Jost maps, ball-homogeneous and harmonic manifolds

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Abstract

Given a real number $\varepsilon > 0$, small enough, an associated Jost map $J_{\varepsilon}$ between two Riemannian manifolds is defined. Then we prove that connected Riemannian manifolds for which the center of mass of each small geodesic ball is the center of the ball (i.e. for which the identity is a $J_{\varepsilon}$ map) are ball-homogeneous. In the analytic case we characterize such manifolds in terms of the Euclidean Laplacian and we show that they have constant scalar curvature. Under some restriction on the Ricci curvature we prove that Riemannian analytic manifolds for which the center of mass of each small geodesic ball is the center of the ball are locally and weakly harmonic.

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0. Introduction

In his paper “Equilibrium maps between metric spaces”, Jost [10] defined $\varepsilon$-equilibrium maps by using the notion of mass center and considering measures

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supported by convex balls of metric spaces, or maps with range in simply connected nonpositively curved Riemannian manifolds. He proved that \( \varepsilon \)-equilibrium maps are also critical points of an energy functional, the so-called \( \varepsilon \)-energy functional.

The use of center of mass to study harmonic maps was also used by Korevaar–Schoen who showed that mollifying a given map (i.e. taking the center of mass with respect to certain measure) essentially decreases energy.

In the present paper Jost maps \( J_\varepsilon \) are revisited and we consider a family of measures induced from the volume element of a Riemannian manifold for which the identity map is a \( J_\varepsilon \) map when \( \varepsilon \) is a real positive number sufficiently small.

It is a contribution to our attempt to find out a description of ball-homogeneous and harmonic manifolds \((M, g)\) using harmonicity of geometric objects living on \((M, g)\).

In fact, the notion of harmonic manifold was introduced in the mathematical literature in 1930 by H.S. Ruse. He dealt with developing a harmonic analysis on Riemannian manifolds similar in some sense to one of Euclidean spaces. For that Ruse attempted to find for the Laplace equation \( \Delta u = 0 \) on a Riemannian manifold solutions which depend only on the geodesic distance. It appeared then that such spherically symmetric harmonic functions \( u \) exist on \((M, g)\) only in a few cases, the cases where the volume density function at each point \( p \), \( \theta_p := \sqrt{|\det g_p|} \) depends only on the geodesic distance function \( d(p, .) \) in a normal coordinate system around \( p \). Due to the symmetry \( \theta_p(q) = \theta_q(p) \) of the volume density function, this amounts to say that \( \theta_p \) itself is a radial function, i.e., for each point \( q \) near \( p \), \( \theta_p(q) = (\sigma \circ d)(p, q) \) for some \( \sigma \) defined on the set of real nonnegative numbers. Although Ruse’s attempt failed, his consideration gave the first historical definition of harmonic manifolds. A Riemannian manifold \((M, g)\) is said to be harmonic if its volume density function at each point \( p \), is a radial function. Nowadays many other definitions exist and they are equivalent, of course [2,6]. For further information on this subject some other references are [1,16,17,23]. What we are looking for is a description of harmonic and ball-homogeneous manifolds using \( J_\varepsilon \) maps (for details see [10,20]).

Let \((M, g)\) be a compact Riemannian manifold and \( \varepsilon > 0 \) a real number that we assume small enough. Denote \( \mu \) the volume element on \((M, g)\) and \( B(x, \varepsilon) \) the ball with center \( x \in M \) and radius \( \varepsilon \). Then \((M, g)\) is said to satisfy the \( \Gamma_\varepsilon \)-property if each point \( x \in M \) is the mass center of the identity map with respect to \( \mu_\varepsilon \) (see Section 3 for precise definition).

Equivalently \((M, g)\) has the \( \Gamma_\varepsilon \)-property if

\[
\forall x \in M, \quad \int_{B(x, \varepsilon)} \exp_x^{-1} y d\mu(y) = 0(\Gamma_\varepsilon),
\]

where \( \exp : TM \to M \) denotes the exponential map.

However the variation (w.r.t. \( \varepsilon \)) of the vector

\[
H(x, \varepsilon) = \int_{B(x, \varepsilon)} \exp_x^{-1} y d\mu(y) \in T_x M
\]
leads to the partial differential equation

\[ \frac{1}{\varepsilon^2} \frac{\partial}{\partial \varepsilon} H(x, \varepsilon) - \frac{1}{\varepsilon} \frac{\partial^2}{\partial \varepsilon^2} H(x, \varepsilon) = \nabla \frac{\partial}{\partial \varepsilon} \hat{m}(x, \varepsilon)(\ast), \]

where \( \hat{m}(x, \varepsilon) \) is the volume of the ball \( B(x, \varepsilon) \) and \( \nabla \) denotes the gradient operator w.r.t. \( x \).

From (\ast) we will deduce in Section 3 that if \( (M, g) \) is connected and satisfies the \( \Gamma_{\varepsilon} \)-property for sufficiently small real number \( \varepsilon > 0 \), then at each point \( x \in M \), the volume of the ball \( B(x, \varepsilon) \) depends only on the radius \( \varepsilon \) and not on the center \( x \); i.e. \( (M, g) \) is ball-homogeneous.

In this same Section 3, we will show that if \( \theta \) is the volume density function of \( (M, g) \) and \( V_y \) is the vector field which associates to each point \( z \in M \) the tangent vector \( V_y(z) = \exp^{-1}_z y \in T_z M \), then the defining identity of the \( \Gamma_{\varepsilon} \)-property yields

\[ \int_{B(x, \varepsilon)} (L_{V_y} \theta^2)(x) \, d\mu(y) = 0 \quad \forall x \in M, \]

where \( (L_{V_y} \theta^2)(x) \) denotes the Lie derivative in the direction \( V_y \) of \( \theta^2 \) at the point \( x \).

In Section 4, the analytic case is considered. We prove that if \( (M, g) \) satisfies the \( \Gamma_{\varepsilon} \)-property for \( \varepsilon > 0 \) small enough, and that its Ricci tensor has some control, say \( \text{Ric} \leq \lambda > 0 \), for some real number \( \lambda \), then:

\[ (L_{V_y} \theta^2)(x) = 0 \quad \forall x \in M \quad \text{and} \quad \forall y \in B(x, \varepsilon). \]

The latter property means that \( (M, g) \) is locally and weakly harmonic [2,5]. Since \( (M, g) \) is assumed to be analytic, it is then infinitesimally harmonic [2,19].

In Section 1, the notions of mass center [11] and equilibrium maps on Riemannian manifolds are reminded. Section 2 is devoted to a quick review of the theory of harmonic manifolds.

1. Mass center and equilibrium maps

1.1. Mass center

Let \( \Omega \) be a compact Riemannian manifold (possibly a finite set of points) and \( \mu \) a measure defined on \( \Omega \). We can suppose \( \mu(\Omega) = 1 \) and consider \( (M, g) \) a complete Riemannian manifold.

Let \( f : (\Omega, \mu) \rightarrow (M, g) \) be a map such that \( f(\Omega) \) is contained in a ball \( B_\varepsilon \) of \( M \) and \( P_f \) be the functional on \( M \) given by

\[ P_f(x) := \frac{1}{2} \int_{\Omega} d^2(x, f(y)) \, d\mu(y), \]

where \( d \) denotes the distance function on \( (M, g) \). It holds:
Proposition 1.1. The notations being as above, we have

\[ \text{grad } P_f(x) = -\int_{\Omega} \exp^{-1} f(y) \, d\mu(y); \]

(ii) For sufficiently small radii \( R > 0 \) the restriction of \( P_f \) to \( B_R \) is strictly convex. It follows then that \( P_f \) admits a unique minimum at a point in \( B_R \). This point is called the mass center of \( f \) w.r.t. \( \mu \) and will be denoted \( c_f \).

The point \( c_f \) can also be characterized as the unique point of \( B_R \) satisfying

\[ \int_{\Omega} \exp^{-1} f(y) \, d\mu(y) = 0. \]

Example 1.1. (i) Let \( M = \mathbb{R}^n \) and \( f : (\Omega, \mu) \to \mathbb{R}^n \) be an integrable map. We have:

\[ P_f(x) = \frac{1}{2} \int_{\Omega} \|x - f(\omega)\|^2 \, d\mu(\omega), \]

where \( \|\| \) denote the euclidean norm on \( \mathbb{R}^n \). Then the mass center of \( f \) w.r.t. \( \mu \) is

\[ c_f = \int_{\Omega} f(\omega) \, d\mu(\omega). \]

(ii) More generally assume that the sectional curvature of \( M \) is bounded by \( s \). Then there exists a positive constant \( \eta(s) \) such that the mass center \( c_f \) of \( f \) satisfies:

\[ (1 - \eta(s)g^2) \|\text{grad } P_f(m)\| \leq d(c_f, m) \leq (1 + \eta(s)g^2) \|\text{grad } P_f(m)\| \quad \forall m \in B_R. \]

We end this part with some invariance properties of the mass center: Let \( f : (\Omega, \mu) \to M \) be a measurable map with image contained in a convex ball of \( M \), \( \phi : (M, g) \to (M, g) \) an isometry, \( \psi : (\Omega, \mu) \to (\Omega, \mu) \) a volume-preserving transformation.

Then \( \phi \circ f \) and \( f \circ \psi \) are also measurable maps with images contained in convex balls of \( M \) and it holds:

\[ c_{\phi \circ f} = \phi(c_f) \quad \text{and} \quad c_{f \circ \psi} = c_f. \]

For more details on the theory of mass center we refer to [11].

1.2. Jost maps \( J_\varepsilon \) (or \( \varepsilon \)-equilibrium maps)

Let \( (M, g) \) be a compact Riemannian manifold equipped with a measure \( \mu \). Suppose that for every point \( x \in M \) and every real number \( \varepsilon > 0 \), there exists a measure \( \mu^\varepsilon_x \) such that the family \( (\mu^\varepsilon_x)_{x \in M} \) satisfies the following symmetry property:
For any integrable function $F : M \times M \to \mathbb{R}$,
\[
\int_M \int_M F(x,y) \, d\mu^c_x(y) \, d\mu(x) = \int_M \int_M F(x,y) \, d\mu^c_y(x) \, d\mu(y) \quad (S).
\]

Let $N$ be a compact and geodesically complete Riemannian manifold, i.e. any two points $p$ and $q$ of $N$ can be joined by a minimizing geodesic [12, p. 172].

A map $f : M \to N$ is said to be a $J_\varepsilon$ map (or an $\varepsilon$-equilibrium map) if for any $x \in M$, $f(x)$ is the center of mass of $f$ w.r.t. the measure $\mu^c_x$.

This means that for any point $x \in M$, we have
\[
\int_M d^2_N(f(x), f(y)) \, d\mu^c_x(y) = \min_{q \in N} P_f(q),
\]
where $P_f(q) = \int_M d^2_N(q, f(y)) \, d\mu^c_x(y)$ and $d_N$ denotes the distance function on $N$.

Equivalently $f$ is a $J_\varepsilon$ map if
\[
\forall x \in M, \quad \int_M \exp_{f(x)}^{-1} f(y) \, d\mu^c_x(y) = 0.
\]

2. Quick review on harmonic manifolds

Historically, an $n$-dimensional Riemannian manifold $(M, g)$ is defined to be harmonic when the Laplace equation $\Delta u = 0$ on $(M, g)$ admitted solutions depending only on the geodesic distance. It is equivalent to say that $(M, g)$ has its volume density function $\theta_p := \sqrt{|\det g_{ij}|}$ spherically symmetric in normal coordinate neighborhood $U$ around each point $p \in M$, where $g_{ij}$, $i,j = 1, \ldots, n$, are the components of the metric $g$ in the normal coordinate system. This means that for every point $q \in M$, $\theta_p(q) = \sigma(d(p,q))$ for some function $\sigma$ defined on the set on nonnegative real numbers.

If $(r, \phi)$ is a polar coordinate system on $U$, then $(M, g)$ is harmonic if its volume density function depends only on $r$ and therefore can be written as $\Omega_p(r)$, for a function $\Omega_p$ depending only on $r$. Note moreover that $\Omega_p(r)$ does not depend on the point $p$, due to the symmetry $\theta_p(q) = \frac{1}{q} \theta_p(q), \forall q \in M$.

The notions of global (respectively local) harmonicity is used to refer the situation when the spherical symmetry property of the volume density function is global (respectively local). From their volume density function, symmetric spaces of rank one are seen to be harmonic manifolds.

Let $(M, g)$ be a harmonic manifold. Ledger’s formula [2, p. 161] gives
\[
\nabla_u \nabla_u \theta_p(r)|_{r=0} := \frac{d^2}{dr^2} \theta_p(r)|_{r=0} = -\frac{2}{3} \text{Ric}(u,u)
\]
for each unit vector $u \in T_p M$. Hence since $\theta_p$, is a function of $r$ alone, for harmonic manifolds the Ricci curvature is constant, i.e. $\text{Ric} = \lambda g$ for some constant $\lambda$. So harmonic manifolds are Einstein spaces. Classically three cases then occur:

1. $\lambda = 0$. In this case $(M, g)$ is isometric to an Euclidean space [15, Theorem 4]; i.e. Ricci flat harmonic manifolds are flat.

2. $\lambda > 0$. Then the Myers–Bonnet theorem yields that $M$ is compact and has a finite fundamental group.

Szabo [18] recently proved that compact harmonic manifolds with finite fundamental group are symmetric spaces of rank one. More generally, he concluded by using the Cheeger–Gromov splitting theorem (stating that the universal covering space $\tilde{M}$ of a complete Riemannian manifold with nonnegative Ricci curvature can be split into the Riemannian product $\tilde{M} = M^k \times \mathbb{R}^{n-k}$, with $M^k$ compact [4]), that complete harmonic manifolds with positive scalar curvature are symmetric of rank one. Thus he proved (for the case $\lambda > 0$) the Lichnerowicz conjecture raised in 1944 asserting that every harmonic space is symmetric of rank one. Before Szabo’s work (1990) the conjecture had been proved only for dimensions less or equal to 4 [14,22].

3. $\lambda < 0$. In this case, the Lichnerowicz conjecture fails. Damek and Ricci constructed [6] a family of nonsymmetric harmonic manifolds, the so-called NA spaces. All of them are homogeneous spaces and they are modeled on Heisenberg type group. Precisely:

The left invariant metrics on the solvable extensions of the Heisenberg type groups define harmonic manifolds.

Since this family of spaces contains, besides the negatively curved rank-one symmetric spaces, a number of nonsymmetric negatively curved spaces, the Lichnerowicz conjecture fails for noncompact harmonic manifolds. For more details we refer to [6,15,18,19].

Finally, as Einstein spaces, harmonic manifolds are real analytic ones in normal and harmonic coordinates by a theorem of DeTurck and Kazdan (second reference in [3, P. 145].

More general curvature conditions have been recently introduced in the theory of harmonic manifolds. Thus a Riemannian manifold is defined to be infinitesimally harmonic when one assumes only that the covariant derivatives $\nabla^{(k)}_u \theta_p = \nabla_u \nabla_u \cdots \nabla_u \theta_p$ w.r.t. each unit vector $u \in T_p M$ define constant functions on $M$. For real analytic Riemannian manifolds, global, local and infinitesimal harmonicity are equivalent [2, pp. 160–178].

The definition of local harmonicity is also based upon the classical mean value property and explains the word “harmonic” in the context under consideration. In Euclidean spaces a harmonic function $f$ (i.e. $\Delta f = 0$) has its mean value on every sphere $S(x, r)$ (with center $x$ and radius $r$) equal to its value at the point $x$; that is

$$(\text{MVf})_{S(x,r)} := \frac{1}{\text{vol} S(x,r)} \int_{S(x,r)} f(y) \, d\sigma(y) = f(x),$$

where $\text{vol}(S(x,r))$ is the volume of $S(x,r)$. 
3. Riemannian manifolds for which the identity is a $J_{\varepsilon}$ map

Let $(M, g)$ be a compact and geodesically complete Riemannian manifold and $\mu$ the Riemannian measure on $(M, g)$. For $x \in M$ and $\varepsilon > 0$ sufficiently small we put $\mu^\varepsilon_x := \mu|_{B(x, \varepsilon)}$ and $\mu^\varepsilon_x := 0$ on $M \setminus B(x, \varepsilon)$, where $B(x, \varepsilon)$ is the ball with center $x$ and radius $\varepsilon$.

For radii $\varepsilon$ sufficiently small, the balls $B(x, \varepsilon)$ are convex. Then the minimum in $M$ of the integral $P(q) = \int_M d^2(q, y) \, d\mu^\varepsilon_x(y) = \int_{B(x, \varepsilon)} d^2(q, y) \, d\mu(y)$ exists and is unique. Although the identity map does not have its image in a convex ball as required in the definition of mass center, we can consider the unique minimum in $M$ of $P(q)$ as the center of mass of the identity map w.r.t. $\mu^\varepsilon_x$.

We say that a compact and geodesically complete Riemannian manifold has the $\Gamma_{\varepsilon}$-property, if the identity is a $J_{\varepsilon}$ map w.r.t. the family $(\mu_x^\varepsilon)_{x \in M}$ of measures. Equivalently, a compact and geodesically complete Riemannian manifold $(M, g)$ has the $\Gamma_{\varepsilon}$-property, if for any $x \in M$,

$$\int_{B(x, \varepsilon)} d^2(x, y) \, d\mu(y) = \min_{q \in M} \int_{B(x, \varepsilon)} d^2(q, y) \, d\mu(y).$$

This means that

$$\forall x \in M, \quad \int_{B(x, \varepsilon)} \exp^{-1}_x y \, d\mu(y) = 0 \quad (\Gamma_{\varepsilon}).$$

As we are going to see, compact Riemannian manifolds with volume-preserving geodesic symmetries satisfy the $\Gamma_{\varepsilon}$-property for sufficiently small $\varepsilon > 0$.

For any two points $x$ and $y$ in $M$, we put $y = \exp_x(ru)$, where $r$ is the geodesic distance between $x$ and $y$ and $u$ the unit vector tangent at $x$ to the minimal geodesic joining $x$ and $y$.

The geodesic symmetry at the point $x$ is the map $s_x : M \to M$ defined for $y = \exp_x(ru)$ by $s_x(y) = \exp_x(-ru)$. We have:

**Proposition 3.1.** Let $(M, g)$ be a compact Riemannian manifold. If for every point $x \in M$ the geodesic symmetry at $x$ is volume-preserving, then $(M, g)$ has the $\Gamma_{\varepsilon}$-property for $\varepsilon > 0$ satisfying $0 < \varepsilon \leq \inf(\text{Inj}(M), \text{Conv}(M))$, where $\text{Inj}(M)$ and $\text{Conv}(M)$ are the injectivity and the convexity radius of $M$, respectively.

**Proof.** Let $(M, g)$ be a compact Riemannian manifold, $x \in M$ and $\varepsilon$ a real number satisfying $0 < \varepsilon \leq \inf(\text{Inj}(M), \text{Conv}(M))$.  


The real number \( e > 0 \) being smaller than the injectivity radius of \( M \), the inverse of the exponential map at \( x \) is a diffeomorphism of the ball \( B(x, \varepsilon) \subset M \) to a ball \( B(0, \rho) \subset T_x M \), and we have

\[
\int_M \exp_x^{-1} y \, d\mu_x^\varepsilon(y) = \int_{B(x, \varepsilon)} \exp_x^{-1} y \, d\mu(y) = \int_{B(0, \rho)} u \, d\text{vol}(u),
\]

where \( d\text{vol} = (\exp_x)^* (d\mu) \) is the volume element on \( T_x M \) induced by the exponential map.

Now we suppose that \((M, g)\) has volume-preserving geodesic symmetries. Then the volume density function \( \theta_x \) at any point \( x \in M \) satisfies the relation

\[
\theta_x(y) = \theta_x(s_x(y)),
\]

where \( s_x \) is the geodesic symmetry at the point \( x \).

It follows that the corresponding volume density function on the tangent space \( T_x M \), \( \tilde{\theta}_x \), is an even function; i.e. \( \tilde{\theta}_x(v) = \tilde{\theta}_x(-v), \forall v \in T_x M \). Hence:

\[
\int_{B(0, \rho)} u \, d\text{vol}(u) = 0.
\]

Thus

\[
\int_M \exp_x^{-1} y \, d\mu_x^\varepsilon(y) = 0
\]
or equivalently,

\[
x = \min_{q \in M} \int_M d^2(q, y) \, d\mu_x^\varepsilon(y).
\]

The real number \( \varepsilon > 0 \) being smaller than the convexity radius of \( M \), the ball \( B(x, \varepsilon) \) is convex. Therefore the point \( x \) is the unique minimum in \( M \) of \( P(q) = \int_M d^2(q, y) \, d\mu_x^\varepsilon(y) \). It follows that the point \( x \) is the center of mass w.r.t. the measure \( \mu_x^\varepsilon \) of the identity map. \( \square \)

Riemannian manifolds with volume-preserving geodesic symmetries are called d’Atri spaces [13]. Compact locally symmetric spaces and compact Riemannian manifolds with radial volume density functions, the so-called harmonic manifolds, are also examples of Riemannian manifolds satisfying the \( G_\varepsilon \)-property. In the following, we show that the product of Riemannian manifolds preserves the property \( (G_\varepsilon) \). Precisely we have:
Theorem 3.1. Let \((M_1, g_1)\) and \((M_2, g_2)\) be two Riemannian manifolds. Suppose that \((M_1, g_1)\) and \((M_2, g_2)\) satisfy the \(\Gamma_\varepsilon\)-property for every \(\varepsilon > 0\) sufficiently small. Then the Riemannian product \((M_1, g_1) \times (M_2, g_2)\) has also the \(\Gamma_\varepsilon\)-property.

Proof. Put \((M, g) = (M_1, g_1) \times (M_2, g_2)\). Let \(x = (x_1, x_2)\) be a point of \(M\), \(X = (X_1, X_2)\) and \(Y = (Y_1, Y_2)\), two vectors in \(T_xM\). There is vectors \(X_1, Y_1 \in T_{x_1}M_1\) and \(X_2, Y_2 \in T_{x_2}M_2\) such that \(X = (X_1, X_2)\) and \(Y = (Y_1, Y_2)\), and we have

\[
g(X, Y) = g_1(X_1, Y_1) + g_2(X_2, Y_2).
\]

For \(y = (y_1, y_2) \in M\), consider the vectors \(U_1 = \exp_{x_1}^{-1}y_1 \in T_{x_1}M_1\) and \(U_2 = \exp_{x_2}^{-1}y_2 \in T_{x_2}M_2\). The curve \(\gamma(t) = (\exp_{x_1} tU_1, \exp_{x_2} tU_2)\) is a geodesic in \(T_xM\) with \(\gamma(1) = (y_1, y_2)\) and \(\gamma(0) = (U_1, U_2)\). It follows that

\[
\exp_{x_1}^{-1}y = (U_1, U_2) = (\exp_{x_1}^{-1}y_1, \exp_{x_2}^{-1}y_2).
\]

For any given vector \(V = (V_1, V_2)\) in \(T_xM\), with \(V_1 \in T_{x_1}M_1\) and \(V_2 \in T_{x_2}M_2\), we then have

\[
g(\exp_{x_1}^{-1}y, V) = g_1(\exp_{x_1}^{-1}y_1, V_1) + g_2(\exp_{x_2}^{-1}y_2, V_2).
\]

Let \(B(x, \varepsilon)\) denote the ball of \(M\) with center \(x = (x_1, x_2)\) and radius \(\varepsilon\), \(B_1(x_1, q_1)\) the ball of \(M_1\) with center \(x_1\) and radius \(q_1\) and \(B_2(x_2, q_2)\) the ball of \(M_2\) with center \(x_2\) and radius \(q_2\). We have

\[
\int_{B(x, \varepsilon)} g(\exp_{x_1}^{-1}y, V) \, d\mu(y) = \int_{B(x, \varepsilon)} g_1(\exp_{x_1}^{-1}y_1, V_1) \, d\mu(y) + \int_{B(x, \varepsilon)} g_2(\exp_{x_2}^{-1}y_2, V_2) \, d\mu(y)
\]

\[
= \int_{B_2(x_2, \varepsilon)} \left( \int_{B_1(x_1, q_1(y_1))} g_1(\exp_{x_1}^{-1}y_1, V_1) \, d\mu_1(y_1) \right) \, d\mu_2(y_2)
\]

\[
+ \int_{B_1(x_1, \varepsilon)} \left( \int_{B_2(x_2, q_2(y_2))} g_2(\exp_{x_2}^{-1}y_2, V_2) \, d\mu_2(y_2) \right) \, d\mu_1(y_1),
\]

with \(q_1(y_2) = \sqrt{\varepsilon^2 - d_2(x_2, y_2)^2}\) and \(q_2(y_1) = \sqrt{\varepsilon^2 - d_1(x_1, y_1)^2}\), where \(\mu\), \(\mu_1\) and \(\mu_2\) denote the Riemannian measures on \((M, g)\), \((M_1, g_1)\) and \((M_2, g_2)\) respectively, and \(d_1\) and \(d_2\) the distance functions on \(M_1\) and \(M_2\) respectively.

If now \((M_1, g_1)\) and \((M_2, g_2)\) satisfy the \(\Gamma_\varepsilon\)-property for every \(\varepsilon > 0\) sufficiently small, then we have

\[
\int_{B_1(x_1, q_1(y_1))} g_1(\exp_{x_1}^{-1}y_1, V_1) \, d\mu_1(dy_1) = \int_{B_2(x_2, q_2(y_2))} g_2(\exp_{x_2}^{-1}y_2, V_2) \, d\mu_2(dy_2) = 0.
\]
Therefore
\[
\int_{B(x,\epsilon)} g(\exp_x^{-1}y, V)\mu(dy) = 0 \quad \text{for any given vector } V \in T_x M.
\]

Thus we get
\[
\int_{B(x,\epsilon)} \exp_x^{-1}y\mu(dy) = 0 \quad \forall x \in M.
\]

This proves the theorem. \(\square\)

We set for \(x \in M\),
\[
H(x, \epsilon) := \int_{B(x,\epsilon)} \exp_x^{-1}y \, d\mu(y) \in T_x M.
\]

In the following we compute the variation of \(H(x, .)\) w.r.t. \(\epsilon\).
Consider \(\psi \in C^\infty(M)\) and a function \(h: \mathbb{R}_+ \to \mathbb{R}\) with compact support. We have
\[
\int_M h(r^2(x,y))\psi(y) \, d\mu(y) = \int_0^\infty h(t^2) \int_{S(x,t)} \psi_t(y) \, d\sigma(y) \, dt,
\]
where \(d\sigma\) is the induced Riemannian measure on the sphere \(S(x,t)\) and \(\psi_t(y) = \psi(y), \forall y \in S(x,t); \text{ i.e. } \psi_t = \psi|_{S(x,t)}\).

Let us take the gradient w.r.t. \(x\) of the two sides of the above equality. We obtain
\[
\int_M \nabla(h(r^2(x,y)))\psi(y) \, d\mu(y) = \int_0^\infty h(t^2) \nabla \left( \int_{S(x,t)} \psi_t(y) \, d\sigma(y) \right) \, dt.
\]

This is equivalent to
\[
\int_M \nabla(h(r^2(x,y)))\psi(y) \, d\mu(y) = \int_0^\infty h(t^2) \nabla(M_x[t,\psi]) \, dt \quad (i),
\]
where \(M_x[t,\psi]\) is the nonnormalized mean-value of the function \(\psi\) on the sphere \(S(x,t)\).
We now compute the left-hand side of equality (i). We have
\[
\int_M \nabla(h(r^2(x,y))\psi(y)) \, d\mu(y) = \int_M h'(r^2(x,y))\nabla r^2(x,y)\psi(y) \, d\mu(y)
\]
\[
= \int_0^\infty h'(t^2) \left( \int_{S(x,t)} \nabla r^2(x,y)\psi_t(y) \, d\sigma(y) \right) \, dt
\]
\[
= \int_0^\infty h'(t^2)M_x[t, \nabla r^2(x,.)\psi] \, dt
\]
\[
= \int_0^\infty \frac{1}{2\sqrt{t}} h'(t)M_x[\sqrt{t}, \nabla r^2(x,.)\psi] \, dt
\]
\[
= -\int_0^\infty h(t) \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} M_x[\sqrt{t}, \nabla r^2(x,.)\psi] \right) \, dt.
\]

The right-hand side of (i) can be written as
\[
\int_0^\infty h(t^2)\nabla(M_x[t, \psi]) \, dt = \int_0^\infty \frac{1}{2\sqrt{t}} h(t)\nabla(M_x[\sqrt{t}, \psi]) \, dt.
\]

Relation (i) then becomes
\[
-\int_0^\infty h(t) \frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} M_x[\sqrt{t}, \nabla r^2(x,.)\psi] \right) \, dt = \int_0^\infty \frac{1}{2\sqrt{t}} h(t)\nabla M_x[\sqrt{t}, \psi] \, dt,
\]

for all functions \( h \in C^\infty(\mathbb{R}_+) \) with compact support.

It follows that
\[
\frac{1}{2\sqrt{t}} \nabla M_x[\sqrt{t}, \psi] = -\frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} M_x[\sqrt{t}, \nabla r^2(x,.)\psi] \right),
\]
\[
= -\frac{\partial}{\partial t} \left( \frac{1}{2\sqrt{t}} \right)M_x[\sqrt{t}, \nabla r^2(x,.)\psi] - \frac{1}{2\sqrt{t}} \frac{\partial}{\partial t}(M_x[\sqrt{t}, \nabla r^2(x,.)\psi])
\]
\[
= \frac{1}{4t\sqrt{t}} M_x[\sqrt{t}, \nabla r^2(x,.)\psi] - \frac{1}{2\sqrt{t}} \frac{\partial}{\partial t}(M_x[\sqrt{t}, \nabla r^2(x,.)\psi]) \cdot \frac{\sqrt{t}}{\partial t}
\]
\[
= \frac{1}{4t\sqrt{t}} M_x[\sqrt{t}, \nabla r^2(x,.)\psi] - \frac{1}{4t} \frac{\partial}{\partial t}(M_x[\sqrt{t}, \nabla r^2(x,.)\psi]).
\]
Thus
\[ \nabla (M_x[t, \psi]) = \frac{1}{2t} M_x[\sqrt{t}, \nabla r^2(x, \cdot) \psi] - \frac{1}{2\sqrt{t}} \frac{\partial}{\partial \sqrt{t}} (M_x[\sqrt{t}, \nabla r^2(x, \cdot) \psi]. \]

Or equivalently,
\[ \nabla M_x[t, \psi] = \frac{1}{2t^2} M_x[t, \nabla r^2(x, \cdot) \psi] - \frac{1}{2t} \frac{\partial}{\partial t} M_x[t, \nabla r^2(x, \cdot) \psi]. \]

Evaluating this equality for \( \psi = 1 \), and using the fact that \( M_x[t, \nabla r^2(x, \cdot)] = 2 \frac{\partial}{\partial t} H(x, t) \), we get
\[ \frac{1}{e^2} \frac{\partial}{\partial e} H(x, e) - \frac{1}{e} \frac{\partial^2}{\partial e^2} H(x, e) = \nabla \frac{\partial}{\partial e} \hat{m}(x, e), \]
where \( \hat{m}(x, e) \) is the volume of the ball \( B(x, e) \) with center \( x \) and radius \( e \).

From the variations of \( H(x, .) \) we obtain the following ball-homogeneity result

**Theorem 3.2.** Let \((M, g)\) be a connected Riemannian manifold. If \((M, g)\) satisfies \( \Gamma_e \)-property for every sufficiently small \( e > 0 \), then \( m(x, e) \) and \( \hat{m}(x, e) \) are independent of the point \( x \in M \), where \( m(x, e) \) and \( \hat{m}(x, e) \) are the volumes of the sphere \( S(x, e) \) and of the ball \( B(x, e) \) respectively; i.e. \((M, g)\) is a ball-homogeneous manifold.

**Proof.** From the variations of \( H(x, e) \) w.r.t. \( e \), we have
\[ \frac{1}{e^2} \frac{\partial}{\partial e} H(x, e) - \frac{1}{e} \frac{\partial^2}{\partial e^2} H(x, e) = \nabla \frac{\partial}{\partial e} \hat{m}(x, e). \]

If \((M, g)\) satisfies \( \Gamma_e \)-property for any sufficiently small \( e > 0 \), then
\[ \frac{\partial^2}{\partial e^2} H(x, e) = \frac{\partial}{\partial e} H(x, e) = 0. \]

Thus
\[ \nabla \frac{\partial}{\partial e} \hat{m}(x, e) = 0. \]

From the connectedness of \( M \), we then obtain
\[ m(x, e) = \frac{\partial}{\partial e} \hat{m}(x, e) = \text{const.} \quad \forall x \in M. \]

Hence \( m(x, e) \) and \( \hat{m}(x, e) \) are independent of the point \( x \in M \). \( \square \)

In the following we examine some consequences of the \( \Gamma_e \)-property on the Lie derivative of the volume density function. We have:
Proposition 3.2. Let \((M, g)\) be a \(n\)-dimensional manifold. We set \(\omega \equiv g^2 = \det g\). If \((M, g)\) satisfies the \(\Gamma^e\)-property, then

\[
\int_{B(x,\varepsilon)} (L_{V_y}\omega)(x) \, d\mu(y) = 0 \quad \forall x \in M,
\]

where \((L_{V_y}\omega)(x)\) is the Lie derivative at \(x\) of \(\omega\) in the direction of tangent vectorfield \(V_y : z \in M \mapsto V_y(z) = \exp_z^{-1} y \in T_z M\).

For the proof we need the following result

Lemma 3.1 ([7, pp. 30–31]). The notations being as in Proposition 3.2, we have for any tangent vectorfield \(X\),

\[
L_X(\omega) = \text{trace}(L_X g) \omega,
\]

where \(L_X(\omega)\) denotes the Lie derivative of \(\omega\) in the direction of \(X\).

Proof of Proposition 3.2. For \(y \in M\), consider the vectorfield \(V_y : z \mapsto V_y(z) = \exp_z^{-1} y\). By the above lemma we have

\[
\frac{1}{\omega(x)} (L_{V_y}\omega)(x) = (\text{trace}(L_{V_y} g))(x).
\]

And

\[
\text{trace}(L_{V_y} g) = 2 \text{ div } V_y.
\]

It follows that

\[
\frac{1}{\omega(x)} \int_{B(x,\varepsilon)} (L_{V_y}\omega)(x) \, d\mu(y) = 2 \int_{B(x,\varepsilon)} (\text{div } V_y)(x) \, d\mu(y).
\]

The divergence of \(V_y\) being computed w.r.t. \(x\), we have

\[
\int_{B(x,\varepsilon)} (\text{div } V_y)(x) \, d\mu(y) = \text{div} \left( \int_{B(x,\varepsilon)} V_y(x) \, d\mu(y) \right).
\]

If \((M, g)\) has the \(\Gamma^e\)-property, then \(\int_{B(x,\varepsilon)} V_y(x) \, d\mu(y) = 0\). Thus

\[
\int_{B(x,\varepsilon)} (L_{V_y}\omega)(x) \, d\mu(y) = 0.
\]

Hence the result. \(\square\)
4. Analytic case

Now assume that \((M, g)\) is a real analytic manifold. The series expansion allows us to obtain some interesting consequences from results in Section 3. For \(x \in M\) consider the map \(\phi^i_x : M \to \mathbb{R}\) defined, for every \(y \in M\), by: \(\phi^i_x(y) = \langle \exp_x^{-1} y, e_i \rangle\), where \((e_i)_{i=1}^n\) is an orthonormal basis of \(T_x M\). The \(\Gamma_e\)-property is equivalent to

\[
\int_{B(x, \varepsilon)} \phi^i_x(y) \, d\mu(y) = 0 \quad \forall x \in M \quad \text{and} \quad \forall i = 1, \ldots, n.
\]

This means that for every \(x \in M\) the mean value \(\hat{M}_x(\varepsilon, \phi^i_x)\) of \(\phi^i_x\) on the ball \(B(x, \varepsilon)\) is, for \(i = 1, \ldots, n\), equal to zero.

Let \(\theta\) be the volume density function of \((M, g)\). We have the following characterization:

**Proposition 4.1.** Let \((M, g)\) be an \(n\)-dimensional analytic Riemannian manifold. \((M, g)\) has the \(\Gamma_e\)-property for every \(\varepsilon > 0\) sufficiently small, if and only if, for every \(i = 1, \ldots, n\),

\[
\hat{A}^k_x(\phi^i_x)(x) = 0 \quad \forall x \in M \quad \text{and} \quad \forall k \in \mathbb{N},
\]

where \(\hat{A}_x\) is the Euclidean Laplace-operator on \(M\) defined in any normal coordinates system \((x^i)_{i=1}^n\) at \(x\) by \(\hat{A}_x = \sum_{k=1}^n \frac{\partial^2}{\partial x^i \partial x^i}\) and \(\hat{A}^k_x = \hat{A}_x(\hat{A}_x^{k-1})\).

The proof of this proposition uses the following result:

**Lemma 4.1** (Gray [8]). Let \(f\) be an analytic function on \(M\). The non-normalized mean-value of \(f\) on the sphere \(S(x, r)\) with center \(x\) and radius \(r\) can be written as follows:

\[
\int_{S(x, r)} f \, d\sigma = 2\pi^\frac{n}{2} r^{n-1} \sum_{k=0}^{\infty} \frac{(r)}{2} \frac{1}{k! \Gamma(\frac{n}{2} + k)} \hat{A}^k_x(f \theta)(x),
\]

where \(\sigma\) is the induced Riemannian measure on \(S(x, r)\).

**Proof of Proposition 4.1.** By Lemma 4.1 and according to the relation

\[
\int_{B(x, \varepsilon)} \phi^i_x(y) \, d\mu(y) = \int_0^\varepsilon \int_{S(x, r)} \phi^i_x(y) \, d\sigma(y) \, dr,
\]

we have

\[
\int_{B(x, \varepsilon)} \phi^i_x(y) \, d\mu(y) = e^n \sum_{k=0}^{\infty} \frac{2\pi^\frac{n}{2} \varepsilon^{2k}}{2k + n} \left(\frac{\varepsilon}{2}\right) \frac{1}{k! \Gamma(\frac{n}{2} + k)} \hat{A}^k_x[\phi^i_x \theta](x).
\]
It follows that \((M, g)\) satisfies the \(\Gamma_\varepsilon\)-property for every sufficiently small \(\varepsilon > 0\) if and only if
\[
\sum_{k=0}^{\infty} \frac{2\pi^n}{2k+n} \left( \frac{\varepsilon}{2} \right)^{2k} \frac{1}{k! \Gamma\left( \frac{1}{2} + k \right)} \tilde{A}_x^k(\phi_x^i \theta)(x) = 0 \quad \forall \varepsilon > 0 \text{ sufficiently small and } \forall x \in M.
\]
And the above equality holds, if and only if
\[
\tilde{A}_x^k(\phi_x^i \theta)(x) = 0 \quad \forall k \in \mathbb{N} \quad \text{and} \quad \forall x \in M.
\]
Hence the result. 

Let now \(\text{Ric}\) and \(\tau\) be the Ricci tensor and the scalar curvature of \((M, g)\) respectively. We have the following properties

**Proposition 4.2.** Let \((M, g)\) be an analytic Riemannian manifold. If \((M, g)\) has the \(\Gamma_\varepsilon\)-property for every \(\varepsilon > 0\) sufficiently small, then:

(i) \(\Lambda(\phi_x^i \theta)(x) = 0 \quad \forall x \in M \text{ and } \forall i = 1, \ldots, n,\)

where \(\Lambda\) is the ordinary Laplace-Beltrami operator on \((M, g)\).

(ii) \(2 \langle \nabla^2 \phi_x^i, \text{Ric} \rangle(x) + 3 \langle \nabla \phi_x^i, \nabla \tau \rangle(x) = 0 \quad \forall x \in M \text{ and } \forall i = 1, \ldots, n.\)

For the proof of this proposition, we need the following result.

**Lemma 4.2 ([8]).** The mean-value of an analytic function \(f\) on the ball \(B(x, r)\),
\[
\tilde{M}_x(r, f),
\]
admits the following expansion:
\[
\tilde{M}_x(r, f) = f(x) + A_f(n + 2)(x)r^2 + B_f(n + 2)(x)r^4 + o(r^6),
\]

with \(A_f(n) = \frac{1}{2n} \Lambda f\) and \(B_f(n) = \frac{1}{24n(n+2)}(3\Lambda^2 f - 2 \langle \nabla^2 f, \text{Ric} \rangle - 3 \langle \nabla f, \nabla \tau \rangle + \frac{4}{n} \tau f).\)

**Proof of Proposition 4.2.** Assume that \((M, g)\) has the \(\Gamma_\varepsilon\)-property for any sufficiently small \(\varepsilon > 0\). Then, for every \(i = 1, \ldots, n,\)
\[
\tilde{M}_x(\varepsilon, \phi_x^i) = 0 \quad \text{for sufficiently small } \varepsilon > 0 \quad \text{and all } x \in M.
\]

By Lemma 4.2, we have, for every \(i = 1, \ldots, n,\):
\[
\phi_x^i(x) + A_{\phi_x^i}(n + 2)(x)\varepsilon^2 + B_{\phi_x^i}(n + 2)(x)\varepsilon^4 + o(\varepsilon^6) = 0 \quad \forall \varepsilon > 0 \text{ sufficiently small},
\]
and by definition
\[
\phi_x^i(x) = 0.
\]
Then

\[ A_{\phi^i}(n+2)(x)e^2 + B_{\phi^i}(n+2)(x)e^4 + o(e^6) = 0 \quad \forall \varepsilon > 0 \] sufficiently small.

Hence, for every \( i = 1, \ldots, n \),

\[ A_{\phi^i}(n+2)(x) = 0 \quad \text{and} \quad B_{\phi^i}(n+2)(x) = 0 \quad \forall x \in M. \]

This proves the result. \( \square \)

From Theorem 3.2 we have:

**Corollary 4.1.** Let \((M, g)\) be an \( n \)-dimensional analytic Riemannian manifold satisfying the \( \Gamma^\varepsilon \)-property for every \( \varepsilon > 0 \) sufficiently small. Then the scalar curvature of \((M, g)\) is constant.

**Proof.** Suppose that \((M, g)\) is analytic and satisfies the \( \Gamma^\varepsilon \)-property for every \( \varepsilon > 0 \) sufficiently small.

Then by Theorem 3.2 the volume \( \hat{m}(x, \varepsilon) \) of the ball with center \( x \in M \) and radius \( \varepsilon \) is independent of the point \( x \). On the other hand, the volume \( \hat{m}(x, \varepsilon) \) admits the following expansion [9,21].

\[ \hat{m}(x, \varepsilon) = \hat{m}_0(\varepsilon) \left\{ 1 - \frac{\tau(x)}{6(n+2)} \varepsilon^2 + O(\varepsilon^4) \right\}, \]

where \( \hat{m}_0(\varepsilon) \) denotes the volume of an \( n \)-dimensional Euclidean ball of radius \( \varepsilon \) and \( \tau \) the scalar curvature of \((M, g)\).

It follows that if \((M, g)\) satisfies the \( \Gamma^\varepsilon \)-property, for every \( \varepsilon > 0 \) sufficiently small, then the scalar curvature \( \tau \) has to be globally constant on \( M \). \( \square \)

From Proposition 3.2 and using the asymptotic expansion of the volume density [5], we get the following “local and weak harmonicity” property for the manifold:

**Corollary 4.2.** Let \((M, g)\) be an analytic connected Riemannian manifold and set \( \omega := 0^2 \). Suppose that there exists a constant \( \lambda > 0 \) such that \( \text{Ric} \geq \lambda \) or \( \text{Ric} \leq -\lambda \). If \((M, g)\) has the \( \Gamma^\varepsilon \)-property, then

\[ (L_{V_y} \omega)(x) = 0 \]

for every \( x \in M \) and for every \( y \in B(x, \varepsilon) \).

**Proof.** For any point \( y \) sufficiently closed to \( x \), we put \( y = \exp_x ru \), where \( u \) is the unit vector tangent to the minimal geodesic joining \( x \) and \( y \) and \( r \) the distance between \( x \) and \( y \).
The asymptotic expansion of $\omega$ at the point $y$ is given for sufficiently small $r$ by

$$
\omega(y) = 1 - \text{Ric}(u, u) \frac{r^2}{3} - \nabla_u \text{Ric}(u, u) \frac{r^3}{6} + o(r^4).
$$

Then

$$
-\nabla_u \omega = \nabla_u \left( \text{Ric}(u, u) \frac{r^2}{3} \right) + \nabla_u \left( \nabla_u \text{Ric}(u, u) \frac{r^3}{6} + \nabla_u(o(r^4)) \right)
$$

$$
= \frac{r^2}{3} \nabla_u \text{Ric}(u, u) + \text{Ric}(u, u) \nabla_u \frac{r^2}{3} + \nabla_u \text{Ric}(u, u) \nabla_u \frac{r^3}{6}
$$

$$
+ \frac{r^3}{6} \nabla^2_{uu} \text{Ric}(u, u) + \nabla_u(0(r^4)).
$$

But

$$
\nabla_u r^2 = \langle \text{grad} r^2, u \rangle
$$

$$
= \langle 2 \exp^{-1} y, u \rangle
$$

$$
= \langle 2ru, u \rangle
$$

$$
= 2r|u|^2
$$

$$
= 2r
$$

and

$$
\nabla_u r^3 = 3r^2.
$$

We then have

$$
-\nabla_u \omega = \frac{r^2}{3} \nabla_u \text{Ric}(u, u) + \frac{2r}{3} \text{Ric}(u, u) + \frac{r^2}{2} \nabla_u \text{Ric}(u, u) + r^3 A_{r,u}
$$

$$
= r \left( \frac{2}{3} \text{Ric}(u, u) + \frac{5r}{6} \nabla_u \text{Ric}(u, u) + r^2 A_{r,u} \right),
$$

where $A_{r,u}$ depends on $r$, on $u$, on curvature terms and their covariant derivatives.

For sufficiently small $r$ the term $\frac{2}{3} \text{Ric}(u, u)$ is dominant in the sum $\frac{2}{3} \text{Ric}(u, u) + \frac{5r}{6} \nabla_u \text{Ric}(u, u) + r^2 A_{r,u}$.

Suppose $\text{Ric} \geq \lambda > 0$. Then there exists $\varepsilon > 0$ such that for all $0 < r \leq \varepsilon$, we have

$$
\frac{2}{3} \text{Ric}(u, u) + \frac{5r}{6} \nabla_u \text{Ric}(u, u) + r^2 A_{r,u} \geq 0.
$$
Hence \( \nabla_u \omega \leq 0 \) for \( 0 < r \leq \varepsilon \) and then
\[
L_{\exp^x v} \omega = \nabla r u \omega \\
= r \nabla_u \omega \\
\leq 0 \forall y \in B(x, \varepsilon). \quad (\ast)
\]

If now \((M, g)\) satisfies the \( I_\varepsilon \)-property, then from Proposition 3.2 and inequality \((\ast)\) we get
\[
(L_{v_y} \omega)(x) = 0 \quad \forall x \in M \quad \text{and} \quad \forall y \in B(x, \varepsilon).
\]

Use an analogous method in the case \( \text{Ric} \leq -\lambda. \quad \square \)

References


