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# DYNAMIC PROGRAMMING AND THE SECRETARY PROBLEM

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**Abstract**—In the secretary problem one seeks to maximize the probability of hiring the best of $N$ candidates who are interviewed in succession and must be accepted or rejected at the interview. A simple dynamic program is formulated and solved. Numerical results are given for secretary problems of small size.

## 1. INTRODUCTION

In the secretary problem a strategy is sought for maximizing the probability of choosing the best among $N$ available candidates. They are interviewed in succession and the decision to hire must be made at the end of an interview before the remaining candidates have been seen. This problem has attracted more attention than is perhaps warranted by its immediate applications, since it involves some clever probabilistic reasoning [1, 2]. For a survey, cf. Ref. [3].

The optimal strategy is known to be this: Let a certain number of candidates pass and after that accept the first who is the best so far. To demonstrate that this is the optimal strategy and to calculate the number of initial candidates to be passed over Lindley [4], who calls this the marriage problem, was the first to introduce a dynamic program. His solution involves, however, an *ad hoc* probabilistic argument. Lindley addresses in fact a more general problem in which expected utility is maximized and this utility need not be zero/one as in his formulation of the secretary problem.

In this paper we propose a dynamic program simpler than Lindley's, solve it and give results of some numerical calculations.

The following terminology will be used:

| | |
|---:|---|
| winning: | choosing the best among all candidates |
| value: | the expected probability of winning |
| viable candidate: | any candidate who is best so far. |

Let $m$ be the number of candidates seen and $n$ the number not yet seen, so that

$$m + n = N$$

is the total number of candidates.

We choose $m$ as our state variable. It is useful to introduce two value functions. Let $v_m$ be the value when $m$ candidates have been observed and the $m$th is not chosen; and $u_m$ be the value when $m$ candidates have been observed and the $m$th is a viable candidate.

When the last observed candidate is not viable, then an optimal policy requires that the search be continued so that $v_m$ applies. A decision rule is needed for the case that the last candidate is viable, and this decision rule can depend only on the state variable $m$ and the total number $N$.

## 2. DYNAMIC PROGRAM

There are two principles of optimality, one for each value function:

$$v_m = \frac{m}{m+1} v_{m+1} + \frac{1}{m+1} u_{m+1}. \tag{1}$$

Equation (1) expresses this: Given that the $m$th is not chosen, the $m+1$th will be either a viable candidate [with probability $1/(m+1)$] or not [with probability $m/(m+1)$]. If not, the value (expected probability of winning) will become $v_{m+1}$, because the $m+1$th will certainly not be chosen. But if so, the value will become $u_{m+1}$ (the value if an optimal decision is made at stage $m+1$):

$$u_m = \max\left[\frac{m}{N}, v_m\right]. \tag{2}$$

Equation (2) expresses this: Given that the $m$th is a viable candidate, there is a decision either to choose it or to continue. If chosen, the value is $m/N$, which is just the probability the chosen one is best among all $N$. If not chosen, the value is $v_m$, as already established.

Starting with $m = N$, equations (1) and (2) may be solved recursively along the following lines. By definition of $u_m$ and $v_m$,

$$v_N = 0, \quad u_N = 1. \tag{3}$$

Now,

$$v_{N-1} = \frac{N-1}{N} v_N + \frac{1}{N} u_N = \frac{1}{N}$$

and

$$u_{N-1} = \max\left[\frac{N-1}{N}, v_{N-1}\right] = \max\left[\frac{N-1}{N}, \frac{1}{N}\right] = \frac{N-1}{N}.$$

Furthermore,

$$v_{N-2} = \frac{N-2}{N-1}\cdot v_{N-1} + \frac{1}{N-1}\cdot u_{N-1}$$

$$= \frac{N-2}{N-1}\cdot\frac{1}{N} + \frac{1}{N-1}\cdot\frac{N-1}{N}$$

$$= \frac{1}{N}\left(\frac{N-2}{N-2} + \frac{N-2}{N-1}\right) = \frac{N-2}{N}\cdot\left(\frac{1}{N-2} + \frac{1}{N-1}\right)$$

and

$$u_{N-2} = \max\left[\frac{N-2}{N}, \frac{N-2}{M}\left(\frac{1}{N-2} + \frac{1}{N-1}\right)\right]$$

$$= \frac{N-2}{N} \max\left(1, \frac{1}{N-2} + \frac{1}{N-1}\right).$$

The pattern is now clear and readily verified:

$$v_m = \frac{m}{N}\cdot\left(\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{N-1}\right) \tag{4}$$

and

$$u_m = \frac{m}{N} \max\left[1, \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{N-1}\right]. \tag{5}$$

## 3. ANALYSIS

The implied optimal policy is as follows. No secretary who is best so far (and hence a viable candidate) will be accepted as long as

$$\sum_{i=m}^{N-1} \frac{1}{i} < 1$$

and the first viable candidate will be chosen when $m \geqslant m^*$. The critical $m = m^*$ is the solution of

$$\sum_{i=m^*}^{N-1} \frac{1}{i} \leqslant 1 < \sum_{i=m^*-1}^{N-1} \frac{1}{i}. \tag{6}$$

If follows that

$$u_0 = v_0 = v_{m^*-1}$$

is the probability of finding the best among $N$ candidates under the optimal strategy. Value and optimal strategy can thus be described simply in terms of a critical number $m^*$:

$$v_0 = v_{m^*-1} = \frac{m^*-1}{m^*} v_{m^*} + \frac{1}{m^*} \max\left[\frac{m^*}{N}, v_{m^*}\right],$$

using equations (1) and (2), and

$$v_0 = \frac{m^*-1}{m^*} \cdot \frac{m^*}{N} \left(\frac{1}{m^*} + \cdots + \frac{1}{N-1}\right) + \frac{1}{m^*} \cdot \frac{m^*}{N},$$

using expressions (4) and (6). Thus,

$$v_0 = \frac{m^*-1}{N} \left(\frac{1}{m^*-1} + \frac{1}{m^*} + \frac{1}{m^*+1} + \cdots + \frac{1}{N-1}\right). \tag{7}$$

## 4. APPROXIMATIONS

For an approximate determination of $m^*$ observe that

$$\sum_{i=m}^{N-1} \frac{1}{i} < \int_m^N \frac{dx}{x} \tag{8}$$

and $\approx$ for sufficiently large $N$. The approximate solution for $m^*$ in formula (6) is therefore

$$\sum_{i=m}^{N-1} \frac{1}{i} = \int_{M^*}^N \frac{dx}{x} = \ln N - \ln m = 1,$$

yielding

$$\frac{m^*}{N} \approx \frac{1}{e} \tag{9}$$

in agreement with the well-known probabilistic solution of the secretary problem [2, 3].

## 5. RESULTS

The dynamic programming (DP) approach proves that this is, in fact, the optimal policy. For we have shown that in equation (2)

$$v_m = \frac{m}{N} \sum_{i=m}^{N-1} \frac{1}{i}$$

Table 1

| | N | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 |
| $m^*-1$ | 0 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 5 | 7 |
| $N/\mathrm{e}$ | 0.37 | 0.74 | 1.1 | 1.47 | 1.84 | 2.2 | 2.58 | 2.94 | 3.31 | 3.68 | 4.41 | 5.52 | 7.36 |
| $v_0$ | 1 | 0.5 | 0.5 | 0.46 | 0.43 | 0.43 | 0.41 | 0.41 | 0.41 | 0.40 | 0.40 | 0.39 | 0.38 |

is a decreasing function of $m$ while $m/N$ is increasing. This shows that the optimal decision is to accept the first best from the $m^*$th candidate on.

For small $N$ the optimal $m^*$ are given in the following Table 1. We also include the approximation $N/\mathrm{e}$ and the probability

$$v_0 = v_0(N)$$

of finding the best person when using the optimal strategy.

$m^* - 1$ is the number of candidates to be passed automatically.

From

$$\int_{m^*-1}^{N} \frac{1}{n}\,\mathrm{d}n > \sum_{i=m^*-1}^{N} \frac{1}{i} > 1$$

we conclude that

$$\ln \frac{N}{m^*-1} > 1$$

or

$$m^* < 1 + \frac{N}{\mathrm{e}}. \tag{10}$$

Table 1 shows, in fact, that for all $N$, except $N = 5$,

$$m^* = \left\{\frac{N}{\mathrm{e}}\right\}, \tag{11}$$

where $\{x\}$ is the smallest integer $\geqslant x$. $v_0$ is decreasing and approaches slowly its limiting value

$$\frac{1}{\mathrm{e}} = 0.3678795.$$

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