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Christoffel–Darboux formula for zonal spherical functions for the Gelfand pair $(U(n), U(n-1))$

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Abstract

In this paper, first we show that an analogy of the Christoffel–Darboux formula holds for the zonal spherical functions for the Gelfand pair $(U(n), U(n-1))$. Next, making use of it, we deal with the problem on point-wise convergence of Fourier expansion by means of the zonal spherical functions for $(U(n), U(n-1))$.

Keywords: Christoffel–Darboux formula; Zonal spherical function; Gelfand pair $(U(n), U(n-1))$

1. Introduction

Let $P_\ell(x)$ be the Legendre polynomial of degree ℓ, and let denote $\sqrt{(2\ell+1)/2}\,P_\ell(x)$ by $p_\ell(x)$. Then the set $\{p_\ell \mid \ell = 0, 1, 2, \ldots\}$ is a complete orthonormal system of the Hilbert space $L^2((-1,1))$ with inner product

$$(f, g) = \int_{-1}^{1} f(x)\overline{g(x)}\,dx.$$

As is well known (cf. [4,6]), under suitable conditions for $f \in L^2((-1,1))$, the following holds:

$$f(x) = \sum_{\ell=0}^{\infty} (f, p_\ell)p_\ell(x). \tag{1}$$

On the other hand, we have

$$\sum_{\ell=0}^{k} (f, p_\ell)p_\ell(x) = \int_{-1}^{1} f(y)\left(\sum_{\ell=0}^{k} \frac{2\ell+1}{2} P_\ell(x)P_\ell(y)\right) dy,$$

and

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$$\sum_{\ell=0}^{k}(2\ell+1)P_\ell(x)P_\ell(y)=\frac{k+1}{x-y}\big(P_{k+1}(x)P_k(y)-P_k(x)P_{k+1}(y)\big). \tag{2}$$

The formula (2) is known as the Christoffel–Darboux formula and plays an important rule in the proof of (1). Notice that the formula (2) follows from the following recurrence relation:

$$(\ell+2)P_{\ell+2}(x)-(2\ell+3)xP_{\ell+1}(x)+(\ell+1)P_\ell(x)=0, \quad \ell\geqslant 0.$$

In general, let $p_\ell(x)$ be a real-valued polynomial of degree ℓ, $\{p_\ell \mid \ell=0,1,2,\ldots\}$ a system of polynomials orthonormal with respect to a positive measure on an interval (a,b), and assume that the system satisfies a recurrence relation of the following form:

$$p_{\ell+2}(x)-(A_\ell x+B_\ell)p_{\ell+1}(x)+C_\ell p_\ell(x)=0, \quad \ell\geqslant 0.$$

Then it is known (cf. [1]) that the Christoffel–Darboux formula holds under some suitable conditions for A_ℓ, B_ℓ and C_ℓ

$$\sum_{\ell=0}^{k}p_\ell(x)p_\ell(y)=\frac{1}{A_k}\frac{p_{k+1}(x)p_k(y)-p_k(x)p_{k+1}(y)}{x-y}, \quad k\geqslant 0. \tag{3}$$

Let us turn to the zonal spherical functions for the Gelfand pair $(U(n), U(n-1))$. They are related closely to the Jacobi polynomials (cf. [5]) and given by the orthogonal functions $G_{p,q}$, $p,q=0,1,2,\ldots$, which have the following generating function (cf. [7]):

$$\big(1-2\operatorname{Re}(wz)+|w|^2\big)^{1-n}=\sum_{p,q=0}^{\infty}G_{p,q}(z)w^p\overline{w}^q, \quad w,z\in\mathbf{C},\ |w|<1,\ |z|\leqslant 1.$$

Besides, we have the following recurrence relations for the system $\{G_{p,q} \mid p,q=0,1,2,\ldots\}$ (cf. [3]):

$$(p+1)\big(G_{p+1,q+1}(z)-\bar{z}\,G_{p+1,q}(z)\big)+(p+n-1)\big(G_{p,q}(z)-z\,G_{p,q+1}(z)\big)=0, \quad p,q\geqslant 0, \tag{4}$$

$$(q+1)\big(G_{p+1,q+1}(z)-z\,G_{p,q+1}(z)\big)+(q+n-1)\big(G_{p,q}(z)-\bar{z}\,G_{p+1,q}(z)\big)=0, \quad p,q\geqslant 0. \tag{5}$$

We are reasonably led to the question whether a situation similar to the Legendre case occurs in the case of the zonal spherical functions for $(U(n), U(n-1))$.

The first purpose of this paper is to show that an analogy of the Christoffel–Darboux formula (3) holds for the system $\{G_{p,q} \mid p,q=0,1,2,\ldots\}$.

The second purpose of this paper is to deal with the problem on point-wise convergence of the Fourier expansion for $f(z)$ defined on the unit open disk $|z|<1$ in $\mathbf{C}$

$$f(z)=\sum_{p,q=0}^{\infty}c_{p,q}G_{p,q}(z),$$

$$c_{p,q}=\frac{(p+q+n-1)p!q!(\Gamma(n-1))^2}{\pi\Gamma(p+n-1)\Gamma(q+n-1)}\int\limits_{|w|<1}f(w)\overline{G_{p,q}(w)}\big(1-|w|^2\big)^{n-2}\,dw,$$

where $dw=du\,dv$ with $w=u+iv$. More precisely, under some conditions for $f(z)$, we shall show that

$$f(z)=\lim_{k\to\infty}\sum_{0\leqslant p,q\leqslant k}c_{p,q}G_{p,q}(z). \tag{6}$$

The first purpose is dealt with in Section 3 and the second one in Section 4.

2. Notation and preliminaries

Throughout this paper let n be a positive integer such that $n\geqslant 2$. We shall use the notation $\mathbf{N}_0$, $\mathbf{C}$, $\mathbf{C}^n$, $U(n)$ for the set of nonnegative integers, the field of complex numbers, the usual n-dimensional complex space and the unitary group of degree n, respectively. For $z\in\mathbf{C}$ let $\operatorname{Re}(z)$ be the real part of z, and $z\mapsto\bar{z}$ the usual conjugation in $\mathbf{C}$.

We denote by $S(\mathbf{C}^n)$ the unit sphere in $\mathbf{C}^n$, and set $e_1 = {}^t(1, 0, \dots, 0) \in S(\mathbf{C}^n)$. Then we have the identification $U(n)/U(n-1) \cong S(\mathbf{C}^n)$ by the mapping $gU(n-1) \mapsto ge_1$, $g \in U(n)$. Further, the zonal spherical functions for $(U(n), U(n-1))$ are given by the functions $G_{p,q}$, $p, q \in \mathbf{N}_0$. That is to say, for each $p, q \in \mathbf{N}_0$ define

$$\varphi_{p,q}(g) = \frac{G_{p,q}((ge_1, e_1))}{G_{p,q}(1)}, \quad g \in U(n),$$

where (,) denotes the canonical inner product on the complex vector space $\mathbf{C}^n$. Then the set $\{\varphi_{p,q} \mid p, q \in \mathbf{N}_0\}$ is equal to the set of all the zonal spherical functions for $(U(n), U(n-1))$. We remark that $G_{q,p}(z) = \overline{G_{p,q}(z)} = G_{p,q}(\bar{z})$.

Let D be the unit open disk $|z| < 1$ in $\mathbf{C}$, and let $L^2(D)$ be the Hilbert space of Lebesgue measurable functions f on D with

$$\|f\|_D = \sqrt{\int_D \left|f(z)\right|^2 \left(1 - |z|^2\right)^{n-2} dz} < \infty,$$

where $dz = dx\,dy$ with $z = x + iy$. The inner product is given by

$$(f, g)_D = \int_D f(z)\overline{g(z)}\left(1 - |z|^2\right)^{n-2} dz.$$

Then the set $\{G_{p,q} \mid p, q \in \mathbf{N}_0\}$ is a complete orthogonal system of $L^2(D)$.

The Jacobi polynomials $P_\ell^{(\alpha,\beta)}(x)$, $\ell = 0, 1, 2, \dots$, are defined by the Rodrigues formula (cf. [2]):

$$P_\ell^{(\alpha,\beta)}(x) = \frac{(-1)^\ell}{2^\ell \ell!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^\ell}{dx^\ell}\left[(1-x)^{\alpha+\ell}(1+x)^{\beta+\ell}\right].$$

A function is assumed to be complex-valued. Denote $\Gamma(\alpha + \ell)/\Gamma(\alpha)$ by $(\alpha)_\ell$, where Γ is the Gamma function.

3. Christoffel–Darboux formula for the system $\{G_{p,q} \mid p, q \in \mathbf{N}_0\}$

In this section we shall show the following theorem, which can be regarded as a Christoffel–Darboux formula for the system $\{G_{p,q} \mid p, q \in \mathbf{N}_0\}$.

Theorem 1. *For an arbitrary positive integer k, we have*

$$\sum_{0 \leqslant p,q \leqslant k} \frac{(p+q+n-1)\,p!\,q!}{\Gamma(p+n-1)\Gamma(q+n-1)}(\bar{z} - w)G_{p,q}(z)G_{p,q}(w)$$
$$= \sum_{p=0}^{k} \frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}\left(G_{p,k+1}(z)G_{p,k}(w) - G_{k,p}(z)G_{k+1,p}(w)\right).$$

Set

$$L_{p,q}(z, w) = \frac{(p+q+n-1)\,p!\,q!}{\Gamma(p+n-1)\Gamma(q+n-1)}(\bar{z} - w)G_{p,q}(z)G_{p,q}(w).$$

Then Theorem 1 may be rewritten in the following:

$$\sum_{0 \leqslant p,q \leqslant k} L_{p,q}(z, w) = \sum_{p=0}^{k} \frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}\left(G_{p,k+1}(z)G_{p,k}(w) - G_{k,p}(z)G_{k+1,p}(w)\right). \tag{7}$$

We shall make some preparations to show the formula (7). First, we show the following lemma.

Lemma 1. *For any $p, q \in \mathbf{N}_0$, we have*

$$(p+q+n)\bar{z}G_{p+1,q}(z) = (p+n-1)G_{p,q}(z) + (q+1)G_{p+1,q+1}(z), \tag{8}$$

$$(p+q+n)zG_{p,q+1}(z) = (q+n-1)G_{p,q}(z) + (p+1)G_{p+1,q+1}(z). \tag{9}$$

Proof. If we add (4) to (5) and solve it for $zG_{p,q+1}(z)$, then we have

$$zG_{p,q+1}(z) = \frac{p+q+2}{p+q+n}G_{p+1,q+1}(z) + \frac{p+q+2n-2}{p+q+n}G_{p,q}(z) - \bar{z}G_{p+1,q}(z).$$

Substituting this into (5), we can obtain (8). Similarly, we can also obtain (9). □

Remark. Another proof of Lemma 1 can be found in [8].

It follows from Lemma 1 the following lemma.

Lemma 2. *For an arbitrary positive integer k, we have*

$$\sum_{1\leqslant p,q\leqslant k} L_{p,q}(z,w) = \sum_{p=1}^{k} \frac{p!}{\Gamma(n-1)\Gamma(p+n-1)}\big(G_{0,p}(z)G_{1,p}(w) - G_{p,1}(z)G_{p,0}(w)\big) + \sum_{p=1}^{k} \frac{p!\,(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}\big(G_{p,k+1}(z)G_{p,k}(w) - G_{k,p}(z)G_{k+1,p}(w)\big).$$

Proof. If we replace p by $p-1$ in (8) and q by $q-1$ in (9), then we have

$$(p+q+n-1)\bar{z}G_{p,q}(z) = (p+n-2)G_{p-1,q}(z) + (q+1)G_{p,q+1}(z), \tag{10}$$

$$(p+q+n-1)zG_{p,q}(z) = (q+n-2)G_{p,q-1}(z) + (p+1)G_{p+1,q}(z). \tag{11}$$

We multiply the recurrence relation (10) by $G_{p,q}(w)$ and denote the resulting equation by $(10)_z G_{p,q}(w)$. Further, we multiply by $G_{p,q}(z)$ the recurrence relation (11) with z and w interchanged, and denote the resulting equation by $(11)_w G_{p,q}(z)$. Then the difference $(10)_z G_{p,q}(w) - (11)_w G_{p,q}(z)$ is given as follows:

$$\begin{aligned}&(p+q+n-1)(\bar{z}-w)G_{p,q}(z)G_{p,q}(w)\\ &\quad= (p+n-2)G_{p-1,q}(z)G_{p,q}(w) - (p+1)G_{p,q}(z)G_{p+1,q}(w)\\ &\qquad - \big\{(q+n-2)G_{p,q}(z)G_{p,q-1}(w) - (q+1)G_{p,q+1}(z)G_{p,q}(w)\big\}, \quad p,q \geqslant 1.\end{aligned}$$

Hence, we obtain that for $p,q \geqslant 1$

$$\begin{aligned}&\frac{(p+q+n-1)p!q!}{\Gamma(p+n-1)\Gamma(q+n-1)}(\bar{z}-w)G_{p,q}(z)G_{p,q}(w)\\ &\quad= I_{p,q}(z,w) - I_{p+1,q}(z,w) - \big(J_{p,q}(z,w) - J_{p,q+1}(z,w)\big),\end{aligned}$$

where

$$I_{p,q}(z,w) = \frac{p!q!}{\Gamma(p+n-2)\Gamma(q+n-1)}G_{p-1,q}(z)G_{p,q}(w),$$

and

$$J_{p,q}(z,w) = \frac{p!\,q!}{\Gamma(p+n-1)\Gamma(q+n-2)}G_{p,q}(z)G_{p,q-1}(w).$$

This completes the proof. □

By the definition of $L_{p,q}(z,w)$, the left-hand side of Theorem 1 is $\sum_{0\leqslant p,q\leqslant k} L_{p,q}(z,w)$, and the left-hand side of Lemma 2 is $\sum_{1\leqslant p,q\leqslant k} L_{p,q}(z,w)$. Let us consider the difference $\sum_{0\leqslant p,q\leqslant k} L_{p,q}(z,w) - \sum_{1\leqslant p,q\leqslant k} L_{p,q}(z,w)$.

$$\begin{aligned}
&\sum_{0\leqslant p,q\leqslant k} L_{p,q}(z,w) - \sum_{1\leqslant p,q\leqslant k} L_{p,q}(z,w) \\
&\quad = \sum_{q=0}^{k} L_{0,q}(z,w) + \sum_{p=0}^{k} L_{p,0}(z,w) - \frac{n-1}{(\Gamma(n-1))^2}(\bar{z}-w)G_{0,0}(z)G_{0,0}(w) \\
&\quad = \sum_{p=1}^{k} \frac{(p+n-1)\,p!}{\Gamma(n-1)\Gamma(p+n-1)}(\bar{z}-w)\big(G_{0,p}(z)G_{0,p}(w) + G_{p,0}(z)G_{p,0}(w)\big) \\
&\qquad + \frac{n-1}{(\Gamma(n-1))^2}(\bar{z}-w)G_{0,0}(z)G_{0,0}(w). \qquad (12)
\end{aligned}$$

Therefore, we obtain

Lemma 3. *For an arbitrary positive integer k, we have*

$$\begin{aligned}
\sum_{0\leqslant p,q\leqslant k} L_{p,q}(z,w) = {} & \sum_{p=1}^{k} \frac{p!}{\Gamma(n-1)\Gamma(p+n-1)}\big\{\big(G_{0,p}(z)G_{1,p}(w) - G_{p,1}(z)G_{p,0}(w)\big) \\
& + (p+n-1)(\bar{z}-w)\big(G_{0,p}(z)G_{0,p}(w) + G_{p,0}(z)G_{p,0}(w)\big)\big\} \\
& + \sum_{p=1}^{k} \frac{p!\,(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}\big(G_{p,k+1}(z)G_{p,k}(w) - G_{k,p}(z)G_{k+1,p}(w)\big) \\
& + \frac{n-1}{(\Gamma(n-1))^2}(\bar{z}-w)G_{0,0}(z)G_{0,0}(w).
\end{aligned}$$

Proof. This follows from Lemma 2 and (12). □

We shall now show the assertion (7). In Lemma 3, replacing $G_{p,0}, G_{0,p}, G_{p,1}, G_{1,p}$ by the explicit expressions, we see that

$$\begin{aligned}
\sum_{0\leqslant p,q\leqslant k} L_{p,q}(z,w) = {} & \sum_{p=1}^{k} \frac{p!\,(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}\big(G_{p,k+1}(z)G_{p,k}(w) - G_{k,p}(z)G_{k+1,p}(w)\big) \\
& + \sum_{p=1}^{k}\big(K_p(z,w) - K_{p+1}(z,w)\big) + \frac{n-1}{(\Gamma(n-1))^2}(\bar{z}-w)G_{0,0}(z)G_{0,0}(w),
\end{aligned}$$

where

$$K_p(z,w) = \frac{(p+n-2)!}{(\Gamma(n-1))^3(p-1)!}\big(z^{p-1}w^p - \bar{z}^p\overline{w}^{p-1}\big),$$

which implies (7).

4. Point-wise convergence of Fourier expansion by the system $\{G_{p,q} \mid p,q \in \mathbf{N}_0\}$

For each $p,q \in \mathbf{N}_0$, let $F_{p,q}$ be the normalization of $G_{p,q}$ with respect to the norm of $L^2(D)$. Then the set $\{F_{p,q} \mid p,q \in \mathbf{N}_0\}$ is a complete orthonormal system of $L^2(D)$. By simple calculations we have

$$F_{p,q}(z) = \sqrt{\frac{(p+q+n-1)p!q!(\Gamma(n-1))^2}{\pi\Gamma(p+n-1)\Gamma(q+n-1)}}\,G_{p,q}(z).$$

In this section, making use of Theorem 1, under some conditions for $f \in L^2(D)$ we shall show that

$$f(z) = \lim_{k\to\infty} \sum_{0\leqslant p,q\leqslant k} (f, F_{p,q})_D F_{p,q}(z),$$

which is equivalent to (6). While the right-hand side of the Christoffel–Darboux formula (2) consists of two terms for any k, that of the Christoffel–Darboux formula (7) consists of $k+1$ terms. Therefore, comparing with the proof of the formula (1), we need some devices to prove the formula above. In what follows, we divide this section into four parts.

4.1. Evaluation of $|f(z)-\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z)|$

Let k be a positive integer, and let us evaluate the absolute value $|f(z)-\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z)|$. By simple calculations we see that

$$\begin{aligned}\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z) &= \int_D f(w)\Bigg(\sum_{0\leqslant p,q\leqslant k}\overline{F_{p,q}(w)}F_{p,q}(z)\Bigg)\big(1-|w|^2\big)^{n-2}\,dw\\ &= \int_D f(w)\Bigg(\sum_{0\leqslant p,q\leqslant k}F_{p,q}(\overline{w})F_{p,q}(z)\Bigg)\big(1-|w|^2\big)^{n-2}\,dw, \qquad (13)\end{aligned}$$

where $dw=du\,dv$ with $w=u+iv$. Further, we have

$$\begin{aligned}\sum_{0\leqslant p,q\leqslant k}F_{p,q}(\overline{w})F_{p,q}(z) &= \frac{(\Gamma(n-1))^2}{\pi}\sum_{0\leqslant p,q\leqslant k}\frac{(p+q+n-1)p!q!}{\Gamma(p+n-1)\Gamma(q+n-1)}G_{p,q}(\overline{w})G_{p,q}(z)\\ &= \frac{(\Gamma(n-1))^2}{\pi(\bar{z}-\overline{w})}\sum_{0\leqslant p,q\leqslant k}L_{p,q}(z,\overline{w}).\end{aligned}$$

On the other hand, since the set $\{F_{p,q}\mid p,q\in\mathbf{N}_0\}$ is a complete orthonormal system of $L^2(D)$, the following holds:

$$\int_D\Bigg(\sum_{0\leqslant p,q\leqslant k}F_{p,q}(\overline{w})F_{p,q}(z)\Bigg)\big(1-|w|^2\big)^{n-2}\,dw=1.$$

Hence, we have

$$\begin{aligned}&f(z)-\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z)\\ &\quad=\int_D\big(f(z)-f(w)\big)\Bigg(\sum_{0\leqslant p,q\leqslant k}F_{p,q}(\overline{w})F_{p,q}(z)\Bigg)\big(1-|w|^2\big)^{n-2}\,dw\\ &\quad=\frac{(\Gamma(n-1))^2}{\pi}\int_D\frac{f(z)-f(w)}{\bar{z}-\overline{w}}\Bigg(\sum_{0\leqslant p,q\leqslant k}L_{p,q}(z,\overline{w})\Bigg)\big(1-|w|^2\big)^{n-2}\,dw.\end{aligned}$$

Therefore, by Theorem 1, we obtain

$$\begin{aligned}&f(z)-\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z)\\ &\quad=\frac{(\Gamma(n-1))^2}{\pi}\sum_{p=0}^{k}\frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}\int_D\frac{f(z)-f(w)}{\bar{z}-\overline{w}}\\ &\qquad\times\big(G_{p,k+1}(z)G_{p,k}(\overline{w})-G_{k,p}(z)G_{k+1,p}(\overline{w})\big)\big(1-|w|^2\big)^{n-2}\,dw. \qquad (14)\end{aligned}$$

Using the formula (14), we can arrive at the following:

$$\begin{aligned}&\Bigg|f(z)-\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z)\Bigg|\\ &\quad\leqslant\frac{(\Gamma(n-1))^2}{\pi}\sum_{p=0}^{k}\frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}(A_{p,k}+B_{p,k}), \qquad (15)\end{aligned}$$

where

$$A_{p,k} = \left| G_{p,k+1}(z) \int\limits_D \frac{f(z) - f(w)}{\bar{z} - \overline{w}} G_{p,k}(\overline{w})\big(1 - |w|^2\big)^{n-2}\, dw \right|,$$

$$B_{p,k} = \left| G_{k,p}(z) \int\limits_D \frac{f(z) - f(w)}{\bar{z} - \overline{w}} G_{k+1,p}(\overline{w})\big(1 - |w|^2\big)^{n-2}\, dw \right|.$$

4.2. Evaluations of $A_{p,k}$ and $B_{p,k}$

Set

$$\varphi(z, w) = \frac{f(z) - f(w)}{\bar{z} - \overline{w}},$$

regard this function as a function of w and suppose that it belongs to $L^2(D)$ as a function of w. Let us evaluate $A_{p,k}$, $B_{p,k}$. By the Riemann–Lebesgue lemma, we see that there exists a positive number M such that the following inequality holds for all $p, q \in \mathbf{N}_0$:

$$\left| \int\limits_D \varphi(z, w)\overline{F_{p,q}(w)}\big(1 - |w|^2\big)^{n-2}\, dw \right| \leqslant M.$$

In other words,

$$\left| \int\limits_D \varphi(z, w) G_{p,q}(\overline{w})\big(1 - |w|^2\big)^{n-2}\, dw \right| \leqslant M \sqrt{\frac{\pi \Gamma(p + n - 1)\Gamma(q + n - 1)}{(p + q + n - 1)p!q!(\Gamma(n - 1))^2}},$$

for all $p, q \in \mathbf{N}_0$. Therefore, we have

$$A_{p,k} \leqslant M \big|G_{p,k+1}(z)\big| \sqrt{\frac{\pi \Gamma(p + n - 1)\Gamma(k + n - 1)}{(p + k + n - 1)p!k!(\Gamma(n - 1))^2}}, \tag{16}$$

$$B_{p,k} \leqslant M \big|G_{k,p}(z)\big| \sqrt{\frac{\pi \Gamma(p + n - 1)\Gamma(k + n)}{(p + k + n)p!(k + 1)!(\Gamma(n - 1))^2}}. \tag{17}$$

4.3. Evaluation of $|G_{p,k}(z)|$

Let $0 \leqslant p \leqslant k$ and let us evaluate $|G_{p,k}(z)|$. Set $z = re^{i\theta}$. As is well known (cf. [5]), the following equality holds:

$$G_{p,k}\big(re^{i\theta}\big) = \frac{(-1)^p e^{i(p-k)\theta}}{(n - 2)!(k - p + 1)_p} r^{k-p} P_p^{(k-p, n-2)}\big(1 - 2r^2\big). \tag{18}$$

On the other hands, it is known that the maximum value of $|P_p^{(k-p,n-2)}(x)|$ in $-1 \leqslant x \leqslant 1$ is given by $(s + 1)_p/p!$, where $s = \max\{k - p, n - 2\}$ (cf. [1,2]). Therefore, we can conclude that

$$\big|G_{p,k}\big(re^{i\theta}\big)\big| \leqslant \begin{cases} \frac{(p+n-2)!}{(n-2)!p!k!}, & k - p \leqslant n - 2, \\ \frac{r^{k-p}}{(n-2)!p!}, & k - p > n - 2. \end{cases} \tag{19}$$

4.4. Conclusion

Let us return to the inequality (15). By (17) and (19), if $k - p \leqslant n - 2$, we have

$$\begin{aligned}
&\frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}B_{p,k}\\
&\quad\leqslant \frac{M(k+1)}{((n-2)!)^2(k+n-2)!}\sqrt{\frac{\pi(p+n-2)!(k+n-1)!}{(p+k+n)p!(k+1)!}}\\
&\quad\leqslant \frac{M(k+1)}{((n-2)!)^2}\sqrt{\frac{\pi(k+n-1)}{(p+k+n)p!(k+1)!}}\\
&\quad\leqslant \frac{M}{((n-2)!)^2}\sqrt{\frac{\pi(k+1)}{k!}}.
\end{aligned}$$

Similarly, if $k-p>n-2$, we have

$$\begin{aligned}
&\frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}B_{p,k}\\
&\quad\leqslant \frac{Mr^{k-p}}{((n-2)!)^2}\sqrt{\frac{\pi(k+1)!(k+n-1)}{(p+n-2)!(k+n-2)!(p+k+n)p!}}\\
&\quad\leqslant \frac{Mr^{k-p}}{((n-2)!)^2}\sqrt{\frac{\pi(k+1)!}{(p!)^2k!}}=\frac{M\sqrt{\pi(k+1)}}{((n-2)!)^2}r^{k-p}\frac{1}{p!},
\end{aligned}$$

where $r=|z|$. Therefore, we obtain

$$\begin{aligned}
&\sum_{p=0}^{k}\frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}B_{p,k}\\
&\quad\leqslant \frac{M(k+1)}{((n-2)!)^2}\sqrt{\frac{\pi(k+1)}{k!}}+\frac{M\sqrt{\pi(k+1)}}{((n-2)!)^2}r^k\sum_{p=0}^{k-n+1}\frac{1}{r^p p!}\to 0,\quad \text{as } k\to\infty.
\end{aligned}\tag{20}$$

Remark. This argument is valid for $0<r<1$. But, since $0^{k-p}=0$ for $k-p>n-2$, the conclusion holds also for $r=0$.

In the same way, by (16) and (19), we have

$$\frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}A_{p,k}\leqslant\begin{cases}\frac{M}{((n-2)!)^2}\sqrt{\frac{\pi}{k!}}, & k+1-p\leqslant n-2,\\ \frac{M\sqrt{\pi}(k+1)}{((n-2)!)^2}r^{k+1-p}\frac{1}{p!}, & k+1-p>n-2.\end{cases}$$

Hence, we can obtain

$$\sum_{p=0}^{k}\frac{p!(k+1)!}{\Gamma(p+n-1)\Gamma(k+n-1)}A_{p,k}\to 0,\quad \text{as } k\to\infty.\tag{21}$$

By (15), (20) and (21), we can arrive at

$$\left|f(z)-\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z)\right|\to 0,\quad \text{as } k\to\infty.$$

That is to say,

$$f(z)=\lim_{k\to\infty}\sum_{0\leqslant p,q\leqslant k}(f,F_{p,q})_D F_{p,q}(z).$$

Theorem 2. *Let $f \in L^2(D)$ and $z \in D$. If f satisfies*

$$\int_D \left|\frac{f(z)-f(w)}{z-w}\right|^2 \left(1-|w|^2\right)^{n-2} dw < \infty,$$

where $dw = du\,dv$ with $w = u + iv$, then the following holds:

$$f(z) = \lim_{k\to\infty} \sum_{0\leqslant p,q\leqslant k} (f, F_{p,q})_D F_{p,q}(z).$$

Example. Set $f(z) = |z|$. By the inequality $||z|-|w|| \leqslant |z-w|$, the function f satisfies the assumptions of Theorem 2 for all $z \in D$. On the other hand, by (18) we have

$$\int_D |w|\overline{G_{p,q}(w)}\left(1-|w|^2\right)^{n-2} dw = \begin{cases} \frac{(-1)^p\pi}{2^{n-1/2}(n-2)!p!}\int_{-1}^1 P_p^{(0,n-2)}(t)(1-t)^{1/2}(1+t)^{n-2}\,dt, & p = q, \\ 0, & p \neq q. \end{cases}$$

Combine this relation with the following connection formula (cf. [1]):

$$P_p^{(\gamma,\beta)}(t) = \frac{(\beta+1)_p}{(\alpha+\beta+2)_p}\sum_{\ell=0}^{p}\frac{(\gamma-\alpha)_{p-\ell}(\alpha+\beta+1)_\ell(\alpha+\beta+2\ell+1)(\beta+\gamma+p+1)_\ell}{(p-\ell)!(\beta+1)_\ell(\alpha+\beta+1)(\alpha+\beta+p+2)_\ell}P_\ell^{(\alpha,\beta)}(t),$$

where $\alpha = 1/2$, $\beta = n-2$ and $\gamma = 0$. Then we obtain

$$\int_D |w|\overline{G_{p,p}(w)}\left(1-|w|^2\right)^{n-2} dw = \frac{\pi^{3/2}(-1)^p(n-1)_p(-1/2)_p}{2(p!)^2\Gamma(p+n+1/2)}.$$

Therefore, we can arrive at

$$|z| = \sum_{p=0}^{\infty} c_p G_{p,p}(z),$$

$$c_p = \frac{\pi^{1/2}(-1)^p(2p+n-1)((n-2)!)^2(n-1)_p(-1/2)_p}{2((p+n-2)!)^2\Gamma(p+n+1/2)}.$$

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