Hamilton-connectivity of 3-domination critical graphs with \( \alpha = \delta + 1 \geq 5 \)

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Abstract

A graph \( G \) is 3-domination critical if its domination number \( \gamma \) is 3 and the addition of any edge decreases \( \gamma \) by 1. Let \( G \) be a 3-domination critical graph with toughness more than one. It was proved that \( G \) is Hamilton-connected for the cases \( \alpha \leq \delta \) [Y.J. Chen, F. Tian, B. Wei, Hamilton-connectivity of 3-domination critical graphs with \( \alpha \leq \delta \), Discrete Math. 271 (2003) 1–12] and \( \alpha = \delta + 2 \) [Y.J. Chen, F. Tian, Y.Q. Zhang, Hamilton-connectivity of 3-domination critical graphs with \( \alpha = \delta + 2 \), European J. Combin. 23 (2002) 777–784]. In this paper, we show \( G \) is Hamilton-connected for the case \( \alpha = \delta + 1 \geq 5 \).

Keywords: Domination-critical graph; Hamilton-connectivity

1. Introduction

Let \( G = (V(G), E(G)) \) be a graph. A graph \( G \) is said to be \( t \)-tough if for every cutset \( S \subseteq V(G) \), \( |S| \geq t \omega(G - S) \), where \( \omega(G - S) \) is the number of components of \( G - S \). The toughness of \( G \), denoted by \( \tau(G) \), is defined to be \( \min\{|S|/\omega(G - S)|S \text{ is a cutset of } G\} \). Let \( u, v \in V(G) \) be any two distinct vertices. We denote by \( p(u, v) \) the length of a longest path connecting \( u \) and \( v \). The codiameter of \( G \), denoted by \( d^*(G) \), is defined to be \( \min\{p(u, v) | u, v \in V(G)\} \).

A graph \( G \) of order \( n \) is said to be Hamilton-connected if \( d^*(G) = n - 1 \), i.e., every two distinct vertices are joined by a hamiltonian path. A graph \( G \) is called \( k \)-domination critical, abbreviated as \( k \)-critical, if \( \gamma(G) = k \) and \( \gamma(G + e) = k - 1 \) holds for any \( e \in E(G) \), where \( \overline{G} \) is the complement of \( G \). The concept of domination critical graphs was introduced by Sumner and Blitch in [11]. Given three vertices \( u, v \) and \( x \) such that \([u, x] \) dominates \( V(G) - \{v\} \) but not \( v \), we will write \([u, x] \rightarrow v \). It was observed in [11] that if \( u, v \) are any two nonadjacent vertices of a 3-critical graph \( G \), then since \( \gamma(G + uv) = 2 \), there exists a vertex \( x \) such that either \([u, x] \rightarrow v \) or \([v, x] \rightarrow u \). If \( U, V \subseteq V(G) \) and \( U \) dominates \( V \), that is, \( V \) is contained in the closed neighborhood of \( U \), we write \( U \succ V \); otherwise we write \( U \prec V \). For notations not defined here, we follow [6].

It was conjectured in [10] that every connected 3-critical graph of order more than 6 has a hamiltonian path. This was proved by Wojcicka [13] who in turn conjectured that every connected 3-critical graph \( G \) with \( \delta(G) \geq 2 \) has a hamiltonian cycle. Wojcicka’s conjecture has now been proved completely, see [8,9,12,2]. It is well known that if a graph \( G \) has a hamiltonian cycle, then \( \tau(G) \geq 1 \) and the converse does not hold in general. However, this is not the case.

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when $G$ is 3-critical. Noting that $\tau(G) < 1$ if $G$ is a connected 3-critical graph with $\delta(G) = 1$, we see that the following theorem is a direct consequence of the validity of Wojciech's conjecture.

**Theorem 1.** Let $G$ be a connected 3-critical graph. Then $G$ has a hamiltonian cycle if and only if $\tau(G) \geq 1$.

For Hamilton-connectivity, it is known that if a graph $G$ is Hamilton-connected, then $\tau(G) > 1$ and the converse need not hold. However, motivated by Theorem 1, Chen et al. [6] posed the following.

**Conjecture 1 (Chen et al. [6]).** A connected 3-critical graph $G$ is Hamilton-connected if and only if $\tau(G) > 1$.

In the same paper they proved that the conjecture is true when $\chi(G) \leq \delta(G)$.

**Theorem 2 (Chen et al. [6]).** Let $G$ be a connected 3-critical graph with $\chi(G) \leq \delta(G)$. Then $G$ is Hamilton-connected if and only if $\tau(G) > 1$.

Let $G$ be a 3-connected 3-critical graph. It is shown in [5] that $\tau(G) \geq 1$ if $\tau(G) = 1$ and only if $G$ belongs to a special infinite family $\mathcal{F}$ described in [5]. Since $\chi(G) = \delta(G) = 3$ for each $G \in \mathcal{F}$, it is easy to obtain that $\tau(G) > 1$ if $\chi(G) = \delta(G) + 1$.

In [7], Chen et al. showed that the conjecture holds when $\chi(G) = \delta(G) + 2$.

**Theorem 3 (Chen et al. [7]).** Let $G$ be a 3-connected 3-critical graph with $\chi(G) = \delta(G) + 2$. Then $G$ is Hamilton-connected.

By a result of Favaron et al. [8] that $\chi(G) \leq \delta(G) + 2$ for any connected 3-critical graph $G$, we can see the conjecture has only one case $\chi(G) = \delta(G) + 1$ unsolved. In this paper, we will show that the conjecture is true when $\chi(G) = \delta(G) + 1 \geq 5$.

The main result of this paper is the following.

**Theorem 4.** Let $G$ be a 3-connected 3-critical graph with $\chi(G) = \delta(G) + 1 \geq 5$. Then $G$ is Hamilton-connected.

Noting that $\tau(G) > 1$ implies $\delta(G) \geq 3$, we can see that the conjecture is still open for the case $\chi(G) = \delta(G) + 1 = 4$. Now, we restate a result due to Chen et al. for later use.

**Theorem 5 (Chen et al. [4]).** Let $G$ be a 3-connected 3-critical graph of order $n$. Then $d^{*}(G) \geq n - 2$.

2. Properties of maximum independent set

In order to prove Theorem 4, we need to use a classical tool—closure operation in hamiltonian theory. In 1976, Bondy and Chvátal defined a (Hamilton-connected) closure operation of a graph.

**Theorem 6 (Bondy and Chvátal [1]).** Let $G$ be a graph of order $n$. Let $a$ and $b$ be nonadjacent vertices of $G$ such that $d(a) + d(b) \geq n + 1$. Then for any two distinct vertices $x$, $y$, $p(x, y) = n - 1$ in $G$ if and only if $p(x, y) = n - 1$ in $G + ab$.

Now, given a graph $G$ of order $n$, repeat the following recursive operation, named Bondy–Chvátal closure operation, as long as possible: for each pair of nonadjacent vertices $a$ and $b$, if $d(a) + d(b) \geq n + 1$, then add the edge $ab$ to $G$. We denote by $cl(G)$ the resulting graph and call it the Bondy–Chvátal (Hamilton-connected) closure of $G$. By Theorem 6 we get the following.

**Theorem 7 (Bondy and Chvátal [1]).** Let $G$ be a graph of order $n$. Then for any two distinct vertices $x$, $y$, $p(x, y) = n - 1$ in $G$ if and only if $p(x, y) = n - 1$ in $cl(G)$.

Let $G$ be a 3-critical graph of order $n$, $\chi(G) = \delta(G) + 1$ and $v_0 \in V(G)$ with $d(v_0) = \delta(G) = k \geq 3$. Suppose $N(v_0) = \{v_1, \ldots, v_k\}$ and $I = \{v_0, w_1, \ldots, w_k\}$ is an independent set. In this section, we will give some properties of $I$ in $G$ and $G^* = cl(G)$. 

The following lemma restates a lemma due to Sumner and Blitch [11], which has proven to be of considerable use in dealing with 3-critical graphs. In [11] they considered the case \( l \geq 4 \), which guarantees \( P(U) \cap U = \emptyset \). For the cases \( l = 2 \) and 3, Lemma 2.1 can be easily verified since \( G \) is a 3-critical graph.

**Lemma 2.1.** Let \( G \) be a connected 3-critical graph and \( U \) an independent set of \( l \geq 2 \) vertices. Then there exist an ordering \( u_1, u_2, \ldots, u_l \) of the vertices of \( U \) and a sequence \( P(U) = (y_1, y_2, \ldots, y_{l-1}) \) of \( l - 1 \) distinct vertices such that \([u_i, y_i] \rightarrow u_{i+1}, 1 \leq i \leq l - 1\).

**Proof.** Since \( I \) is an independent set of order at least 4, by Lemmas 2.1 and 2.2, we may assume without loss of generality that
\[
[w_i, u_i] \rightarrow w_{i+1} \quad \text{for} \quad 1 \leq i \leq k - 1.
\]
(2.1)
By (2.1), it is easy to obtain the following:
\[
v_j v_{j+1} \in E(G) \quad \text{for} \quad 1 \leq j \leq k - 2.
\]
(2.2)

**Lemma 2.3.** If \( w_i v_k \notin E(G) \) with \( i \neq 1 \), then \( G[N(v_0) - \{v_{i-1}, v_k\}] \) is a clique. If \( w_1 v_k \notin E(G) \), then \( G[N(v_0) - \{v_k\}] \) is a clique.

**Proof.** Let \( w_i, v_m \in N(v_0) - \{v_{i-1}, w_k\} \) with \( l \leq m - 1 \). If \( l = m - 1 \), then \( w_i v_m \in E(G) \) by (2.2). If \( l \leq m - 2 \), then since \( w_i v_m \notin E(G) \), there is some vertex \( z \) such that \([w_i, z] \rightarrow v_{m+1} \) or \([w_{m+1}, z] \rightarrow w_i \). Since \( k \geq 3 \), by Lemma 2.2 we have \( z \in N(v_0) \). Since \( w_i v_k \notin E(G) \), we have \( z \neq v_k \). By (2.1), either \([w_i, v_m] \rightarrow w_{m+1} \) or \([w_{m+1}, v_i] \rightarrow w_{i+1} \). In both cases, we have \( v_i v_m \in E(G) \) and hence \( G[N(v_0) - \{v_{i-1}, v_k\}] \) is a clique. As for the latter part, the proof is similar. \( \square \)

**Lemma 2.4.** If \( w_i v_k \notin E(G) \) with \( i \neq 1 \), then \([w_1, v_{j-1}] \rightarrow w_j \) for \( j \geq 3 \) and \( j \neq i \).

**Proof.** Since \( w_i v_j \notin E(G) \), by Lemma 2.2, there is some \( z \in N(v_0) \) such that \([w_i, z] \rightarrow v_{j-1} \) or \([v_{j-1}, z] \rightarrow w_i \). By (2.1) and the assumption, we can see that \([w_j, z] \rightarrow w_i \) is impossible for any \( z \in N(v_0) \) and hence \([w_1, v_{j-1}] \rightarrow w_j \). \( \square \)

**Lemma 2.5.** If \([v_0, z] \rightarrow w_i \) for some \( i \) with \( l \leq i \leq k - 1 \), then \( z \notin N(v_0) \) and if \([v_0, v_i] \rightarrow w_k \) for some \( v_i \in N(v_0) \), then \( l = k - 1 \).

**Proof.** If \( i = 1 \) and \( z \in N(v_0) \), then \( z = v_k \) by (2.1). Thus, we have \( \{v_2, v_k\} \rightarrow V(G) \) by Lemma 2.3, a contradiction. If \( i \geq 2 \) and \( z \in N(v_0) \), then by (2.1) we have \( z = v_{i-1} \) or \( v_k \) and \( N(v_0) - \{v_{i-1}, v_k\} \subseteq N(w_i) \). If \( z = v_{i-1} \), then \( w_i v_k \notin E(G) \) for otherwise \( \{v_{i-1}, w_i\} \rightarrow V(G) \). Since \([w_i, v_i] \rightarrow v_{i+1}, v_i v_k \in E(G) \). By Lemma 2.4, we have \([w_1, v_i] \rightarrow v_{i+1}, v_{i+1} \rightarrow v_k \), which implies \( v_i v_k \in E(G) \). Thus by Lemma 2.3, we have \( \{v_{i-1}, v_k\} \rightarrow V(G) \), a contradiction. If \( z = v_k \) and \( i \neq 2 \), then by Lemma 2.3 we have \( \{v_{i-2}, v_k\} \rightarrow V(G) \), a contradiction. If \( z = v_k \) and \( i = 2 \), then by Lemma 2.4 we have \([w_1, v_2] \rightarrow w_3 \), which implies \( v_2 v_3 \in E(G) \) and hence \( \{v_2, v_k\} \rightarrow V(G) \) by Lemma 2.3, also a contradiction. Thus, \( z \notin N(v_0) \).

If \([v_0, v_i] \rightarrow w_k \) for some \( v_i \in N(v_0) \), then by (2.1), we have \( l = k - 1 \) or \( k \). If \( l = k \), then by Lemma 2.3, we have \( \{v_{k-2}, v_k\} \rightarrow V(G) \), a contradiction. \( \square \)

**Lemma 2.6.** If \([v_0, v_{k-1}] \rightarrow w_k \), then \( N(v_k) \cap \{v_1, \ldots, v_k, w_k\} \) is a clique. If \( \{v_{k-1}, v_k\} \subseteq N(v_0) \), then \( \{v_{k-1}, v_k\} \rightarrow V(G) \).

**Proof.** By (2.1), we have \( N(v_0) - \{v_{k-1}, w_k\} \subseteq N(v_k) \). If \( w_k v_k \notin E(G) \), then since \([v_0, v_{k-1}] \rightarrow w_k \), we have \( \{v_{k-1}, v_k\} \rightarrow V(G) \) and hence \( w_k v_k \notin E(G) \). By Lemma 2.3, \( G[N(v_0) - \{v_{k-1}, v_k\}] \) is a clique. Thus, if \( v_{k-1} v_k \in \)
Proof. Let $d(w_k) \geq 0$ and if $d(w_k) = 0$, then $d(w_k) \geq n - \delta$.

Proof. By the assumption, we may assume $[w_3, z] \rightarrow v_0$, which implies $z \in N(v_0) \cap N(w_2) \cap U$. If $d(w_3) = \delta$, then $N(u_3) = [z]$ by (2.3). Since $[w_3, z] \rightarrow v_0$, by (2.1) and Lemma 2.7 we have $V(G) - \{w_3, v_k\} \subseteq \mathcal{N}[v_3]$. By Lemma 2.6, $w_3 v_k \in E(G)$. Thus, $[v_2, w_3] \rightarrow V(G)$, a contradiction. Since $k \geq 4$ and $[w_2, v_2] \rightarrow w_3$, by (2.1) and Claim 2.1, we have $|N(w_2) \cap N(v_2)| \geq 2$. By (2.3), $w_2 v_2 \in E(G)$. Thus, we have $d(w_2) + d(v_2) \geq n + 1$ and the conclusion follows. □

Claim 2.3. For any $u \in N_U(w_k)$, either $u v_2 \in E(G)$ or $u w_3 \in E(G)$.

Proof. Suppose $u \in N_U(w_k)$ and $w_2, w_3 \notin N(u)$. By Lemma 2.2, there is some vertex $z \in N(v_0)$ such that $[w_3, z] \rightarrow u$ or $[u, z] \rightarrow w_3$. If $[u, z] \rightarrow w_3$, then we must have $z = v_2$, which is impossible since $[u, v_2] \notin N_k$ by Lemmas 2.6 and 2.7. If $[w_3, z] \rightarrow u$, then since $[w_2, v_2] \rightarrow w_3$ and $u w_3 \notin E(G)$, we have $z \neq v_2$. By (2.1) and Lemma 2.6, we can see $z \in N(v_0) - \{v_2\}$ is also impossible, a contradiction. □

Claim 2.4. $v_{k-1} \in N^*(w_k)$.

Proof. Since $[v_0, v_{k-1}] \rightarrow w_k$, by Lemma 2.7 we have $d(v_{k-1}) = n - 3$. Noting that $d(w_k) \geq \delta + 4$, we have $d(v_{k-1}) + d(w_k) \geq n + 1$ and hence $v_{k-1} \in N^*(w_k)$. □

Claim 2.5. If $d(w_2) = \delta + 1$ and $d(w_3) = \delta$, then $v_k \in N^*(w_k)$.

Proof. Let $N(w_k) \cap U = U_3$ and $U_4 = U - U_3$. By (2.1) and Lemma 2.6, we have $v_{k-1}, v_k \notin N(w_k)$ and hence $|U_3| \geq 2$. By the assumption, there are some $z_1 \in U$ such that $[w_1, z_1] \rightarrow v_0$ for $i = 1, 2$. If $z_1 \neq z_2$, then $d_U(v_1) = 2$. If $k = 4$, then $w_3 v_3 \in E(G)$ by the assumption and if $k \geq 5$, then $w_3 v_3 \in E(G)$ by (2.3). By (2.1) and Lemma 2.6, $N(v_0) - \{v_2, v_3\} \subseteq N(w_3)$. Thus we have $d(w_3) \geq \delta + 1$ and hence we may assume $z_1 = z_2 = u_1$. Obviously, $u_1 \in U_3$.
Since \(d(w_2) = \delta + 1\) and \(d(w_3) = \delta\), by Claim 2.3, we have \(|U_3| = 2\) and \(N_{U}(w_2) = U_3\). Since \([w_2, u_1] \to v_0\), \(v_{k-1} \in N(w_2) \cap N(u_1)\) and \(w_{2u_1} \in E(G)\), we have \(d(u_1) + d(w_2) \geq n\), which implies \(d(u_1) \geq n - \delta - 1\). We now show \([w_k, v_k] \to v_{k-1}\). If \(U_4 = \emptyset\), then by (2.1) and Lemma 2.6, \([w_k, v_k] \to v_{k-1}\). If \(U_4 \neq \emptyset\), then since \(w_{1u_3} \in E(G)\) and \(d(w_3) = \delta\), we have \(N(w_3) \cap U_1 = \emptyset\). For any \(u \in U_4\), by Lemma 2.2, there is some vertex \(z \in N(v_0)\) such that \([u, z] \to w_3\) or \([w_3, z] \to u\). If \([w_3, z] \to u\), then since \([w_2, v_2] \to w_3\) and \(u \notin N(w_2)\), we have \(z \neq v_2\). By (2.1) and Lemma 2.6, \(z \notin N(v_0) \setminus \{v_2\}\), a contradiction. If \([u, z] \to w_3\), then by (2.1) and Lemma 2.6, \(z = v_2\). Since \(v_2, v_k \notin E(G)\) by Lemma 2.6, we have \(v_k u \in E(G)\) and hence \(U_4 \subseteq N(v_0)\). Thus, \([w_k, v_k] \to v_{k-1}\). Since \(d(v_{k-1}) = n - 3, d(v_k) \geq n - \delta\) by Claim 2.2 and \(d(u_1) \geq n - \delta - 1\), we have \(v_{k-1}, v_2, u_1 \in N^*(v_k)\). By Claim 2.4, \(v_{k-1} \in N^*(w_k)\). By Lemmas 2.6 and 2.7, \(v_{k-1}, v_2, u_1 \notin N(v_k)\). Thus, we have \(d^*(w_k) + d^*(v_k) \geq n + 1\) and hence \(v_k \in N^*(w_k)\).

**Claim 2.6.** For any \(u \in U_2\), we have \([u, v_1] \to w_1\).

**Proof.** Since \(uw_1 \notin E(G)\), there exists some vertex \(z\) such that \([w_1, z] \to u\) or \([u, z] \to w_1\). In order to dominate \(v_0\), we have \(z \in N(v_0)\). Thus by (2.1) and Lemma 2.6, it is easy to see \([w_1, z] \to u\) is impossible. If \([u, z] \to w_1\), then by the assumption we have \(z \neq v_0\). By (2.1) and Lemma 2.6, we have \(z = v_1\), that is, \([u, v_1] \to w_1\).

**Claim 2.7.** For any \(u \in U_2, N(v_0) \subseteq N(u)\).

**Proof.** Since \([w_1, v_1] \to w_2\) and \(u \in U_2\), we have \(v_1 \in N(u)\). By Lemmas 2.4 and 2.6, we have \(v_1 \in N(u)\) for \(2 \leq i \leq k - 2\). By Lemma 2.6 and Claim 2.6, we have \(v_k \in N(u)\). We now show \(v_{k-1} \in N(u)\). Since \(w_1w_k \notin E(G)\), by Lemma 2.2, there exists some vertex \(z \in N(v_0)\) such that \([w_1, z] \to w_k\) or \([w_k, z] \to w_1\). By (2.1) and Lemma 2.6, we can see \([w_k, z] \to w_1\) is impossible. Thus we have \([w_1, z] \to w_k\). By Claim 2.6 we have \(w_1v_1 \notin E(G)\). By Lemma 2.6, we have \(z \neq v_k\) since \([w_1, v_k] \notin v_1\). By (2.1), we have \(z = v_{k-1}\) which implies \(v_{k-1} \in N(u)\).

**Claim 2.8.** If \(U_2 \neq \emptyset\), then \(N_{U}(w_k) \subseteq N(w_1) \cap N(w_2)\).

**Proof.** Let \(u \in N_{U}(w_k)\) and \(w \in \{w_1, w_2\}\). If \(uw \notin E(G)\), then there is some vertex \(z\) such that \([u, z] \to w\) or \([w, z] \to u\). If \([w, z] \to u\), then \(z \in N(v_0)\). By Claim 2.6, \(w_1v_1 \notin E(G)\), which implies \([w_2, v_1] \to u\) cannot occur. Thus, by (2.1) and Lemma 2.6 we see that \([w, z] \to u\) is impossible. If \([u, z] \to w\), then by the assumption, \(z \neq v_0\). By Lemma 2.6, \(z \neq v_k\). If \(z \in N(v_0) \setminus \{v_k\}\), then \([u, z] \notin v_k\) by Lemmas 2.6 and 2.7. Thus, \(z \notin N(v_0)\), a contradiction.

We first show that \(w_1v_1 \in E(G^*)\).

If \(w_1v_1 \in E(G)\), then \(w_1v_1 \in E(G^*)\). If \(\delta \geq 5\), then by Lemma 2.7, Claim 2.1 and \([w_1, v_1] \to w_2\), we have \(d(w_1) + d(v_1) \geq n + 1\) and hence \(w_1v_1 \in E(G^*)\). Thus, we may assume that \(w_1v_1 \notin E(G)\) and \(\delta = 4\).

If \(|N(w_1) \cap N(v_1) \cap U| \geq 2\), then by Lemma 2.7 and \([w_1, v_1] \to w_2\), we have \(d(w_1) + d(v_1) \geq n + 1\) and hence \(w_1v_1 \in E(G^*)\). Thus by Claim 2.1 we may assume

\[
N(w_1) \cap N(v_1) \cap U = \{u_1\}. \tag{2.4}
\]

By the assumption, we let \([w_1, z] \to v_0\). If \(z \neq u_1\), then \(z \in U_2\) by (2.4). This is impossible since \([w_1, z] \notin w_k\) by Claim 2.8 and hence we have

\[
[w_1, u_1] \to v_0. \tag{2.5}
\]

If \(U_2 \neq \emptyset\), we let \(u \in U_2\). If \(u' \in U_2\) and \(uu' \notin E(G)\), then there is some vertex \(z\) such that \([u, z] \to u'\) or \([u', z] \to u\). By symmetry we may assume \([u, z] \to u'\). By Claim 2.7, \(z \notin N(v_0)\). If \(z = v_0\), then \([u, z] \notin w_1\), a contradiction. Hence \(U_2\) is a clique. If \(u' \in U_1\) and \(uu' \notin E(G)\), then by Claim 2.6 we have \([u', u_1] \to v_1\), which implies \([u', u_1] \to v_1\) by (2.4). By (2.5), \([u, u_1] \in E(G)\). Thus, \(U \subseteq N[u]\) for any \(u \in U_2\). By Claim 2.6, \(U_2 \subseteq N(w_2)\). Thus by Claim 2.7, we have \(d(u) \geq n - \delta - 1\). If \(d(w_1) \geq \delta + 2\), then \(w_1v_1 \in E(G^*)\), which implies \(w_1v_1 \in E(G^*)\). If \(d(w_1) \leq \delta + 1\), then by (2.1) and Lemma 2.6 we have \(|U_1| \leq 2\). By Lemma 2.6 and the assumption, we have \(d_U(w_k) \geq 2\). Thus by Claim 2.8 we have \(U_1 \subseteq N(w_k) \subseteq N(w_2)\) and hence \(U \subseteq N(w_2)\). Thus, we have \(w_1w_2 \in E(G^*)\) and hence \(w_1v_1 \in E(G^*)\).
Thus, by Claim 2.7 we have such that \( d \leq 0 \) and hence \( z \in N(v_0) \). By Lemma 2.7, \( z = v_k \). This is impossible since \( \{w_1, v_k\} \not\subset G \) by Claim 2.6. Thus we have \( w_1 \to v_1 \). Since \( U_2 = \emptyset \) and \( N(v_0) - \{v_1\} \subset N(w_1) \), we have \( z \in \{w_2, \ldots, v_k\} \). In this case, \( z = w_2 \), that is, \( \{w_2, v_1\} \to w_1 \). By (2.5), \( u_1 w_2 \in E(G) \). Thus by (2.4), we have \( U \subset N(w_2) \). By (2.1) and Lemmas 2.4 and 2.6, \( v_2, v_3, v_4 \in N(w_1) \cap N(w_2) \). Thus, if \( |U| \geq 4 \), then \( d(w_1) + d(w_2) \geq n + 1 \), which implies \( w_1 w_2 \in E(G^*) \) and hence \( w_1 v_1 \in E(G^*) \). If \( |U| \leq 3 \), then \( n \leq 12 \). After an easy but tedious check, we can show \( w_1 v_1 \in E(G^*) \).

Next, we show \( U \subset N^*(w_1) \). If \( U_2 = \emptyset \), then \( U \subset N(w_1) \subset N^*(w_1) \) and hence we assume \( U_2 \neq \emptyset \). Let \( u \in U_2 \). Suppose \( u' \in V(G) - N[v_0] \) and \( u' \not\subset G \). Obviously, \( u' \not\subset G \) and hence there is some \( z \) such that \( [u', z] \to u \) or \( [u, z] \to u' \). If \( [u', z] \to u \), then \( z \notin N(v_0) \) by Claim 2.7 and hence \( z = v_0 \). In this case, \( u' \in U \). Since \( [v_0, v_k] \to w_k \), \( v_k \in N(u') \). By Claim 2.6, \( v_1 u' \in E(G) \). Thus we have \( d(u') \geq n - \delta - 1 \). By the assumption, there exists some \( z' \) such that \( [w_1, z'] \to v_0 \). By Lemma 2.7 and Claim 2.7, \( z' \in U_1 \) and hence \( U_1 \cap U_2 \neq \emptyset \). By Claim 2.6, \( w_2 \in E(u) \). Thus, by Claim 2.7 we have \( d(u) \geq \delta + 2 \), which implies \( u' \in N^*(u) \) and hence \( [u', z] \to u' \) is impossible. Thus we always have \( [u, z] \to u' \). By Claim 2.8, \( w_k \notin N(u) \). Thus we have \( z \neq v_0 \) since \( [v_0, v_k] \notin \{w_1, w_k\} \) and hence \( z \in N(v_0) \). If \( V(G) - N[v_0] \) contains \( \delta \) vertices, say \( u'_1, u'_2, \ldots, u'_k \), that are not adjacent to \( u \) in \( G^* \), then there are \( z_{u'_i} \in N(v_0) \) such that \( [u, z_{u'_i}] \to u'_i \) for \( 1 \leq i \leq k \). Clearly, if \( i \neq j \), then \( z_{u'_i} \neq z_{u'_j} \) since \( u'_i \neq u'_j \). This is impossible since \( \{u, v_k\} \notin w_k \) and \( [u, v_k] \notin w_k \). Therefore, \( V(G) - N[v_0] \) contains at most \( \delta - 1 \) vertices that are not adjacent to \( u \) in \( G^* \) and hence \( d(u) \geq n - \delta - 1 \) since \( N(v_0) \subset N(u) \) by Claim 2.7. By Claim 2.6, \( w_1 u \in E(G) \). By Lemma 2.6 and the assumption, \( d(U) \geq 2 \) which implies \( d(U) \geq 2 \) by Claim 2.8. Thus by (2.1) and Lemma 2.6 we have \( d(w_1) \geq \delta + 1 \) and hence \( d^*(w_1) \geq \delta + 2 \) since \( w_1 v_1 \in E(G^*) \). This implies \( d^*(w_1) + d^*(u) \geq n + 1 \) and thus \( U \subset N^*(w_1) \).

Finally, we show \( N^*[w_1] = V(G) \). Since \( w_1 v_1 \in E(G^*) \) and \( U \subset N^*(w_1) \), by (2.1), we have \( d^*(w_1) \geq n - \delta - 1 \). By Claim 2.2, \( d(w_2) \geq \delta + 1 \). If \( d(w_2) \geq \delta + 2 \), then by Claim 2.4, we have \( w_k \in N^*(w_1) \), which implies \( d^*(w_1) \geq n - \delta + 1 \) and hence \( N^*[w_1] = V(G) \). If \( d(w_2) \geq \delta + 1 \) and \( d(w_3) \geq \delta + 1 \), then by Claim 2.2 we have \( d^*(w_3) \geq \delta + 2 \). Thus \( w_3, w_2 \in N^*(w_1) \) and hence \( N^*[w_1] = V(G) \). If \( d(w_2) = \delta + 1 \) and \( d(w_3) = \delta + 1 \), then \( d^*(w_2) = \delta + 2 \) by Claims 2.4 and 2.5. Thus, \( w_k, w_2 \in N^*(w_1) \) and hence \( N^*[w_1] = V(G) \). \( \square \)

3. Some lemmas

Let \( G \) be a graph of order \( n \), and \( x, y \) vertices of \( G \) such that the longest \((x, y)\)-path is of length \( n - 2 \). Let \( P = P_{xy} \) be an \((x, y)\)-path of length \( n - 2 \) and suppose the orientation of \( P \) is from \( x \) to \( y \). We denote by \( x_P \) the only vertex not in \( P \) and let \( d(x_P) = k \) with

\[
N(x_P) = X = \{x_1, x_2, \ldots, x_k\}, \text{ indices following the orientation of } P; \quad A = X^+ = \{a_1, a_2, \ldots, a_i\}, \text{ where } a_i = x_i^+ \in V(P) \text{ and } s \geq k - 1; \quad B = X^- = \{b_1, b_2, \ldots, b_k\}, \text{ where } b_i = x_i^- \in V(P) \text{ and } t \leq 2; \quad P_i = a_i \overrightarrow{P} b_j, \text{ where } 1 \leq i \leq k - 1.
\]

Furthermore, we let \( P_0 = x \overrightarrow{P} b_1 \) if \( x \notin X \) and \( P_k = a_k \overrightarrow{P} y \) if \( y \notin X \). In this section, we will establish some lemmas. It is worth noting that all lemmas in this section except the last one do not depend on the 3-critical property of \( G \).

**Definition.** A vertex \( v \in P_i \) (1 \( \leq i \leq k \)) is called an \( A \)-vertex if \( G[V(P_i) \cup \{x_i+1\}] \) contains a hamiltonian \((v, x_i+1)\)-path, and \( v \in P_i \) (0 \( \leq i \leq k - 1 \)) is a \( B \)-vertex if \( G[V(P_i) \cup \{x_i\}] \) contains a hamiltonian \((x_i, v)\)-path, where \( x_{k+1} = y \) and \( x_0 = x \).

From the definition, we can see that each \( a_i \) is an \( A \)-vertex and each \( b_i \) is a \( B \)-vertex. Let \( u_i \in P_i \) be an \( A \)-vertex and \( Q_i \) a given hamiltonian \((u_i, x_{i+1})\)-path in \( G[V(P_i) \cup \{x_{i+1}\}] \). Suppose the orientation of \( Q_i \) is from \( u_i \) to \( x_{i+1} \). We have the following two lemmas.

**Lemma 3.1.** If \( u_i \in P_i \) and \( u_j \in P_j \) are two \( A \)-vertices (\( B \)-vertices, respectively) with \( i \neq j \), then \( x_P u_i \notin E(G) \) and \( u_i u_j \notin E(G) \). In particular, both \( A \cup \{x_P\} \) and \( B \cup \{x_P\} \) are independent sets.

**Proof.** If \( x_P u_i \in E(G) \), then \( x \overrightarrow{P} x_P u_i \overrightarrow{Q_i} x_{i+1} \overrightarrow{P} y \) is a hamiltonian \((x, y)\)-path. Assume \( i < j \). If \( u_i u_j \in E(G) \), then the \((x, y)\)-path \( x \overrightarrow{P} x_P x_P x_P \overrightarrow{Q_i} u_i u_j \overrightarrow{Q_j} x_{j+1} \overrightarrow{P} y \) is hamiltonian, a contradiction. \( \square \)
Lemma 3.2. Let $u_i \in P_i$, $u_j \in P_j$ be A-vertices with $i < j$, $Q = \overrightarrow{Q_i}x_{i+1}\overrightarrow{P_j}x_j$ and $R = \overrightarrow{Q_j}x_{j+1}\overrightarrow{P_i}y$. If $v \in N_Q(u_i)$, then $v^- \notin N(u_j)$ and if $v \in N(Q) \cap (x\overrightarrow{P_i} \cup R)$, then $v^+ \notin N(u_j)$. In particular, let $a_i, a_j \in A$ with $i < j$ and $v \in N(a_i)$, then $v^- \notin N(a_j)$ if $v \in x\overrightarrow{P_i}x_j$ and $v^+ \notin N(a_j)$ if $v \in x\overrightarrow{P_j}y$.

Proof. If $v \in N_Q(u_i)$ and $v^- \in N(u_j)$, then the $(x, y)$-path $x\overrightarrow{P_i}x_jx\overrightarrow{P}x_jy$ is hamiltonian, a contradiction. As for the latter case, the proof is similar. □

By symmetry of $A$ and $B$, Lemma 3.2 still holds if we exchange $A$ and $B$.

Lemma 3.3. Let $u, v \in a_i\overrightarrow{P_j}b_j$ with $j \geq i + 1$ and $G[a_i\overrightarrow{P_j}b_j]$ contain a hamiltonian $(u, v)$-path $Q$. Suppose that $w \in x\overrightarrow{P_i}x_jx_j\overrightarrow{P}y$ and $u \in E(G)$. If $w^- \notin E(G)$ if $w^- \in x\overrightarrow{P_j}x_j\overrightarrow{P}y$, and $w^+ \notin E(G)$ if $w^+ \in x\overrightarrow{P_j}x_j\overrightarrow{P}y$. In particular, let $a_i \in A$ and $b_j \in B$ with $j \geq i + 1$. Suppose that $v \in x\overrightarrow{P_i}x_jx_j\overrightarrow{P}y$ and $a_i \in E(G)$. Then $v^- \notin E(G)$ if $v^- \in x\overrightarrow{P_i}x_jx_j\overrightarrow{P}y$ and $v^+ \notin E(G)$ if $v^+ \in x\overrightarrow{P_j}x_j\overrightarrow{P}y$.

Proof. Suppose that $w \in x\overrightarrow{P_i}x_j$. If $w^- \in x\overrightarrow{P_i}x_j$ and $w^+ \notin E(G)$, then the $(x, y)$-path $x\overrightarrow{P}w\overrightarrow{Q}uw\overrightarrow{P}x_jx_j\overrightarrow{P}y$ is hamiltonian, and if $w^- \in x\overrightarrow{P_i}x_j$ and $w^- \notin E(G)$, then the $(x, y)$-path $x\overrightarrow{P}wu\overrightarrow{Q}vw^+\overrightarrow{P}x_jx_j\overrightarrow{P}y$ is hamiltonian, a contradiction. As for the case $w \in x\overrightarrow{P_j}y$, the proof is similar. □

Lemma 3.4. Let $(u, u^+) \in V(P_i)$. If $u^+ \in E(G)$ for some $l \geq i + 1$, then $b_j \notin E(G)$ for all $j < l$.

Proof. If $b_j \in E(G)$ for some $l \geq i$, then the $(x, y)$-path $x\overrightarrow{P}b_jx\overrightarrow{P}x_jx_j\overrightarrow{P}u^+a_i\overrightarrow{P}y$ is hamiltonian, a contradiction. □

Lemma 3.5. Let $z \in V(G) - N[x_P]$. If $|N(z) \cap A| \geq 2$, then $z^+z^- \notin E(G)$.

Proof. Let $a_i, a_m \in N(z)$ with $l < m$ and $z \in P_j$. If $z^-z^+ \in E(G)$, then the $(x, y)$-path $x\overrightarrow{P}z^-z^+\overrightarrow{P}x_jx_j\overrightarrow{P}y$ is hamiltonian if $j < l$, $x\overrightarrow{P}x_jx_j\overrightarrow{P}z^+z^-\overrightarrow{P}a_la_m\overrightarrow{P}y$ is hamiltonian if $l \leq j < m$, and $x\overrightarrow{P}x_jx_j\overrightarrow{P}z^+z^-\overrightarrow{P}y$ is hamiltonian if $m \leq j$, a contradiction. □

Lemma 3.6. Let $z^-, z^- \in P_i, w, w^- \in P_j$ with $i > j$ and $k \geq 4$. If $|A - N(z)| \leq 1$ and $A \subseteq N(w)$, then $z^-w^+ \notin E(G)$.

Proof. Suppose to the contrary $z^-w^- \in E(G)$. If $i = j$ and $w \in x\overrightarrow{P}z$, then $a_iz \notin E(G)$ for otherwise $w$ is an A-vertex, which contradicts Lemma 3.1 since $A \subseteq N(w)$. Hence we have $A - \{a_i\} \subseteq N(z)$. Noting that $A \subseteq N(w)$ and $k \geq 4$, we have $w \notin z^- \overrightarrow{P}y$ by Lemma 3.2. Thus, the $(x, y)$-path $x\overrightarrow{P}w^-z^-\overrightarrow{P}w^+\overrightarrow{P}x_jx_j\overrightarrow{P}y$ is hamiltonian if $i = 1$, $x\overrightarrow{P}x_jx_j\overrightarrow{P}a_la_m\overrightarrow{P}w^-z^-\overrightarrow{P}w^+\overrightarrow{P}y$ is hamiltonian if $i = 2$, and $x\overrightarrow{P}x_jx_j\overrightarrow{P}a_la_m\overrightarrow{P}w^-z^-\overrightarrow{P}a_la_m\overrightarrow{P}y$ is hamiltonian if $j \geq 3$, a contradiction. If $i = j$ and $z \in x\overrightarrow{P}w$, then since $a_i \in E(G), z \in A$, which contradicts Lemma 3.1 since $|A - N(z)| \leq 1$. If $i \neq j$, then since $a_j \in E(G), w^-$ is an A-vertex. Since $z^-w^- \in E(G)$, by Lemma 3.1, $za_i \notin E(G)$. Thus, $x\overrightarrow{P}x_jx_j\overrightarrow{P}a_la_m\overrightarrow{P}w^-z^-\overrightarrow{P}a_la_m\overrightarrow{P}y$ is a hamiltonian $(x, y)$-path if $i < j$, and $x\overrightarrow{P}x_jx_j\overrightarrow{P}w^+a_la_m\overrightarrow{P}y$ is a hamiltonian $(x, y)$-path if $i > j$, also a contradiction. □

Lemma 3.7. Let $z^-, z^- \in P_i, w^- \in P_j$ with $i > j$ and $k \geq 4$. If $|A \cup B - N(z)| \leq 1$ and $|A - N(w)| \leq 1$, then $w^-z^- \notin E(G)$.

Proof. We first show the following claim.

Claim 3.1. Let $u^-, u \in P_i, v^- \in P_m$ and $h \neq l, m$. If $u^--v^- \in E(G)$, then either $ua_l \notin E(G)$ or $vb_{h+1} \notin E(G)$.

Proof. Assume without loss of generality $v \in u\overrightarrow{P}y$. If $ua_l, vb_{h+1} \in E(G)$, then $u \neq v^-$ by Lemma 3.3. Thus the $(x, y)$-path $x\overrightarrow{P}x_lx_jx_{l+1}\overrightarrow{P}u^-v^-\overrightarrow{P}ua_l\overrightarrow{P}b_{h+1}v\overrightarrow{P}y$ is hamiltonian if $h < l$, $x\overrightarrow{P}u^-v^-\overrightarrow{P}x_{l+1}x_jx_{l+1}\overrightarrow{P}u\overrightarrow{P}b_{h+1}v\overrightarrow{P}y$ is hamiltonian if $l < h < m$, and $x\overrightarrow{P}u^-v^-\overrightarrow{P}ua_l\overrightarrow{P}b_{h+1}v\overrightarrow{P}x_jx_{l+1}\overrightarrow{P}y$ is hamiltonian if $m < h$, a contradiction. □
By Lemma 3.6, we may assume \( B \subseteq N(z) \). If \( w^-z^- \in E(G) \), then by Claim 3.1, \( a_i w \notin E(G) \) for \( l \neq i, j \). Noting \( k \geq 4 \) and \( |A - N(w)| \leq 1 \), we have \( i \neq j \) and \( wa_i, wa_j \in E(G) \). Since \( wa_j \in E(G) \), \( w^- \) is an A-vertex. If \( za_j \in E(G) \), then \( z^- \) is also an A-vertex which contradicts Lemma 3.1 since \( i \neq j \) and \( w^-z^- \in E(G) \). Hence, \( za_j \notin E(G) \), which implies \( za_j \in E(G) \) since \( |A \cup B \setminus N(z)| \leq 1 \). If \( j < k \), then \( w^- P \) \( a_j w^- P b_{j+1} \) is a hamiltonian path in \( G(V(P_j)) \), which contradicts Lemma 3.3 since \( w^-z^- \), \( zb_{j+1} \in E(G) \), and hence we have \( i < j \) and \( j = k \) by Lemma 3.3. In this case, the \( (x, y) \)-path \( x P \) \( x P j P j P x j P x P j P x P j P \) \( P \) \( z a j P \) \( w^- P \) \( w^- P a j \) \( w^- P y \) is hamiltonian, a contradiction. \( \square \)

**Lemma 3.8 (Chen et al. [3]).** Let \( z \in V(P) - X \) and \( v \in A \cup B \). If \( d(x, P) = k \geq 4 \) and \( A \cup B - \{ v \} \subseteq N(z) \), then \( A \cup \{ z^+ \} \) is an independent set if \( z^+ \in V(P) \) and \( B \cup \{ z^- \} \) is an independent set if \( z^- \in V(P) \).

**Lemma 3.9 (Chen et al. [6]).** Let \( u, v \notin V(P) \) and \( \{ u, v \} \supset V(P) \). If \( u a_1, v b_{i+1} \in E(G) \), where \( b_{k+1} = y \) if \( i = k \), then there is some \( w \in V(P_1) \) such that \( uw, vw^+ \in E(G) \).

Let \( z \in P_j \) and \( \{ a_i, z \} \to x_p \). We have the following five lemmas (3.10–3.14).

**Lemma 3.10.** If \( 2 \leq i \leq j \) and \( z^+ \in V(P) \), then \( A \cup \{ x_p, z^+ \} \) is an independent set.

**Proof.** Since \( za_1 \in E(G) \), we have \( a_i z^+ \notin E(G) \) for \( 2 \leq i \leq j \) by Lemma 3.2. If \( a_1 z^+ \in E(G) \) or \( a_i z^+ \in E(G) \) for some \( l \geq j + 1 \), then by Lemmas 3.3 or 3.4, we have \( b_2 z \notin E(G) \) and hence \( b_2 a_i \notin E(G) \). By Lemma 3.9, there is some \( w \in P_1 \) such that \( w, u_+ a_i \in E(G) \). Thus, the \( (x, y) \)-path \( x P \) \( x P j P j P x j P x P j P x P j P x P j P \) \( P \) \( w^+ a_i \) \( P a \) \( w^+ P y \) is hamiltonian if \( a_1 z^+ \in E(G) \), and \( x P \) \( u z \) \( P a_i w^+ P x j P x j P P a_i z^+ P y \) is hamiltonian if \( a_1 z^+ \in E(G) \) for some \( l \geq j + 1 \), a contradiction. If \( z \in B \), then \( z = b_{j+1} \). By Lemma 3.1 we have \( a_i b_{j+1}, b_{i+1} \in E(G) \). By Lemma 3.9, there is some \( w \in P_1 \) such that \( w b_{j+1} \), \( w a^+ \in E(G) \), which contradicts Lemma 3.3. Thus, \( z \notin B \) and hence \( z^+ x_p \notin E(G) \), which implies \( A \cup \{ x_p, z^+ \} \) is an independent set. \( \square \)

**Lemma 3.11.** If \( 2 \leq i \leq j \) and \( |A| \geq 3 \), then \( B \cup \{ z^-, x_p \} \) is an independent set.

**Proof.** Since \( A - \{ a_i \} \subseteq N(z) \) and \( 2 \leq i \leq j \), we have \( b_2 z^- \notin E(G) \) for \( l \neq i, j + 1 \) by Lemma 3.3. If \( b_1 z^- \in E(G) \) or \( z^- b_{j+1} \in E(G) \), then by Lemmas 3.2 or 3.1, we have \( b_2 \notin N(z) \). Since \( \{ a_i, z \} \to x_p \), we have \( b_2 a_i \in E(G) \). By Lemma 3.9, there is some \( u \in P_1 \) such that \( u z, u^+ a_i \in E(G) \). Thus, the \( (x, y) \)-path \( x P \) \( b_1 z^- P a_i u^+ P x j P x j P P u z P y \) is hamiltonian if \( b_1 z^- \in E(G) \), and \( x P \) \( u z \) \( P b_{j+1} z^- P a_i u^+ P x j P x j P P u z P y \) is hamiltonian if \( b_{j+1} z^- \in E(G) \), a contradiction. Since \( |A| \geq 3 \) and \( \{ a_i, z \} \to x_p \), by Lemma 3.1 we have \( z \notin A \) which implies \( z^- x_p \notin E(G) \). Thus, by Lemma 3.1 we can see that \( B \cup \{ z^-, x_p \} \) is an independent set. \( \square \)

**Lemma 3.12.** If \( j + 1 < i \), then \( A \cup \{ z^+, x_p \} \) is an independent set.

**Proof.** Since \( a_{j+1} z \in E(G) \), by Lemma 3.2 we have \( a_i z^+ \notin E(G) \) for all \( l \neq j + 1 \). If \( a_{j+1} z^+ \in E(G) \), then by Lemma 3.3 we have \( b_{j+2} z \notin E(G) \) and hence \( a_{j+2} \in E(G) \). By Lemma 3.9, there is some \( u \in P_{j+1} \) such that \( u z, u^+ a_i \in E(G) \). Thus, the \( (x, y) \)-path \( x P \) \( z u P \) \( a_{j+1} z^+ P x j P x j P P u^+ a_i P y \) is hamiltonian, a contradiction. If \( z \in B \), then \( z = b_{j+1} \). Since \( \{ a_i, z \} \to x_p \) and \( j + 1 < i \), there is some \( u \in P_{j+1} \) such that \( u z, u^+ a_i \in E(G) \), which contradicts Lemma 3.4. Hence \( z \notin B \) which implies \( z^+ x_p \notin E(G) \). Thus, \( A \cup \{ z^+, x_p \} \) is an independent set by Lemma 3.1. \( \square \)

**Lemma 3.13.** Let \( |A| \geq 3 \). If \( j + 1 < i \) and \( z^- \in V(P) \), then \( B \cup \{ z^-, x_p \} \) is an independent set.

**Proof.** Since \( a_{j+1} z \in E(G) \), we have \( b_{j+1} z \notin E(G) \) for \( l \neq j + 1 \) by Lemma 3.3 and 3.4. If \( b_{j+1} z \in E(G) \), then \( z \) is a B-vertex. By Lemma 3.1 we have \( z b_{j+2} \notin E(G) \), which implies \( a_{j+2} \in E(G) \). By Lemma 3.9, there is some \( w \in P_{j+1} \) such that \( z w, w^+ a_i \in E(G) \). Thus, the \( (x, y) \)-path \( x P \) \( z^- b_{j+1} P z w P x j P x j P P w^+ a_i P y \) is hamiltonian, a contradiction. Since \( |A| \geq 3 \) and \( \{ a_i, z \} \to x_p \), we have \( z \notin A \) by Lemma 3.1 and hence \( z^- x_p \notin E(G) \). Thus, \( B \cup \{ z^-, x_p \} \) is an independent set. \( \square \)

The following two lemmas can be extracted from [6]: Lemma 3.14 is extracted from the Case 2 of Lemma 2.8(2) and Lemma 3.15 from Lemma 2.9 in [6].
Lemma 3.14 (Chen et al. [6]). If \( j = i - 1 \geq 1 \), \( d(x_p) = k \geq 4 \) and \( \{x, y\} \subseteq N(x_Q) \) for any longest \((x, y)\)-path \( Q \), then \( B \cup \{z^-, x_P\} \) is an independent set.

Lemma 3.15 (Chen et al. [6]). Suppose that \( P \) is a longest \((x, y)\)-path such that \(|X \cap \{x, y\}| \) is as small as possible and that for this path, \( d(x_P) = k \geq 4 \). If \( G \) is 3-critical, then there exists an independent set \( I \) such that either \( \{x_P \} \cup A \subseteq I \) or \( \{x_P \} \cup B \subseteq I \) and \(|I| \geq k + 1 \).

4. Proof of Theorem 4

Let \( G \) be a 3-connected 3-critical graph with \( \chi(G) = \delta(G) + 1 \geq 5 \). If \( G \) is not Hamilton-connected, then by Theorem 5, there are two vertices \( x, y \in V(G) \) such that \( p(x, y) = n - 2 \). Among all the longest \((x, y)\)-paths, we choose \( P \) such that \(|\{x, y\} \cap N(x_P)| \) is as small as possible. Choose an orientation of \( P \) such that \(|A| \geq |B| \). Assume without loss of generality that the orientation is from \( x \) to \( y \). We still use the notations given in Section 3.

Since \( \chi(G) = \delta(G) + 1 \geq 5 \), by the choice of \( P \) and Lemma 3.15, \( d(x_P) = k = \delta \geq 4 \). We first show the following claims.

Claim 4.1. Let \( z \in P_j \) and \([a_i, z] \to x_P \). If \(|A| = k = j = i - 1 \geq 1 \), then \( B \cup \{z^-, x_P\} \) is an independent set.

Proof. Let \( U = N[x_P] \cup A \). By Lemmas 2.1 and 2.2, we may assume that \([a_{i_l} x_{j_l}] \to a_{i_{l+1}} \) for \( 1 \leq l \leq k - 1 \). Thus, noting that \(|A| = k \), we have

\[
d_{U}(x_l) \geq \delta \quad \text{for any } x_l \in N(x_P).
\] (4.1)

Assume \( b_1 \in B \) and \( b_2 z^- \in E(G) \). Since \( A - \{a_i\} \subseteq N(z) \), by Lemma 3.3, \( l \in \{1, j + 1, i + 1\} \). If \( j = 1 \), then \( i = 2 \). Since \( a_3 z \in E(G) \), by Lemma 3.4, \( l \neq 1 \) and hence \( l \in \{2, 3\} \). If \( l = 2 \) or \( 3 \), then by Lemma 3.2 we have \( b_4 z \notin E(G) \) and hence \( a_2 b_4 \in E(G) \). Since \( z a_3, a_2 b_4 \in E(G) \), by Lemma 3.1 we have \(|P_1| \geq 2 \) and \(|P_2| \geq 2 \), which implies \( b_2 b_3 \notin U \). Thus we have \( d(x_2) \geq \delta + 1 \) and \( d(x_3) \geq \delta + 1 \) by (4.1). If \( l = 2 \), then \( Q = x P z b_2 P z a_3 P b_4 a_2 P x P x P x P x P x P x P z P y \) is an \((x, y)\)-path of length \( n - 2 \) with \( d(x_Q) = d(x_2) \geq \delta + 1 \) and if \( l = 3 \), then \( R = x P z b_2 P b_3 P a_3 P z P x P x P x P x P x P x P x P x P y \) is an \((x, y)\)-path of length \( n - 2 \) with \( d(x_R) = d(x_3) \geq \delta + 1 \). Since \( \chi(G) = \delta(G) + 1 \), by Lemma 3.1 we have \( x \in N(x_Q) \) if \( l = 2 \) and \( y \in N(x_3) \) if \( l = 3 \). If \( y \neq a_k \), then \( d(x_k) \geq \delta + 2 \) if \( l = 2 \) and \( d(x_3) \geq \delta + 2 \) if \( l = 3 \), which implies \( x(x) = \delta(G) + 2 \) by Lemma 3.1, a contradiction. Hence \( y = a_k \). Thus, \( x P z b_2 P z a_3 P x P x P x P x P a_k \) is a hamiltonian \((x, y)\)-path if \( l = 2 \) and \( x P z b_3 P z a_3 P x P x P x P x P a_k \) is a hamiltonian \((x, y)\)-path if \( l = 3 \), a contradiction. Hence we have \( j \geq 2 \).

Since \( l \in \{1, j + 1, i + 1\} \), we have \( b_2 z \notin E(G) \) by Lemma 3.2 and hence \( b_2 a_i \in E(G) \). If \( l = 1 \), then since \([a_i, z] \to x_P \), we have \( z x_1 \in E(G) \) or \( a_i x_1 \in E(G) \). Thus, \( x P b_2 z b_3 P x P x P x P z x_1 P b_2 a_i P y \) is a hamiltonian \((x, y)\)-path if \( z x_1 \in E(G) \) and \( x P b_2 z b_3 P x P x P x P z x_1 P a_i P y \) is a hamiltonian \((x, y)\)-path if \( a_i x_1 \in E(G) \). If \( l = j + 1 \), then \( Q = x P x P x P x P z b_j P z a_1 P b_2 a_i P y \) is an \((x, y)\)-path of length \( n - 2 \) with \( x_0 = x_{j+1} \). Since \(|P_j| \geq 2 \), \( b_{j+1} \notin U \) which implies \( d(x_{j+1}) \geq \delta + 1 \) by (4.1). Since \( \chi(G) = \delta(G) + 1 \), by Lemma 3.1 we have \( x x_{j+1} \in E(G) \) and \( x = x_1 \).

If \( x P x P x P z b_j P z a_1 P b_2 a_i P y \) is a hamiltonian \((x, y)\)-path. If \( l = i + 1 \), then since \([a_i, z] \to x_P \), we have \( z x_{j+1} \in E(G) \) or \( a_i x_{j+1} \in E(G) \). Thus, \( x P b_2 a_i P b_1 z b_j P b_2 x P x P x P x P x P x P z x_{j+1} P y \) in the former case and \( x P x P x P x P z b_j P z a_1 P b_2 a_i x_{j+1} P y \) in the latter case, is a hamiltonian \((x, y)\)-path, a contradiction. Therefore, \( B \cup \{z^-\} \) is an independent set. On the other hand, since \( k \geq 4 \) and \([a_i, z] \to x_P \), by Lemma 3.1, we have \( z \notin A \) and hence \( z^- x P \notin E(G) \). Thus by Lemma 3.1, \( B \cup \{z^-, x_P\} \) is an independent set. □

Claim 4.2. Let \( I = \{x_P\} \cup W \) with \(|I| = k + 1 \geq 5 \) be an independent set. If \( W = A \) or \( I \) is obtained by one of the Lemmas 3.8 and 3.10–3.15, then \([x_P, x_I] \to w \) is impossible for any \( x_I \in X \) and \( w \in W \).

Proof. If \([x_P, x_I] \to w \) for some \( w \in W \) and \( x_I \in X \), then by Lemmas 2.5 and 2.8, \( W \) contains a vertex \( w' \) such that \( V(G) \subseteq N^*[w'] \). If \( W = A \), then by Lemma 3.1, \( G^* \) contains a hamiltonian \((x, y)\)-path and hence \( p(x, y) = n - 1 \) by Theorem 7, a contradiction. If \( I \) is obtained by one of the Lemmas 3.8 and 3.10–3.15, then by the proofs of these lemmas, we can see that \( G^* \) contains a hamiltonian \((x, y)\)-path, which implies \( p(x, y) = n - 1 \) by Theorem 7, also a contradiction. □

If \( N(x_P) \cap \{x, y\} = \emptyset \), then \(|A| = |B| = k \). By Lemmas 2.1 and 2.2, we may assume \([a_{i_l} x_{j_l}] \to a_{i_{l+1}} \) for \( 1 \leq l \leq k - 1 \). Since \( k \geq 4 \), by Lemma 2.5 there is some \( a_i \) with \( i \geq 2 \) and a vertex \( z \in V(G) - N[x_P] \) such that \( \{x_P, z\} \to a_i \) or
Thus, there is some vertex $z_i, z_j$ such that $[x_p, z] \to x_p$. If $[x_p, z] \to a_i$, then $x \geq \delta + 2$ by Lemma 3.8 and if $[a_i, z] \to x_p$, then $x \geq \delta + 2$ by Lemmas 3.10–3.14 and Claim 4.1, a contradiction. Thus, $|N(x_p) \cap \{x, y\}| \geq 1$. By the choice of the orientation of $P$, we have $x = x_1$.

**Claim 4.3.** For any $a_i \in A$ and any $z \in V(G) - N[x_p]$, $[x_p, z] \to a_i$ is impossible.

**Proof.** Suppose to the contrary there is some $z \in V(G) - N[x_p]$ such that $[x_p, z] \to a_i$. Since $x = x_1$, by Lemma 3.8, $B \cup \{x_p, z^+\}$ is an independent set, and if $|A| = k - 1$, then $A \cup \{x_p, z^+\}$ is also an independent set. Noting that $A \cup \{x_p\}$ or $A \cup \{x_p, z^+\}$ is a maximum independent set and $k \geq 4$, by Claim 4.2, there are some $a_j \in A$ with $j \neq 1$, and $w \in V(G) - N[x_p]$ such that $[x_p, w] \to a_j$ or $[a_j, w] \to x_p$. In both cases, we have $w \neq z$ and $|A - N(w)| \leq 1$. By Lemma 3.8 or Lemmas 3.11, 3.13, 3.14 and Claim 4.1, $B \cup \{x_p, w^–\}$ is an independent set. By Lemma 3.7, $w^–z^-\notin E(G)$. Thus, $B \cup \{x_p, z^–, w^-\}$ is an independent set of order $k + 2$, a contradiction. □

If $|A| = k - 1$, then Lemma 3.15 and the symmetry of $A$ and $B$, we may assume that $G$ contains an independent set $I$ such that $A \cup \{x_p\} \subseteq I$ and $|I| = k + 1$. If $|A| = k$, then $A \cup \{x_p\}$ is a maximum independent set. Thus, by Claim 4.2, $[x_p, x_i] \to a$ is impossible for any $a \in A$ and $x_i \in X$. Since $A \cup \{x_p\}$ is an independent set by Lemma 3.1 and $G$ is 3-critical, by Claim 4.3 we may assume in the following proof that $[a_i, z_i] \to x_p$ for all $a_i \in A$.

We now consider the following two cases separately.

**Case 1:** $|N(x_p) \cap \{x, y\}| = 1$.

Let $w \in P_I$ and $w_{a_i} \in E(G)$. If $a_i \bar{P}$ or $w \subseteq N[a_i]$, say, $v \in a_i \bar{P}$ or $v$ is the last vertex that is not adjacent to $a_i$ along $a_i \bar{P}$, then since $w_{a_i} \in E(G)$, $v$ is an $A$-vertex. Thus, $A \cup \{x_p, v\}$ is an independent set of order $k + 2$ by Lemma 3.1 and hence we have

$$a_i \bar{P} w \subseteq N[a_i] \quad \text{if } w \in P_I \text{ and } w_{a_i} \in E(G).$$

(4.2)

Since $z = \delta + 1$, by Lemmas 3.10–3.14 and Claim 4.1, we have $z_i \in P_{i-1}$ or $z_i = y$ for $2 \leq i \leq k$. If there are two vertices $z_i$ and $z_j$ such that $z_i \in P_{i-1}$ and $z_j \in P_{j-1}$, then both $B \cup \{x_p, z_i^–\}$ and $B \cup \{x_p, z_j^–\}$ are independent sets by Claim 4.1. Since $a_i \in E(G)$, $z_i^–$ and $z_j^–$ are $A$-vertices and hence $z_i^– z_j^– \notin E(G)$ by Lemma 3.1, which implies $B \cup \{x_p, z_i^–, z_j^–\}$ is an independent set of order $k + 2$, a contradiction. Thus, noting that $k \geq 4$, there exist at least two vertices $z_i, z_j$ with $i, j \neq 1$ such that $z_i = z_j = y$, which implies $A \subseteq N(y)$ and $B \cup \{y^-\}$ is an independent set by Lemma 3.11. If there is some $z_i$ with $i \geq 2$ such that $z_i \notin y$, then $z^- y^- \notin E(G)$ by Lemma 3.6 and hence $B \cup \{x_p, z_i^–, y^-\}$ is an independent set of order $k + 2$, a contradiction. Thus, we have $z_i = y$ for $2 \leq i \leq k$. By (4.2), $P_k \subseteq N[a_k]$, which implies each vertex of $P_k - \{y\}$ is an $A$-vertex. Let $z_i \in P_j$. If $z_i \neq y$, then $j \leq k - 1$. Since $a_{i+j} \in E(G)$, we have $b_i z_i^– \notin E(G)$ for $l \neq j + 1$ by Lemmas 3.3 and 3.4. Since $z_i a_k, a_l y \in E(G)$ and $[a_i, z_1] \to x_p$, by Lemma 3.9 there is some vertex $w \in P_k$ such that $w_{z_1}, w^+ a_i \in E(G)$, which implies $z_i^– b_{l+j} \in E(G)$ by Lemma 3.3. By Lemma 3.6, $z_i^– z^- \notin E(G)$ and hence $B \cup \{x_p, z_i^–, y^-\}$ is an independent set of order $k + 2$, a contradiction. Thus, $z_i = y$ and hence we have

$$z_i = y \quad \text{for } 1 \leq i \leq k.$$  

(4.3)

Since $A \subseteq N(y)$, by Lemma 3.1, we have $y \neq a_k$ and hence $y^– x_p \notin E(G)$. If there is some $z \in V(G) - N[x_p]$ such that $[x_p, z] \to y^–$, then $z \notin y$. By Lemma 3.8, $A \cup \{x_p, z^+\}$ is an independent set of order $k + 2$, a contradiction. Since $B \cup \{y^-\}$ is a maximum independent set, by Claim 4.2, there is no vertex $x_i \in X$ such that $[x_p, x_i] \to y^-$. Thus, there is some vertex $z \in P_i$ such that $[y^–, z] \to x_p$. If $z \neq y$, then since $a_k y \in E(G)$, all vertices of $a_k \bar{P} y^–$ are $A$-vertices by (4.2), which implies $z \notin P_k$ since otherwise $[y^–, z] \notin A - \{a_k\}$ by Lemma 3.1. Since $y^–$ is an $A$-vertex, we have $A - \{a_k\} \subseteq N(z)$, which implies $b_{i+j} z^- \notin E(G)$ for $l \neq i + 1$. If $z^- b_{i+1} \in E(G)$, then $z$ is a $B$-vertex. Thus, noting that $B \cup \{y^-\}$ is an independent set, we can see $[y^–, z] \notin B - \{b_{i+1}\}$, a contradiction. Thus we have $z^- b_i \notin E(G)$ for $2 \leq i \leq k$. Since $y^–$ is an $A$-vertex, $k \geq 4$ and $[y^–, z] \to x_p$, we have $z \notin A$ and hence $z^- x_p \notin E(G)$. By Lemma 3.6, $y^- z^- \notin E(G)$. Thus, $B \cup \{x_p, y^–, z^-\}$ is an independent set of order $k + 2$, also a contradiction. Thus we have $z = y$, that is,

$$[y, y^–] \to x_p.$$  

(4.4)

By Lemma 3.1, (4.2) and (4.3), $P_k \subseteq N[y]$. By Lemma 3.11, (4.3) and (4.4), $A \cup B \subseteq N(y)$. For $1 \leq i \leq k - 1$, if there is some $u \in P_i$ such that $u y \notin E(G)$, then $u^+, u^- \in P_i$ since $A \cup B \subseteq N(y)$. By (4.3), $A \subseteq N(u)$. By Lemma 3.5, we have...
Thus, since \( u \cup v \notin E(G) \). By Lemma 3.6, \( u \cup v \notin E(G) \). If \( u \cup \bar{v} \notin E(G) \), then \( u \cup v \notin E(G) \), and hence \( u \cup \bar{v} \notin E(G) \) is hamiltonian and hence \( u \cup \bar{v} \notin E(G) \). By Lemma 3.3, \( u \cup \bar{v} \notin E(G) \) for \( l \neq i + 1 \), which implies \( B \cup \{x, v, u, \bar{v}, y\} - \{b_{i+1}\} \) is an independent set of order \( k + 2 \), a contradiction. Thus, we have \( P_l \subseteq N[y] \) for \( 1 \leq i \leq k - 1 \) and hence \( \{x, y\} \sim V(G) \), a contradiction.

Case 2: \( |N(x) \cap \{x, y\}| = 2 \).

In this case, we let \( z_2 \in P_l \).

Suppose \( i = 1 \), \( l \geq 3 \) and \( z_1 \in P_j \). Assume \( z_1 \neq z_2 \). If \( j \neq 1 \), then \( z_2 \cup z_1 \notin E(G) \) for otherwise the \((x, y)\)-path \( axP_{xy}P_{yv}P_{vz}P_{z}z_1 \) is hamiltonian. If \( j = 1 \) and \( z_2 \cup z_1 \notin E(G) \), then \( z_2 \) is an \( A \)-vertex if \( z_2 \in x \cup u \) and \( z_2 \) an \( A \)-vertex if \( z_2 \in x \cup v \cup P \cup z_1 \). By Lemma 3.1, \( z_2 \) or \( z_2 \notin E(G) \) and hence \( B \cup \{x, z_2, z_1\} \) is an independent set of order \( k + 2 \) by Lemmas 3.11, 3.13 and 3.14. Therefore, we have

\[ z_2 = z_2 \quad \text{for } 3 \leq l \leq k - 1 \text{ if } i = 1. \tag{4.5} \]

If \( i > 2 \), then \( A \cup \{x, z_2\} \) is an independent set by Lemma 3.10. If \( i = 1 \), then by (4.5) and Lemma 3.12, \( A \cup \{x, z_2\} \) is an independent set. By Lemmas 3.11 and 3.14, \( B \cup \{x, z_2\} \) is an independent set. Thus, both \( B \cup \{x, z_2\} \) and \( A \cup \{x, z_2\} \) are independent sets.

If there is some \( w \in V(G) - N[x, y] \) such that \( \{x, w\} \rightarrow z_2 \) \((\{x, w\} \rightarrow z_1 \), respectively), then \( w \notin z_2 \). By Lemma 3.8, \( B \cup \{x, w\} \rightarrow z_2 \) for either set. Because Lemma 3.7 we have \( w \cup z_2 \notin E(G) \) and hence \( B \cup \{x, w, z_2\} \) is an independent set of order \( k + 2 \), a contradiction. Thus, noting that both \( B \cup \{x, z_2\} \) and \( A \cup \{x, z_2\} \) are independent sets, by Claim 4.2, we may assume \( z_2 \cup w \cup x \rightarrow y \) and \( z_2 \cup w \cup x \rightarrow y \).

Let \( w_1 \in P_j \). If \( w_1 \notin z_2 \), then since \( k \geq 4 \), \( A \cup z_2 \) is an independent set and \( z_2 \to x \). If \( w_1 \notin z_2 \), then since \( B \cup z_2 \) is an independent set, we have \( B \cup z_2 \) is an independent set. Thus, both \( B \cup z_2 \) and \( A \cup z_2 \) are independent sets. However, \( B \cup z_2 \) is an independent set of order \( k + 2 \), a contradiction. Hence \( B \cup z_2 \) is an independent set. By Lemma 3.6, \( z_2 \) or \( z_2 \notin E(G) \). Thus by Lemma 3.1, \( B \cup \{x, z_2\} \) is an independent set of order \( k + 2 \), a contradiction. Hence we have \( w_1 = z_2 \), that is,

\[ z_2 \rightarrow x \tag{4.6} \]

If \( w_2 \notin z_2 \), then since \( B \cup \{z_2, x\} \) is an independent set, we have \( B \subseteq N(w_2) \). By (4.6), we have \( A \subseteq N(z_2) \) is an independent set. Thus, \( B \cup \{z_2, x\} \) is an independent set of order \( k + 2 \), a contradiction. Noting that \( z_2 \in N(a_1) \), we have \( a_1 \cap z_2 \subseteq N[a_1] \). By symmetry, we have \( z_2 \cap b_{i+1} \subseteq N[b_{i+1}] \). If \( z_2 \cap b_{i+1} \cap \bar{z}_2 \neq 0 \), then since \( a_1 \cap z_2 \subseteq N[a_1] \), \( z_2 \) is an \( A \)-vertex and if \( z_2 \cap b_{i+1} \neq 0 \), then since \( z_2 \cap b_{i+1} \subseteq N[b_{i+1}] \), \( z_2 \) is a \( A \)-vertex, which contradicts Lemma 3.1. Thus, \( A \cup \{x, z_2\} \) is an independent set of order \( k + 2 \), a contradiction. Hence we have \( w_2 = z_2 \), that is,

\[ z_2 \rightarrow x \tag{4.7} \]

By (4.6) and (4.7), \( A \cup B \subseteq N(z_2) \). If there is some vertex \( v \in a_1 \cap z_2 \) such that \( v \cup a_1 \notin E(G) \) and \( v \cup a_1 \notin E(G) \), then \( v \) is an \( A \)-vertex. If \( v \cup z_2 \notin E(G) \), then \( z_2 \) is an \( A \)-vertex, which contradicts Lemma 3.1. Thus, \( A \cup \{x, v, z_2\} \) is an independent set of order \( k + 2 \), a contradiction. Noting that \( z_2 \in N(a_1) \), we have \( a_1 \cap z_2 \subseteq N[a_1] \). By symmetry, we have \( z_2 \cap b_{i+1} \subseteq N[b_{i+1}] \). If \( N(z_2) \cap a_1 \cap z_2 \neq 0 \), then since \( a_1 \cap z_2 \subseteq N[a_1] \), \( z_2 \) is an \( A \)-vertex and if \( N(z_2) \cap z_2 \cap b_{i+1} \neq 0 \), then since \( z_2 \cap b_{i+1} \subseteq N[b_{i+1}] \), \( z_2 \) is a \( B \)-vertex, which contradicts Lemma 3.1 since \( A \cup B \subseteq N(z_2) \). Thus, we have

\[ N(z_2) \cap a_1 \cap z_2 \neq 0 \quad \text{and} \quad N(z_2) \cap z_2 \cap b_{i+1} = 0. \tag{4.8} \]

Assume \( z_1 \in P_j \) and \( z_1 \neq z_2 \). Since \( [a_1, z_1] \to x \) and \( k \geq 4 \), by Lemma 3.1 we have \( z_1 \notin A \), which implies \( z_1 \in a_1 \cap b_{i+1} \cap \bar{z}_2 \). Thus, \( a_1 \cap b_{i+1} \cap \bar{z}_2 \) is an independent set of order \( k + 2 \), a contradiction. Thus, by Lemma 3.1 and 3.9, there is some vertex \( w = P_{k-1} \) such that \( w \cup a_1 \cup z_1 \in E(G) \), which contradicts Lemma 3.3. Hence, \( B \cup \{x, y\} \) is an independent set. If \( k = k - 1 \), then \( i = k - 1 \) for otherwise \( a_1 \cup z_1 \notin b_{i-1} \), if \( z_1 \in a_{k-1} \cap b_{i-1} \) by Lemma 3.10 and (4.8), and \( a_1 \cup z_1 \notin b_{i-1} \) if \( z_1 \in z_2 \cap b_{i-1} \) by (4.8) and Lemma 3.1 since \( z_2 \) is an \( A \)-vertex. Since \( a_1 \cup z_1 \in E(G) \), we have \( b_{i-1} \notin E(G) \) for \( l \neq 2, k \) by Lemma 3.3. If \( b_{i-1} \notin E(G) \), then \( c_{i-1} \notin E(G) \) by Lemma 3.2 which implies \( a_1 \cap b_{i-1} \in E(G) \). Since \( a_1 \cup z_1 \to x \), we can see
that either \(a_1x_3 \in E(G)\) or \(z_1x_3 \in E(G)\). Thus, the \((x, y)-\)path \(xx_Px_3\overrightarrow{P}x_3a_1\overrightarrow{P}b_2z_1\overrightarrow{P}a_3z_1\overrightarrow{P}y\) is hamiltonian in the former case, and \(xx_Px_3\overrightarrow{P}b_3a_1\overrightarrow{P}b_2z_1\overrightarrow{P}x_3z_1\overrightarrow{P}y\) is hamiltonian in the latter case, a contradiction. If \(z_1b_k \in E(G)\), then \(z_1\) is a \(B\)-vertex. By (4.8), \(z_1^+\) is a \(B\)-vertex, which implies \(z_1^+z_1 \notin E(G)\) by Lemma 3.1 and hence \(\{a_1, z_1\} \neq z_2^+\), a contradiction. Thus, \(B \cup \{x_P, z_1\}\) is an independent set. By (4.6) and (4.7), we have \(A \cup B \subseteq N(z_2)\), which implies \(z_1^+z_2^+ \notin E(G)\) by Lemma 3.7. Thus, \(B \cup \{x_P, z_1^+, z_2^+\}\) is an independent set of order \(k + 2\) and hence we have \(z_1 = z_2\).

By (4.5), we have \(z_l = z_2\) for \(l \geq 3\) if \(i = 1\). If \(i \geq 2\) and there is some \(z_l\) with \(l \geq 3\) such that \(z_l \neq z_2\), then \(B \cup \{x_P, z_l^+\}\) is an independent set by Lemmas 3.11, 3.13 and 3.14. By (4.6), \(A \subseteq N(z_2)\) and hence \(z_2^+z_l^+ \notin E(G)\) by Lemma 3.6.

Thus, \(B \cup \{x_P, z_2^+, z_l^+\}\) is an independent set of order \(k + 2\), a contradiction. Thus we have

\[
z_l = z_2 \quad \text{for} \quad l \neq 2.
\]

By (4.6)–(4.8) we have \(P_l \subseteq N[z_2]\) and \(A \cup B \subseteq N(z_2)\). Let \(l \neq i\). If there is some \(u \in P_l\) such that \(uz_2 \notin E(G)\), then \(u^+, u^- \notin N(x_P)\) and \(A \subseteq N(u)\) by (4.9). By Lemma 3.3, \(b_mu^+, b_mu^- \notin E(G)\) for \(m \neq l + 1\). By Lemma 3.5, \(u^+u^- \notin E(G)\). By Lemma 3.7, \(u^+z_l^- \notin E(G)\). If \(u^+z_l^- \in E(G)\), then the \((x, y)-\)path \(x\overrightarrow{P}x_{i-1}\overrightarrow{P}a_{l-1}\overrightarrow{P}a_1u\overrightarrow{P}a_{l+2}\overrightarrow{P}y\) is hamiltonian if \(l < i\) and if \(i > l\), then \(x\overrightarrow{P}x_{i+1}\overrightarrow{P}a_{l+2}\overrightarrow{P}u_1\overrightarrow{P}a_{l+2}\overrightarrow{P}\overrightarrow{P}z_2a_1\overrightarrow{P}a_{l+2}\overrightarrow{P}u_1\overrightarrow{P}a_{l+2}\overrightarrow{P}\overrightarrow{P}a_1u\overrightarrow{P}a_{l+2}\overrightarrow{P}y\) is hamiltonian, a contradiction. Thus, we have \(u^+z_l^- \notin E(G)\), which implies \(B \cup \{x_P, u^+, u^-, z_l^+\} \setminus \{b_{l+1}\}\) is an independent set of order \(k + 2\), a contradiction. Therefore, we have \(P_l \subseteq N[z_2]\) for \(l \neq i\), which implies \(\{x_P, z_2\} \supset V(G)\), a contradiction.

The proof of Theorem 4 is complete.

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