The Bishop–Phelps–Bollobás theorem for operators

María D. Acosta a,1, Richard M. Aron b,*,2, Domingo García c,2, Manuel Maestre c,2

a Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain
b Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA
c Departamento de Análisis Matemático, Universidad de Valencia, Valencia, Spain

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Abstract

We prove the Bishop–Phelps–Bollobás theorem for operators from an arbitrary Banach space \( X \) into a Banach space \( Y \) whenever the range space has property \( \beta \) of Lindenstrauss. We also characterize those Banach spaces \( Y \) for which the Bishop–Phelps–Bollobás theorem holds for operators from \( \ell_1 \) into \( Y \). Several examples of classes of such spaces are provided. For instance, the Bishop–Phelps–Bollobás theorem holds when the range space is finite-dimensional, an \( L_1(\mu) \)-space for a \( \sigma \)-finite measure \( \mu \), a \( C(K) \)-space, a uniformly convex Banach space.

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1. Introduction and preliminaries

The celebrated Bishop–Phelps theorem states that the set of norm attaining functionals on a Banach space is norm dense in the dual space. The study of when a theorem of this type holds in the vector valued case has produced a theory with deep and elegant results. Lindenstrauss

* Corresponding author.
E-mail addresses: dacosta@ugr.es (M.D. Acosta), aron@math.kent.edu (R.M. Aron), domingo.garcia@uv.es (D. García), manuel.maestre@uv.es (M. Maestre).

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in [6] proved that for certain Banach spaces $X$ and $Y$, the subset of norm attaining operators from $X$ into $Y$ is not norm dense in the space of all continuous and linear operators $L(X,Y)$. Nevertheless, there are also remarkable situations in which a Bishop–Phelps theorem for operators does hold, such as when the domain space is reflexive [6] or, more generally, when it has the Radon–Nikodým property [4].

Given a Banach space $X$, we denote the unit sphere of $X$ by $S_X$ and the closed unit ball by $B_X$. $X^*$ will be the topological dual of $X$. Bollobás in [3, Theorem 16.1], [2, Theorem 1] proved a “quantitative version” of the Bishop–Phelps theorem [1] (known as the Bishop–Phelps–Bollobás theorem) that can be stated as follows.

Let $\varepsilon > 0$ be arbitrary. If $x \in B_X$ and $x^* \in S_{X^*}$ are such that $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, then there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

Bollobás proved this result in order to be able to apply it to the study of the numerical range of an operator.

For a Banach space $X$, we let $\Pi(X)$ denote the subset of $X \times X^*$ given by $\Pi(X) := \{(x,x^*): x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$. Given a bounded function $\Phi : S_X \to X$, its numerical range is $V(\Phi) := \{x^*(\Phi(x)): (x,x^*) \in \Pi(X)\}$. Of course, the properties of the set $\Pi(X)$ play a crucial role in the study of numerical range. Roughly speaking, what Bollobás proved was that the ordered pairs of $X \times X^*$ that “almost belong” to $\Pi(X)$ can be approximated, in the product norm, by elements of $\Pi(X)$. The numerical range of an operator allows the recovery of some properties of the operator. Thanks, among other things, to the Bishop–Phelps–Bollobás theorem, the theory of numerical range is far richer than one might expect at first glance (see [3, Section 17]). Since this theory studies operators from a Banach space into itself it may be of interest to consider possible extensions of the Bishop–Phelps–Bollobás theorem to operators between two Banach spaces.

Moreover, since it is false in general that for every pair of Banach spaces $X$ and $Y$, the subset of norm attaining operators from $X$ into $Y$ is norm dense in the space $L(X,Y)$, we cannot expect a version of the Bishop–Phelps–Bollobás theorem for operators to hold in full generality. That is why we introduce the following property.

**Definition 1.1.** Let $X$ and $Y$ be real or complex Banach spaces. We say that the couple $(X,Y)$ satisfies the Bishop–Phelps–Bollobás property for operators (or that the Bishop–Phelps–Bollobás theorem holds for all bounded operators from $X$ into $Y$) if given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t \to 0} \beta(t) = 0$ such that for all $T \in SL(X,Y)$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in SL(X,Y)$ that satisfy the following conditions:

$$
\|Su_0\| = 1, \quad \|u_0 - x_0\| < \beta(\varepsilon), \quad \text{and} \quad \|S - T\| < \varepsilon.
$$

Note that an independent concept, the Bishop–Phelps–Bollobás property for a pair $(X,Y)$, has been studied for closed subspaces $X \subset Y$ by M. Martín, J. Merí, and R. Payá in [7] in their work on intrinsic and spatial numerical range.

In the study of the Bishop–Phelps theorem for operators between Banach spaces two kind of questions are usually considered:

1. For which $X$ is it true that for every Banach space $Y$, the norm attaining operators are dense in $L(X,Y)$?
(2) For which \( Y \) is it true that for every Banach space \( X \), the norm attaining operators are dense in \( L(X, Y) \)?

Schachermayer in [9] introduced property \( \alpha \) as a sufficient condition on a Banach space \( X \) to fulfill (1). A sufficient condition for (2) was given by Lindenstrauss [6] introducing property \( \beta \). These two properties generalize in some sense the geometric situations of the classical Banach spaces \( \ell_1 \) and \( c_0 \), respectively. Our aim will be to study whether these properties still work not just for the Bishop–Phelps theorem for operators but for the Bishop–Phelps–Bollobás theorem for operators.

In Section 2, we prove that the pair \( (X, Y) \) has the Bishop–Phelps–Bollobás property for operators for every Banach space \( X \) whenever \( Y \) has property \( \beta \). This implies a general positive result about the Bishop–Phelps–Bollobás theorem for operators between Banach spaces whenever the range space is fixed. Looking at the dual case, we concentrate on \( \ell_1 \), since it is the typical example of space having property \( \alpha \). In Section 4, we characterize when the Bishop–Phelps–Bollobás theorem holds for operators from \( \ell_1 \) into \( Y \). In order to do this, we introduce property AHSP in Section 3 and show that there are many spaces having this property, including finite-dimensional normed spaces, \( L_1(\mu) \) for every \( \sigma \)-finite measure \( \mu \), \( C(K) \) for any compact Hausdorff space \( K \), and every uniformly convex Banach space. A consequence of our study is that property \( \alpha \) of Schachermayer is no longer a sufficient condition for a Banach space \( X \) to satisfy that the pair \( (X, Y) \) has the Bishop–Phelps–Bollobás property for operators for every Banach space \( Y \). In Section 5, we show that a version of the Bishop–Phelps–Bollobás theorem holds when \( X = \ell_1^n \) and \( Y \) is uniformly convex. Finally, following Lindenstrauss’ fundamental paper [6], it seems reasonable to ask if there is a version of the Bishop–Phelps–Bollobás theorem that involves the second duals of \( X \) and \( Y \). In Section 6, we provide an example to show that no such result holds in general.

2. A positive result

In this section we will provide a partial positive result concerning the Bishop–Phelps–Bollobás theorem for operators under an additional assumption. The result will use an isometric condition on the range space \( Y \), called property \( \beta \), that was introduced by Lindenstrauss [6].

**Definition 2.1.** A Banach space \( Y \) is said to have property \( \beta \) (of Lindenstrauss) if there are two sets \( \{y_\alpha: \alpha \in \Lambda\} \subset S_Y \), \( \{y^*_\alpha: \alpha \in \Lambda\} \subset S_{Y^*} \) and \( 0 \leq \rho < 1 \) such that the following conditions hold:

1. \( y^*_\alpha(y_\alpha) = 1 \).
2. \( |y^*_\alpha(y_\beta)| \leq \rho < 1 \) if \( \alpha \neq \beta \).
3. \( \|y\| = \sup_\alpha \{|y^*_\alpha(y)|\} \), for all \( y \in Y \).

Clearly, \( c_0(\Lambda) \) and \( \ell_\infty(\Lambda) \) satisfy the above property for \( \{y_\alpha: \alpha \in \Lambda\} = \{e_\alpha: \alpha \in \Lambda\} \) and \( \{y^*_\alpha: \alpha \in \Lambda\} \) the biorthogonal functionals, and \( \rho = 0 \) in this case.

**Theorem 2.2.** Let \( X \) and \( Y \) be Banach spaces such that \( Y \) has property \( \beta \). Then the pair \( (X, Y) \) has the Bishop–Phelps–Bollobás property for operators. Indeed, if \( T \in S_L(X, Y), \varepsilon > 0 \)
and $x_0 \in S_X$ satisfy $\|T(x_0)\| > 1 - \frac{\varepsilon^2}{4}$, then for each real number $\eta$ such that $\eta > \frac{\rho}{1 - \rho} (\varepsilon + \frac{\varepsilon^2}{4})$, there are $S \in L(X,Y)$, $z_0 \in S_X$ such that:

$$\|Sz_0\| = \|S\|, \quad \|z_0 - x_0\| < \varepsilon, \quad \|S - T\| < \eta + \varepsilon + \frac{\varepsilon^2}{4}.$$ 

**Proof.** Since $Y$ has property $\beta$, there is $\alpha_0 \in \Lambda$ such that $|y^*_{\alpha_0}(T(x_0))| > 1 - \frac{\varepsilon^2}{4}$. By the Bishop–Phelps–Bollobás theorem, there exist $z^*_0 \in S_{X^*}$ and $z_0 \in S_X$ such that $|z^*_0(z_0)| = 1$, $\|z_0 - x_0\| < \varepsilon$ and $\|z^*_0 - \frac{T^t(y^*_0)}{\|T^t(y^*_0)\|}\| < \varepsilon$ (see [3, Theorem 1]). Hence we obtain that

$$\|z^*_0 - T^t(y^*_0)\| \leq \|z^*_0 - \frac{T^t(y^*_0)}{\|T^t(y^*_0)\|}\| + \|\frac{T^t(y^*_0)}{\|T^t(y^*_0)\|} - T^t(y^*_0)\|< \varepsilon + \|T^t(y^*_0)\| - 1 \leq \varepsilon + \frac{\varepsilon^2}{4}.$$ 

For a real number $\eta$ satisfying $\eta > \frac{\rho}{1 - \rho} (\varepsilon + \frac{\varepsilon^2}{4})$, we define the operator $S \in L(X,Y)$ by

$$S(x) = T(x) + \left[ (1 + \eta)z^*_0(x) - T^t(y^*_0)(x) \right]y^*_{\alpha_0} \quad (x \in X).$$

Note that $S$ is a rank one perturbation of $T$, and so $S - T$ is compact. Thus for all $y^* \in Y^*$,

$$S^t(y^*) = T^t(y^*) + y^*(y^*_{\alpha_0})\left[ (1 + \eta)z^*_0 - T^t(y^*_0) \right].$$

Since the set $\{y^*_\alpha: \alpha \in \Lambda\}$ is norming for $Y$ it follows that $\|S\| = \sup_\alpha \|S^t(y^*_\alpha)\|$. Let us estimate the norm of $S$. Clearly,

$$S^t(y^*_0) = (1 + \eta)z^*_0,$$

and thus

$$\|S\| \geq \|S^t(y^*_0)\| = (1 + \eta)\|z^*_0\| = 1 + \eta.$$ 

On the other hand, for $\alpha \neq \alpha_0$, by the choice of $\eta$, we obtain

$$\|S^t(y^*_\alpha)\| \leq 1 + \rho\left( \|z^*_0 - T^t(y^*_\alpha)\| + \eta\|z^*_0\| \right) < 1 + \rho \left( \varepsilon + \frac{\varepsilon^2}{4} + \eta \right) < 1 + \eta.$$ 

Therefore,

$$\|S\| = \|S^t(y^*_0)\| = (1 + \eta)\|z^*_0\| = (1 + \eta)\|z^*_0(z_0)\| = \|y^*_\alpha(Sz_0)\| \leq \|Sz_0\| \leq \|S\|,$$
so $S$ attains its norm at $z_0$, and moreover we have that

$$\|z_0 - x_0\| < \epsilon \quad \text{and} \quad \|S - T\| < \eta + \frac{\epsilon^2}{4}. \quad \square$$

Since property $\beta$ is not restrictive at all from an isomorphic point of view (see [8, Theorem 1]), we deduce the following consequence.

**Corollary 2.3.** For every Banach space $Y$, there is a space $Z$ isomorphic to $Y$ such that the Bishop–Phelps–Bollobás theorem holds for the operators from any other Banach space $X$ to $Z$. In fact, the function that controls the distance between the original operator $T$ and its norm attaining approximation $S$ depends just on $Y$.

Now, we are going to prove that for finite-dimensional spaces, the Bishop–Phelps–Bollobás theorem holds for operators. More precisely, the following result is true.

**Proposition 2.4.** Let $X$ and $Y$ be finite-dimensional Banach spaces. For every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $T \in SL(X, Y)$, there is a linear operator $R \in SL(X, Y)$ such that the following conditions hold:

(i) $\|R - T\| < \epsilon$, and

(ii) for all $x \in SX$ satisfying $\|T(x)\| > 1 - \delta$, there is $\tilde{x} \in SX$ such that $\|R(\tilde{x})\| = 1$ and such that $\|x - \tilde{x}\| < \epsilon$.

In other words, we have a Bishop–Phelps–Bollobás theorem for finite-dimensional spaces $X$ and $Y$ that is uniform in the following sense. Given $X, Y$ and $\epsilon$, there is a $\delta$ such that for any $T: X \to Y$ there is $R: X \to Y$, as above, with $\|T - R\| < \epsilon$ and such that for any unit vector $x$ at which $T$ is within $\delta$ of attaining the norm, there is a unit vector $\tilde{x}$ within $\epsilon$ of $x$ at which $R$ attains its norm. That is, the same $R$ “works” for all such $x$. On the other hand, unlike the classical Bishop–Phelps–Bollobás theorem, the constant $\delta$ depends not only on $\epsilon$ but also on $X$ and $Y$. This is true, even in the case when $Y = \mathbb{R}$ or $\mathbb{C}$.

**Proof.** The proof is by contradiction. If the result is false for some $\epsilon_0$ then for every $n$, we can find $T_n \in SL(X, Y)$ such that for all $R \in SL(X, Y)$ with $\|T_n - R\| \leq \epsilon_0$, there is $x_{n,R} \in SX$ satisfying $\|T_n(x_{n,R})\| > 1 - \frac{1}{n}$ and such that $\text{dist}(x_{n,R}, NA(R)) \geq \epsilon_0$ (where $NA(R) = \{z \in SX: \|R(z)\| = 1\}$). By taking subsequences, we may assume that $(T_n) \to T_0 \in SL(X, Y)$. Putting $x_n := x_{n,T_0}$, we can also assume that $(x_n) \to x_0 \in SX$. Now, $\|T_0(x_0)\| = 1$ although for all large $n$, $\epsilon_0 \leq \text{dist}(x_n, NA(T_0)) \leq \|x_n - x_0\| \to 0$, which is the desired contradiction. \quad \square

**Remark 2.5.** We note that in general it is not true that any $R \in SL(X, Y)$ such that $\|T - R\| < \epsilon$ will automatically satisfy condition (ii) of the above proposition. Indeed let $X = (\mathbb{R}^2, \max), Y = \mathbb{R}$, $T(x, y) = \frac{x + y}{1 + \epsilon}$ and $S(x, y) = \frac{x}{1 + \epsilon} + \frac{y}{1 + \epsilon}$. It is easy to verify that $\|T\| = \|S\| = 1$ and $\|S - T\| < 2\epsilon$. However, the distance between the set of points where $T$ attains the norm and those points where $S$ attains its norm is 2.
3. Property AHSP and examples

In order to give versions of Bishop–Phelps theorem for operators from a fixed Banach space $X$, Schachermayer in [9, Definition 1.2] introduced the isometric property $\alpha$, which has a certain duality relationship with property $\beta$. The most typical example of a space having property $\alpha$ is $\ell_1$. Our aim will be to characterize when the Bishop–Phelps–Bollobás theorem holds for operators from $\ell_1$ into an arbitrary Banach space $Y$. In this section we introduce the awkwardly named property AHSP (standing for approximate hyperplane series property) that we use to get such a characterization, and we show the richness of this property by proving that several classes of spaces enjoy it.

**Definition 3.1.** A Banach space $X$ is said to have property AHSP if for every $\varepsilon > 0$ there exists $0 < \eta < \varepsilon$ such that for every sequence $(x_k) \subset S_X$ and every convex series $\sum_{k=1}^\infty \alpha_k$ with
\[ \left\| \sum_{k=1}^\infty \alpha_k x_k \right\| > 1 - \eta, \]
there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k : k \in A\}$ satisfying

1. $\sum_{k \in A} \alpha_k > 1 - \eta$, and
2. (a) $\|z_k - x_k\| < \varepsilon$ for all $k \in A$,
   (b) $x^*(z_k) = 1$ for a certain $x^* \in S_{X^*}$ and all $k \in A$.

It is immediate that the above property holds if it is satisfied just for finite convex combinations (instead of infinite convex series). In Definition 3.1 we can consider sequences $(x_k)_{k=1}^\infty$ of vectors in the unit ball of $X$. A characterization of property AHSP is the following.

**Remark 3.2.** A Banach space $X$ has AHSP if and only if for every $\varepsilon > 0$ there exist $\gamma(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0^+} \gamma(\varepsilon) = 0$ such that for every sequence $(x_k)_{k=1}^\infty \subset B_X$ and every convex series $\sum_{k=1}^\infty \alpha_k$ satisfying
\[ \left\| \sum_{k=1}^\infty \alpha_k x_k \right\| > 1 - \eta(\varepsilon), \]
there exist a subset $A \subset \mathbb{N}, \{z_k : k \in A\} \subset S_X$ and $x^* \in S_{X^*}$ such that

1. $\sum_{k \in A} \alpha_k > 1 - \gamma(\varepsilon)$,
2. $\|z_k - x_k\| < \varepsilon$ for all $k \in A$, and
3. $x^*(z_k) = 1$ for all $k \in A$.

Geometrically, $X$ has AHSP if whenever we have a convex series of vectors in $B_X$ whose norm is very close to 1, then a preponderance of these vectors are uniformly close to unit vectors that lie in the same hyperplane $(x^*)^{-1}(1)$ where $x^*$ has norm 1.

The following elementary lemma will be very useful to check that some Banach spaces have property AHSP.

Lemma 3.3. Let \( \{c_n\} \) be a sequence of complex numbers with \( |c_n| \leq 1 \) for every \( n \), and let \( \eta > 0 \) be such that for a convex series \( \sum \alpha_n \), \( \text{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta \). Then for every \( 0 < r < 1 \), the set \( A := \{ i \in \mathbb{N} : \text{Re} \ c_i > r \} \), satisfies the estimate

\[
\sum_{i \in A} \alpha_i \geq 1 - \frac{\eta}{1 - r}.
\]

Proof. By the assumption we have that

\[
1 - \eta \leq \text{Re} \sum_{i=1}^{\infty} \alpha_i c_i = \sum_{i=1}^{\infty} \alpha_i \text{Re} \ c_i \leq \sum_{i \in A} \alpha_i + r \sum_{i \notin A} \alpha_i = (1 - r) \sum_{i \in A} \alpha_i + r.
\]

Then we obtain that

\[
\sum_{i \in A} \alpha_i \geq 1 - \frac{\eta - r}{1 - r} = 1 - \frac{\eta}{1 - r}.
\]

Lemma 3.4. Let \( X \) be a finite-dimensional normed space. Then for every \( \varepsilon > 0 \), there is \( \delta > 0 \) such that whenever \( x^* \in S_{X^*} \), there exists \( y^* \in S_{Y^*} \) such that \( \text{dist}(x, D(y^*)) < \varepsilon \) for all \( x \in \{ x \in S_X : \text{Re} \ x^*(x) > 1 - \delta \} \), where \( D(y^*) := \{ y \in B_Y : y^*(y) = 1 \} \).

Proof. We argue by contradiction. Assume that there is some positive real number \( \varepsilon_0 \) not satisfying the above condition. Thus, for every positive \( \delta > 0 \) there exists \( x_n^* \in S_{X^*} \) such that for all \( y^* \in S_{Y^*} \), \( \text{dist}(x, D(y^*)) < \varepsilon \) for all \( x \in \{ x \in S_X : \text{Re} \ x^*(x) > 1 - \delta \} \). Hence, we can find sequences \( (r_n) \to 1 \), \( (x_n^*) \subset S_{X^*} \) such that for all \( y^* \in S_{Y^*} \), \( [x \in S_X : \text{dist}(x, D(y^*)) \geq \varepsilon_0] \neq \emptyset \). By compactness of the unit sphere, we may assume \( (x_n^*) \to x^* \) for some \( x^* \in S_{X^*} \). By the previous condition there is a sequence \( (x_n) \subset S_X \) so that \( r_n < \text{Re} x_n^*(x_n) \leq 1 \) for every \( n \) and such that for all \( n \in \mathbb{N} \),

\[
\text{dist}(x_n, D(x^*)) \geq \varepsilon_0. \tag{3.1}
\]

We may also assume that \( (x_n) \) converges to some \( x \in S_X \). Since \( (x_n^*(x_n)) \to 1 \) and both sequences are convergent, it follows that \( x^*(x) = 1 \); that is \( x \in D(x^*) \). We obtain that \( \text{dist}(x_n, D(x^*)) \leq ||x_n - x|| \) for every \( n \). Since \( (x_n) \) converges to \( x \), this inequality contradicts (3.1). \( \square \)

Lemma 3.4 should be compared with the Bishop–Phelps–Bollobás theorem that is valid in finite-dimensional spaces. Here, the functional \( y^* \) depends only on \( x^* \), whereas in the general case \( y \) and \( y^* \) depend on the choice of \( x \) and \( x^* \). Note that this strengthened version comes at the cost of having \( \delta \) depend not only on \( \varepsilon \) but also on the particular space \( X \). The condition appearing in Lemma 3.4 is a strengthening of property AHSP, as we will check below.

Proposition 3.5. Every finite-dimensional normed space has AHSP.
Proof. If $X$ is a finite-dimensional normed space, then we have just seen that for each $\varepsilon > 0$, there is $\delta > 0$ satisfying the condition in Lemma 3.4. We may assume that $\delta < \varepsilon < 1$. Now assume that

$$\left\| \sum_{k=1}^{\infty} \alpha_k y_k \right\| > 1 - \delta^2$$

for some convex series $\sum \alpha_k y_k$ of elements $\{y_k\}$ in $B_X$. If $\operatorname{Re} x^* (\sum_{k=1}^{\infty} \alpha_k y_k) > 1 - \delta^2$ for some $x^* \in S_X^*$, then the subset

$$G := \{ n \in \mathbb{N} : \operatorname{Re} x^* (y_n) > 1 - \delta \}$$

is such that $\sum_{k \in G} \alpha_k > 1 - \delta$ in view of Lemma 3.3. Hence the above lemma provides an element $y^* \in S_X^*$ and a subset $\{z_k : k \in G\} \subset S_X$ such that $y^*(z_k) = 1$ for all $k \in G$ with $\|y_k - z_k\| < \varepsilon$, as we wanted to show.

Now we will show that some classical Banach spaces have AHSP.

**Proposition 3.6.** For every $\sigma$-finite measure $\mu$, the real or complex space $L_1(\mu)$ has AHSP.

**Proof.** The following proof for the complex case works for real $L_1(\mu)$ as well. Assume that $0 < \varepsilon < 1$ and take

$$s(\varepsilon) := \sqrt{\frac{4 \varepsilon}{4 + \varepsilon^2}}, \quad r(\varepsilon) := \frac{4 + \varepsilon (s(\varepsilon) - 1)}{4}$$

and $\eta(\varepsilon) := \varepsilon (1 - r(\varepsilon))$. (3.2)

Note that $0 < s(\varepsilon) < r(\varepsilon) < 1$ and so $\eta(\varepsilon) > 0$.

Assume that $(f_n)$ is a sequence in $B_{L_1(\mu)}$ such that a certain convex series $\sum_n \alpha_n f_n$ satisfies $\left\| \sum_n \alpha_n f_n \right\| > 1 - \eta(\varepsilon)$. We choose a functional $x^*$ in the unit sphere of the dual of $L_1(\mu)$ such that $\operatorname{Re} x^* (\sum_n \alpha_n f_n) > 1 - \eta(\varepsilon)$. We may assume that $x^*$ is an extreme point of the unit ball of $(L_1(\mu))^*$, and we denote the corresponding function by $h \in L_\infty(\mu)$. Since $x^*$ is an extreme point, we may assume that $|h| = 1$. By using a convenient isometry we may also assume that the function $h \in L_\infty(\mu)$ that represents the functional $x^*$ is the constant function, $h \equiv 1$.

Now we define

$$A := \{ n \in \mathbb{N} : \operatorname{Re} x^* (f_n) > r(\varepsilon) \} = \left\{ n \in \mathbb{N} : \int \Omega \operatorname{Re} f_n d\mu > r(\varepsilon) \right\}.$$

By Lemma 3.3 we know that

$$\sum_{i \in A} \alpha_i > 1 - \frac{\eta(\varepsilon)}{1 - r(\varepsilon)},$$

and so we take $\gamma(\varepsilon) := \frac{\eta(\varepsilon)}{1 - r(\varepsilon)} = \varepsilon$ to see that property (i) of Remark 3.2 holds. Letting $E_n := \{ t \in \Omega : \operatorname{Re} f_n(t) > s(\varepsilon)|f_n(t)| \}$ for each $n \in A$, we clearly have
\[ r(\varepsilon) < \int_{\Omega} \Re f_n d\mu = \int_{E_n} \Re f_n d\mu + \int_{\Omega \setminus E_n} \Re f_n d\mu \]
\[ \leq \int_{E_n} \Re f_n d\mu + \int_{\Omega \setminus E_n} s(\varepsilon) |f_n| d\mu \]
\[ \leq \int_{E_n} \Re f_n d\mu + s(\varepsilon) \left( 1 - \int_{E_n} |f_n| d\mu \right) \]
\[ \leq \int_{E_n} \Re f_n d\mu + s(\varepsilon) \left( 1 - \int_{E_n} \Re f_n d\mu \right) \]
\[ = (1 - s(\varepsilon)) \int_{E_n} \Re f_n d\mu + s(\varepsilon). \]

Then we obtain
\[ \int_{E_n} \Re f_n d\mu > \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)}. \quad (3.3) \]

Hence
\[ \int_{\Omega \setminus E_n} |f_n| d\mu \leq 1 - \int_{E_n} |f_n| d\mu \leq 1 - \int_{E_n} \Re f_n d\mu < 1 - \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)}. \quad (3.4) \]

If \( t \in E_n \) we have \( (\Re f_n(t))^2 > s(\varepsilon)^2((\Re f_n(t))^2 + (\Im f_n(t))^2) \) and so \( s(\varepsilon)|\Im f_n(t)| \leq \sqrt{1 - s(\varepsilon)^2}|\Re f_n(t)| \). Hence we obtain the following upper-estimate:
\[ \int_{E_n} |\Im f_n| d\mu \leq \sqrt{\frac{1 - s(\varepsilon)^2}{s(\varepsilon)^2}}. \quad (3.5) \]

For each \( n \in A \), we define \( g_n \in L_1(\mu) \) by
\[ g_n := \frac{(\Re f_n) \chi_{E_n}}{\|(\Re f_n) \chi_{E_n}\|_1} \quad (n \in A). \]

It is clear that \( \|g_n\|_1 = 1 \) and also \( x^*(g_n) = \int_{\Omega} g_n d\mu = 1 \) for every \( n \in A \). This shows that property (iii) of Remark 3.2 holds.

To complete the proof, we prove property (ii) of Remark 3.2, by finding an upper-estimate of \( \|g_n - f_n\|_1 \). We have
\[ \|g_n - f_n\|_1 \leq \|g_n - f_n \chi_{E_n}\|_1 + \|f_n \chi_{\Omega \setminus E_n}\|_1 \]
\[ \leq \|g_n - (\Re f_n) \chi_{E_n}\|_1 + \|(\Re f_n - f_n) \chi_{E_n}\|_1 + \|f_n \chi_{\Omega \setminus E_n}\|_1 \]
\[ \leq 1 - \| \text{Re} f_n \chi E_n \|_1 + \| (\text{Im} f_n) \chi E_n \|_1 + 1 - \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)} \quad \text{(by (3.4))} \]
\[ \leq 1 - \| \text{Re} f_n \chi E_n \|_1 + \sqrt{\frac{1 - s(\varepsilon)^2}{s(\varepsilon)^2}} + 1 - \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)} \quad \text{(by (3.5))} \]
\[ \leq \sqrt{\frac{1 - s(\varepsilon)^2}{s(\varepsilon)^2}} + 2 \left( 1 - \frac{r(\varepsilon) - s(\varepsilon)}{1 - s(\varepsilon)} \right) \quad \text{(by (3.3) and (3.2))} \]
\[ = \varepsilon. \]

Hence we have proved that \( L_1(\mu) \) has AHSP. \( \square \)

**Proposition 3.7.** The real or complex spaces \( C(K) \) have AHSP for any compact Hausdorff space \( K \).

**Proof.** Once again, the proof will only deal with \( \mathbb{C} \)-valued functions on \( K \) and it is valid in both cases. Fix \( 0 < \varepsilon < 1 \), and let \( (f_k)_{k=1}^\infty \subset B_{C(K)} \) and a convex series \( (\alpha_k)_{k=1}^\infty \) satisfy
\[ \left\| \sum_{k=1}^\infty \alpha_k f_k \right\| > 1 - \left( \frac{\varepsilon}{4} \right)^4. \]

Consider a point \( t_0 \in K \) and a scalar \( \lambda, |\lambda| = 1 \), satisfying
\[ 1 \geq \text{Re} \left( \lambda \sum_{k=1}^\infty \alpha_k f_k(t_0) \right) > 1 - \left( \frac{\varepsilon}{4} \right)^4. \]

We take \( A := \{ k \in \mathbb{N} : \text{Re}(\lambda f_k(t_0)) > 1 - (\frac{\varepsilon}{4})^2 \} \) and \( \delta = \varepsilon^2/4^2 \). Then
\[ 1 - \left( \frac{\varepsilon}{4} \right)^4 < \text{Re} \left( \lambda \sum_{k=1}^\infty \alpha_k f_k(t_0) \right) = \sum_{k=1}^\infty \alpha_k \text{Re}(\lambda f_k(t_0)) \]
\[ \leq \sum_{k \in A} \alpha_k + \left( 1 - \left( \frac{\varepsilon}{4} \right)^2 \right) \sum_{k \notin A} \alpha_k. \]

As \( \sum_{k=1}^\infty \alpha_k = 1 \), we obtain \( 1 - (\frac{\varepsilon}{4})^4 < 1 - (\frac{\varepsilon}{4})^2 \sum_{k \notin A} \alpha_k \). Hence \( \sum_{k \notin A} \alpha_k > (\frac{\varepsilon}{4})^2 \) and so
\( \sum_{k \in A} \alpha_k > 1 - (\frac{\varepsilon}{4})^2 \), which means that condition (i) in Remark 3.2 is satisfied.

For each \( k \in A \), we choose a function \( u_k \in C(K) \) such that
\[ \text{supp } u_k \subset |f_k|^{-1}((1 - \delta, 1]], \quad 0 < u_k < 1, \quad u_k(t_0) = 1. \]

If we define \( g_k \) on \( K \) by \( g_k := \lambda(f_k + u_k(-f_k + \frac{f_k}{|f_k|})) \) on \text{supp } u_k \) and \( g_k = \lambda f_k \) on \( K \setminus \text{supp } u_k \), then \( g_k \) is continuous on \( K \). Also, \( g_k \) is in the unit sphere of \( C(K) \) since \( f_k \) is in the unit ball,
\[ |f_k + u_k(-f_k + \frac{f_k}{|f_k|})| \leq |f_k| + |1 - |f_k|| = 1, \]
and $|g_k(t_0)| = 1$. In addition, $\|g_k - \lambda f_k\| < \delta$ since this function is zero outside the set $\text{supp} u_k$ and for $t \in \text{supp} u_k$ we know that

$$|g_k(t) - \lambda f_k(t)| \leq |1 - |f_k(t)|| < \delta.$$  \hfill (3.6)

Writing $a := 2(\frac{\varepsilon}{4})^2$, we see that

$$\text{Re} g_k(t_0) > \text{Re} \lambda f_k(t_0) - \delta > 1 - \left(\frac{\varepsilon}{4}\right)^2 - \delta = 1 - a,$$

and so

$$|\text{Im} g_k(t_0)| < \sqrt{2a}.$$

Hence

$$|g_k(t_0) - 1| < \sqrt{a^2 + 2a}. \hfill (3.7)$$

Now for every $k \in A$ we set $h_k := \mu_k \bar{\lambda} g_k$, where $\mu_k = \overline{g_k(t_0)}$, so that $h_k \in S_{C(K)}$. The element $x^* = \lambda \delta_0$ is an element of $S_{C(K)^*}$ and satisfies $x^*(h_k) = 1$ for all $k \in A$. This shows that (iii) in Remark 3.2 holds. Finally, (ii) in Remark 3.2 holds as well. Indeed, in view of (3.7), (3.6) and the choice of $\delta$, for every $k \in A$,

$$\|h_k - f_k\| = \|\mu_k \bar{\lambda} g_k - f_k\| = \|\mu_k g_k - \lambda f_k\|$$

$$\leq \|\mu_k g_k - g_k\| + \|g_k - \lambda f_k\|$$

$$\leq |\mu_k - 1| + \|g_k - \lambda f_k\|$$

$$\leq \sqrt{a^2 + 2a} + \delta = \sqrt{\frac{\varepsilon^4}{4^3} + \frac{\varepsilon^2}{4} + \left(\frac{\varepsilon}{4}\right)^2} < \varepsilon.$$  \hfill \Box

In the above proof we need only that every point $t_0$ in $K$ has a basis of compact neighborhoods. Hence the same argument shows that $C_0(\Omega)$, the Banach space of continuous functions on $\Omega$ that vanish at $\infty$, also has AHSP for any locally compact space $\Omega$.

We now show that many spaces that are completely different from $C(K)$ and $L_1(\mu)$ also have AHSP. To do so, we recall that a Banach space $X$ is uniformly convex if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

for all $u, v \in B_X$ such that $\frac{\|u + v\|}{2} > 1 - \delta$, we have $\|u - v\| < \varepsilon.$

In such a case, the modulus of convexity is given by

$$\delta(\varepsilon) := \inf \left\{1 - \frac{\|u + v\|}{2} : u, v \in B_X, \|u - v\| \geq \varepsilon \right\}.$$

**Proposition 3.8.** A uniformly convex Banach space has AHSP.
Proof. Let $X$ be a uniformly convex Banach space, let $\varepsilon > 0$ be arbitrary, and let $\delta = \delta(\varepsilon)$ be as in the definition of uniformly convex.

Let us fix $0 < \varepsilon < 1$ and take $r(\varepsilon) = 1 - \delta(\varepsilon)$, $\eta(\varepsilon) = \frac{\varepsilon(1-r(\varepsilon))}{2}$ and $\gamma(\varepsilon) = \frac{\varepsilon}{2}$. Assume that $\{x_n: n \in \mathbb{N}\} \subset B_X$ is a subset such that for some convex series $\sum_{n=1}^{\infty} \alpha_n x_n$, $\|\sum_{n=1}^{\infty} \alpha_n x_n\| > 1 - \eta(\varepsilon)$. We choose a functional $x^* \in S_{X^*}$ such that $\text{Re} x^*(\sum_{n=1}^{\infty} \alpha_n x_n) > 1 - \eta(\varepsilon)$ and let $A = \{n \in \mathbb{N}: \text{Re} x^*(x_n) > r(\varepsilon)\}$. By Lemma 3.3 we know that $\sum_{n=1}^{\infty} \alpha_n > 1 - \frac{\eta(\varepsilon)}{r(\varepsilon)} = 1 - \frac{\varepsilon}{2}$. For $n, m \in A$ we have that $\|x_n + x_m\| \geq |x^*(x_n + x_m)| > 2r(\varepsilon) = 2 - 2\delta(\varepsilon)$ and, by using the uniform convexity of $X$, we obtain $\|x_n - x_m\| < \varepsilon$. Since $A \neq \emptyset$, we can choose $n_0 \in A$ and define $z_n = x_{n_0}$ for every $n \in A$. Hence we have that

$$\|z_n - x_n\| < \varepsilon, \quad \text{for all } n \in A, \quad \text{and} \quad \sum_{n \in A} \alpha_n > 1 - \frac{\varepsilon}{2} = 1 - \gamma(\varepsilon).$$

Finally, if we choose a functional $x^* \in S_{X^*}$ such that $x^*(x_{n_0}) = 1$, we see that the three requirements for property AHSP have been met. \qed

The following proposition shows that every strictly convex Banach space which is not uniformly convex fails AHSP. In particular, the reflexive space $X = \bigoplus_2 \ell^n_\infty$ does not satisfy AHSP (see [5, Theorems 9.18, 9.14 and 8.17]).

Proposition 3.9. A strictly convex Banach space having AHSP is uniformly convex.

Proof. Recall that a Banach space $Z$ is said to be strictly convex if every point of its unit sphere is an extreme point of the unit ball. Assume that the Banach space $X$ has AHSP. By assumption, for $\varepsilon > 0$ small enough such that $\gamma(\varepsilon) < 1/2$, we have the following. Whenever $x, y \in B_X$, $\|x + y\| > 2 - 2\eta(\varepsilon)$, then there exist $u, v \in S_X$ such that $\|u - x\| < \varepsilon$, $\|v - y\| < \varepsilon$, and $\|u + v\| = 2$. If we use the strict convexity of $X$, it follows that $u = v$ so $\|x - y\| < 2\varepsilon$. It follows that $X$ is uniformly convex. \qed

4. Operators from $\ell_1$ into a Banach space

In this section we are going to characterize those Banach spaces $Y$ having the property that the Bishop–Phelps–Bollobás theorem holds for operators from $\ell_1$ into $Y$. To do so, we will use the property AHSP that was introduced in the previous section.

Theorem 4.1. A Banach space $Y$ is such that the couple $(\ell_1, Y)$ has the Bishop–Phelps–Bollobás property for operators if, and only if, $Y$ satisfies AHSP.

Proof. Our proof will be given for the case of complex Banach spaces. (In fact, the case of real Banach spaces is simpler and gives a better order of approximation.)

Let $Y$ be a Banach space with AHSP. Given $\varepsilon > 0$, we will use the functions $\gamma(\varepsilon)$ and $\eta(\varepsilon)$ satisfying the conditions of Remark 3.2. We can assume that $\varepsilon$ is small enough such that $0 < \gamma(\varepsilon) < 1$. Given $T \in S_{L(\ell_1, Y)}$, we take $x_0 = (x_0(n))_{n=1}^{\infty} \in S_{\ell_1}$, such that $\|T x_0\| > 1 - \eta(\varepsilon)$. By composing with an isometry, we may assume that $x_0(n) = \text{Re} x_0(n) \geq 0$ for every positive integer $n$.

By the assumptions on $T$ and $x_0$, we can apply AHSP to the convex series $\sum x_0(n)$ and for the elements $x_n = T(e_n)$, $n \in \mathbb{N}$, where $(e_n)$ is the canonical basis of $\ell_1$. 

Hence, there is a subset $A \subset \mathbb{N}$ and $\{y_n: n \in A\} \subset Y$ such that
\[
\sum_{n \in A} x_0(n) > 1 - \gamma(\varepsilon), \quad \|y_n - x_n\| < \varepsilon, \quad \text{for all } n \in A,
\]  
(4.8)
and
\[
\left\| \sum_{n \in A} x_0(n) y_n \right\| = \sum_{n \in A} x_0(n). \tag{4.9}
\]

There is a linear bounded operator $S$ of norm 1 from $\ell_1$ to $Y$ such that
\[
S(e_n) = \begin{cases} 
    y_n & \text{if } n \in A, \\
    T(e_n) & \text{if } n \notin A.
\end{cases}
\]

In view of (4.8) we obtain that
\[
\|T - S\| = \sup_n \|S(e_n) - T(e_n)\| = \sup_{n \in A} \|y_n - x_n\| \leq \varepsilon.
\]

Since $\gamma(\varepsilon) < 1$, in view of (4.8), then $A \neq \emptyset$. If $P_A(x_0) = \sum_{n \in A} x_0(n)e_n$, then the element $z_0 = \frac{P_A(x_0)}{\|P_A(x_0)\|} \in S_{\ell_1}$ is such that
\[
\|x_0 - z_0\| \leq \|x_0 - P_Ax_0\| + \left| 1 - \|P_Ax_0\| \right| < 2\gamma(\varepsilon).
\]

Also, by using (4.9), we know that
\[
\|Sz_0\| = \frac{\left| \sum_{n \in A} x_0(n)y_n \right|}{\|P_Ax_0\|} = \frac{\left| \sum_{n \in A} x_0(n)y_n \right|}{\sum_{n \in A} x_0(n)} = 1.
\]

Hence, by taking $\beta(\varepsilon) = 2\gamma(\varepsilon)$, we obtain that $(\ell_1, Y)$ satisfies the Bishop–Phelps–Bollobás property for operators.

Conversely, assume that $Y$ is a complex Banach space such that $(\ell_1, Y)$ satisfies the Bishop–Phelps–Bollobás property for operators. Given $0 < \rho < 1$, we choose $s$ such that $0 < s < 1$ and $0 < \sqrt{2(1 - s)} < \frac{\rho}{2}$.

Let $\eta(\varepsilon)$ and $\beta(\varepsilon)$ be the positive numbers that appear in the definition of the Bishop–Phelps–Bollobás property for operators. Choose $\varepsilon = \varepsilon(\rho)$ such that $0 < \varepsilon < \frac{\rho}{2} < 1$ and $\frac{\beta(\varepsilon)}{1 - s} < \frac{\rho}{2}$. 
Let \((y_n) \subset S_Y\) be a sequence and let \(\sum \alpha_n\) be a convex series such that
\[
\left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| > 1 - \eta(\varepsilon).
\]

There is a bounded linear operator \(T : \ell_1 \to Y\) such that \(T(e_n) = y_n\) for all \(n\). We have \(\|T\| = 1\) and the element \(x_0 = \sum_{n=1}^{\infty} \alpha_n e_n \in S_{\ell_1}\) satisfies that
\[
\left\| T(x_0) \right\| = \left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| > 1 - \eta(\varepsilon). \tag{4.10}
\]

We apply the assumption that \((\ell_1, Y)\) satisfies the Bishop–Phelps–Bollobás property to obtain a norm one operator \(S \in L(\ell_1, Y)\) and an element \(u_0 \in S_{\ell_1}\) such that
\[
\| Su_0 \| = 1, \quad \| u_0 - x_0 \| < \beta(\varepsilon), \quad \| S - T \| < \varepsilon.
\]

It then follows that
\[
\sum_{n=1}^{\infty} (\alpha_n - \text{Re} u_0(n)) \leq \sum_{n=1}^{\infty} |u_0(n) - \alpha_n| = \| u_0 - x_0 \| < \beta(\varepsilon), \tag{4.11}
\]
and so
\[
\sum_{n=1}^{\infty} \text{Re} u_0(n) > 1 - \beta(\varepsilon). \tag{4.12}
\]

Let us consider the set
\[
A := \{ n \in \mathbb{N} : \text{Re} u_0(n) > s \left| u_0(n) \right| \}.
\]

By using (4.12) we obtain that
\[
1 - \beta(\varepsilon) < \sum_{n=1}^{\infty} \text{Re} u_0(n)
\]
\[
= \sum_{n \in A} \text{Re} u_0(n) + \sum_{n \notin A} \text{Re} u_0(n)
\]
\[
\leq \sum_{n \in A} \text{Re} u_0(n) + s \sum_{n \notin A} |u_0(n)|
\]
\[
= \sum_{n \in A} \text{Re} u_0(n) + s \left( 1 - \sum_{n \in A} |u_0(n)| \right)
\]
\[
\leq \sum_{n \in A} \text{Re} u_0(n) + s \left( 1 - \sum_{n \in A} \text{Re} u_0(n) \right).
\]
So
\[ \sum_{n \in A} \text{Re} u_0(n) > 1 - \frac{\beta(\varepsilon)}{1 - s}. \] (4.13)

Hence
\[
\sum_{n \in A} \alpha_n \geq \sum_{n \in A} \text{Re} u_0(n) - \|u_0 - x_0\|
\geq 1 - \frac{\beta(\varepsilon)}{1 - s} - \|u_0 - x_0\|
> 1 - \frac{\beta(\varepsilon)}{1 - s} - \beta(\varepsilon).
\] (4.14)

We take \( \gamma(\rho) := \beta(\varepsilon) + \frac{\beta(\varepsilon)}{1 - s} < \rho \) and so \( \lim_{t \to 0} \gamma(t) = 0 \). Now, if \( z \in \mathbb{C} \) satisfies \( |z| = 1 \) and \( \text{Re} z > t > 0 \), then we know that
\[ |1 - z|^2 = 1 + |z|^2 - 2 \text{Re} z < 2(1 - t). \]

Thus, for \( n \in A \), by the choice of \( s \), it follows that
\[ \left| 1 - \frac{u_0(n)}{|u_0(n)|} \right|^2 < 2(1 - s) < \frac{\rho^2}{4}. \] (4.15)

If we write \( z_n := S(e_n) \), then
\[ 1 = \|S(u_0)\| = \left\| \sum_{n=1}^{\infty} u_0(n)z_n \right\|. \]

Hence, there is an element \( y^* \in S_{Y^*} \) such that
\[ u_0(n)y^*(z_n) = |u_0(n)| \] (4.16)

for all \( n \in \mathbb{N} \). Thus, for all \( n \in A \), \( z_n \) belongs to \( SY \). Also we know that for \( n \in A \) we have
\[ ||z_n - y_n|| = \|S(e_n) - T(e_n)\| < \varepsilon < \frac{\rho}{2}, \]
and so
\[
\left\| \frac{u_0(n)}{|u_0(n)|}z_n - y_n \right\| \leq \left| \frac{u_0(n)}{|u_0(n)|} - 1 \right| + ||z_n - y_n||
< \frac{\rho}{2} + \frac{\rho}{2} = \rho \quad \text{(by (4.15))}.
\]
In view of (4.14), the previous inequality and (4.16), we have checked that
\[ \sum_{n \in A} \alpha_n > 1 - \gamma(\rho), \quad \left\| \frac{u_0(n)}{|u_0(n)|} \epsilon_n - y_n \right\| < \rho, \quad \text{and} \quad y^* \left( \frac{u_0(n)}{|u_0(n)|} \epsilon_n \right) = 1, \quad \text{for all} \ n \in A, \]
and so \( Y \) satisfies AHSP. \( \square \)

**Remark 4.2.** Using Theorem 2.2, we see that the couple \((\ell_1, Y)\) has the Bishop–Phelps–Bollobás property for operators whenever \( Y \) has property \( \beta \). Hence, property \( \beta \) implies AHSP in view of Theorem 4.1. However, it is false that Theorem 4.1 can be deduced from Theorem 2.2, as can be seen by taking pairs \((\ell_1, Y)\) for \( Y = C(K) \) for non-scattered \( K \), \( Y = L_1(\mu) \), or \( Y \) is a uniformly convex space. On the other hand, taking \( X = \ell_1 \) and \( Y \) a space failing AHSP, we see that \( X \) having property \( \alpha \) of Schachermayer is not sufficient to conclude that \((X, Y)\) satisfies the Bishop–Phelps–Bollobás property for operators.

5. **Bishop–Phelps–Bollobás theorem for operators from \( \ell_n^\infty \) into a uniformly convex Banach space**

Our aim in this section is to show that for every \( n \in \mathbb{N} \) and for every uniformly convex space \( Y \), the pair \((\ell_n^\infty, Y)\) satisfies the Bishop–Phelps–Bollobás property for operators.

Let us begin with the following result for operators from \( c_0 \) into a uniformly convex Banach space. In order to state it, let us recall that for \( A \) a Banach space, on the other hand, taking \( X = \ell_1 \) and \( Y \) a space failing AHSP, we see that \( X \) having property \( \alpha \) of Schachermayer is not sufficient to conclude that \((X, Y)\) satisfies the Bishop–Phelps–Bollobás property for operators.

**Lemma 5.1.** Let \( Y \) be a uniformly convex Banach space with modulus of convexity \( \delta(\varepsilon) \). Let \( \varepsilon > 0 \). If \( T \in S_{L(c_0, Y)} \), and \( A \subset \mathbb{N} \) has the property that \( \|TP_A\| > 1 - \delta(\varepsilon) \), then we have that \( \|T(I-P_A)\| \leq \varepsilon \).

**Proof.** Since \( Y \) is uniformly convex, for each \( \varepsilon > 0 \), there is \( 0 < \delta(\varepsilon) < 1 \) such that whenever \( u \) and \( v \) are in \( B_X \), satisfying \( \|u + v\| \geq 2 - 2\delta(\varepsilon) \), it follows that \( \|u - v\| < \varepsilon \). Assume that \( T \in L(c_0, Y) \) satisfies \( \|T\| = 1 \) and let \( A \subset \mathbb{N} \) with \( \|TP_A\| > 1 - \delta(\varepsilon) \). Choose \( x_0 \in P_A(c_0) \cap S_{c_0} \) such that \( \|TP_A(x_0)\| > 1 - \delta(\varepsilon) \).

Since \( 1 = \|T\| \geq \|T(x_0 \pm z)\| \) for every element \( z \in B_{c_0} \) whose support lies outside \( A \), we obtain that \( \|T(x_0) \pm T(I-P_A)(y)\| \leq 1 \) for any \( y \in B_{c_0} \). Also, we have that
\[ \|T(x_0 + (I-P_A)(y)) + T(x_0 - (I-P_A)(y))\| = \|2T(x_0)\| = \|2TP_A(x_0)\| \geq 2 - 2\delta(\varepsilon). \]

Thus, by using the uniform convexity of \( Y \) we obtain that
\[ \|2T(I-P_A)(y)\| = \|(T(x_0 + T(I-P_A)(y)) - (T(x_0 - T(I-P_A)(y))\| < 2\varepsilon. \]

Since \( y \) is an arbitrary element of the unit ball of \( c_0 \), we finally get that \( \|T(I-P_A)\| \leq \varepsilon. \) \( \square \)

Now, we are ready to prove the promised result that \((\ell_n^\infty, Y)\) satisfies the Bishop–Phelps–Bollobás property for operators for every \( n \) whenever \( Y \) is a uniformly convex Banach space. Unfortunately, our method involves constants that depend on \( n \), and we do not know whether the result can be extended to, say, \((c_0, Y)\) or \((\ell_\infty, Y)\) if \( Y \) is uniformly convex.
Theorem 5.2. Let $Y$ be a uniformly convex Banach space with modulus of convexity $\delta(\varepsilon)$. Let $n \in \mathbb{N}$, $0 < \varepsilon < 1$, $0 < \varepsilon' < \varepsilon$ with $\varepsilon' + \frac{\varepsilon'}{1 + \varepsilon'} < \min\{\delta(\varepsilon), \frac{3}{2}(\varepsilon + \varepsilon^2/3)\}$. For any $x_0 \in B_{\ell^2_n}$ and $T \in S_{L(\ell^2_n, Y)}$ such that $\|Tx_0\| > 1 - \varepsilon'$, there exist $z_0 \in B_{\ell^2_n}$ and $V \in S_{L(\ell^2_n, Y)}$ such that

$$\|Vz_0\| = 1, \quad \|z_0 - x_0\| < \varepsilon^{1/4} + \varepsilon^{1/3}, \quad \|V - T\| \leq \varepsilon + 6n(\sqrt{\varepsilon} + \varepsilon^{1/6}) + \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}}\right).$$

Proof. Let $T \in L(\ell^2_n, Y)$ be a norm one operator and $x_0 \in B_{\ell^2_n}$ satisfying $\|Tx_0\| > 1 - \varepsilon'$. By composing with an isometry on $\ell^2_n$ if necessary, we may assume that $x_0(i) > 0$ for each $i \leq n$. Let $y^* \in S_Y$ be such that $y^*T(x_0) = \text{Re}(T^iy^*)(x_0) > 1 - \varepsilon'$.

Define

$$E := \{i \leq n: \text{Re}(T^iy^*)(e_i)x_0(i) > (1 - \varepsilon^{1/3})|(T^iy^*)(e_i)|\}$$

$$\subset \{i \leq n: \text{Re}(T^iy^*)(e_i) > 0, x_0(i) > 1 - \varepsilon^{1/3}\}.$$ 

Since $T^iy^* \in (\ell^2_n)^* \equiv \ell^1_n$ and $\|T^iy^*\| \leq \|y^*\| = 1$,

$$\sum_{k=1}^n|(T^iy^*)(e_k)| \leq 1.$$ 

If $A := \sum_{i \notin E}|(T^iy^*)(e_i)|$, we will check that $A < \frac{\varepsilon'}{\varepsilon^{1/3}}$. Indeed,

$$1 - \varepsilon' < \text{Re}(T^iy^*)(x_0) = \sum_{i=1}^n\text{Re}(T^iy^*)(e_i)x_0(i)$$

$$\leq \sum_{i \in E}|(T^iy^*)(e_i)x_0(i)| + \sum_{i \notin E}\text{Re}(T^iy^*)(e_i)x_0(i)$$

$$\leq \sum_{i \in E}|(T^iy^*)(e_i)| + (1 - \varepsilon^{1/3})\sum_{i \notin E}|(T^iy^*)(e_i)|$$

$$\leq 1 - A + (1 - \varepsilon^{1/3})A = 1 - \varepsilon^{1/3}A,$$

so

$$A < \frac{\varepsilon'}{\varepsilon^{1/3}}.$$ \hspace{1cm} (5.17)

Thus,

$$1 - \varepsilon' < \text{Re}(T^iy^*)(x_0) \leq \sum_{i \in E}\text{Re}(T^iy^*)(e_i)x_0(i) + \sum_{i \notin E}|(T^iy^*)(e_i)x_0(i)|$$

$$\leq \sum_{i \in E}\text{Re}(T^iy^*)(e_i) + \sum_{i \notin E}|(T^iy^*)(e_i)| < \sum_{i \in E}\text{Re}(T^iy^*)(e_i) + \frac{\varepsilon'}{\varepsilon^{1/3}}.$$
and so

$$\sum_{i \in E} \text{Re}(T^i y^*)(e_i) > 1 - \varepsilon' - \frac{\varepsilon'}{\varepsilon^{1/3}}.$$ 

By the choice of $\varepsilon'$ we have

$$\|TP_E\| \geq \|TP_E \left( \sum_{i \in E} e_i \right) \| \geq \left| (T^i y^*)(\sum_{i \in E} e_i) \right|$$

$$> 1 - \left( \varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}} \right) > 1 - \delta(\varepsilon). \quad (5.18)$$

By Lemma 5.1 we obtain that

$$\|T(I - P_E)\| \leq \varepsilon. \quad (5.19)$$

Setting $e_0 = \sum_{i \in E} e_i$ in $B_{PE}(\ell_\infty^n)$ and

$$x_0^* = \sum_{i \in E} \frac{1}{|E|} e_i^*$$

in $(\ell^n_\infty)^*$, by the definition of $E$, we have that

$$\|P_E(x_0) - e_0\| < \varepsilon^{1/3} \quad \text{and} \quad x_0^*(e_0) = 1. \quad (5.20)$$

Define the operator $S: \ell^n_\infty \to Y$ by

$$S(x) := TP_E(x) + 3n\left(\sqrt{\varepsilon + \varepsilon^{1/6}}\right)x_0^*(P_E(x)) \frac{T(e_0)}{\|T(e_0)\|} \quad (x \in \ell^n_\infty).$$

Let $\tau = \frac{\sqrt{\varepsilon}}{2|E|}$. We claim that $\|e - e_0\| < \varepsilon^{1/4}$ for all $e \in \text{Ext}(B_{PE}(\ell_\infty^n))$ satisfying $|x_0^*(e)| < \tau$. Indeed, if $|x_0^*(e) - 1| < \tau$ then $|\sum_{i \in E} e(i) - |E|| < \tau |E|$, and so $\text{Re}(1 - e(i)) < \tau |E|$ for all $i \in E$. Hence $|e(i) - 1| = \sqrt{2 - 2 \text{Re}(e(i))} < \sqrt{2\tau |E|} = \varepsilon^{1/4}$ for all $i \in E$, and the claim follows. By (5.18) we obtain

$$\|S(e_0)\| = \|TP_E(e_0)\| + 3n\left(\sqrt{\varepsilon + \varepsilon^{1/6}}\right) \geq 1 - \left( \varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}} \right) + 3n\left(\sqrt{\varepsilon + \varepsilon^{1/6}}\right) \quad (5.21)$$

and we also know that

$$\|S(e)\| \leq 1 + 3n\left(\sqrt{\varepsilon + \varepsilon^{1/6}}\right)(1 - \tau), \quad (5.22)$$

for all $e \in \text{Ext}(B_{PE}(\ell_\infty^n))$ such that $|x_0^*(e)| < 1 - \tau$. By the choice of $\varepsilon'$, the upper bound in (5.22) is less than the lower bound in (5.21), so the operator $S = S \circ P_E$ attains its norm at some point $e$ in $\text{Ext}(B_{PE}(\ell_\infty^n))$ with $1 - |x_0^*(e)| < \tau$. So, by the claim above, $S$ attains its norm at $\lambda e$.
for some number $\lambda$ of modulus one such that $\|\lambda e - e_0\| < \varepsilon^{1/4}$. Hence $S$ also attains its norm at $z_0 = \lambda e + (I - P_E)(x_0)$ and by (5.20) we have

$$\|z_0 - x_0\| = \|\lambda e - P_E(x_0)\| \leq \|\lambda e - e_0\| + \|e_0 - P_E(x_0)\| < \varepsilon^{1/4} + \varepsilon^{1/3}.$$  

From the definition of $S$ and by (5.21) and (5.22) we have

$$1 - \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}}\right) + 3n\left(\sqrt[3]{\varepsilon} + \varepsilon^{1/6}\right) \leq \|S\| \leq 1 + 3n\left(\sqrt[3]{\varepsilon} + \varepsilon^{1/6}\right)$$

and putting $V := \frac{S}{\|S\|}$ it follows that

$$\|T - V\| 
\leq \|T - S\| + \|S - V\| 
\leq \|T(I - P_E)\| + \|TP_E - S\| + \frac{S - S}{\|S\|} 
\leq \|T(I - P_E)\| + 3n\left(\sqrt[3]{\varepsilon} + \varepsilon^{1/6}\right) + \|S\| - 1 
\leq \varepsilon + 3n\left(\sqrt[3]{\varepsilon} + \varepsilon^{1/6}\right) + \|S\| - 1 \quad \text{(by (5.19))} 
\leq \varepsilon + 3n\left(\sqrt[3]{\varepsilon} + \varepsilon^{1/6}\right) + \max\left\{\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}}, 3n\left(\sqrt[3]{\varepsilon} + \varepsilon^{1/6}\right)\right\} 
\leq \varepsilon + 6n\left(\sqrt[3]{\varepsilon} + \varepsilon^{1/6}\right) + \left(\varepsilon' + \frac{\varepsilon'}{\varepsilon^{1/3}}\right),$$

and the proof is complete. \(\square\)

6. Final remark

In his work on a general vector valued result of Bishop–Phelps type, Lindenstrauss [6, Theorem 1] proved the denseness of the subset of operators between Banach spaces whose second adjoints attain their norms. Thus, instead of asking whether or not every pair of Banach spaces $(X, Y)$ has the Bishop–Phelps–Bollobás property for operators, one could begin by asking the following question:

*Is there a function $\gamma : \mathbb{R}^+ \to (0, 1)$, $\lim_{t \to 0} \gamma(t) = 0$, such that the following holds; for all $T \in S_{L(X,Y)}$ and $x_0 \in S_X$ with $\|Tx_0\| > 1 - \gamma(\varepsilon)$, there exist $S \in S_{L(X,Y)}$ and $x_0^{**} \in S_{X^{**}}$ satisfying,*

$$\|Sx_0^{**}\| = 1, \quad \|S - T\| < \varepsilon, \quad \|x_0^{**} - x_0\| < \varepsilon?$$

Unfortunately, even this question has a negative answer in general. We will use the original idea of Lindenstrauss to show that.

**Lemma 6.1.** Let $Y$ be a strictly convex Banach space.

(a) Let $T : \ell_\infty \to Y$ be an operator such that $T(e_n) \neq 0$ for all $n$. If $T$ attains its norm at a point $z \in B_{\ell_\infty}$, then $|z(n)| = 1$, for all $n \in \mathbb{N}$.

(b) If $T : c_0 \to Y$ is an operator attaining its norm, then $T$ is a finite rank operator.
Proof. (a) Suppose that there exists a point \( z \in S_{\ell_\infty} \) at which \( T \) attains its norm. If we assume that there exists \( n \) so that \( |z(n)| < 1 \), then \( \|z \pm (1 - |z(n)|)e_n\| \leq 1 \) and so, by convexity,

\[
\|T\| = \|T(z)\| = \|T(z \pm (1 - |z(n)|)e_n)\|.
\]

Since \( Y \) is strictly convex we get that \( T(e_n) = 0 \). This is a contradiction.

(b) Let \( z \in S_{c_0} \) be such that \( T \) attains its norm at \( z \). Since there exists an \( n_0 \) with \( |z(n)| < 1 \) for all \( n \geq n_0 \), the above argument implies that \( T(e_n) = 0 \) for all \( n \geq n_0 \). \( \square \)

The argument of the proof of part (b) of the above lemma actually shows that if for some operator \( T : c_0 \to Y \) the Bishop–Phelps–Bollobás theorem holds, then \( T \) can be approximated by finite-rank operators and so it is compact.

By taking second adjoints we obtain the following proposition.

**Proposition 6.2.** Let \( T_0 : c_0 \to Y \) be an isomorphism. Assume that \( Y^{**} \) is strictly convex and \( T \in L(c_0, Y) \) is such that

\[
\|T - T_0\| < \inf_n \{\|T_0(e_n)\|\}.
\]

Then

\[
\{y \in B_{\ell_\infty} : \|T^u(y)\| = \|T\|\} \subset \{y \in B_{\ell_\infty} : |y(n)| = 1, \text{ for all } n \in \mathbb{N}\}.
\]

**Example 6.3.** Applying the above proposition to \( X = c_0, Y \) any Banach space isomorphic to \( c_0 \) such that \( Y^{**} \) is strictly convex and \( T_0 = I \), the identity mapping, gives a negative answer to the above question. Indeed, given \( T \in L(c_0, Y) \) such that \( \|T - I\| < \inf_n \{\|e_n\|_Y\} \), then \( |z(n)| = 1 \), for all \( n \in \mathbb{N} \) and all \( z \in B_{\ell_\infty} \) with \( \|T^u(z)\| = \|T\| \). So \( \text{dist}(z, B_{c_0}) = 1 \).

**References**