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Monomial Modular Representations and Construction of the Held Group

R. T. Curtis

School of Mathematics and Statistics, University of Birmingham,

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Monomial representations of familiar finite groups over finite fields are used to construct (infinite) semi-direct products of free products of cyclic groups by groups of monomial automorphisms. Finite homomorphic images of these *progenitors* in which the actions on the group of automorphisms and on the cyclic components are faithful are sought. The smallest non-trivial images of this type are often sporadic simple groups. The technique is demonstrated by three examples over the fields $\mathbf{Z}_3$, $\mathbf{Z}_5$, and $\mathbf{Z}_7$, which produce the Mathieu group M_{11}, the unitary group $U_3(5):2$, and the Held group, respectively.

1. INTRODUCTION

In [8, 9] we showed how the Mathieu group M_{12} is generated by five elements of order 3 which are permuted under conjugation within M_{12} by a subgroup isomorphic to the alternating group A_5, and how the Mathieu group M_{24} is generated by seven involutions which are similarly permuted by a subgroup isomorphic to the linear group $L_3(2)$. Thus M_{12} is a homomorphic image of a free product of five cyclic groups of order 3, extended by a group of automorphisms permuting five generators of these cyclic groups evenly; and M_{24} is a homomorphic image of a free product of seven cyclic groups of order 2, extended by a group of permutations of the generators isomorphic to $L_3(2)$.

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In the notation used below M_{12} and M_{24} are thus homomorphic images of

$$3^{\star 5} : A_5 \qquad \text{and} \qquad 2^{\star 7} : L_3(2)$$

respectively. In more recent work, see [10–12], we showed how several of the sporadic simple groups can be defined and constructed by hand as images of similarly defined groups of shape $2^{\star n} : \mathcal{N}$ where $\mathcal{N}$ is a group of permutations acting transitively on the n involutory symmetric generators.

In the present work we generalize this approach to the case when the symmetric generators are of prime order greater than 2, and the action of $\mathcal{N}$ is to permute them and raise them to some power; that is, $\mathcal{N}$ has a proper monomial action on the n cyclic subgroups. Thus we take a monomial representation of a familiar group, change the base field to a prime field of order p having roots of unity of the necessary order, and interpret each element as an automorphism of a free product of cyclic groups of order p through its matrix representation. We then form the semi-direct product of the free product by this group of automorphisms and seek its smallest non-collapsing homomorphic image which is generated by the images of the cyclic groups.

2. MONOMIAL MODULAR REPRESENTATIONS

A *monomial matrix* is a matrix in which there is precisely one non-zero term in each row, and one in each column; thus a monomial matrix is a product of a non-singular diagonal matrix and a permutation matrix. We require that the set of non-zero entries should be closed under multiplication and inversion and so lie in, say, the group of units of a ring. In this paper we restrict our attention to the case when this ring is a field F, and so a *monomial representation* of a group G is a homomorphism from G into $GL_n(F)$, the group of non-singular $n \times n$ matrices over the field F, in which the image of every element of G is a monomial matrix over F. Thus the action of the image of a monomial representation on the underlying vector space is to permute the vectors of a basis while multiplying them by scalars. Every monomial representation of G in which this permutation action is transitive is obtained by inducing a linear representation of a subgroup H up to G. If this linear representation is trivial we obtain the permutation representation of G acting on the cosets of H. Otherwise we obtain a proper monomial representation. Now an ordinary linear representation of H Is a homomorphism of H onto C_m, say, a cyclic multiplicative subgroup of the complex numbers $\mathbb{C}$, and the resulting monomial matrices will involve complex mth roots of unity. But we can similarly define a linear representation into a finite field F which possesses mth roots of unity.

EXAMPLE 2.1. Consider the double cover of S_4, the symmetric group on four letters, in which the transpositions lift to involutions. This group G, which is isomorphic to the general linear group $GL_2(3)$ and whose ordinary character table is displayed in Table I, is often denoted by $2^{\cdot}S_4^+$. The isoclinic group, see [5, page xxiii], in which transpositions lift to elements of order 4, is denoted by $2^{\cdot}S_4^-$. Now a subgroup H of index 4 in G is isomorphic to $2 \times S_3$. One of the three non-trivial linear representations of H maps the involution in its center to 1, and so the resulting monomial representation is not faithful. The other two representations map this involution to -1 and give rise to equivalent monomial representations of degree 4, whose character is highlighted in Table I.

Now G may be shown to have the following presentation

$$\left\langle x, y \mid y^3 = (xy)^4, x^2 = 1 \right\rangle,$$

TABLE I
Character Tables of $2^{\cdot}S_4^+$ and $2^{\cdot}A_5$

	24	8	3	4	4
	1A	2A	3A	2B	4A
+	1	1	1	1	1
+	1	1	1	−1	−1
+	2	2	−1	0	0
+	3	−1	0	1	−1
+	3	−1	0	−1	1
	1	4	3	2	8
	2		6		8
o	2	0	−1	0	$i\sqrt{2}$
o	2	0	−1	0	$-i\sqrt{2}$
+	**4**	**0**	**1**	**0**	**0**

	60	4	3	5	5
	1A	2A	3A	5A	B*
+	1	1	1	1	1
+	3	−1	0	−b5	*
+	3	−1	0	*	−b5
+	4	0	1	−1	−1
+	5	1	−1	0	0
	1	4	3	5	5
	2		6	10	10
−	2	0	−1	b5	*
−	2	0	−1	*	b5
−	4	0	1	−1	−1
−	**6**	**0**	**0**	**1**	**1**

and the above representation is generated by

$$x = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix}, \qquad y = \begin{bmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & -1 & \cdot & \cdot \end{bmatrix}.$$

Clearly these matrices will give a faithful representation over any field of characteristic other than 2. So, in particular, they give a 3-modular monomial representation of the group.

EXAMPLE 2.2. Consider now $G \cong 2^{\cdot}A_5 \cong SL_2(5)$, the double cover of the alternating group on five letters. A subgroup of index 5 has shape $2^{\cdot}A_4$ and its derived group contains its centre. Thus any linear representation would have the centre in its kernel, and the resulting monomial representation of G would not be faithful. A subgroup H of index 6 and order 20 has shape $H \cong 2^{\cdot}D_{10}$ or $H \cong (2 \times 5)^{\cdot}2$ and presentation $\langle u, v \mid u^5 = v^4 = 1,\ u^v = u^{-1} \rangle$. The quotient $H/H' \cong C_4$ and $H' \cong C_5$. To obtain a faithful monomial representation we must induce a linear representation of H which maps a generator of this cyclic group of order 4 onto a primitive 4th root of unity in our field. The character of the two resulting equivalent representations is highlighted in Table I, and a presentation for the group is

$$\left\langle x, y \mid x^5 = y^4 = (xy)^3 = 1 = \left[x, y^2\right] \right\rangle;$$

over $\mathbb{C}$ the representation is given by

$$x = \left[\begin{array}{ccc|ccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{array}\right], \qquad y = \left[\begin{array}{ccc|ccc} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \hline \cdot & \cdot & \cdot & i & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{array}\right].$$

Now the smallest finite field which contains primitive 4th roots of unity is $GF_5 \cong Z_5$, the integers modulo 5. Thus a faithful 5-modular representation of G is obtained by replacing i by 2 and $-i$ by 3 in the matrix representing y.

EXAMPLE 2.3. As our last example of a monomial representation we take $G \cong 3^{\cdot}A_7$, the triple cover of the alternating group on seven letters. We seek a *faithful* monomial representation of this group. Now a subgroup of index 7 in G is isomorphic to $3^{\cdot}A_6$, which being perfect has the center of G in the kernel of its unique linear representation; thus G has a unique seven-dimensional monomial representation, which is the permutation representation lifted from A_7. However, G possesses two classes of subgroups of index 15 which are isomorphic to $3 \times L_2(7)$ and which are fused by the outer automorphism. Such a subgroup H possesses linear representations onto C_3, the group of complex cube roots of unity. Inducing such representations up to G we obtain two faithful monomial representations of G of degree 15, whose characters are the second pair of faithful irreducible characters of degree 15 in Table II. In order to construct this representation explicitly we take the following presentation of G:

$$\left\langle x, y \mid x^7 = y^4 = \left(xy^2\right)^3 = \left(x^3 y\right)^3 = (yx)^5 \cdot \left(xyx^2 y^{-1}\right)^2 = 1 \right\rangle \cong 3^{\cdot}A_7.$$

Coset enumeration over $\langle x \rangle$ verifies the order, the group is visibly perfect (abelianizing reduces it to the trivial group), and the homomorphism $x \mapsto (1234567)$, $y \mapsto (1724)(56)$ demonstrates that it has A_7 as an image. It is particularly suitable for our purposes as $(x, y^2) \cong L_3(2)$, the subgroup in which we are interested. Induction to the full group, as described above,

TABLE II
The Character Table of $3^{\cdot}A_7$

	2520	24	36	9	4	5	12	7	7
	1A	2A	3A	3B	4A	5A	6A	7A	B**
+	1	1	1	1	1	1	1	1	1
+	6	2	3	0	0	1	−1	−1	−1
o	10	−2	1	1	0	0	1	b7	**
o	10	−2	1	1	0	0	1	**	b7
+	14	2	2	−1	0	−1	2	0	0
+	14	2	−1	2	0	−1	−1	0	0
+	15	−1	3	0	−1	0	−1	1	1
+	21	1	−3	0	−1	1	1	0	0
+	35	−1	−1	−1	1	0	−1	0	0
	1	2	3	3	4	5	5	7	7
	3	6			12	15	6	21	21
	3	6			12	15	6	21	21
o2	6	2	0	0	0	1	2	−1	−1
o2	15	−1	0	0	−1	0	2	1	1
o2	**15**	**3**	**0**	**0**	**1**	**0**	**0**	**1**	**1**
o2	21	1	0	0	−1	1	−2	0	0
o2	21	−3	0	0	1	1	0	0	0
o2	24	0	0	0	0	−1	0	b7	**
o2	24	0	0	0	0	−1	0	**	b7

then gives us the representation:

$$x = \left[\begin{array}{ccccccc|ccccccc|c}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right] ;$$

$$y = \left[\begin{array}{ccccccc|ccccccc|c}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \omega & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{\omega} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \overline{\omega} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{\omega} \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \overline{\omega} & \cdot & \cdot & \cdot \\
\cdot & \omega & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \overline{\omega}\,\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right] .$$

Of course such a matrix may, as mentioned previously, be written as a non-singular diagonal matrix followed by a permutation matrix:

$$u = \big(I_{15}, (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15)\big),$$

$$v = \Big(\mathrm{diag}\big(\omega, \omega, 1, \overline{\omega}, \overline{\omega}, 1, \overline{\omega}, \overline{\omega}, \omega, \overline{\omega}, 1, 1, \omega, 1, \omega\big),$$

$$(1,11,8,12)(2,6,13,9)(4,10,5,3)(7,15)(14)\Big).$$

For our purposes we require a finite field with cube roots of unity. If we insist on a prime field, then we may take Z_7, the integers modulo 7, and replace ω and $\overline{\omega}$ by 2 and 4 respectively; or we may simply interpret ω and $\overline{\omega}$ as elements in the Galois field of order 4 in the usual way.

3. SYMMETRIC GENERATION AND PROGENITOR GROUPS

As in [10–12] we adopt the notation $m^{\star n}$ to mean $C_m \star C_m \star \cdots \star C_M$ (n times), a free product of n copies of the cyclic group of order m. Let $F = T_0 \star T_1 \star \cdots \star T_{n-1}$ be such a group, with $T_i = \langle t_i \rangle \cong C_m$. Certainly permutations of the set of *symmetric generators* $\mathscr{T} = \{t_0, t_1, \dots, t_{n-1}\}$ induce automorphisms of F. But raising a given t_i to a power of itself coprime to m, while fixing the others, also gives rise to an automorphism of F. Together these generate the group $\mathscr{M}$ of all *monomial automorphisms* of F which is a wreath product $H_r \wr S_n$, where H_r is an abelian group of order $r = \phi(m)$, the number of positive integers less than m and coprime to it. A split extension of the form

$$\mathscr{P} \cong m^{\star n} : \mathscr{N},$$

where $\mathscr{N}$ is a subgroup of $\mathscr{M}$, which acts transitively on the set of cyclic subgroups $\bar{\mathscr{T}} = \{T_0, T_1, \dots, T_{n-1}\}$, is called a *progenitor*. $\mathscr{N}$ is known as the *control subgroup* and its elements are *monomial permutations* or, more informally, *permutations*. Of course, $\mathscr{N}$ may simply permute the set of elements $\mathscr{T}$, as will always be the case when $m = 2$, and a wealth of interesting homomorphic images arise from this case. The more general case involving proper monomial action allows further fascinating possibilities. Note that, since $\mathscr{P} = \langle \mathscr{N}, \mathscr{T} \rangle$ and the action of $\mathscr{N}$ on $\mathscr{T}$ by conjugation is well-defined, elements of $\mathscr{P}$ may be put into a canonical form by gathering the permutations on the left-hand side. Thus every element of $\mathscr{P}$ may be written, essentially uniquely, as a permutation in $\mathscr{N}$ followed by a word in the symmetric generators $\mathscr{T}$. In particular, if we seek homomorphic images of $\mathscr{P}$, as we shall be doing, the relators by which we must factor will have form πw, for $\pi \in \mathscr{N}$ and w a word in the elements of $\mathscr{T}$.

EXAMPLE 3.1. Our monomial representation over GF_3 in Example 2.1 shows how the group $2^{\cdot}S_4^+$ can act faithfully as control subgroup in the progenitor

$$\mathscr{P} \cong 3^{\star 4} : 2^{\cdot}S_4^+ .$$

To obtain a presentation for this subgroup we must adjoint an element of order 3, t_0 say, to the control subgroup $\mathscr{N} \cong 2^{\cdot}S_4^+$ such that $T_0 = \langle t_0 \rangle$ has just four images under its action. Specifically we require

$$x \sim (t_0\ \ t_1)(t_2)\left(t_3\ \ t_3^{-1}\right),\ y \sim \left(t_0\ \ t_0^{-1}\right)\left(t_1\ \ t_2^{-1}\ \ t_3\ \ t_1^{-1}\ \ t_2\ \ t_3^{-1}\right),$$

under conjugation, or more concisely

$$x \sim (0\ \ 1)(2)(3\ \ \bar{3}),\ y \sim (0\ \ \bar{0})\left(1\ \ \bar{2}\ \ 3\ \ \bar{1}\ \ 2\ \ \bar{3}\right).$$

This is easily seen to be achieved by

$$\left\langle x, y, t \mid y^3 = (xy)^4,\ x^2 = 1 = t^3 = t^y t = \left[t^{xy}, t\right]\right\rangle \cong 3^{\star 4} : 2^{\cdot}S_4^+,$$

since $N_{\mathscr{N}}(\langle t_0 \rangle) = \langle y, x^{y^{-1}x} \rangle \cong 2 \times S_3$.

EXAMPLE 3.2. The five-modular representation of $SL_2(5)$, obtained from the matrices in Example 2.2 by replacing i by 2 and $-i$ by 3, defines the progenitor

$$\mathscr{P} \cong 5^{\star 6} : SL_2(5),$$

in which

$$x \sim (t_\infty)(t_0\ \ t_1\ \ t_2\ \ t_3\ \ t_4), \qquad y \sim \left(t_\infty\ \ t_0\ \ t_\infty^{-1}\ \ t_0^{-1}\right)\left(t_1\ \ t_4^{-1}\ \ t_1^{-1}\ \ t_4\right)$$
$$\left(t_2\ \ t_2^2\ \ t_2^4\ \ t_2^3\right)\left(t_3\ \ t_3^3\ \ t_3^4\ \ t_3^2\right).$$

For our second example, though, we prefer to extend this to a progenitor

$$\mathscr{P} \cong 5^{\star(6+6)} : \left(SL_2(5) : 2\right) \cong 5^{\star(6+6)} : 2^{\cdot}S_5,$$

by adjoining the outer automorphism of the control subgroup. The associated 5-modular monomial representation has degree $(6 + 6) = 12$, with the two 6-dimensional subspaces being interchanged by the outer automorphism. As presentation for the extended control subgroup we take

$$2^{\cdot}S_5 \cong \left\langle x, y \mid x^5 = y^4 = \left(xy^2\right)^3, \left(x^3y\right)^2 = 1\right\rangle.$$

To obtain the progenitor from this we must adjoin an element r of order 5 which is normalized by a subgroup of the control subgroup of order 20 as described in Example 2.2 above:

$$5^{\star\{6+6\}} : 2^{\cdot}S_5 \cong \left\langle x, y, r \mid x^5 = y^4 = \left(xy^2\right)^3, \left(x^3y\right)^2 = 1 = r^5 = r^x r,\right.$$
$$\left. r^{y^2x^2y^2} = \left(r^{y^2x^2}\right)^3\right\rangle.$$

This follows from the fact that $\langle x, (y^2x^2)y^2(y^2x^2)^{-1}\rangle$ is isomorphic to such a subgroup of order 20. If we let the two sets of six symmetric generators be $\mathscr{R} = \{r_\infty, r_0, \ldots, r_4\}$ and $\mathscr{S} = \{s_\infty, s_0, \ldots, s_4\}$, then the action on these satisfying the above presentation may be given by:

$$x^2 = (r_\infty)(r_0\ r_1\ \ldots\ r_4)(s_\infty)(s_0\ s_1\ \ldots\ s_4)$$
$$y = \big(r_\infty\ s_2\ r_4\ s_1^{-1}\ \ldots\big)\big(r_0\ s_0^2\ r_0^{-2}\ s_0\ \ldots\big)$$
$$\big(r_1\ s_\infty^{-2}\ r_2\ s_4^{-2}\ \ldots\big)\big(r_3\ s_3\ r_3^2\ s_3^2\ \ldots\big),$$

where $x^5 = y^4$ inverts all the symmetric generators and r corresponds to the symmetric generator r_∞. It should be noted that $[r_\infty, x] = r_\infty^{-1}r_\infty^{-1} = r_\infty^{-2}$, and so the derived group of the progenitor contains all the symmetric generators. It also contains the derived group of the control subgroup and so has shape $5^{\star(6+6)}:SL_2(5)$. By a similar argument this group itself is perfect.

EXAMPLE 3.3. The monomial modular representation obtained in Example 2.3 defines the progenitor

$$\mathscr{P} \cong 7^{\star 15} : 3^{\cdot}A_7$$

in the same way. As in the above example, however, we prefer to extend the control subgroup to $3^{\cdot}S_7$. Note that this group does not possess a subgroup of index 15 as the two classes of subgroups isomorphic to $L_3(2)$ are fused by the outer automorphism, see page 3 of the ATLAS [5]. Recall, moreover, that the outer automorphism of $3^{\cdot}A_7$ inverts a central element of order three, and so in the monomial representation obtained by inducing a non-trivial linear representation of $3 \times L_3(2)$ up to $3^{\cdot}S_7$ such a "central" element will be represented by $\mathrm{diag}(\omega^{15}, \overline{\omega}^{15})$ or its conjugate. Translating this into the language of progenitors, in

$$7^{\star(15+15)} : (3^{\cdot}S_7)$$

a "central" element of order 3 will, by conjugation, square one set of 15 symmetric generators while fourth-powering the other. In keeping with the rest of this work, we choose to take a symmetric presentation for the control subgroup $3^{\cdot}S_7$.

LEMMA 3.3.1.

$$\mathscr{G} \cong \frac{2^{\star 7} : L_3(2)}{\big[(0,1,2,3,4,5,6)t_0\big]^6} \cong 3^{\cdot}S_7, \tag{1}$$

where the control subgroup $\mathscr{L}$ is taken to be all permutations of the set $\{0, 1, 2, 3, 4, 5, 6\}$ *which preserve the set of 7 triples given by* $\{1, 2, 4\}$ *and its translates.*

Proof. First observe, by for instance abelianizing the group, that the progenitor $\mathscr{P} \cong 2^{\star 7} : L_3(2)$ has derived group $\mathscr{P}'$ with $|\mathscr{P} : \mathscr{P}'| = 2$. This, in fact, holds for any progenitor $2^{*n} : \mathscr{N}$ in which $\mathscr{N}$ is simple and acts transitively on the n symmetric generators, and the derived group is given by

$$\mathscr{P}' = \langle \mathscr{N}, t_i t_j \rangle,$$

the subgroup generated by $\mathscr{N}$ together with words of even length in the symmetric generators. Moreover, we have

$$[t_0 t_1, (1,2)(3,6)] = t_1 t_0 (t_0 t_1)^{(1,2)(3,6)} = t_1 t_0 t_0 t_2 = t_1 t_2,$$

and so $\mathscr{P}'' = \mathscr{P}'$; i.e., the derived group is perfect. Clearly an analogous argument holds whenever $\mathscr{N}$ is simple and doubly transitive. [In fact, a little thought shows us that $\mathscr{P}'$ is perfect whenever $\mathscr{N}$ is simple and *primitive*.]. Our next step is to show that the symmetric group S_7 is a homomorphic image of $\mathscr{G}$. To see this let $t_0 = (1,3)(2,6)(4,5)$, and note that this element, which preserves the three lines of the 7-point plane which pass through 0, is centralized in $\mathscr{L}$ by a subgroup isomorphic to S_4. Thus, under conjugation by elements of $\mathscr{L}$, t_0 has just seven images. Now $L_3(2)$ is maximal in S_7 and so $\langle \mathscr{L}, t_0) \rangle \cong S_7$, which is thus an image of the progenitor $\mathscr{P}$. To show that it is an image of $\mathscr{G}$ we must check that the additional relation holds. But

$$(0,1,2,3,4,5,6)t_0 = (0,1,2,3,4,5,\ 6)(1,3)(2,6)(4,5) = (0,3,5,2,1,6),$$

which has order 6 as required. To complete the proof we show that $|\mathscr{G}| = 3 \times |S_7|$. Then $\mathscr{G}'$ will be a perfect group of order $3 \times |A_7|$, having A_7 as a homomorphic image, which can only be the triple cover $3^{\cdot}A_7$. The outer automorphism necessarily inverts the central elements of order 3 to form $3^{\cdot}S_7$. We can do this manually using the double coset enumeration techniques of [11] and [12] to obtain the Cayley diagram shown in Fig. 1. Alternatively, we build a presentation for $\mathscr{G}$ and perform a mechanical (single) coset enumeration over $\mathscr{N}$. Thus we take

$$\mathscr{N} \cong L_3(2) \cong \langle x, y \mid x^7 = y^3 = (xy)^2 = [x, y]^4 = 1 \rangle,$$

where x and y correspond to $(0,1,2,3,4,5,6)$ and $(4,3,2)(6,5,1)$ respectively. Now $\langle y, (xy)^{x^2} \rangle = \mathscr{L}^0 \cong S_4$, the stabilizer in $\mathscr{L}$ of 0, and so we have

$$\mathscr{G} \cong \langle x, y \mid x^7 = y^3 = (xy)^2 = [x, y]^4 = 1,$$
$$t^2 = [t, y] = \left[t, (xy)^{x^2}\right] = (xt)^6 = 1 \rangle.$$

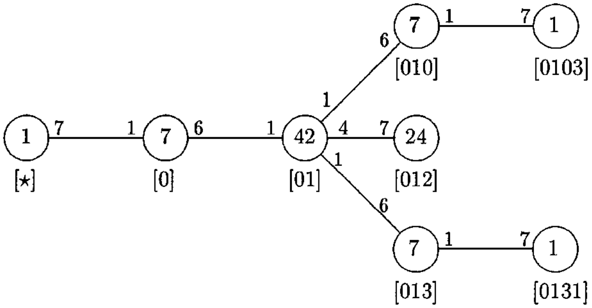


FIG. 1. The Cayley diagram of $3^{\cdot}S_7$ over $L_3(2)$.

Since S_7 is an image of this group we know that $\langle x, y\rangle \cong L_3(2)$, and a computer coset enumeration over $\langle x, y\rangle$ returns the index 90 as required. ∎

From Fig. 1 we see that every element of $\mathscr{N} \cong 3^{\cdot}S_7$ can be expressed, not necessarily uniquely, as a permutation of $L_3(2)$ acting on 7 letters, followed by a word of length less than or equal to four in the 7 symmetric generators. Now when the normal subgroup of order 3 is factored out the three single point orbits in Fig. 1 are fused to give suborbits of lengths $1 + 7 + 14 + 8$, and so we can deduce that the "central" elements of order three lie in the double cosets [0103] $(=\mathscr{L}t_0t_1t_0t_3\mathscr{L})$ and [0131]. The images of the t_i in S_7 are given by $t_0 = (1,3)(2,6)(4,5)$ as above, and $t_i = t_0^{x^i}$, and so we find that $t_0t_1t_0t_3 \equiv (2,6)(4,5)$ modulo $Z(\mathscr{N}')$. Thus with $z = (2,6)(4,5)t_0t_1t_0t_3$ we have

$$Z(\mathscr{N}') = \langle z\rangle = \langle x^3yx^{-2}t^{xt}t^{x^3}\rangle.$$

To obtain a presentation for the progenitor

$$7^{\star(15+15)} : (3^{\cdot}S_7)$$

we must adjoint an element s of order 7, which is centralized by our $\mathscr{L} \cong L_3(2)$ and mapped to its square under conjugation by a central element of $\mathscr{N}'$. This is achieved by adjoining s with

$$s^7 = [s, x] = [s, y] = s^{tx^2txtx^{-1}}s^3 = 1,$$

where the monomial action of x, y and t is given by

$$x = \big(I_{30}, (1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7)(8,\ 9,\ 10,\ 11,\ 12,\ 13,\ 14)(15)\\ (16,\ 17,\ 18,\ 19,\ 20,\ 21,\ 22)(23,\ 24,\ 25,\ 26,\ 27,\ 28,\ 29)(30)\big).$$

$$y = \big(\mathrm{diag}(1,1,1,1,1,1,1,4,2,2,4,1,1,1,1,2,4,4,1,4,2,4,4,2,2,\\ 1,2,4,2,1),\\ (1,6,5)(2,4,3)(7)(8)(9,14,11)(10)(12,13,15)\\ (16,29,27)(17,26,25)(18,24,19)(20,23,22)(21)(28)(30)\big).$$

and

$$t = \big(\mathrm{diag}(1,4,1,2,4,2,1,1,4,2,2,2,2,4,1,1,2,4,4,4,4,2,1,1,2,\\ 4,2,1,4,1),\\ (1,16)(2,22)(3,23)(4,20)(5,27)(6,29)(7,30)(8,28)\\ (9,25)(10,21)(11,26)(12,19)(13,18)(14,17)(15,24)\big).$$

If, as above, we let the two sets of symmetric generators be denoted by $\mathscr{R} = \{r_1, r_2, \ldots, r_{15}\}$ and $\mathscr{S} = \{s_1, s_2, \ldots, s_{15}\}$, this becomes

$$x\colon (r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ r_7)(r_8\ r_9\ r_{10}\ r_{11}\ r_{12}\ r_{13}\ r_{14})(r_{15})\\ (s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7)(s_8\ s_9\ s_{10}\ s_{11}\ s_{12}\ s_{14})(s_{15}),$$

$$y\colon (r_1\ r_6\ r_5)(r_2\ r_4\ r_3)(r_7)(r_8\ r_8^4\ r_8^2)(r_9\ r_{14}^2\ r_{11}^2)(r_{10}\ r_{10}^2\ r_{10}^4)\\ (r_{12}\ r_{13}\ r_{15})(s_1\ s_{14}^2\ s_{12}^4)(s_2\ s_{11}^4\ s_{10}^4)(s_3\ s_9^4\ s_4)(s_5\ s_8^4\ s_7^2)\\ (s_6\ s_6^2\ s_6^4)(s_{13}\ s_{13}^4\ s_{13}^2)(s_{15}),$$

and

$$t\colon (r_1\ s_1)(r_2\ s_7^4)(r_3\ s_8)(r_4\ s_5^2)(r_5\ s_{12}^4)(r_6\ s_{14}^2)(r_7\ s_{15})(r_8\ s_{13})\\ (r_9\ s_{10}^4)(r_{10}\ s_6^2)(r_{11}\ s_{11}^2)(r_{12}\ s_4^2)(r_{13}\ s_3^2)(r_{14}\ s_2^4)(r_{15}\ s_9),$$

where the generator s used in the presentation corresponds to s_{15}. It is worth stressing that our progenitor $\mathscr{P}$ is a uniquely defined group whose structure depends only on the control subgroup $3^{\cdot}S_7$. We may observe further that, since for instance $[r_8, y] = r_8^{-1} r_8^3 = r_8^3$, the derived group of $\mathscr{P}$ contains the symmetric generators. It also contains the derived group of the control subgroup and so is isomorphic to

$$7^{\star(15+15)} : 3^{\cdot}A_7$$

of index 2 of $\mathscr{P}$. By a similar argument, the derived group of $\mathscr{P}$ is perfect. Moreover, the derived group of the control subgroup, which is isomorphic to $3^{\cdot}A_7$, acts non-identically on $\bar{\mathscr{R}} = \{\langle r_i \rangle \mid 1 \leq i \leq 15\}$ and $\bar{\mathscr{S}} = \{\langle s_i \rangle \mid 1 \leq i \leq 15\}$. Indeed, the subgroup $\langle x, y \rangle \cong L_2(7)$ mentioned above as commuting with $s = s_{15}$, has orbits of lengths $(1 + 14) + (7 + 8)$ on $\bar{\mathscr{R}} \cup \bar{\mathscr{S}}$, explicitly

$$\{R_1, R_2, \ldots, R_7\} \cup \{R_8, R_9, \ldots, R_{15}\} \cup \{S_1, S_2, \ldots, S_{14}\} \cup \{S_{15}\}.$$

In seeking interesting homomorphic images of $\mathscr{P}$, therefore, we should consider what the images of $\langle r_1, s_{15} \rangle$, $\langle r_{15}, s_{15} \rangle$, and $\langle s_1, s_{15} \rangle$ might be.

4. FINITE HOMOMORPHIC IMAGES OF THE PROGENITORS

As has been demonstrated above, these three infinite progenitors have a lot of structure imposed upon them; in particular, the second and third examples have perfect derived groups. It is now our intention to seek finite homomorphic images. The three control subgroups chosen in our examples have normal subgroups, of orders 2, 2, and 3 respectively, which raise the symmetric generators to some non-trivial power by conjugation. Thus, if such a normal subgroup is factored out by a homomorphism, we see that the symmetric generators (whose orders in each case divide a prime number) must also be mapped to the identity. But we see that every non-trivial normal subgroup of our control subgroups contains this normal subgroup. So, for interesting images we may assume that the homomorphism acts faithfully on the control subgroup. Similarly we require that the homomorphism acts faithfully on the cyclic subgroups T_i. Thus the homomorphic image of $\langle \mathscr{T} \rangle$ or $\langle \mathscr{R} \cup \mathscr{S} \rangle$ is a group possessing all the automorphisms induced by the monomial permutations in $\mathscr{N}$. These automorphisms may be inner or outer depending on which additional relators we factor by.

EXAMPLE 4.1. In order to obtain a non-trivial finite image of

$$3^{\star 4} : 2^{\cdot} S_4$$

it is natural to specify the order of xt, which is to say the order of $(0\ 1)(2)(3\ \bar{3})t_0$. It is readily shown, either manually or computationally, that if this order is less than 5 the homomorphism does not act faithfully on both the control subgroup and the symmetric generators, and so the group collapses. However, if this element has order 5, mechanical coset enumeration over $\mathscr{N}$ shows that the resulting image has order 7,920. It is, of course, the smallest sporadic simple group, the Mathieu group M_{11}.

Thus we have

$$\frac{3^{\star 4} : 2^{\cdot}S_4}{\left[(0\ 1)(3\ \bar{3})t_0\right]^5} \cong M_{11},$$

and a presentation is given by

$$\left\langle x, y, t \mid y^3 = (xy)^4, x^2 = t^3 = t^y t = [t^{xy}, x] = (xt)^5 = 1 \right\rangle \cong M_{11}.$$

As permutations on 11 letters we may choose

$$\begin{aligned}
x &= (1\ X)(2\ 3)(6\ 8)(7\ 9) &&\sim [0\ 1][3\ \bar{3}]\\
y &= (0\ X\ 1)(2\ 7\ 4\ 8\ 9\ 5)(3\ 6) &&\sim [0\ \bar{0}]\left[1\ \bar{2}\ 3\ \bar{1}\ 2\ \bar{3}\right]\\
t_0 &= (1\ 4\ 5)(2\ 8\ 0)(7\ X\ 9)\\
t_1 &= (X\ 4\ 5)(3\ 6\ 0)(9\ 1\ 7)\\
t_2 &= (1\ 2\ 8)(3\ 6\ X)(4\ 0\ 5)\\
t_3 &= (0\ 9\ 7)(3\ 6\ 1)(2\ X\ 8),
\end{aligned}$$

where X stands for 10, and the action on the symmetric generators is written in square brackets. Invariably these highly symmetric generating sets carry a great deal of information about the structure of the group. In this case we have

$$\langle t_i, t_j \rangle \cong A_5, \langle t_i, t_j, t_k \rangle \cong L_2(11),$$

for i, j, k distinct, and

$$\langle t_0 t_1 t_0, xy \rangle \cong M_{10} \cong A_6^{\cdot}2.$$

A systematic investigation of progenitors with small control subgroups such as this one has been carried out by Hammas, Bray, and the author [1, 11, 12, 15].

EXAMPLE 4.2. An involution in the outer half of the control subgroup $2^{\cdot}S_5$ is given by

$$x^3 y = \left(r_\infty\ s_2^{-1}\right)\left(r_0\ s_1\right)\left(r_1\ s_0^{-2}\right)\left(r_2\ s_\infty^2\right)\left(r_3\ s_4^2\right)\left(r_4\ s_3^{-1}\right),$$

which thus normalizes the subgroup $\langle r_\infty, s_2^{-1} \rangle$. Furthermore the element $(y^2)^{x^2 y^2}$ squares r_∞ and cubes s_2, and so we seek a homomorphic image of

$$5^{\star 2} : \mathscr{K} \text{ where } D_8 \cong \mathscr{K} = \left\langle \begin{pmatrix} 2 & \cdot \\ \cdot & 3 \end{pmatrix}, \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \right\rangle.$$

If $\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} r_\infty$ is given order 3 this becomes S_5, and factoring the progenitor $5^{\star(6+6)} : 2^{\cdot}S_5$ by the corresponding relator (namely $(x^3yr)^3$) gives the triple cover $3^{\cdot}U_3(5)$; the centre may be factored out by adjoining the relator $(y^3r)^7$, where r is mapped onto r_∞. This example, complete with its outer automorphism of order 2, was described fully in [12] where it was used in the construction of the Higman–Sims group. Again the symmetric generators yield much information about the subgroup structure and a representative of each class of maximal subgroups in terms of the symmetric generators is given in [12]. In particular

$$\langle r_\infty, r_0, r_2 \rangle = \langle r_\infty, r_0, r_2, s_1, s_3, s_4 \rangle \cong A_7.$$

EXAMPLE 4.3. In order to find a non-trivial finite homomorphic image G of

$$\mathscr{P} \cong 7^{\star(15+15)} : 3^{\cdot}S_7$$

we recall the obvious put powerful Lemma 1 of [10, p. 383]:

$$\mathscr{N} \cap \langle t_i, t_j \rangle \le C_{\mathscr{N}}(\mathscr{N}^{ij}),$$

where we are identifying $T_i = \langle t_i \rangle$ and $\mathscr{N}$ with their isomorphic images in G, and where $\mathscr{N}^{ij}$ denotes the centralizer in $\mathscr{N}$ of $\langle t_i, t_j \rangle$. Now

$$C_{\mathscr{N}}(\langle s_{15}, r_7 \rangle) = \left\langle y, (xy)^{x^2} \right\rangle \cong S_4$$

and the centralizer in $\mathscr{N}$ of this subgroup is just $\langle z, t \rangle \cong S_3$, where $\langle z \rangle = Z(\mathscr{N}')$; and so we seek a group generated by two elements of order 7, normalized by an element of order 3 which squares one of them and fourth-powers the other, and an involution which interchanges them. The lemma asserts that both these automorphisms may be inner, i.e., that we can have $\langle z, t \rangle \le \langle s_{15}, r_7 \rangle$. We seek a minimal example in which this holds. In the language of progenitors we are thus seeking a homomorphic image of

$$7^{\star 2} : \mathscr{K} \qquad \text{where } S_3 \cong K = \left\langle \begin{pmatrix} 2 & \cdot \\ \cdot & 4 \end{pmatrix}, \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \right\rangle,$$

which is generated by the images of the two symmetric generators of order 7. Such a group is clearly perfect and, since we are looking for a minimal example, we may assume it is simple. Now a presentation of this progenitor is given by

$$\left\langle u, a, b \mid u^7 = a^3 = b^2 = (ab)^2 = u^a u^{-2} = 1 \right\rangle.$$

If we require the element ub to have order 3, we may manually construct the Cayley diagram of

$$\mathscr{L} = \left\langle u, a, b \mid u^7 = a^3 = b^2 = (ab)^2 = u^a u^{-2} = (ub)^3 = 1 \right\rangle$$

over $\mathscr{K} = \langle u, a \rangle$ to obtain index 8. When the 8 cosets are suitably labeled we find that the elements u, a, and b correspond to the linear fractional maps $x \mapsto x + 1$, $x \mapsto 2x$, and $x \mapsto -1/x$, respectively, in $PSL_2(7)$. So

$$\mathscr{L} \cong L_2(7)$$

is the smallest possibility. A corresponding relator by which to factor our progenitor $\mathscr{P}$ is $(ts)^3$, and thus we consider

$$\mathscr{He} = \frac{7^{\star(15+15)} : 3^{\cdot}S_7}{(ts)^3}$$

where $s = s_{15}$ and t is as given on page 12. As a presentation we have

$$\begin{aligned} \mathscr{He} = \langle x, y, t, s \mid x^7 = y^3 = (xy)^2 = [x, y]^4 = t^2 \\ = [t, y] = \left[t, (xy)^{x^2}\right] = (xt)^6 = s^7 \\ = [s, x] = [s, y] = s^{tx^2txtx^{-1}t}s^3 = (st)^3 = 1\rangle. \end{aligned}$$

Using the coset enumerator in the MAGMA package [4] we find that the subgroup $\langle x, y, t \rangle \cong 3^{\cdot}S_7$ has index 266,560 and so $\mathscr{He}$ has order 4,030,200. The group is easily seen to be the Held sporadic simple group, see [16, 17] and [5, p. 104]. This presentation can be extended to the automorphism group of $\mathscr{He}$ by adjoining a further involutory generator a which commutes with x, y, and t and inverts s; thus a inverts all the symmetric generators.

4.1. *Subgroups of $\mathscr{He}$ Generated by Subsets of the Symmetric Generators*

As was mentioned in the previous examples, the symmetric generators often contain a great deal of information about the subgroup structure of the group they generate. In order to investigate the subgroups generated by subsets of our 30 generators of order 7, we need to be conversant with the action of S_7 on 30 letters. Now

$$A_7 \leq A_8 \cong L_4(2),$$

and the 30 letters may be taken to be the 15 one-dimensional subspaces (the *points*) and the 15 three-dimensional subspaces (the *hyperplanes*) of a four-dimensional vector space over Z_2. These actions are seen clearly in the Mathieu group M_{24}: the subgroup fixing an octad, a point in it, and a point outside it is isomorphic to A_7, acting simultaneously on the 7 remaining points of the fixed octad and the 15 remaining *points* of the complementary 16-ad. The *hyperplanes* then correspond to the 15 octads disjoint from the fixed octad and containing the fixed point of the 16-ad; thus points are contained in 7 hyperplanes. The outer elements of S_7 correspond to polarities which interchange points and hyperplanes. One may readily read off the action of elements of A_7 on points and hyperplanes from the Miracle Octad Generator of [7], and the correspondence with our generators $\mathscr{R} \cup \mathscr{S}$ is given in Table III together with the action of the polarity t.

As seen above, the stabilizer in S_7 of the hyperplane s_{15}, say, is isomorphic to $L_3(2)$ with orbits of lengths 7 and 8 on the points. Fixing a point in the 7-orbit reduces this to a subgroup isomorphic to S_4. In our construction, with s_{15} as the hyperplane and r_7 as the point, we imposed a relation which forced $\langle r_7, s_{15} \rangle \cong L_3(2)$, and thus

$$\left\langle s, t, y, (xy)^{x^2} \right\rangle \cong L_3(2) \times S_4,$$

a maximal subgroup of $\mathscr{H}e$. If instead we fix a point in the 8-orbit we are left with a Frobenius group of shape $7:3$, which normalizes the point but does not centralize it. Thus

$$C_{\mathscr{N}}(\langle r_{15}, s_{15} \rangle) = \langle x \rangle \cong C_7.$$

In fact, $\langle r_{15}, s_{15} \rangle \cong 7^{1+2}$, an extra-special group, and

$$\begin{aligned} N_{\mathscr{H}e}(\langle r_{15}, s_{15} \rangle) = \langle r_{15}, s_{15} \rangle : N_{\mathscr{N}}(\langle r_{15}, s_{15} \rangle) &= \left\langle s, y^{x^2 y^{-1}}, \left(x^{yx^{-2}} t^{x^3} \right)^3 \right\rangle \\ &= \left\langle s_{15}, (0,4,5)(1,6,2), (0,3,2,6,4,1,5) t_1 t_6 t_3 \right\rangle \\ &\cong 7^{1+2} : (S_3 \times 3), \end{aligned}$$

a further maximal subgroup. In addition, the control subgroup $\mathscr{N}$ itself is maximal, as is the normalizer of a symmetric generator:

$$\begin{aligned} N_{\mathscr{G}}(\langle s \rangle) &= \langle s, x, y, z \rangle \\ &= \left\langle s_{15}, (0,1,2,3,4,5,6), (6,5,1)(4,3,2), (6,2)(4,5) t_0 t_1 t_0 t_3 \right\rangle \\ &\cong 7:3 \times L_3(2). \end{aligned}$$

TABLE III

The 30 Letters Permuted by S_6 Showing the Polarity t

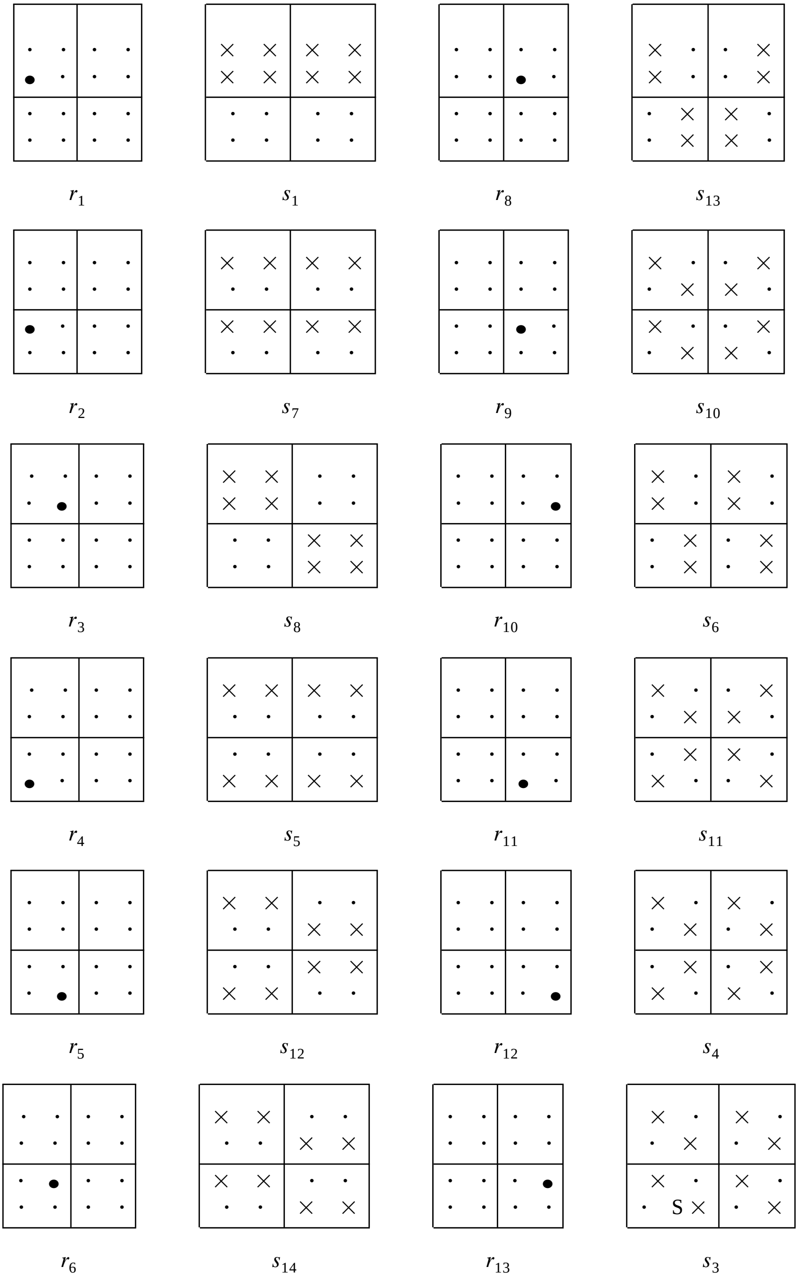
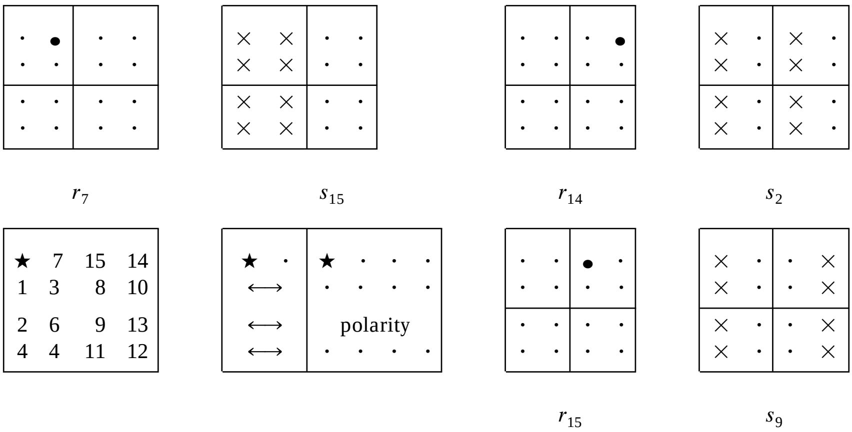


TABLE III
(Continued)



This leaves us with possibly the most interesting case: the four-dimensional space containing the 15 points and 15 hyperplanes also contains 35 two-dimensional subspaces, each of which contains 3 points and is contained in 3 hyperplanes. A set of 6 such symmetric generators is thus normalized by a subgroup of $\mathcal{N}$ of order $(3 \times 7!)/35 = 2^4 3^3$, of which a subgroup isomorphic to V_4 centralizes all six generators. We can see from Table III that $D = \{r_1, r_3, r_7, s_1, s_8, s_{15}\}$ is such a set. So $\mathcal{D} = \langle D \rangle$ is a group generated by six elements of order 7 normalized by $\mathcal{N}_D$, a group of monomial automorphisms of order 432, of which a subgroup isomorphic to V_4 acts trivially. Indeed we can see from the equations toward the end of Section 3 that

$$t:(r_1\ \ s_1)(r_3\ \ s_8)(r_7\ \ s_{15})$$

$$yx:(r_1\ \ r_7)(r_3)(s_1\ \ s_8^2)(s_{15})$$

$$y^{x^2}:(r_1 r_7\ \ r_3)(s_1\ \ s_1^2\ \ s_1^4)(s_8\ \ s_8^4\ \ s_8^2)(s_{15}),$$

and a coset enumeration over $\langle s, t, yx, y^{x^2} \rangle$ gives index 8,330. We can, of course, investigate the structure of this subgroup through the resulting permutation action but, in the spirit of the present work, note that it is a homomorphic image of

$$7^{\star(3+3)}:(2^2 \cdot 3^{1+2}:2^2),$$

where the monomial automorphisms in $\mathcal{N}_D$ are generated by

$$\left[\begin{array}{ccc|ccc} \cdot & \cdot & 4 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 4 \end{array}\right]; \qquad \left[\begin{array}{ccc|ccc} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{array}\right];$$

$$(u) \qquad\qquad\qquad (v)$$

$$\left[\begin{array}{ccc|ccc} \cdot & \cdot & 4 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 4 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot \end{array}\right];$$

$$(w)$$

This automorphism group is defined (modulo the central V_4) by

$$\left\langle u, v, w \mid u^3 = v^2 = w^2 = (vw)^2 = 1, u^v u^w = [u, v] \right\rangle \cong 3^{1+2} : 2^2.$$

If we let q stand for the symmetric generator s_8^2 in the above we see that $q^u = q^2$ and $[q, w] = 1$. The additional relator by which we factored $\mathscr{P}$ becomes $(vq)^3$, and we find

$$\left\langle u, v, w, q \mid u^3 = v^2 = w^2 = (vw)^2 = q^7 = q^u q^{-2} = \right.$$
$$\left. [q, w] = (vq)^3 = 1, u^v u^w = [u, v] \right\rangle \cong L_3(4) : S_3,$$

which can be shown by explicitly exhibiting permutations of $L_3(4) : S_3$ which satisfy the presentation, together with a coset enumeration over, for instance, $\langle u, v, w \rangle$. The full subgroup of index 8,330 has shape $2^{2\cdot} L_3(4) : S_3$, and the outer automorphism of $\mathscr{H}e$ which commutes with $\mathcal{N}$ and inverts the symmetric generators completes this to $2^{2\cdot} L_3(4) : (S_3 \times 2)$.

From page 104 of the ATLAS [5] we see that representatives of five of the conjugacy classes of maximal subgroups of $\mathscr{H}e$ can be described naturally in terms of the control subgroup and symmetric generators. In Table IV we exhibit this information by giving, in the last four cases, a subgroup of $\mathcal{N}$, its orbits on the symmetric generators, and the normalizer of the subgroup generated by the symmetric generators it fixes. Each of these is maximal.

TABLE IV
Normalizers of Subgroups Generated by Symmetric Generators

Subgroup of $\mathcal{N} \cong 3^{\cdot}S_7$	Orbits on (15 + 15) Symmetric Generators	Corresponding Maximal Subgroup
$3^{\cdot}S_7$	*transitive*	$3^{\cdot}S_7$
$L_3(2)$	(1 + 14) + (7 + 8)	$7:3 \times L_3(2)$
C_7	(1 + 7 + 7) + (1 + 7 + 7)	$7^{1+2}:(S_3 \times 3)$
S_4	(1 + 6 + 8) + (1 + 6 + 8)	$S_4 \times L_3(2)$
V_4	$(1^3 + 4^3) + (1^3 + 4^3)$	$2^{2\cdot}L_3(4):\mathrm{S}_3$

5. OTHER MAXIMAL SUBGROUPS OF $\mathcal{He}$

Although the permutation representation of $\mathcal{He}$ on 8,330 letters emerges naturally in this approach, the minimal action on 2,058 letters with point stabilizer isomorphic to $S_4(4):2$ (whose order is not divisible by 7) seems less promising. In fact, we can obtain copies of this maximal subgroup as follows. First note that the preimage in $3^{\cdot}S_7$ of the centralizer of a transposition in S_7 is a group isomorphic to $S_3 \times S_5$. As an example of this we take

$$\mathcal{N}_T = \langle (6,5,1)(4,3,2)t_2, (2,6)(4,5), t_0 \rangle = \langle yt^{x^2}, (xy)^{x^{-2}}, t \rangle \cong S_3 \times S_5.$$

It turns out that $\langle \mathcal{N}_T, r_7 s_{15}^{-2} = s^t s^{-2} \rangle \cong S_4(4):2$, and so we may readily obtain the permutation representation of minimal degree. Unfortunately it is not quite as easily as usual to extend this to a maximal subgroup of $\mathrm{Aut}\,\mathcal{He} \cong \mathcal{He}:2$, as the outer half of $S_4(4):4$ contains no involutions. However, for completeness we mention that

$$\langle (6,5,1)(4,3,2)t_2, as_{15}r_1^3 r_3^{-1} \rangle = \langle yt^{x^2}, as(s^{tx})^3 (s^{tx^3})^{-1} \rangle \cong S_4(4):4.$$

Now a subgroup of $\mathcal{N}$ isomorphic to $3^{\cdot}S_6$ is given by $\langle y, xtx^4 \rangle$. If we adjoin the element $s_{15}^2 r_7^2 s_{15}^{-1}$ to this we obtain the maximal subgroup

$$\langle y, xtx^4, s^2 (s^t)^2 s^{-1} \rangle \cong 2^6:3^{\cdot}S_6.$$

Of course, the two conjugacy classes of subgroups of this shape are interchanged by our outer automorphism a.

Our element t is a representative of the conjugacy class of involutions denoted by $2B$ in the ATLAS [5]. Its centralizer is given by

$$\langle y, r_1^2 s_1^{-1} r_1^4, r_1^2 s_1^3 r_1^3 \rangle \cong 2^{1+6}:L_3(2),$$

where $r_1 = s^{tx}$ and $s_1 = s^{txt}$. This maximal subgroup is normalized by the outer automorphism a. The maximal subgroups of $\mathscr{He}$ which were worked out by Butler, see [2], are listed on page 104 of the ATLAS [5]. We have given generators for a representative of each class except those of shape $7^2:2^{\cdot}L_2(7)$ and $5^2:4^{\cdot}A_4$. In fact, the former of these is simply $N_{\mathscr{He}}(\langle x, s_{15}t_{15}^{-1}\rangle)$ and the latter is $N_{\mathscr{He}}(Syl_5(\mathscr{He}))$, and so representatives can be readily obtained using MAGMA. In addition, Aut$\mathscr{He}$ has two classes of maximal subgroups known as novelties whose intersections with $\mathscr{He}$ are not maximal in the simple group. These subgroups, which have shapes $(S_5 \times S_5):2$ and $2^{4+4}.(S_3 \times S_3).2$, were found by Wilson in [21].

6. CONCLUSION

In each of the three examples described above, we started with some familiar group and took a faithful monomial representation of it over a field of prime order p. We used this representation to interpret our initial group as automorphisms of a free product of cyclic groups of order p, and constructed the semi-direct product of this free product by the resulting group of automorphisms. We then sought non-collapsing images of this semi-direct product which are generated by the images of the cyclic subgroups. In each case the smallest such image is a simple group, and in two of the three cases it is sporadic. Moreover, in each case the images of the cyclic groups afford a great deal of information about the subgroup structure of the image group. Of course, there is no need to limit ourselves to prime fields. Indeed, if we take nominal representations over GF_4, the Galois field of order 4, then the cyclic subgroups are replaced by Klein fourgroups and we can obtain revealing constructions of, for example, the Hall–Janko group and the Suzuki group.

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