we give simple but very rough sufficient conditions on the coefficients of the homogeneous polynomial from (1) such that relation (13) is satisfied. As usual, we define for $\boldsymbol{a} = (a_0, a_1, \ldots, a_k) \in \mathbb{R}^{k+1}$

$$\|\boldsymbol{a}\|_2 := \sqrt{a_0^2 + a_1^2 + \cdots + a_k^2} \quad \text{and} \quad \|\boldsymbol{a}\|_1 := |a_0| + |a_1| + \cdots + |a_k|.$$

Corollary 2.7. *Let* $n, m, k \in \mathbb{N}_0$ *be such that* $n+m \geq 3$ *and* $1 \leq k \leq n+m-2$. *Let* $a_i \in \mathbb{R},\ i = 0, \ldots, k,\ l \in \{0, 1, \ldots, k-1\}$ *be the smallest index such that* $a_l \neq 0$ *and let* M_{k-l} *be given by* (12).

(a) *Put* $\boldsymbol{a} := (a_0, a_1, a_2, \ldots, a_k)$. *If*

$$(1 + M_{k-l})\|\boldsymbol{a}\|_2^2 - a_0^2 - M_{k-l}\|\boldsymbol{a}\|_1^2 \geq 0, \tag{16}$$

then $\sum_{i=0}^{k} a_i P_{n+m-i,i}(x,y)$ *is a min–max polynomial on* $\mathcal{D}$.

(b) *Suppose that we have* $\pm a_{4j}, \mp a_{4j+2} \in \mathbb{R}_0^+$ *and* $\pm a_{4j+1},\ \mp a_{4j+3} \in \mathbb{R}_0^+,\ j = 0, 1, 2, \ldots$, *and put* $\boldsymbol{a}_e := (a_0, a_2, \ldots)$ *and* $\boldsymbol{a}_o := (a_1, a_3, \ldots)$. *If*

$$\|\boldsymbol{a}_e\|_2^2 + \|\boldsymbol{a}_o\|_2^2 - a_0^2 - 2M_{k-l}\|\boldsymbol{a}_e\|_1\|\boldsymbol{a}_o\|_1 \geq 0, \tag{17}$$

then $\sum_{i=0}^{k} a_i P_{n+m-i,i}(x,y)$ *is a min–max polynomial on* $\mathcal{D}$.

Inequality (16) gives a rough description of the wide parameter space covered by our approach. Concerning explicit examples, let us consider the case of two arbitrary given coefficients (conditions for several coefficients can be derived from (16) also but they become a little bit more lengthy).

Corollary 2.8. (a) *Let us have* $n, m, k, l \in \mathbb{N}_0,\ n+m \geq 5,\ 3 \leq k \leq n+m-2,\ 1 \leq l \leq k-2$, *and let* $b \in \mathbb{R}$ *be such that* $|b| \leq 1/(M_{k-l} + \sqrt{M_{k-l}^2 - 1})$ *or* $|b| \geq M_{k-l} + \sqrt{M_{k-l}^2 - 1}$. *Then* $P_{n+m-l,l}(x,y) + bP_{n+m-k,k}(x,y)$ *is a min–max polynomial on* $\mathcal{D}$.

(b) *Let us have* $n, m, k \in \mathbb{N}_0,\ n+m \geq 4,\ 2 \leq k \leq n+m-2$, *and let* $b \in \mathbb{R}$ *be such that* $|b| \geq 2M_k$. *Then* $P_{n+m,0}(x,y) + bP_{n+m-k,k}(x,y)$ *is a min–max polynomial on* $\mathcal{D}$.

If the coefficients satisfy the sign condition of Corollary 2.7(b), then (17) will give better bounds than (16). For example, in the above case of two given coefficients it follows by (17) that $P_{n+m-l,l}(x,y) + bP_{n+m-k,k}(x,y)$, $k-l$ even, is a min–max polynomial if $(-1)^{(k-l)/2-1}b \leq 0$, in agreement with (5). Let us further exemplify this for the case of three coefficients:

Corollary 2.9. *Let us have* $n, m, l, k, j \in \mathbb{N}_0,\ n+m \geq 5,\ 3 \leq k \leq n+m-2,\ 1 \leq l < j < k$, *and let* l, j *be even (odd) and* k *odd (even). Let* $b, c \in \mathbb{R}$ *be such that* $(-1)^{(j-l)/2}b \geq 0$ *and* $|c| \leq (1+b^2)/(M_{k-l}(1+|b|) + \sqrt{M_{k-l}^2(1+|b|)^2 - (1+b^2)})$ *or* $|c| \geq M_{k-l}(1+|b|) +$

$\sqrt{M_{k-l}^2(1+|b|)^2-(1+b^2)}$. *Then* $P_{n+m-l,l}(x,y)+bP_{n+m-j,j}(x,y)+cP_{n+m-k,k}(x,y)$ *is a min–max polynomial on* $\mathcal{D}$.

Finally we note that the min–max polynomials from Theorem 2.3 have an extremal signature σ of the form (4), where φ_j, $j=1,\ldots,2(n+m)$, are the zeros of $\sin((n+m)\varphi-\alpha)$, for some $\alpha\in[0,2\pi)$.

Notation 2.10. If $\mathcal{P}$ is a homogeneous polynomial of degree $n+m, n, m \in \mathbb{N}_0, n+m\geq 3$, defined by (1), then let $V_{n+m-1}[\mathcal{P}]$ denote the space of best approximations; i.e.,

$$V_{n+m-1}[\mathcal{P}] := \{p^*\in \Pi^2_{n+m-1} : \|\mathcal{P}-p^*\| = \inf_{p\in\Pi^2_{n+m-1}} \|\mathcal{P}-p\|\}. \tag{18}$$

In [2, p. 61] there is given an upper bound for the dimension of $V_{n+m-1}[\mathcal{P}]$, understood as the dimension of the affine space generated by this non-empty convex set. In what follows we give a sufficient condition on the coefficients of $\mathcal{P}$ for which this upper bound is also a lower bound.

Corollary 2.11. *Let* $n,m,k\in\mathbb{N}_0$ *be such that* $n+m\geq 3$ *and* $1\leq k\leq n+m-2$, *and* $a_i\in\mathbb{R}$, $i=0,\ldots,k$ *be given. Suppose that*

$$\sum_{i=1}^{k} a_i^2 - 2\sum_{i=1}^{k-1}\left(\sum_{j=1}^{k-i} a_j a_{j+i}\right)\Sigma_i(x,y) - 2a_0\sum_{i=1}^{k} a_i\sigma_i(x,y) > 0, \tag{19}$$

for all $(x,y)\in\mathcal{D}$. *Then*

$$\dim V_{n+m-1}\left[\sum_{i=0}^{k} a_i x^{n+m-i}y^i\right] = \binom{n+m-1}{2}. \tag{20}$$

For monomials $x^n y^m, n, m\in\mathbb{N}, n+m\geq 3$, this corollary follows immediately from [3, Theorem 2.2]; see also [2].

3. Proofs

Let us briefly outline the idea of the proof of Theorem 2.3. We will demonstrate that linear combinations of $P_{n,m}$'s satisfy the following quadratic equation (a bivariate Pell type equation), which is the keystone for our investigations.

Theorem 3.1. *Let* $n,m,k\in\mathbb{N}_0$ *be such that* $n+m\geq 3, 1\leq k\leq n+m-2$, *and* $a_i\in\mathbb{R}$, $i=0,\ldots,k$, *be given. Furthermore, let* $\sigma_i(x,y)$ *and* $\Sigma_i(x,y)$, $i\in\mathbb{N}$, *be given by* (10) *and* (9) *respectively. Then the following identity holds:*

$$\left(\sum_{i=0}^{k} a_i P_{n+m-i,i}(x,y)\right)^2 + \left(\sum_{i=0}^{k} a_i P_{n+m-(i+1),i+1}(x,y)\right)^2$$
$$= \frac{\left(\sum_{i=0}^{[k/2]}(-1)^i a_{2i}\right)^2 + \left(\sum_{i=0}^{[(k-1)/2]}(-1)^i a_{2i+1}\right)^2}{2^{2(n+m-1)}}$$

$$
\begin{aligned}
&- \frac{1 - x^2 - y^2}{2^{2(n+m-1)}} \left[\left(a_0 U_{n+m-1}(x) + \sum_{i=1}^{k} a_i p_{n+m-i,i}(x, y) \right)^2 \right. \\
&+ \left(\sum_{i=1}^{k} a_i p_{n+m-i-1,i-1}(x, y) \right)^2 \\
&\left. + \sum_{i=1}^{k} a_i^2 - 2 \sum_{i=1}^{k-1} \left(\sum_{j=1}^{k-i} a_j a_{j+i} \right) \Sigma_i(x, y) - 2a_0 \sum_{i=1}^{k} a_i \sigma_i(x, y) \right] \qquad (21)
\end{aligned}
$$

where

$$p_{n,m}(x, y) := xU_{n-2}(x)U_m(y) + yU_{n-1}(x)U_{m-1}(y). \qquad (22)$$

By the quadratic equation (21) it is easy to see that if the term in square brackets in the RHS of (21) is nonnegative on $\mathcal{D}$, then the polynomial $\sum_{i=0}^{k} a_i P_{n+m-i,i}(x, y)$ takes on its global extremal values on $\partial\mathcal{D}$ at the zeros of $\sum_{i=0}^{k} a_i P_{n+m-(i+1),i+1}(x, y)$. Now if $E_{n,m}$ denotes the minimum deviation given by (14),

$$E^e_{n,m} := \frac{\sum_{i=0}^{[k/2]} (-1)^i a_{2i}}{2^{n+m-1}} \quad \text{and} \quad E^o_{n,m} := \frac{\sum_{i=0}^{[(k-1)/2]} (-1)^i a_{2i+1}}{2^{n+m-1}},$$

and $\alpha \in [0, 2\pi)$ is uniquely determined by $\cos\alpha = E^e_{n,m}/E_{n,m}$ and $\sin\alpha = E^o_{n,m}/E_{n,m}$, then with the help of (3) it can be shown that on $\partial\mathcal{D}$ the following relation holds: $\sum_{i=0}^{k} a_i P_{n+m-(i+1),i+1}(\cos\varphi, \sin\varphi) = E_{n,m} \sin((n + m)\varphi - \alpha)$. Thus the points $(\cos\varphi_j, \sin\varphi_j)$, $j = 1, \ldots, 2(n + m)$, where φ_j are the zeros of $\sin((n + m)\varphi - \alpha)$, build the support of an extremal signature; recall (4). Hence $\sum_{i=0}^{k} a_i P_{n+m-i,i}(x, y)$ is a min–max polynomial on $\mathcal{D}$ if condition (13) is satisfied. Thus Theorem 2.3 follows.

The proof of Theorems 2.3 and 3.1 requires only elementary tools once one has found the right ansatz, that is, that one has discovered that the $P_{n,m}$'s are min–max polynomials and that linear combinations of such polynomials are good candidates for being min–max polynomials, since the $P_{n,m}$'s are quite resistant to perturbations.

To derive (21) let us first show that the polynomials $G_{n,m}$ and $P_{n,m}$ satisfy a quadratic equation.

Remark 3.2. In the following, we use frequently the following identities involving the Chebyshev polynomials of the second kind, which are very easy to verify:

$$
\begin{aligned}
&\text{(a) } U_n(x)U_{n-2}(x) = U^2_{n-1}(x) - 1, \\
&\text{(b) } U^2_n(x) + U^2_{n-2}(x) - 2 = (4x^2 - 2)U^2_{n-1}(x), \\
&\text{(c) } U_n(x) + U_{n-4}(x) = (4x^2 - 2)U_{n-2}(x), \\
&\text{(d) } U_n(x)U_{n-k-2}(x) = U_{n-1}(x)U_{n-k-1}(x) - U_k(x),
\end{aligned} \qquad (23)
$$

where $n, k \in \mathbb{Z}$.

Proposition 3.3. (a) *Let us have* $n \in \mathbb{N}, m \in \mathbb{N}_0$. *Then*

$$[G_{n,m}(x, y)]^2 + [G_{n-1,m+1}(x, y)]^2$$

$$= \frac{1}{2^{2(n+m-1)}} - \frac{1-x^2-y^2}{2^{2(n+m-1)}}\left[U_{n-1}^2(x)U_m^2(y) + U_{n-2}^2(x)U_{m-1}^2(y)\right]. \tag{24}$$

(b) *Let us have* $n, m \in \mathbb{N}, n \geq 2$. *Then*

$$[P_{n,m}(x,y)]^2 + [P_{n-1,m+1}(x,y)]^2$$
$$= \frac{1}{2^{2(n+m-1)}} - \frac{1-x^2-y^2}{2^{2(n+m-1)}}\left[1 + p_{n,m}^2(x,y) + p_{n-1,m-1}^2(x,y)\right], \tag{25}$$

where $p_{n,m}(x,y)$ *is given by* (22).

Proof. (a) By (2) and (23)(a) and (23)(b), we have

$$\begin{aligned}
&2^{2(n+m)}\{[G_{n,m}(x,y)]^2 + [G_{n-1,m+1}(x,y)]^2\}\\
&= U_n^2(x)U_m^2(y) + U_{n-2}^2(x)U_{m-2}^2(y) + U_{n-1}^2(x)U_{m+1}^2(y) + U_{n-3}^2(x)U_{m-1}^2(y)\\
&\quad + 2U_n(x)U_{n-2}(x)U_m(y)U_{m-2}(y) + 2U_{n-1}(x)U_{n-3}(x)U_{m+1}(y)U_{m-1}(y)\\
&= U_{n-1}^2(x)(U_{m+1}^2(y) + U_{m-1}^2(y) - 2) + U_{n-2}^2(x)(U_m^2(y) + U_{m-2}^2(y) - 2)\\
&\quad + (U_n^2(x) + U_{n-2}^2(x) - 2)U_m^2(y) + (U_{n-1}^2(x) + U_{n-3}^2(x) - 2)U_{m-1}^2(y) + 4\\
&= 4 + [(4x^2-2) + (4y^2-2)]\left[U_{n-1}^2(x)U_m^2(y) + U_{n-2}^2(x)U_{m-1}^2(y)\right].
\end{aligned}$$

Thus, relation (24) is proved.

(b) By the definition (6) of $P_{n,m}(x,y)$ we obtain

$$\begin{aligned}
&[P_{n,m}(x,y)]^2 + [P_{n-1,m+1}(x,y)]^2 = \frac{1}{4}x^2\left\{[G_{n-1,m}(x,y)]^2 + [G_{n-2,m+1}(x,y)]^2\right\}\\
&\quad + \frac{1}{2}xy(G_{n,m-1}(x,y) + G_{n-2,m+1}(x,y))G_{n-1,m}(x,y)\\
&\quad + \frac{1}{4}y^2\left\{[G_{n,m-1}(x,y)]^2 + [G_{n-1,m}(x,y)]^2\right\}.
\end{aligned} \tag{26}$$

As an immediate consequence of the definition (2) of $G_{n,m}(x,y)$ and of (23)(c) it follows that for $n, m \in \mathbb{N}_0,\ n \geq 2$,

$$G_{n,m}(x,y) + G_{n-2,m+2}(x,y) = -\frac{1-x^2-y^2}{2^{n+m-2}}U_{n-2}(x)U_m(y). \tag{27}$$

Combining relation (26) with relation (24) written for $G_{n-1,m}(x,y)$ and $G_{n-2,m+1}(x,y)$, relation (27) for $G_{n,m-1}(x,y)$ and $G_{n-2,m+1}(x,y)$, the definition (2) of $G_{n-1,m}(x,y)$ and relation (24) written for $G_{n,m-1}(x,y)$ and $G_{n-1,m}(x,y)$, and taking into consideration notation (22), we derive immediately relation (25). □

Remark 3.4. For $n \in \mathbb{N}$, by (6) one obtains easily that $P_{n,1}(x,y) = G_{n,1}(x,y)$, which combined with (7) and (24) yields

$$[P_{n,0}(x,y)]^2 + [P_{n-1,1}(x,y)]^2 = \frac{1}{2^{2(n-1)}} - \frac{1-x^2-y^2}{2^{2(n-1)}}U_{n-1}^2(x). \tag{28}$$

To derive the quadratic equation (21) for linear combinations we need also expressions for $P_{n,m}(x,y)P_{n-i,m+i}(x,y) + P_{n-1,m+1}(x,y)P_{n-i-1,m+i+1}(x,y)$.

Let $S_0(x, y) := 0$ and for $i \in \mathbb{N}$ let

$$S_i(x, y) := \frac{2[(-1-(-1)^i)/2]^{i/2-1} - Q_{i,i}(x, y)}{2(1-x^2-y^2)}, \tag{29}$$

where the polynomials $Q_{i,i}$ are given by (8).

Proposition 3.5. (a) *Let $n, m, i \in \mathbb{N}_0$ be such that $1 \le i \le n-1$. Then*

$$\begin{aligned} &G_{n,m}(x, y)G_{n-i,m+i}(x, y) + G_{n-1,m+1}(x, y)G_{n-i-1,m+i+1}(x, y) \\ &\quad = \frac{[(-1-(-1)^i)/2]^{i/2}}{2^{2(n+m-1)}} - \frac{1-x^2-y^2}{2^{2(n+m-1)}}[r_{n,m}(x, y)r_{n-i,m+i}(x, y) \\ &\qquad + r_{n-1,m-1}(x, y)r_{n-i-1,m+i-1}(x, y) - S_i(x, y)], \end{aligned} \tag{30}$$

where $r_{n,m}(x, y) := U_{n-1}(x)U_m(y)$.

(b) *Let $n, m, i \in \mathbb{N}$ be such that $1 \le i \le n-2$. Then*

$$\begin{aligned} &P_{n,m}(x, y)P_{n-i,m+i}(x, y) + P_{n-1,m+1}(x, y)P_{n-i-1,m+i+1}(x, y) \\ &\quad = \frac{[(-1-(-1)^i)/2]^{i/2}}{2^{2(n+m-1)}} - \frac{1-x^2-y^2}{2^{2(n+m-1)}}[p_{n,m}(x, y)p_{n-i,m+i}(x, y) \\ &\qquad + p_{n-1,m-1}(x, y)p_{n-i-1,m+i-1}(x, y) - \Sigma_i(x, y)], \end{aligned} \tag{31}$$

where $p_{n,m}(x, y)$ is given by (22).

Proof. (a) Let us consider first the case where i is even. We begin with the obvious identity

$$\begin{aligned} &G_{n,m}(x, y) + (-1)^{i/2-1}G_{n-i,m+i}(x, y) \\ &\quad = \sum_{j=0}^{i/2-1}(-1)^j\left[G_{n-2j,m+2j}(x, y) + G_{n-2j-2,m+2j+2}(x, y)\right] \\ &\quad = -\frac{1-x^2-y^2}{2^{n+m-2}}\sum_{j=0}^{i/2-1}(-1)^jU_{n-2j-2}(x)U_{m+2j}(y), \end{aligned}$$

where for the last equality we have used (27). Equivalently,

$$\begin{aligned} G_{n-i,m+i}(x, y) &= (-1)^{i/2}G_{n,m}(x, y) \\ &\qquad + \frac{1-x^2-y^2}{2^{n+m-2}}(-1)^{i/2}\sum_{j=0}^{i/2-1}(-1)^jU_{n-2j-2}(x)U_{m+2j}(y). \end{aligned}$$

Using this expression and the corresponding one for $G_{n-i-1,m+i+1}(x, y)$, as well as the definition (2) of $G_{n,m}(x, y)$ and $G_{n-1,m+1}(x, y)$, we have

$$\begin{aligned} &G_{n,m}(x, y)G_{n-i,m+i}(x, y) + G_{n-1,m+1}(x, y)G_{n-i-1,m+i+1}(x, y) \\ &\quad = (-1)^{i/2}\left\{[G_{n,m}(x, y)]^2 + [G_{n-1,m+1}(x, y)]^2 + \frac{1-x^2-y^2}{2^{2(n+m-1)}}q(x, y)\right\} \end{aligned} \tag{32}$$

where

$$q(x, y) := \sum_{j=0}^{i/2-1}(-1)^jU_n(x)U_{n-2j-2}(x)U_m(y)U_{m+2j}(y)$$

$$+\sum_{j=0}^{i/2-1}(-1)^j U_{n-2}(x)U_{n-2j-2}(x)U_{m-2}(y)U_{m+2j}(y)$$
$$+\sum_{j=0}^{i/2-1}(-1)^j U_{n-1}(x)U_{n-2j-3}(x)U_{m+1}(y)U_{m+2j+1}(y)$$
$$+\sum_{j=0}^{i/2-1}(-1)^j U_{n-3}(x)U_{n-2j-3}(x)U_{m-1}(y)U_{m+2j+1}(y).$$

Applying relation (23)(d) to $U_n(x)U_{n-2l-2}(x)$, $U_{n-2}(x)U_{n-2l-4}(x)$, $U_{m-1}(y)U_{m+2l-1}(y)$ and $U_{m+1}(y)U_{m+2l+1}(y)$, the polynomial $q(x, y)$ defined above becomes

$$\begin{aligned} q(x,y) =\ & U_{n-1}(x)U_m(y)\sum_{j=0}^{i/2-1}(-1)^j Q_{n-2j-1,m+2j+2}(x,y)\\ & + U_{n-2}(x)U_{m-1}(y)\sum_{j=0}^{i/2-1}(-1)^j Q_{n-2j-2,m+2j+1}(x,y)\\ & + \sum_{j=0}^{i/2-1}(-1)^j (U_{n-2}(x)U_{n-2j-2}(x) - U_{n-1}(x)U_{n-2j-3}(x))U_{2j}(y)\\ & + \sum_{j=0}^{i/2-1}(-1)^j U_{2j}(x)(U_{m-1}(y)U_{m+2j+1}(y) - U_m(y)U_{m+2j}(y))\\ =:\ & A(x,y)+B(x,y)+C(x,y)+D(x,y), \end{aligned}$$

where the polynomials $Q_{n,m}$ are given by (8).

After cancellations in $A(x, y)$ and $B(x, y)$ and by relation (23)(d) in the case of $C(x, y)$ and $D(x, y)$, we obtain

$$\begin{aligned} A(x,y) &= U_{n-1}^2(x)U_m^2(y) + (-1)^{i/2-1}U_{n-1}(x)U_{n-i-1}(x)U_m(y)U_{m+i}(y)\\ B(x,y) &= U_{n-2}^2(x)U_{m-1}^2(y) + (-1)^{i/2-1}U_{n-2}(x)U_{n-i-2}(x)U_{m-1}(y)U_{m+i-1}(y)\\ C(x,y) &= D(x,y) = -\sum_{j=0}^{i/2-1}(-1)^j U_{2j}(x)U_{2j}(y), \end{aligned}$$

and consequently the expression of $q(x, y)$. This expression combined with the quadratic equation (24) of the Gearhart polynomials yields relation (30) if we prove that for i even, $i \geq 2$,

$$S_i(x,y) = (-1)^{i/2-1}2\sum_{j=0}^{i/2-1}(-1)^j U_{2j}(x)U_{2j}(y). \tag{33}$$

But this can be easily verified by induction and with the help of the formula

$$Q_{n,m}(x,y) + Q_{n+2,m+2}(x,y) = -4(1-x^2-y^2)U_n(x)U_m(y), \tag{34}$$

for $n, m \in \mathbb{N}_0$, obtained from (8) and (23)(c).

Concerning the case where i is odd, for $i = 1$ relation (30) follows by (27) and the definition (2) of $G_{n-1,m}(x, y)$. When $i \geq 3$, the proof runs in a similar way to that when i is even, except that by induction one shows now that for i odd, $i \geq 3$,

$$S_i(x, y) = (-1)^{(i-3)/2} 2 \sum_{j=0}^{(i-3)/2} (-1)^j U_{2j+1}(x) U_{2j+1}(y). \quad (35)$$

(b) By definition (6) of $P_{n,m}(x, y)$, we get

$$\begin{aligned} &P_{n,m}(x, y) P_{n-i,m+i}(x, y) + P_{n-1,m+1}(x, y) P_{n-i-1,m+i+1}(x, y) \\ &= \frac{x^2}{4} \left(G_{n-1,m}(x, y) G_{n-i-1,m+i}(x, y) + G_{n-2,m+1}(x, y) G_{n-i-2,m+i+1}(x, y) \right) \\ &+ \frac{xy}{4} \left[\left(G_{n,m-1}(x, y) G_{n-i-1,m+i}(x, y) + G_{n-1,m}(x, y) G_{n-i-2,m+i+1}(x, y) \right) \right. \\ &+ \left. \left(G_{n-1,m}(x, y) G_{n-i,m+i-1}(x, y) + G_{n-2,m+1}(x, y) G_{n-i-1,m+i}(x, y) \right) \right] \\ &+ \frac{1}{4} y^2 \left(G_{n,m-1}(x, y) G_{n-i,m+i-1}(x, y) + G_{n-1,m}(x, y) G_{n-i-1,m+i}(x, y) \right). \end{aligned}$$

This expression coupled with (30) and the fact that

$$\Sigma_i(x, y) = ((-1 - (-1)^i)/2)^{i/2-1} + xy(S_{i+1}(x, y) + S_{i-1}(x, y)) + (x^2 + y^2) S_i(x, y),$$

as one can easily verify, yields the desired relation (31). □

For $i \in \mathbb{N}$, let

$$s_i(x, y) := \begin{cases} \dfrac{2(-1)^{i/2-1} y - Q_{i,i-1}(x, y)}{2(1 - x^2 - y^2)}, & \text{if } i \text{ even} \\ \dfrac{2(-1)^{(i-3)/2} x - Q_{i,i-1}(x, y)}{2(1 - x^2 - y^2)}, & \text{if } i \text{ odd.} \end{cases} \quad (36)$$

Proposition 3.6. (a) *Let us have $n, i \in \mathbb{N}$, $1 \leq i \leq n - 1$. Then*

$$\begin{aligned} &G_{n,0}(x, y) G_{n-i,i-1}(x, y) + G_{n-1,1}(x, y) G_{n-i-1,i}(x, y) \\ &= \frac{-Q_{i,i-1}(x, y) - 2(1 - x^2 - y^2) s_i(x, y)}{2^{2(n-1)}} \\ &- \frac{1 - x^2 - y^2}{2^{2n-3}} [U_{n-1}(x) U_{n-i-1}(x) U_{i-1}(y) - s_i(x, y)]. \end{aligned} \quad (37)$$

(b) *Let us have $n, i \in \mathbb{N}$, $1 \leq i \leq n - 2$. Then*

$$\begin{aligned} &P_{n,0}(x, y) P_{n-i,i}(x, y) + P_{n-1,1}(x, y) P_{n-i-1,i+1}(x, y) \\ &= \frac{[(-1 - (-1)^i)/2]^{i/2}}{2^{2(n-1)}} - \frac{1 - x^2 - y^2}{2^{2(n-1)}} [U_{n-1}(x) p_{n-i,i}(x, y) - \sigma_i(x, y)], \end{aligned} \quad (38)$$

where $p_{n-i,i}(x, y)$ is defined by (22).

Proof. (a) First let us consider the case where i is even. For $i = 2$, by (2) and (23)(a) and (23)(c), we have

$$\begin{aligned} &G_{n,0}(x, y) G_{n-2,1}(x, y) + G_{n-1,1}(x, y) G_{n-3,2}(x, y) \\ &= \frac{1}{2^{2n-2}} \Big[U_n(x) U_{n-2}(x) y - U_{n-2}^2(x) y - U_{n-1}(x) U_{n-3}(x) y \end{aligned}$$

$$
\begin{aligned}
&+ U_{n-1}(x)U_{n-3}(x)4y^3 + U_{n-1}(x)U_{n-5}(x)y\Big] \\
&= \frac{1}{2^{2n-2}}\Big[(U_{n-1}^2(x)-1)y - (U_{n-1}(x)U_{n-3}(x)+1)y - U_{n-1}(x)U_{n-3}(x)y \\
&+ U_{n-1}(x)U_{n-3}(x)4y^3 + U_{n-1}(x)((4x^2-2)U_{n-3}(x) - U_{n-1}(x))y\Big] \\
&= -\frac{y}{2^{2n-3}} - \frac{1-x^2-y^2}{2^{2n-3}}[U_{n-1}(x)U_{n-3}(x)U_1(y)].
\end{aligned}
$$

Since $s_2(x, y) = 0$, relation (37) when $i = 2$ is proved. The case where $i \geq 4$ is handled like in Proposition 3.5, noting that for i even, $i \geq 4$,

$$
s_i(x, y) = (-1)^{i/2-1} 2 \sum_{j=1}^{i/2-1} (-1)^j U_{2j}(x)U_{2j-1}(y)
$$

which can be proved by induction.

Next we consider the case where i is odd. For $i = 1$, by (2) and (23)(c) and (d), we have

$$
\begin{aligned}
&G_{n,0}(x, y)G_{n-1,0}(x, y) + G_{n-1,1}(x, y)G_{n-2,1}(x, y) \\
&= \frac{1}{2^{2n-1}}\big[U_n(x)U_{n-1}(x) - U_n(x)U_{n-3}(x) \\
&-U_{n-1}(x)U_{n-2}(x) + U_{n-2}(x)U_{n-3}(x) + 4U_{n-1}(x)U_{n-2}(x)y^2\big] \\
&= \frac{1}{2^{2n-1}}\big[U_n(x)U_{n-1}(x) - (U_{n-1}(x)U_{n-2}(x) - U_1(x)) \\
&-U_{n-1}(x)U_{n-2}(x) + (U_{n-1}(x)U_{n-4}(x) + U_1(x)) + 4U_{n-1}(x)U_{n-2}(x)y^2\big] \\
&= \frac{x}{2^{2n-3}} - \frac{1-x^2-y^2}{2^{2n-3}}U_{n-1}(x)U_{n-2}(x),
\end{aligned}
$$

and thus, since $s_1(x, y) = 0$, relation (37) when $i = 1$ is proved. In the case $i \geq 3$, the relation follows by the same procedure as the one indicated in the proof of Proposition 3.5, except that here, for i odd, $i \geq 3$,

$$
s_i(x, y) = (-1)^{(i-3)/2} 2 \sum_{j=0}^{(i-3)/2} (-1)^j U_{2j+1}(x)U_{2j}(y).
$$

(b) The proof is similar to that of Proposition 3.5, except that in this case one has to verify that

$$
\sigma_i(x, y) = ((-1-(-1)^i)/2)^{i/2-1} + xs_{i+1}(x, y) + ys_i(x, y). \quad \square
$$

Proof of Theorem 3.1. The identity (21) is a direct consequence of Proposition 3.3(b), Remark 3.4 and of Propositions 3.5(b) and 3.6(b). $\square$

What remains to be derived are estimates for $\Sigma_i(x, y)$ and $\sigma_i(x, y)$. To this end, we show first the following:

Proposition 3.7. (a) *Let us have $n, m \in \mathbb{N}$. For all $(x, y) \in \mathcal{D}$ it holds that*

$$
|Q_{n,m}(x, y) - 2(1-x^2-y^2)U_n(x)U_{m-2}(y)| \leq 2. \tag{39}
$$

(b) *Let us have* $n, m \in \mathbb{N}$. *For all* $(x, y) \in \mathcal{D}$ *it holds that*

$$|Q_{n,m}(x, y) - (1 - x^2 - y^2)(Q_{n,m}(x, y) + 4xyU_{n-1}(x)U_{m-1}(y))| \le 2. \tag{40}$$

Proof. (a) By the definition (8) of the polynomials $Q_{n,m}$ and the well-known recurrence relation of the Chebyshev polynomials of the second kind, we have

$$\begin{aligned} Q_{n,m}(x, y) + Q_{n+2,m}(x, y) &= (U_{n+2}(x) + U_n(x))U_{m-2}(y) + (U_n(x) + U_{n-2}(x))U_m(y) \\ &= 2xU_{n+1}(x)U_{m-2}(y) + 2xU_{n-1}(x)U_m(y) \\ &= 2xQ_{n+1,m}(x, y) \end{aligned}$$

and, similarly,

$$Q_{n,m-2}(x, y) + Q_{n,m}(x, y) = 2yQ_{n,m-1}(x, y).$$

With the help of these relations and of (34) we can write

$$\begin{aligned} &Q_{n,m}(x, y) - 2(1 - x^2 - y^2)U_n(x)U_{m-2}(y) \\ &\quad = \frac{1}{2}(Q_{n,m}(x, y) + Q_{n+2,m}(x, y)) + \frac{1}{2}(Q_{n,m-2}(x, y) + Q_{n,m}(x, y)) \\ &\quad = xQ_{n+1,m}(x, y) + yQ_{n,m-1}(x, y). \end{aligned} \tag{41}$$

One can show, by analogy with Proposition 3.3(a), that for $n, m \in \mathbb{N}_0$,

$$\begin{aligned} &[Q_{n+1,m+1}(x, y)]^2 + [Q_{n,m}(x, y)]^2 \\ &\quad = 4 - (1 - x^2 - y^2)\left[4U_n^2(x)U_{m-1}^2(y) + 4U_{n-1}^2(x)U_m^2(y)\right]. \end{aligned} \tag{42}$$

Now, by (41), by applying the Schwarz inequality and using afterwards (42), we obtain that for all $(x, y) \in \mathcal{D}$, relation (39) holds.

(b) With the help of the well-known recurrence relation of the Chebyshev polynomials of the second kind, of (34) and of (41), we get

$$\begin{aligned} &Q_{n,m}(x, y) - (1 - x^2 - y^2)(Q_{n,m}(x, y) + 4xyU_{n-1}(x)U_{m-1}(y)) \\ &\quad = x[xQ_{n,m}(x, y) + yQ_{n+1,m+1}(x, y)] + y[xQ_{n-1,m-1}(x, y) + yQ_{n,m}(x, y)]. \end{aligned} \tag{43}$$

Furthermore, like for Proposition 3.3(b), we have that for $n, m \in \mathbb{N}_0$ it holds that

$$\begin{aligned} &[xQ_{n+1,m+1}(x, y) + yQ_{n+2,m+2}(x, y)]^2 + [xQ_{n,m}(x, y) + yQ_{n+1,m+1}(x, y)]^2 \\ &\quad = 4 - (1 - x^2 - y^2)\left[4 + 4q_{n,m}^2(x, y) + 4q_{n-1,m+1}^2(x, y)\right], \end{aligned} \tag{44}$$

where $q_{n,m}(x, y) := xU_n(x)U_{m-1}(y) + yU_{n+1}(x)U_m(y)$.

By a simple application of Schwarz's inequality in (43) and by (44), we conclude that (40) holds, and thus the proposition is proved. □

Proposition 3.8. *Let us have* $i \in \mathbb{N}$. *Then the following estimates hold:*

$$|\Sigma_i(x, y)| \le M_i \quad \textit{and} \quad |\sigma_i(x, y)| \le M_i, \tag{45}$$

for all $(x, y) \in \mathcal{D}$, *where* M_i *is given by* (12).

Furthermore, if i *is even,*

$$0 \le (-1)^{i/2-1}\Sigma_i(x, y) \quad \textit{and} \quad 0 \le (-1)^{i/2-1}\sigma_i(x, y), \tag{46}$$

for all $(x, y) \in \mathcal{D}$.

Proof. By (9) and (29) we see that

$$\Sigma_i(x, y) = \frac{1}{2}[Q_{i,i}(x, y) + 4xyU_{i-1}(x)U_{i-1}(y)] + S_i(x, y), \tag{47}$$

and by (10) and (29) that

$$\sigma_i(x, y) = U_i(x)U_{i-2}(y) + S_i(x, y). \tag{48}$$

We use the equivalent representations of $S_i(x, y)$, given by (33) when i is even and (35) when i is odd, to show that

$$|S_i(x, y)| \le \frac{i^3}{3} \tag{49}$$

for all $(x, y) \in \mathcal{D}$. The well-known fact that $|U_n(x)| \le n + 1,\ n \in \mathbb{N}_0, -1 \le x \le 1$, leads to the very rough estimate

$$|U_n(x)U_m(y)| \le (n + 1)(m + 1), \tag{50}$$

when $(x, y) \in \mathcal{D}, n, m \in \mathbb{N}_0$. Now, if i is even, by (33), we have

$$|S_i(x, y)| \le 2 \sum_{j=0}^{i/2-1} |U_{2j}(x)U_{2j}(y)| \le 2 \sum_{j=0}^{i/2-1} (2j + 1)^2 = \frac{(i - 1)i(i + 1)}{3} \le \frac{i^3}{3},$$

for any $(x, y) \in \mathcal{D}$. The case where i is odd is handled likewise.

If as usual $T_n(\cos x) := \cos n \arccos x,\ n \in \mathbb{N}_0, -1 \le x \le 1$, then from the well-known relation $T_{n+1}^2(x) + (1 - x^2)U_n^2(x) = 1$, we obtain immediately that

$$y^2U_n^2(x) = 1 - [T_{n+1}^2(x) + (1 - x^2 - y^2)U_n^2(x)] \le 1,$$

for all $(x, y) \in \mathcal{D}$. Similarly we deduce that $x^2U_m^2(y) \le 1$ for $(x, y) \in \mathcal{D}$, and combining the two inequalities we obtain that

$$|xyU_n(x)U_m(y)| \le 1 \tag{51}$$

for all $(x, y) \in \mathcal{D}$.

By (47), (42), (51) and (49), we get

$$|\Sigma_i(x, y)| \le \frac{i^3}{3} + 3, \tag{52}$$

for all $(x, y) \in \mathcal{D}$. Also, by (48), (50) and (49), we obtain that

$$|\sigma_i(x, y)| \le \frac{i^3}{3} + i^2 - 1, \tag{53}$$

for all $(x, y) \in \mathcal{D}$. Noting that by (9), $\Sigma_1(x, y) = 2xy$, and hence $|\Sigma_1(x, y)| = 2\sqrt{x^2y^2} \le 2\frac{x^2+y^2}{2} \le 1$, and taking into consideration notation (12), we can rewrite the inequalities (52) and (53) as (45).

Concerning (46), this is an immediate consequence of (9) and (40) and (10) and (39). □

Proof of Proposition 2.1. Using the definition (6) of $P_{n,m}$ we get by Schwarz's inequality and by Proposition 3.3(a) that $|P_{n,m}(x, y)| \le 1/2^{n+m-1}$ for all $(x, y) \in \mathcal{D}$. Moreover, (3) and (6) imply that

$$P_{n,m}(\cos\varphi,\sin\varphi)=\begin{cases}\dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\cos(n+m)\varphi, & \text{if } m \text{ even},\\ \dfrac{(-1)^{[m/2]}}{2^{n+m-1}}\sin(n+m)\varphi, & \text{if } m \text{ odd}.\end{cases}$$

Hence, $|P_{n,m}(x,y)|$ attains its maximum on $\mathcal{D}$ at the points $(\cos\varphi_j,\sin\varphi_j)$, $j=1,\dots,2(n+m)$, where φ_j's are the zeros of $\sin(n+m)\varphi$ when m is even and the zeros of $\cos(n+m)\varphi$ when m is odd, which form the support of an extremal signature. Thus the assertion is proved. □

Proof of Corollary 2.4. If one puts in Theorem 2.3 $a_0,\dots,a_{m-1}=0$ and $a_{m+\nu}=b_\nu$ for $\nu=0,\dots,l$ then condition (13) becomes (15) and Corollary 2.4 follows. □

Proof of Corollary 2.5. If $i=0$, condition (13) is $b^2-2ab\sigma_1(x,y)\ge 0$, for all $(x,y)\in\mathcal{D}$, which holds for any $a,b\in\mathbb{R}$, since by (10), $\sigma_1(x,y)=0$.

If $i\ge 1$, then condition (13) is $a^2+b^2-2ab\Sigma_1(x,y)\ge 0$ for all $(x,y)\in\mathcal{D}$. Since by (9), $\Sigma_1(x,y)=2xy$, and using the fact that on the disc $|xy|\le 1/2$, we get $a^2+b^2-2ab\Sigma_1(x,y)\ge(|a|-|b|)^2\ge 0$ for all $(x,y)\in\mathcal{D}$.

The assertion now follows by Theorem 2.3. □

Proof of Corollary 2.6. In order to apply Theorem 2.3 in this case, we need those values of b and c for which

$$1+b^2+c^2-2b\Sigma_1(x,y)-2c\Sigma_2(x,y)-2bc\Sigma_1(x,y)\ge 0,$$

for all $(x,y)\in\mathcal{D}$. By (10), this is equivalent to

$$1+b^2+c^2-2c-4bxy-4bcxy-4cx^2-4cy^2-16cx^2y^2\ge 0,$$

for all $(x,y)\in\mathcal{D}$. The values for b and c in the statement of the corollary were the output of the following command in Mathematica:

Resolve[ForAll[x, $-1\le x\le 1$, ForAll[y, $-1\le y\le 1$,
Implies[$x^2+y^2\le 1$, $1+b^2+c^2-2c-4bxy-4bcxy$
$-4cx^2-4cy^2-16cx^2y^2\ge 0$]]], b, c, Reals]; .

The assertion is thus proved. □

Proof of Corollary 2.7. (a) Taking into consideration the bound M_i from (45) for $\Sigma_i(x,y)$ and $\sigma_i(x,y)$, we have

$$\sum_{i=1}^{k}a_i^2-2\sum_{i=1}^{k-1}\left(\sum_{j=1}^{k-i}a_ja_{j+i}\right)\Sigma_i(x,y)-2a_0\sum_{i=1}^{k}a_i\sigma_i(x,y)$$
$$\ge\sum_{i=1}^{k}a_i^2-M_{k-l}2\sum_{i=0}^{k-1}\left(\sum_{j=i+1}^{k}|a_j|\right)|a_i|=\|\boldsymbol{a}\|_2^2-a_0^2-M_{k-l}(\|\boldsymbol{a}\|_1^2-\|\boldsymbol{a}\|_2^2).$$

Therefore, condition (13) is obviously satisfied if $\|\boldsymbol{a}\|_2^2-a_0^2-M_{k-l}(\|\boldsymbol{a}\|_1^2-\|\boldsymbol{a}\|_2^2)\ge 0$, that is, if (16) holds.

Concerning (b), in addition to (45) we use also (46), and thus we have for the expression in (13)

$$\sum_{i=1}^{k} a_i^2 - 2\sum_{i=1}^{k-1}\left(\sum_{j=1}^{k-i} a_j\right) \Sigma_i(x, y) - 2a_0 \sum_{i=1}^{k} a_i\sigma_i(x, y)$$
$$\geq \|\boldsymbol{a}_e\|_2^2 + \|\boldsymbol{a}_o\|_2^2 - a_0^2 - M_{k-l}\|\boldsymbol{a}_e\|_1\|\boldsymbol{a}_o\|_1,$$

which gives condition (17). □

Proof of Corollary 2.8. The statements are simple verifications of condition (16) from Corollary 2.7(a) with $a_i = 0$ if $i \in \{0, \dots, k\} \setminus \{l, k\}, a_l = 1$ and $a_k = b$ for statement (a) and $a_i = 0$ if $i \in \{1, \dots, k-1\}, a_0 = 1$ and $a_k = b$ for statement (b). □

Proof of Corollary 2.9. The assertion follows immediately by (17) in Corollary 2.7(b) with $a_i = 0,\ i \in \{0, \dots, k\} \setminus \{l, j, k\}, a_l = 1, a_j = b, a_k = c.$ □

Proof of Corollary 2.11. If assumption (19) is satisfied, then by Theorem 2.3, the polynomial $F(x, y) := \sum_{i=0}^{k} a_i P_{n+m-i,i}(x, y)$ is a min–max polynomial on $\mathcal{D}$ and we let $E_{n,m}$ denote its minimum deviation, defined by (14). For any $q \in \Pi^2_{n+m-3}$, the polynomial $F(x, y) + (1 - x^2 - y^2)q(x, y)$ has the form of a min–max polynomial and hence

$$\|F + (1 - x^2 - y^2)q\| \geq E_{n,m}. \tag{54}$$

Since (19) is assumed to hold on the compact set $\mathcal{D}$, there exists $\varepsilon > 0$ such that

$$\sum_{i=1}^{k} a_i^2 - 2\sum_{i=1}^{k-1}\left(\sum_{j=1}^{k-i} a_j a_{j+i}\right) \Sigma_i(x, y) - 2a_0 \sum_{i=1}^{k} a_i\sigma_i(x, y) \geq 2^{2(n+m-1)}\varepsilon,$$

for all $(x, y) \in \mathcal{D}$. Thus, by relation (21), we have

$$(E_{n,m})^2 - (F(x, y))^2 \geq \varepsilon(1 - x^2 - y^2), \tag{55}$$

for all $(x, y) \in \mathcal{D}$. Furthermore, for any polynomial $q \in \Pi^2_{n+m-3}$ such that $\|q\| \leq \sqrt{(E_{n,m})^2 + \varepsilon} - E_{n,m}$, it holds, using $\|F\| = E_{n,m}$, that

$$|2F(x, y)q(x, y) + (1 - x^2 - y^2)q^2(x, y)| \leq 2\|F\|\,\|q\| + \|q\|^2 \leq \varepsilon, \tag{56}$$

for all $(x, y) \in \mathcal{D}$. Combining (55) and (56), it follows that

$$|F(x, y) + (1 - x^2 - y^2)q(x, y)|^2 \leq (E_{n,m})^2$$

for all $(x, y) \in \mathcal{D}$, and hence that

$$\|F + (1 - x^2 - y^2)q\| \leq E_{n,m}. \tag{57}$$

Therefore, by (54) and (57), the polynomial $F + (1 - x^2 - y^2)q$ is a min–max polynomial on $\mathcal{D}$, for every $q \in \Pi^2_{n+m-3}$ such that $\|q\| \leq \sqrt{(E_{n,m})^2 + \varepsilon} - E_{n,m}$, which implies that

$$\dim V_{n+m-1}\left[\sum_{i=0}^{k} a_i x^{n+m-i} y^i\right] \geq \dim \Pi^2_{n+m-3}. \tag{58}$$

On the other hand, as we already mentioned, the extremal signature in this case is formed of $2(n + m)$ points on $\partial\mathcal{D}, (x_i, y_i), i = 1, 2, \dots, 2(n + m)$. Then according to Möller

[6, p. 43], this extremal signature, obtained by Shapiro, is a primitive one and the rank of the matrix $(x_i^k y_i^l)_{\substack{i=\overline{1,2(n+m)},\\ k+l=\overline{0,n+m-1}}}$ is $2(n+m)-1$. But since all min–max polynomials agree on the points forming the support of a primitive extremal signature (see for example [9, Theorem 1]), we have that

$$\dim V_{n+m-1}\left[\sum_{i=0}^{k} a_i x^{n+m-i} y^i\right] \leq \dim \Pi^2_{n+m-1} - (2(n+m)-1); \tag{59}$$

see also [2] for this bound in a more general setting. Since $\dim \Pi_N^2 = \binom{N+2}{2}$, $N \in \mathbb{N}_0$, by (58) and (59), taking into account that $\dim \Pi^2_{n+m-1} - (2(n+m)-1) = \dim \Pi^2_{n+m-3}$, relation (20) follows. $\square$

Remark 3.9. Checking the proof of Theorem 2 in [12] it turns out that the given sufficient condition, that is, the inequality above (9) in [12], has to be modified as follows, using the notation from there:

$$\| f(\cos\varphi, \sin\varphi)\| \leq \frac{1}{2^n}\left(\frac{n}{2} - 1\right) \|A_n(f,\varphi)\|, \tag{60}$$

i.e., the factor $1/2^n$ is missing in [12]. As a consequence the condition will be satisfied very rarely, if at all. For instance, for the case of monomials $f(x, y) = x^k y^{n-k}$ it is not difficult to show that $|f(\sqrt{2}/2, \sqrt{2}/2)| \geq 1/2^n(n-2)$ which implies that the corrected condition (60) does not hold if $n \geq 3$. In the case of a homogeneous polynomial $f(x, y) = a_k x^k y^{n-k} + a_l x^l y^{n-l}$, a_k, a_l fixed, condition (60) does not hold if $n \geq n_0$, n_0 sufficiently large, which can be shown by bounding the norm of f from below by its value at $(\sqrt{3}/2, 1/2)$ or $(1/2, \sqrt{3}/2)$, depending on the relations between k, l and $[n/2]$.

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