Chirality and projective linear groups

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Abstract

In recent years the term 'chiral' has been used for geometric and combinatorial figures which are symmetrical by rotation but not by reflection. The correspondence of groups and polytopes is used to construct infinite series of chiral and regular polytopes whose facets or vertex-figures are chiral or regular toroidal maps. In particular, the groups $PSL_2(\mathbb{Z}_m)$ are used to construct chiral polytopes, while $PSL_2(\mathbb{Z}_m[i])$ and $PSL_2(\mathbb{Z}_m[\omega])$ are used to construct regular polytopes.

1. Introduction

Abstract polytopes are combinatorial structures that generalize the classical polytopes. We are particularly interested in those that possess a high degree of symmetry. In this section, we briefly outline some definitions and basic results from the theory of abstract polytopes. For details we refer to [9, 22, 25, 27].

An (*abstract*) *polytope* $\mathscr{P}$ *of rank* n, or an *n-polytope*, is a partially ordered set with a strictly monotone rank function rank($\cdot$) with range $\{-1, 0, \ldots, n\}$. The elements of $\mathscr{P}$ with rank j are called *j-faces* of $\mathscr{P}$. The maximal chains (totally ordered subsets) of $\mathscr{P}$ are called *flags*. We require that $\mathscr{P}$ have a smallest (-1)-face F_{-1}, a greatest n-face F_n and that each flag contains exactly $n+2$ faces. Furthermore, we require that $\mathscr{P}$ be strongly flag-connected and that $\mathscr{P}$ have the following homogeneity property: whenever $F \leqslant G$, rank(F)$= j-1$ and rank(G)$= j+1$, then there are exactly two j-faces H with $F < H < G$.

If F and G are faces of $\mathscr{P}$ with $F \leqslant G$, we shall call $G/F := \{H \mid F \leqslant H \leqslant G\}$ a *section* of $\mathscr{P}$. We shall not distinguish between a face F and the section F/F_{-1}, which itself is

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a polytope with the same rank as F. The faces of rank 0, 1 and $n-1$ are called *vertices*, *edges* and *facets*, respectively. If F is a face, then the polytope F_n/F is called the *co-face* of $\mathscr{P}$ at F, or the *vertex-figure* of $\mathscr{P}$ at F if F is a vertex.

The *dual* $\tilde{\mathscr{P}}$ of a polytope $\mathscr{P}$ is obtained from $\mathscr{P}$ by reversing the partial order, while leaving the set of faces unchanged. We call $\mathscr{P}$ *self-dual* if $\tilde{\mathscr{P}}$ is isomorphic to $\mathscr{P}$. For a more refined notion of self-duality, see Section 3.

A polytope $\mathscr{P}$ is said to be *regular* if its group of automorphisms is transitive on the flags. For a regular $\mathscr{P}$, its group $A(\mathscr{P})$ is generated by involutions $\rho_0, \dots, \rho_{n-1}$, where ρ_j is the unique automorphism keeping fixed all but the j-face of fixed *base* flag $\Phi := \{F_{-1}, F_0, \dots, F_n\}$ of $\mathscr{P}$. These *distinguished generators* satisfy the relations

$$(\rho_j \rho_k)^{p_{jk}} = 1 \quad (j, k = 0, \dots, n-1), \tag{1}$$

where $p_{jj} = 1$, $p_{jk} = p_{kj} =: p_{j+1}$ if $k = j+1$, and $p_{jk} = 2$ otherwise. The generators also satisfy the intersection property

$$\langle \rho_j \,|\, j \in J \rangle \cap \langle \rho_j \,|\, j \in K \rangle = \langle \rho_j \,|\, j \in J \cap K \rangle \quad \text{for all } J, K \subseteq \{0, \dots, n-1\}. \tag{2}$$

Properties (1) and (2) characterize the groups of regular polytopes. Namely, if A is a group generated by involutions $\rho_0, \dots, \rho_{n-1}$ which satisfy (1) and (2), then A is a group of a regular polytope [9, 25] and $\rho_0, \dots, \rho_{n-1}$ are the distinguished generators for its group. Such a group is called a *C-group*, and the polytope $\mathscr{P}$ is said to be of type $\{p_1, \dots, p_{n-1}\}$.

The Coxeter group abstractly defined by relations (1) is denoted by $[p_1, \dots, p_{n-1}]$. This group is the automorphism group of the universal polytope $\{p_1, \dots, p_{n-1}\}$ (cf. [25]).

For a regular n-polytope $\mathscr{P}$, we define the rotation

$$\sigma_j := \rho_{j-1} \rho_j \quad (j = 1, \dots, n-1).$$

Then $\sigma_1, \dots, \sigma_{n-1}$ generate the *rotation subgroup* $A^+(\mathscr{P})$ of $A(\mathscr{P})$, which is of index at most 2 in $A(\mathscr{P})$. When the index is 2 we shall say that $\mathscr{P}$ is *directly regular*. The rotations σ_j satisfy the relations

$$\sigma_j^{p_j} = 1 \quad (1 \leqslant j \leqslant n-1); \tag{3}$$

$$(\sigma_j \sigma_{j+1} \cdot \dots \cdot \sigma_k)^2 = 1 \quad (1 \leqslant j < k \leqslant n-1).$$

By $[p_1, \dots, p_{n-1}]^+$ we denote the group abstractly defined by (3); this is the rotation group of the universal polytope $\{p_1, \dots, p_{n-1}\}$.

Now let $\mathscr{P}$ be a polytope of rank $n \geqslant 3$. Then $\mathscr{P}$ is said to be *chiral* if $\mathscr{P}$ is not regular, but if for some *base flag* $\Psi := \{F_{-1}, F_0, \dots, F_n\}$ of $\mathscr{P}$ there exist automorphisms $\sigma_1, \dots, \sigma_{n-1}$ of $\mathscr{P}$ such that σ_j fixes all faces in $\Psi \setminus \{F_{j-1}, F_j\}$ and cyclically permutes consecutive j-faces of $\mathscr{P}$ in the (polygonal) rank 2 section F_{j+1}/F_{j-2} of $\mathscr{P}$. For a chiral polytope the σ_j's can be chosen in such a way that, if F_j' denotes the j-face of $\mathscr{P}$ with $F_{j-1} < F_j' < F_{j+1}$ and $F_j' \neq F_j$, then $\sigma_j(F_j') = F_j$ (and thus $\sigma_j(F_{j-1}) = F_{j-1}'$) for $j = 1, \dots, n-1$. The corresponding automorphisms $\sigma_1, \dots, \sigma_{n-1}$ generate $A(\mathscr{P})$ and

satisfy relations (3), with $p_1, \ldots, p_{n-1}$ given by the *type* $\{p_1, \ldots, p_{n-1}\}$ of $\mathcal{P}$. The elements $\sigma_1, \ldots, \sigma_{n-1}$ are called the *distinguished generators* of $A(\mathcal{P})$.

It is not hard to see that all sections of a chiral polytope $\mathcal{P}$ must be directly regular or chiral polytopes. In particular, the $(n-2)$-faces and the co-faces at edges are directly regular. For a detailed discussion of chiral polytopes, we refer to [27].

A group A with distinguished generators $\sigma_1, \ldots, \sigma_{n-1}$ must necessarily satisfy a certain intersection property which is stated below for $n = 4$ (see [27]). Conversely, if A is a group generated by $\sigma_1, \ldots, \sigma_{n-1}$ satisfying relations (3) and this intersection property, then A is the group of a chiral polytope or the rotation group of a directly regular polytope. This polytope is directly regular if and only if there exists an involutory group automorphism $\rho: A \to A$ such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2 \sigma_2$ and $\rho(\sigma_j) = \sigma_j$ for $j = 3, \ldots, n-1$ [27, Theorem 1]. Furthermore, it is properly self-dual (in the sense of Section 3) if and only if there exists an involutory automorphism ρ such that $\rho \sigma_1 \rho^{-1} = \sigma_3^{-1}$ and $\rho \sigma_2 \rho^{-1} = \sigma_2^{-1}$.

This paper deals mostly with rank 4 polytopes. If $\sigma_1, \sigma_2, \sigma_3$ are the distinguished generators of the group of a rank 4 chiral polytope $\mathcal{P}$, then the intersection property takes the form

$$\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = 1 = \langle \sigma_2 \rangle \cap \langle \sigma_3 \rangle,$$

$$\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle$$

(see [27, Lemma 11]).

2. Toroidal maps and locally toroidal polytopes of rank 4

Polytopes of rank 3 are (essentially) maps on surfaces. Note that in [6] the term 'regular' has been used for two kinds of maps: maps which are regular in our sense (reflexible maps); and maps which are chiral in our sense (irreflexible maps). The regular and chiral toroidal maps (maps on the torus) are all of type $\{4, 4\}_{(b,c)}$, $\{3, 6\}_{(b,c)}$ or $\{6, 3\}_{(b,c)}$ (cf. [6, pp. 101–109]).

Consider the regular tessellation $\{4, 4\}$ of the Euclidean plane. Let $[4, 4] = \langle \rho_0, \rho_1, \rho_2 \rangle$ and $[4, 4]^+ = \langle \sigma_1, \sigma_2 \rangle$, in the notation of Section 1. In either of these groups the translations $X = \rho_1 \rho_0 \rho_1 \rho_2 = \sigma_1^{-1} \sigma_2$ and $Y = \rho_0 \rho_1 \rho_2 \rho_1 = \sigma_1 \sigma_2^{-1}$ generate an abelian subgroup. Regarding X and Y as unit translations along the Cartesian coordinate axes, $X^b Y^c$ translates the origin $(0, 0)$ to the point (b, c). The orbit of $(0, 0)$ under $\langle X, Y \rangle$ is the set of vertices of $\{4, 4\}$ (which is $\mathbb{Z}[i]$). For a given pair of integers (b, c) the square

$$(b, c), (0, 0), (-c, b), (b - c, b + c)$$

is a fundamental region for the translation subgroup $\langle X^b Y^c, X^{-c} Y^b \rangle$. Identifying opposite edges of the square (see Fig. 1), we obtain the toroidal map $\mathcal{M} := \{4, 4\}_{(b,c)}$. Note that we made a slight change in the notation of [6] where $\{4, 4\}_{(b,c)}$ is denoted by $\{4, 4\}_{b,c}$.
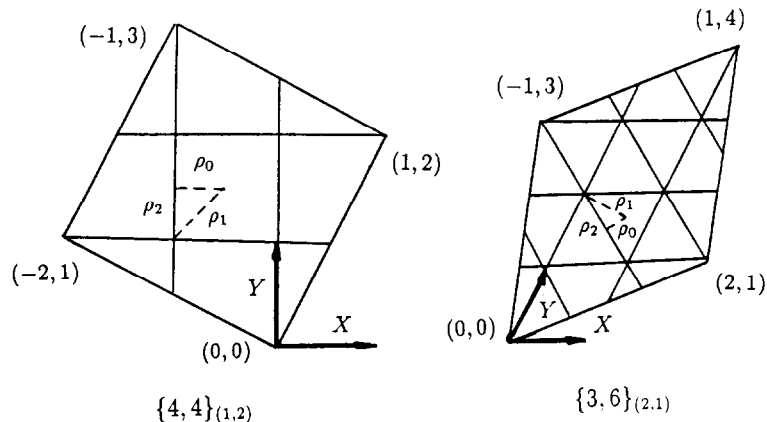
Fig. 1.

It is easy to see that if $b=0$ or $c=0$, or if $b=c$ (equivalently, if $bc(b-c)=0$), then $\{4,4\}_{(b,c)}$ is regular; otherwise, it is chiral. Also note that $\{4,4\}_{(b,c)}=\{4,4\}_{(-c,b)}$; in particular, $\{4,4\}_{(b,0)}=\{4,4\}_{(0,b)}$. If $bc(b-c)\neq 0$, then $\{4,4\}_{(b,c)}$ and $\{4,4\}_{(c,b)}$, although isomorphic, are distinct. In a sense the map $\{4,4\}_{(c,b)}$ is a mirror image of $\{4,4\}_{(b,c)}$. We will say that one is the *enantiomorphic* form of the other, or that the two maps are the two enantiomorphic forms of the same underlying isomorphism type of toroidal map.

Note that conjugation by σ_2 maps X to Y^{-1} and Y to X, so that $X^b Y^c=1$ implies $X^{-c}Y^b=1$. Hence, $A^+(\mathcal{M})=:[4,4]_{(b,c)}^+$ has the following presentation:

$$\sigma_1^4=\sigma_2^4=(\sigma_1\sigma_2)^2=(\sigma_1^{-1}\sigma_2)^b(\sigma_1\sigma_2^{-1})^c=1. \tag{4}$$

Let us now consider the Euclidean tessellation $\{3,6\}$. Let $[3,6]=\langle\rho_0,\rho_1,\rho_2\rangle$ and $[3,6]^+=\langle\sigma_1,\sigma_2\rangle$. The translations $X=(\rho_0\rho_1\rho_2)^2=\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2=\sigma_1^{-1}\sigma_2^2$ and $Y=(\rho_1\rho_2\rho_0)^2=\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}=\sigma_2\sigma_1^{-1}\sigma_2$ generate again an abelian subgroup. Now regarding X and Y as unit translations along the oblique axes inclined at $\pi/3$, the orbit of $(0,0)$ under $\langle X, Y\rangle$ is the set of vertices of $\{3,6\}$ (which is $\mathbb{Z}[\omega]$). For a given pair of integers (b,c), consider $\langle X^b Y^c, X^{-c}Y^{b+c}\rangle$ whose fundamental region is the parallelogram

$$(b,c),(0,0),(-c,b+c),(b-c,b+2c),$$

with coordinates understood relative to the basis X, Y. Identifying opposite edges of this parallelogram (Fig. 1), we obtain the map $\mathcal{N}:=\{3,6\}_{(b,c)}$.

As before, if $bc(b-c)=0$ then $\mathcal{N}$ is regular; otherwise, $\mathcal{N}$ is chiral. Also $\{3,6\}_{(b,c)}=\{3,6\}_{(-c,b+c)}$ and $\{3,6\}_{(b,0)}=\{3,6\}_{(0,b)}$. When $\mathcal{N}$ is chiral, the maps $\{3,6\}_{(b,c)}$ and $\{3,6\}_{(c,b)}$ are enantiomorphic.

The conjugation by σ_2 maps X to Y and Y to YX^{-1}. Hence, $X^b Y^c=1$ implies $X^{-c}Y^{b+c}=1$, hence $A^+(\mathcal{N})=:\{3,6\}_{(b,c)}^+$ has the following presentation:

$$\sigma_1^3=\sigma_2^6=(\sigma_1\sigma_2)^2=(\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2)^b(\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1})^c=1. \tag{5}$$

The dual of $\mathcal{N} = \{3,6\}^+_{(b,c)}$ is given by $\tilde{\mathcal{N}} = \{6,3\}_{(b,c)}$, where the index (b,c) of the latter refers to the same translations X, Y as above. In particular, $\{6,3\}_{(b,c)} = \{6,3\}_{(-c,b+c)}$ and $\{6,3\}_{(b,0)} = \{6,3\}_{(0,b)}$. Again, in the chiral case the maps $\{6,3\}_{(b,c)}$ and $\{6,3\}_{(c,b)}$ are enantiomorphic. By dualizing (i.e. by replacing ρ_0, ρ_1, ρ_2 by ρ_2, ρ_1, ρ_0, and σ_1, σ_2 by $\sigma_2^{-1}, \sigma_1^{-1}$) we find that $A^+(\tilde{\mathcal{N}}) =: [6,3]^+_{(b,c)}$ has the following presentation:

$$\sigma_1^6 = \sigma_2^3 = (\sigma_1\sigma_2)^2 = (\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^b(\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2)^c = 1. \tag{6}$$

Let $\mathcal{P}$ be either a regular or chiral 4-polytope. We say that $\mathcal{P}$ is *locally toroidal* if its facets and its vertex-figures are maps on the 2-sphere or on the torus, and either its facets or its vertex-figures, or both, are actually toroidal. Then locally toroidal rank 4 polytopes are necessarily of type $\{4,4,3\}$, $\{3,4,4\}$, $\{4,4,4\}$, $\{3,6,3\}$, $\{6,3,p\}$ or $\{p,3,6\}$ with $p = 3,4,5$ or 6. In this paper we will construct infinite families of such polytopes.

3. Enantiomorphic forms of chiral polytopes

It is a well-known phenomenon that certain objects (or more exactly, their isomorphism types) occur in two enantiomorphic (mirror image) geometric forms. Examples are the chiral toroidal maps described in Section 2. As we shall see below, each isomorphism type of chiral polytope occurs in two enantiomorphic forms. We begin with the following observation.

Let $\mathcal{L}$ be a regular n-polytope, $\Psi := \{G_{-1}, G_0, \dots, G_{n-1}, G_n\}$ its base flag, and $\alpha_0, \dots, \alpha_{n-1}$ the distinguished generators of $A(\mathcal{L})$. For $i = 0, \dots, n-1$ denote by G'_i the i-face of $\mathcal{L}$ with $G_{i-1} < G'_i < G_{i+1}$ and $G'_i \neq G_i$, and let $\Psi^i := (\Psi \setminus \{G_i\}) \cup \{G'_i\}$ be the flag i-adjacent to Ψ. For $i = 1, \dots, n-1$, let $\beta_i := \alpha_{i-1}\alpha_i$. Then β_i cyclically permutes consecutive i-faces of $\mathcal{L}$ in the (polygonal) 2-section G_{i+1}/G_{i-2} of $\mathcal{L}$, and the 'orientation' of β_i is such that $\beta_i(G'_i) = G_i$ (and $\beta_i(G_{i-1}) = G'_{i-1}$). The elements $\beta_1, \dots, \beta_{n-1}$ are the distinguished generators of $A^+(\mathcal{L})$ defined with respect to the base flag Ψ.

Now, consider changing the base flag to Ψ^k for some fixed k. Clearly, $\alpha_k(\Psi) = \Psi^k$, so that $\alpha_k\beta_1\alpha_k, \dots, \alpha_k\beta_{n-1}\alpha_k$ are the distinguished generators of $A^+(\mathcal{L})$ defined with respect to the new base flag Ψ^k. Note that

$$\alpha_k\beta_i\alpha_k = \begin{cases} \beta_i & \text{if } i \leq k-2 \text{ or } i \geq k+3, \\ \beta_i\beta_{i+1}^2 & \text{if } i = k-1, \\ \beta_i^{-1} & \text{if } i = k, k+1, \\ \beta_{i-1}^2\beta_i & \text{if } i = k+2. \end{cases} \tag{7}$$

For example, if $k = 0$, the new generators of $A^+(\mathcal{L})$ are

$$\beta_1^{-1}, \beta_1^2\beta_2, \beta_3, \dots, \beta_{n-1}.$$

We are particularly interested in the case when $\mathcal{L}$ is directly regular. Then $A^+(\mathcal{L})$ has precisely two orbits on the flags. (Once a base flag has been fixed, these are the set of even flags and the set of odd flags; see [27].) Passing from one flag to another flag in the same orbit implies that the corresponding sets of distinguished generators of $A^+(\mathcal{L})$ are related by conjugation in $A^+(\mathcal{L})$. However, if we pass from one flag to a new flag in the other orbit, then the sets of distinguished generators of $A^+(\mathcal{L})$ are related by conjugation in $A(\mathcal{L})$ but not in $A^+(\mathcal{L})$. The above change from Ψ to Ψ^k illustrates this case; here the change to new generators is realized by an involutory group automorphism of $A^+(\mathcal{P})$.

Now let $\mathcal{P}$ be a chiral or directly regular n-polytope of type $\{p_1,\ldots,p_{n-1}\}$, $\Phi := \{F_{-1}, F_0, \ldots, F_n\}$ its base flag, and $\sigma_1, \ldots, \sigma_{n-1}$ the distinguished generators of $A^+(\mathcal{P})$ defined with respect to Φ. Again, the distinguished generators of $A^+(\mathcal{P})$ belonging to flags in the same orbit of $A^+(\mathcal{P})$ are related by conjugation in $A^+(\mathcal{P})$. As above, the change from Φ to the k-adjacent flag Φ^k results in a change from $\sigma_1, \ldots, \sigma_{n-1}$ to the new generators given on the right-hand side of (7). (Note that in the chiral case the left-hand side of (7) makes no sense.) This can be seen either directly, or by relating $\mathcal{P}$ to the universal (directly regular) n-polytope $\mathcal{L} = \{p_1, \ldots, p_{n-1}\}$. Note that for different k's the corresponding sets of generators are related by conjugation in $A^+(\mathcal{P})$. It follows that up to conjugation in $A^+(\mathcal{P})$ there are precisely two sets of distinguished generators of $A^+(\mathcal{P})$, namely $\sigma_1, \ldots, \sigma_{n-1}$ and $\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1}$, belonging to Φ and Φ^0, respectively. Recall from [27] that the polytope $\mathcal{P}$ is regular if and only if there exists an involutory group automorphism of $A^+(\mathcal{P})$ carrying one set into the other.

These considerations motivate the following definition. By an *oriented chiral* or *oriented directly regular* polytope $\mathcal{P}$, we mean a chiral or directly regular polytope together with a distinguished orbit $\{\Phi\}$ of flags under the action of $A^+(\mathcal{P})$. Here we use the notation $(\mathcal{P}, \{\Phi\})$, or simply $(\mathcal{P}, \Phi)$ with appropriate identifications modulo $A^+(\mathcal{P})$ understood. Each (isomorphism type of) chiral or directly regular polytope $\mathcal{P}$ gives rise to two oriented chiral or oriented directly regular polytopes; if one of them is $(\mathcal{P}, \Phi)$, then the other is $(\mathcal{P}, \Phi^0)$ (or $(\mathcal{P}, \Phi^k)$ for any k). We say that these two oriented polytopes are the two *enantiomorphic forms* of $\mathcal{P}$. The orbits $\{\Phi\}$ and $\{\Phi^0\}$ are also called the two *orientations* of $\mathcal{P}$. In the case of a directly regular polytope $\mathcal{P}$, we shall later identify the two enantiomorphic forms $(\mathcal{P}, \Phi)$ and $(\mathcal{P}, \Phi^0)$. For now, note that by the above remarks each oriented chiral or oriented regular polytope $(\mathcal{P}, \Phi)$ comes along with a distinguished choice of generators of $A^+(\mathcal{P})$ (unique up to conjugation in $A^+(\mathcal{P})$). If $\mathcal{P}$ is directly regular, then these sets of generators are equivalent under an involutory group automorphism of $A^+(\mathcal{P})$.

Yet another view of enantiomorphism is obtained by relating $\mathcal{P}$ to the universal polytope $\mathcal{L} = \{p_1, \ldots, p_{n-1}\}$. Let $\beta_1, \ldots, \beta_{n-1}$ be as above, so that $A^+(\mathcal{L}) = [p_1, \ldots, p_{n-1}]^+ = \langle \beta_1, \ldots, \beta_{n-1} \rangle$; note that here implicitly we are considering $\mathcal{L}$ as an oriented directly regular polytope. Let $(\mathcal{P}, \Phi)$ and $(\mathcal{P}, \Phi^0)$ be the two enantiomorphic forms of $\mathcal{P}$ with corresponding generators $\sigma_1, \ldots, \sigma_{n-1}$ and

$\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1}$, respectively. Then $\beta_i \mapsto \sigma_i$ $(i=1,\ldots,n-1)$ and $\beta_1 \mapsto \sigma_1^{-1}$, $\beta_2 \mapsto \sigma_1^2\sigma_2$, $\beta_i \mapsto \sigma_i$ $(i=3,\ldots,n-1)$ define two surjective homomorphisms $f: A^+(\mathscr{L}) \mapsto A^+(\mathscr{P})$ and $g: A^+(\mathscr{L}) \mapsto A^+(\mathscr{P})$ with kernels N_f and N_g, respectively. However, $g(\beta) = f(\alpha_0\beta\alpha_0)$ for each $\beta \in A^+(\mathscr{L})$, with $\alpha_0, \ldots, \alpha_{n-1}$ as above. It follows that $N_g = \alpha_0 N_f \alpha_0$ $(=\alpha_k N_f \alpha_k$ for each $k)$. Hence, distinguishing the two enantiomorphic forms $(\mathscr{P}, \Phi)$ and $(\mathscr{P}, \Phi^0)$ of a chiral or directly regular polytope $\mathscr{P}$ is equivalent to distinguishing the groups in a pair (N_f, N_g) of normal subgroups of $A^+(\mathscr{L})$ which are related by conjugation with α_0 or any other α_k). As an example, the two translation subgroups of $[4,4]$ defining the two enantiomorphic forms $\{4,4\}_{(b,c)}$ and $\{4,4\}_{(c,b)}$ are conjugates by α_0. In the general situation, if $\mathscr{P}$ is directly regular, then f is the restriction to $A^+(\mathscr{L})$ of the homomorphism $\tilde{f}: A(\mathscr{L}) \rightarrow A(\mathscr{P})$ which maps $\alpha_0, \ldots, \alpha_{n-1}$ to the distinguished generators of $A(\mathscr{P})$; since $\mathscr{P}$ is directly regular, its kernel is again N_f and thus N_f is normal in $A(\mathscr{L})$. It follows that for a directly regular $\mathscr{P}$ the two subgroups N_f and N_g coincide.

Note that, in the above interpretation of enantiomorphism, the universal $\{p_1, \ldots, p_{n-1}\}$ can be replaced by any directly regular polytope $\mathscr{L}$ which covers $\mathscr{P}$ (i.e. for which f and g exist).

Recall that the dual $\tilde{\mathscr{P}}$ of an abstract polytope $\mathscr{P}$ is obtained by reversing the partial order of $\mathscr{P}$ while keeping all faces of $\mathscr{P}$. For an oriented chiral or oriented directly regular polytope $(\mathscr{P}, \{\Phi\})$, the dual is defined as $(\tilde{\mathscr{P}}, \{\Phi\})$, with flags given in reverse order. If $\sigma_1, \ldots, \sigma_{n-1}$ are the distinguished generators for $A^+(\mathscr{P})$ with respect to $(\mathscr{P}, \{\Phi\})$, then $\sigma_{n-1}^{-1}, \sigma_{n-2}^{-1}, \ldots, \sigma_1^{-1}$ are the distinguished generators with respect to $(\tilde{\mathscr{P}}, \{\Phi\})$. Note that the 'orientation' of the generators is correct; in fact, in the notation of Section 1, we have $\sigma_{n-i}^{-1}(F'_{n-i-1}) = F_{n-i-1}$ for $i = 1, \ldots, n-2$.

Let $(\mathscr{P}_1, \{\Phi_1\})$ and $(\mathscr{P}_2, \{\Phi_2\})$ be two oriented chiral or oriented directly regular polytopes, and let $\varphi: \mathscr{P}_1 \rightarrow \mathscr{P}_2$ be an isomorphism of abstract polytopes. Then φ maps $\{\Phi_1\}$ onto $\{\Phi_2\}$ or $\{\Phi_2^0\}$, and thus maps $\{\Phi_1^0\}$ onto $\{\Phi_2^0\}$ or $\{\Phi_2\}$, respectively; in fact, in the notation of [27], φ maps the set of even flags of $\mathscr{P}_1$ onto the set of even flags or the set of odd flags of $\mathscr{P}_2$. We call φ *proper* (or an *isomorphism of oriented polytopes*) if it preserves orientations, i.e. $\{\varphi(\Phi_1)\} = \{\Phi_2\}$; otherwise, φ is *improper*.

Note that for any chiral or directly regular polytope $\mathscr{P}$ the identity map is an improper isomorphism between the two enantiomorphic forms of $\mathscr{P}$. If $\mathscr{P}$ is directly regular, then the automorphism α_0 of $\mathscr{P}$ with $\alpha_0(\Phi) = \Phi^0$ is a proper isomorphism of $(\mathscr{P}, \Phi)$ onto $(\mathscr{P}, \Phi^0)$. Hence, since properly isomorphic oriented polytopes can be identified, we can identify the two enantiomorphic forms $(\mathscr{P}, \Phi)$ and $(\mathscr{P}, \Phi^0)$ for any directly regular polytope.

Let $\{\mathscr{P}_1, \Phi_1\}$ and $\{\mathscr{P}_2, \Phi_2\}$ be as above. A duality (incidence reversing bijection) $\varphi: \mathscr{P}_1 \rightarrow \mathscr{P}_2$ is *proper* or *improper* if it preserves or changes the orientations, respectively. A self-dual chiral or directly regular polytope $\mathscr{P}$ is *properly self-dual* if $\mathscr{P}$ admits a duality φ (and thus only dualities) which preserves the two orbits of $A^+(\mathscr{P})$; then φ must be proper. Note that any self-dual directly regular polytope is indeed properly self-dual, since it possesses a polarity (duality of order 2) which fixes the base flag. In the general situation, if $(\mathscr{P}, \Phi)$ is an oriented chiral or oriented directly regular

polytope such that $\mathscr{P}$ is properly self-dual, then $\mathscr{P}$ admits a polarity ω which fixes Φ and thus induces an involutory group automorphism $A^+(\mathscr{P}) \to A^+(\mathscr{P})$ with $\sigma_i \to \omega \sigma_i \omega = \sigma_{n-i}^{-1}$; note that for $(\mathscr{P}, \Phi^0)$ the polarity $\omega \sigma_1 \cdots \sigma_{n-1}$ fixes Φ^0 and induces a corresponding group automorphism for the generators σ_1^{-1}, $\sigma_1^2 \sigma_2$, $\sigma_3, \ldots, \sigma_{n-1}$. Conversely, if for an oriented chiral or oriented directly regular polytope $(\mathscr{P}, \Phi)$ there exists a group automorphism $A^+(\mathscr{P}) \mapsto A^+(\mathscr{P})$ with $\sigma_i \mapsto \sigma_{n-i}^{-1}$, then $\mathscr{P}$ is properly self-dual.

Let $(\mathscr{P}, \Phi)$ be an oriented chiral or oriented directly regular polytope, and let $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ and $-1 \leqslant i < j - 1 \leqslant n - 1$. First note that $(F_j/F_i, \{F_i, F_{i+1}, \ldots, F_j\})$ is again an oriented chiral or oriented regular polytope (but is not necessarily chiral if $\mathscr{P}$ is chiral); this is a particular instance of a *section* of $(\mathscr{P}, \Phi)$. More generally, if F is an i-face of $\mathscr{P}$, G a j-face of $\mathscr{P}$ with $F < G$, and Ψ a flag of G/F equivalent to $\{F_i, F_{i+1}, \ldots, F_j\}$ under an element φ (say) of $A^+(\mathscr{P})$, then $(G/F, \Psi)$ is an oriented chiral or oriented directly regular polytope called a *section* of $(\mathscr{P}, \Phi)$. Note that in this situation φ becomes a proper isomorphism between $(G/F, \Psi)$ and $(F_j/F_i, \{F_i, F_{i+1}, \ldots, F_j\})$. We also use terms like *face, facet, co-face* or *vertex-figure* of $(\mathscr{P}, \Phi)$ for sections $(G/F, \Psi)$ of $(\mathscr{P}, \Phi)$, where G/F is a face, facet, co-face or vertex-figure of $\mathscr{P}$, respectively.

In the above situation, if $i \geqslant 1$ or $j \leqslant n-2$, F_j/F_i and G/F are directly regular polytopes. In this case there exists an element $\tau \in A^+(\mathscr{P})$ which induces a proper isomorphism between $(G/F, \Psi)$ and $(G/F, \Psi^0)$, with Ψ^0 the (0-adjacent) flag of G/F differing from Ψ in an $(i+1)$-face of $\mathscr{P}$. This is in agreement with our identification of the two enantiomorphic forms for directly regular polytopes. In fact, if $i+2 \leqslant j \leqslant n-2$, then $\tau := \sigma_{i+2} \sigma_{i+3} \cdots \sigma_{n-1}$ maps Φ to $(\Phi^{i+1})^{n-1}$ and thus maps $(F_j/F_i, \{F_i, F_{i+1}, \ldots, F_j\})$ to $(F_j/F_i, \{F_i, F_{i+1}, \ldots, F_j\}^0)$; if $j-2 \geqslant i \geqslant 1$, then $\tau := \sigma_1 \sigma_2 \cdots \sigma_{i+1}$ has a similar effect. Note that if $i=0$ or $j=n-1$ such an element τ cannot exist in $A^+(\mathscr{P})$.

From now on we often simplify notation and write $\mathscr{P}$ for an oriented chiral or directly regular polytope $(\mathscr{P}, \Phi)$, with a specification of the orientation $\{\Phi\}$ understood. By $\bar{\mathscr{P}}$ we denote the oppositely oriented polytope $(\mathscr{P}, \Phi^0)$, with the convention that $\bar{\mathscr{P}} = \mathscr{P}$ if $\mathscr{P}$ is directly regular. Then for all $\mathscr{P}$ we have $(\bar{\bar{\mathscr{P}}}) = \mathscr{P}$. Note that terms like section, face, etc., always refer to the chosen orientation.

4. Classes of polytopes

In this section we briefly discuss the problem of amalgamating two rank n-polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$. In particular, we elaborate on the corresponding discussion in [27, Section 6].

For two regular n-polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$, we denote by $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ the class of all regular $(n+1)$-polytopes $\mathscr{P}$ with facets isomorphic to $\mathscr{P}_1$ and vertex-figures isomorphic to $\mathscr{P}_2$. Each nonempty class $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ contains a universal member denoted by $\{\mathscr{P}_1, \mathscr{P}_2\}$. In particular, the polytope $\{\mathscr{P}_1, \mathscr{P}_2\}$ is directly regular if and only if both $\mathscr{P}_1$ and $\mathscr{P}_2$ are directly regular.

The definition of classes is more subtle for chiral polytopes and does not simply carry over from regular polytopes. We need to distinguish the two enantiomorphic forms of chiral or directly regular polytopes. This distinction is more or less irrelevant if at least one of the polytopes $\mathcal{P}_1$, $\mathcal{P}_2$ is directly regular, but is essential if both $\mathcal{P}_1$ and $\mathcal{P}_2$ are chiral.

Let $(\mathcal{P}_1, \Phi_1)$ and $(\mathcal{P}_2, \Phi_2)$ be oriented chiral or oriented directly regular n-polytopes. By $\langle (\mathcal{P}_1, \Phi_1), (\mathcal{P}_2, \Phi_2) \rangle^{\mathrm{ch}}$, or simply $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$, we denote the class of all oriented chiral $(n+1)$-polytopes $(\mathcal{P}, \Phi)$ whose facets and vertex-figures are properly isomorphic to $(\mathcal{P}_1, \Phi_1)$ and $(\mathcal{P}_2, \Phi_2)$, respectively. For this class to be nonempty it is necessary that the vertex-figures of $(\mathcal{P}_1, \Phi_1)$ are properly isomorphic (as oriented polytopes) to the facets of $(\mathcal{P}_2, \Phi_2)$. However in general this will not be sufficient.

Now, let $(\mathcal{P}, \Phi)$ be in $\langle (\mathcal{P}_1, \Phi_1), (\mathcal{P}_2, \Phi_2) \rangle^{\mathrm{ch}}$. Then the oppositely oriented polytope $(\mathcal{P}, \Phi^0) = (\mathcal{P}, \Phi^1)$ is in $\langle (\mathcal{P}_1, \Phi_1^1), (\mathcal{P}_2, \Phi_2^0) \rangle^{\mathrm{ch}}$, so that both facets and vertex-figures get changed to the oppositely oriented polytopes. Hence, in the notation of Section 3 we have $\mathcal{P} \in \langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$ if and only if $\bar{\mathcal{P}} \in \langle \bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2 \rangle^{\mathrm{ch}}$. If we adopt the short form

$$\overline{\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}} := \{ \bar{\mathcal{P}} \mid \mathcal{P} \in \langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}} \},$$

then this can be written as

$$\overline{\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}} = \langle \bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2 \rangle^{\mathrm{ch}}.$$

In particular, if $\mathcal{P}_2$ is directly regular, then $\overline{\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}} = \langle \bar{\mathcal{P}}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$. A similar equality holds if $\mathcal{P}_1$ is directly regular. However, if both $\mathcal{P}_1$ and $\mathcal{P}_2$ are chiral, we cannot in general relate $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$ and $\langle \bar{\mathcal{P}}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$.

For further reference we recall [27, Theorem 2], the following result. (Note that [27] does not elaborate on the notion of oriented chiral polytopes.)

Theorem 4.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be oriented chiral or directly regular n-polytopes but not both directly regular. Assume that $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}} \neq \emptyset$. Then there exists an oriented chiral $(n+1)$-polytope in $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$ such that any other $\mathcal{P}$ in $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$ is obtained from it by suitable identifications. This polytope is denoted by $\{\mathcal{P}_1, \mathcal{P}_2\}^{\mathrm{ch}}$ and is called the universal (oriented) chiral $(n+1)$-polytope with facet type $\mathcal{P}_1$ and vertex-figure type $\mathcal{P}_2$.*

To give an example that different enantiomorphic forms indeed matter, consider the classes $\langle \{4,4\}_{(1,3)}, \{4,4\}_{(1,3)} \rangle^{\mathrm{ch}}$ and $\langle \{4,4\}_{(3,1)}, \{4,4\}_{(1,3)} \rangle^{\mathrm{ch}}$. Here the Coxeter–Todd coset enumeration algorithm found that the universal $\{\{4,4\}_{(1,3)}, \{4,4\}_{(1,3)}\}^{\mathrm{ch}}$ and $\{\{4,4\}_{(3,1)}, \{4,4\}_{(1,3)}\}^{\mathrm{ch}}$ have groups of order 960 and 2000, respectively (cf. [3]).

Note that for universal polytopes we have $\overline{\{\mathcal{P}_1, \mathcal{P}_2\}^{\mathrm{ch}}} = \{\bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2\}^{\mathrm{ch}}$. If either $\mathcal{P}_1$ or $\mathcal{P}_2$ is directly regular, then $\overline{\{\mathcal{P}_1, \mathcal{P}_2\}^{\mathrm{ch}}} = \{\mathcal{P}_1, \bar{\mathcal{P}}_2\}^{\mathrm{ch}}$ or $\overline{\{\mathcal{P}_1, \mathcal{P}_2\}^{\mathrm{ch}}} = \{\bar{\mathcal{P}}_1, \mathcal{P}_2\}^{\mathrm{ch}}$. Note that if both $\mathcal{P}_1$ and $\mathcal{P}_2$ are directly regular, then $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle^{\mathrm{ch}}$ does not contain a universal member, since the natural candidate for this member is directly regular.

5. Generating special linear groups

Let R be a commutative ring with identity 1 and G a subgroup of the units R^* of R. Let $L_2^G(R)$ be the group of 2×2 matrices with entries in R and determinants in G, and $PL_2^G(R)$ be the quotient of $L_2^G(R)$ by its centre. Note that the centre $C_2^G(R)$ of $L_2^G(R)$ consists of all the matrices λI (I being the 2×2 identity matrix) with $\lambda^2 \in G$. Then $C_2(R) := C_2^{\langle 1 \rangle}(R)$, $SL_2(R) := L_2^{\langle 1 \rangle}(R)$ and $PSL_2(R) := PL_2^{\langle 1 \rangle}(R)$. Furthermore, $GL_2(R) = L_2^{R^*}(R)$ and $PGL_2(R) = PL_2^{R^*}(R)$. Let $\hat{P}L_2^G(R)$ denote the quotient of $L_2^G(R)$ by $\{\pm I\}$; then $\hat{P}SL_2(R) = SL_2(R)/\{\pm I\}$ and $\hat{P}GL_2(R) = GL_2(R)/\{\pm I\}$. Note that since $\{\pm I\} \subset C_2^G(R)$, we have a projection $\hat{P}L_2^G(R) = PL_2^G(R)$.

Let $J \subseteq R$ be an ideal. Then the natural ring epimorphism $R \to R/J$, $r \mapsto r + J$, induces a group homomorphism $\varphi : L_2^G(R) \to L_2^{G_J}(R/J)$, where $G_J = \{g + J / g \in G\}$. Note that $\varphi(C_2^G(R))$ is a subgroup of $C_2^{G_J}(R/J)$, and hence for each subgroup H of $C_2^{G_J}(R/J)$ containing $\varphi(C_2^G(R))$ the map φ induces a homomorphism $\varphi_H : PL_2^G(R) \to L_2^{G_J}(R/J)/H$.

We will be particularly interested in the following two rings of complex numbers: the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}, i^2 + 1 = 0\}$, and the ring of Eisenstein integers $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}, \omega^2 + \omega + 1 = 0\}$. In these rings, $\langle i \rangle = \{\pm 1, \pm i\}$ and $\langle \pm \omega \rangle = \{\pm 1, \pm \omega, \pm \bar{\omega}\}$ are the groups of units, respectively, each containing $\langle -1 \rangle = \{\pm 1\}$.

For later reference we will need the following lemmas.

Lemma 5.1. *Let*

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad and \quad C = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}.$$

(a) $SL_2(\mathbb{Z}[i])$ *is generated by* A, B *and* C.

(b) $L_2^{\langle -1 \rangle}(\mathbb{Z}[i])$ *is generated by* A, B, C *and*

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) $L_2^{\langle i \rangle}(\mathbb{Z}[i])$ *is generated by* A, B, C *and*

$$\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. Bianchi [1] noticed that

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = AC\,AC^{-1}\,AC.$$

To complete the proof of (a), see [11, p. 75]. Then (b) is trivial and (c) follows since

$$\begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad \square$$

Lemma 5.2. *Let*

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad and \quad C = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}.$$

(a) $SL_2(\mathbb{Z}[\omega])$ *is generated by* A, B *and* C.

(b) $L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega])$ *is generated by* A, B, C *and*

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof. Again, Bianchi [1] noticed that

$$\begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix} = (CAC^{-1}BA^{-1}CA)^2.$$

To complete the proof of (a), see [11, pp. 75–76]. Then (b) follows trivially. $\square$

For the sake of completeness we give a proof of the following lemma.

Lemma 5.3. *Let* R *be a finite commutative ring with identity. Then* $SL_2(R)$ *is generated by the elementary matrices of the form*

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad and \quad \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \quad with \ a, b \in R.$$

Proof. We first note that whenever $v \in R^*$, then

$$\begin{bmatrix} v & 0 \\ 0 & v^{-1} \end{bmatrix}$$

has the required from since

$$\begin{bmatrix} v & 0 \\ 0 & v^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v^{-1}-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ v-1 & 1 \end{bmatrix}.$$

This settles the case of diagonal matrices. Now, let

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL_2(R)$$

and $\beta\gamma \neq 0$; then $\alpha\delta - \beta\gamma = 1 = 1_R$.

Let $J(R)$ be the radical of R. Then by a general fact for finite commutative rings, $R/J(R)$ is the direct sum of simple commutative rings (see for example [2, p. 208]) and hence

$$R/J(R) \cong F_1 \oplus \cdots \oplus F_k, \tag{8}$$

where each F_j is a finite field. For each $r \in R$, let $\bar{r} = r + J(R)$, and write $\bar{r} = (r_1, \ldots, r_k)$ corresponding to (8). Let $\mathrm{supp}(\bar{r}) := \{j \mid r_j \neq 0\}$. Note that since $\alpha\delta - \beta\gamma = 1$, $\alpha_j\delta_j - \beta_j\gamma_j = 1_{F_j}$ for all $j = 1, \ldots, k$, where $\bar{1} = (1_{F_1}, \ldots, 1_{F_k})$. Also note that $|\mathrm{supp}(\bar{\alpha}) \cup \mathrm{supp}(\bar{\beta})| = k$.

Assume for the moment that $|\mathrm{supp}(\bar{\alpha})| = k$. Then, since each F_i is a field, $\bar{\alpha}$ is invertible in $R/J(R)$ say $\bar{\alpha}^{-1} = \lambda + J(R)$. Then $\alpha\lambda - 1 \in J(R)$, and since $J(R)$ consists of nilpotent elements, $\alpha \in R^*$. Hence, since $\delta - \alpha^{-1}\beta\gamma = \alpha^{-1}$,

$$A \begin{bmatrix} 1 & -\alpha^{-1}\beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha^{-1}\gamma & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix},$$

and A has the required form.

Now let $|\mathrm{supp}(\bar{\alpha})| = m < k$ and, without loss of generality, assume $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m, 0, \ldots, 0)$ where $\alpha_j \neq 0$ for $j = 1, \ldots, m$. Then $\tilde{\beta} = (\beta_{m+1}, \ldots, \beta_k)$ is an invertible element of $F_{m+1} \oplus \cdots \oplus F_k$, and let $(\beta'_{m+1}, \ldots, \beta'_k)$ be its inverse. Let $\rho \in R$ such that $\bar{\rho} = (0, \ldots, 0, \beta'_{m+1}, \ldots, \beta'_k)$. Then

$$A \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} \alpha + \beta\rho & \beta \\ \gamma + \delta\rho & \delta \end{bmatrix}.$$

Here $\overline{\alpha + \beta\rho} = (\alpha_1, \ldots, \alpha_m, 1_{F_{m+1}}, \ldots, 1_{F_k})$ and hence $|\mathrm{supp}(\overline{\alpha + \beta\rho})| = k$. We now complete the proof by the above argument applied to the matrix on the right-hand side. $\square$

Let R be a finite local ring (every element of R is either a unit or nilpotent) with q elements s of which are units. In subsequent sections we will make use of the following formula $|SL_2(R)| = qs(2q - s)$.

For each integer $m \geqslant 2$, let $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ denote the ring of integers mod m. Let

$$\mathbb{Z}_m[i] := \mathbb{Z}[i]/m\mathbb{Z}[i], \qquad \mathbb{Z}_m[\omega] := \mathbb{Z}[\omega]/m\mathbb{Z}[\omega].$$

Note that here we are not requiring the equations $x^2 + 1 = 0$ or $x^2 + x + 1 = 0$, respectively, to be irreducible over $\mathbb{Z}_m$. Instead, we use the notation $\mathbb{Z}_m[i]$ and $\mathbb{Z}_m[\omega]$ (in any case) to mean the quotient rings of $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ by the ideals $m\mathbb{Z}_m[i]$ and $m\mathbb{Z}_m[\omega]$, respectively.

Under the projections $\mathbb{Z}[i] \to \mathbb{Z}_m[i]$ and $\mathbb{Z}[\omega] \to \mathbb{Z}_m[\omega]$ units are mapped onto units. In particular, if $m \geqslant 3$ then $\mathbb{Z}[i]^* = \langle i \rangle \cong \langle i + m\mathbb{Z}[i] \rangle \subset \mathbb{Z}_m[i]^*$ and similarly $\mathbb{Z}[\omega]^* = \langle \pm\omega \rangle \cong \langle \pm\omega + m\mathbb{Z}[\omega] \rangle \subset \mathbb{Z}_m[\omega]^*$. If $m = 2$ then $\mathbb{Z}_2[i]$ is a ring with four elements two of which are units, $1 + 2\mathbb{Z}[i]$ and $i + 2\mathbb{Z}[i]$; it is not the field $GF(2^2)$, since $2\mathbb{Z}[i]$ is not a maximal ideal. On the other hand, $\mathbb{Z}_2[\omega]$ is indeed the field with four elements. If $m = p$ is a prime with $p \equiv -(\mathrm{mod}\, 4)$, then -1 is a quadratic

non-residue mod p and thus $x^2 + 1 = 0$ is irreducible over $\mathbb{Z}_p$. It follows that in this case $\mathbb{Z}_p[i] \cong GF(p^2)$. Similarly, if $m = p$ is a prime with $p \equiv 2 \pmod{3}$, then -3 is a quadratic non-residue mod p, and $x^2 + x + 1 = 0$ is irreducible over $\mathbb{Z}_p$ and thus $\mathbb{Z}_p[\omega] \cong GF(p^2)$.

Below we shall slightly abuse notation and write r for a unit $r + m\mathbb{Z}[i]$ of $\mathbb{Z}_m[i]$ or $r + m\mathbb{Z}[\omega]$ of $\mathbb{Z}_m[\omega]$. This is mainly used with $r = i$ or $r = -1$. For example, $\langle i \rangle$ will denote the subgroup $\langle i + m\mathbb{Z}[i] \rangle$ of $\mathbb{Z}_m[i]$. Also we write $\varphi_m : L_2^{\langle i \rangle}(\mathbb{Z}[i]) \to L_2^{\langle i \rangle}(\mathbb{Z}_m[i])$ and $\psi_m : L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$ for the induced homomorphisms.

Lemma 5.4. *The canonical homomorphisms*

  (a) $\varphi_{m,H} : PL_2^{\langle i \rangle}(\mathbb{Z}[i]) \to L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ *with* $\langle iI \rangle \leqslant H \leqslant C_2^{\langle i \rangle}(\mathbb{Z}_m[i])$; *and*

  (b) $\psi_{m,K} : PL_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K$ *with* $\{\pm I\} \leqslant K \leqslant C_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$, *are epimorphisms.*

Proof. To prove that $\varphi_{m,H}$ is an epimorphism it is sufficient to show that the restriction $SL_2(\mathbb{Z}[i]) \to SL_2(\mathbb{Z}_m[i])$ of φ_m is surjective. By the previous lemma this is true since

$$\begin{bmatrix} 1 & \alpha + \beta i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^\alpha \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}^\beta$$

and

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}.$$

A similar proof applies to $\psi_{m,K}$. □

For the rings $\mathbb{Z}_m[i]$ and $\mathbb{Z}_m[\omega]$ the *conjugation* $\alpha + \beta i \to \alpha - \beta i$ and $\alpha + \beta\omega \to \alpha + \beta\omega^2$, respectively, define involutory ring automorphisms. As usual we write $\bar{x}$ for the conjugate of an element x. A subgroup H of $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])$ or K of $L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$ is called *conjugation invariant* if for each matrix A in H or K the conjugate matrix (with entries conjugate to those of A) is again in H or K, respectively.

Now let us assume that -1 is a quadratic residue mod m; i.e., there exists an element $\tilde{i} \in \mathbb{Z}_m$ such that $\tilde{i}^2 \equiv -1$. Then the ring homomorphism $\mathbb{Z}_m[i] \to \mathbb{Z}_m$, $\alpha + \beta i \mapsto \alpha + \beta\tilde{i}$, induces the homomorphism $\Phi_m : L_2^{\langle i \rangle}(\mathbb{Z}_m[i]) \to L_2^{\langle i \rangle}(\mathbb{Z}_m)$. Similarly, if the polynomial $x^2 + x + 1 = 0$ is reducible in $\mathbb{Z}_m$, we have the homomorphism $\Psi_m : L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega]) \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m)$ induced by $\alpha + \beta\omega \mapsto \alpha + \beta\tilde{\omega}$, where $\tilde{\omega} \in \mathbb{Z}_m$ with $\tilde{\omega}^2 + \tilde{\omega} + 1 \equiv 0$. As before let us abuse the notation by omitting tilde.

The following is a consequence of Lemmas 5.3 and 5.4.

Lemma 5.5. (a) *Assuming that* $x^2 + 1 \equiv 0 \pmod{m}$ *is solvable, the canonical homomorphism* $\Phi_{m,H} : PL_2^{\langle i \rangle}(\mathbb{Z}[i]) \to L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$ *induced by* $\Phi_m\varphi_m$ *is an epimorphism whenever* $\langle iI \rangle \leqslant H \leqslant C_2^{\langle i \rangle}(\mathbb{Z}_m)$.

  (b) *Assuming that* $x^2 + x + 1 \equiv 0 \pmod{m}$ *is solvable, the canonical homomorphism* $\Psi_{m,K} : PL_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ *induced by* $\Psi_m\psi_m$ *is an epimorphism whenever* $\{\pm I\} \leqslant K \leqslant C_2^{\langle -1 \rangle}(Z_m)$.

6. Hyperbolic honeycombs and the inversive plane

The regular and chiral polytopes which we construct in this paper are all finite and locally toroidal. They can be derived from the universal polytopes which are isomorphic to regular honeycombs in the three-dimensional hyperbolic space $\mathbb{H}^3$; see Section 2.

The absolute of the hyperbolic space $\mathbb{H}^3$ is an inversive plane as can be seen from Poincaré's half-space model for $\mathbb{H}^3$. The absolute is then an extension of the Euclidean plane by the point at infinity. The reflection in a hyperbolic plane induces the inversion in the circle which is the intersection of that plane by the absolute. Conversely, the inversion in a circle of the inversive plane induces the corresponding hyperbolic reflection in $\mathbb{H}^3$. This then implies that a group of displacements in $\mathbb{H}^3$ is isomorphic to a group of Möbius transformations over $\mathbb{C}$.

The symmetry group $[p,q,r]$ of a honeycomb $\{p,q,r\}$ is generated by reflections ρ_v in four planes R_v (say) which form an orthoscheme [5, p. 188], a simplex with dihedral angles $\sphericalangle(R_0,R_1)=\pi/p$, $\sphericalangle(R_1,R_2)=\pi/q$, $\sphericalangle(R_2,R_3)=\pi/r$, and the remaining three angles $\pi/2$. By the above-mentioned isomorphism, $[p,q,r]$ can be represented by a group of Möbius transformations generated by the inversions in four circles cutting one another at the same angles as the corresponding reflection planes. For the complete list of generating inversions one is referred to [31]. Here we only require five of the ten possible groups.

As before, let i denote a fourth root of unity in $\mathbb{C}$, so that $i^2+1=0$, and ω a cube root of unity, i.e. $\omega^2+\omega+1=0$. Then

$$[4,4,3]=\langle\rho_0(z)=\bar{z},\quad \rho_1(z)=i\bar{z},\quad \rho_2(z)=1-\bar{z},\quad \rho_3(z)=1/\bar{z}\rangle,$$

$$[6,3,3]=\langle\rho_0(z)=\bar{z},\quad \rho_1(z)=-\overline{\omega z},\quad \rho_2(z)=1-\bar{z},\quad \rho_3(z)=1/\bar{z}\rangle.$$

In terms of its generators $\sigma_1=\rho_0\rho_1$, $\sigma_2=\rho_1\rho_2$ and $\sigma_3=\rho_2\rho_3$ the corresponding rotation subgroups are given by

$$[4,4,3]^+=\langle\sigma_1(z)=-iz,\quad \sigma_2(z)=-iz+i,\quad \sigma_3(z)=1-1/z\rangle,$$

$$[6,3,3]^+=\langle\sigma_1(z)=-\omega z,\quad \sigma_2(z)=(z-1)/\omega,\quad \sigma_3(z)=1-1/z\rangle.$$

By simplex dissection, we know that the group $[4,4,4]$ of $\{4,4,4\}$ is a subgroup of index 3 in $[4,4,3]$. In terms of the generators ρ'_v (say) of $[4,4,3]$, the generating reflections ρ_v for $[4,4,4]$ are

$$\rho_0=\rho'_1,\quad \rho_1=\rho'_0,\quad \rho_2=\rho'_2\rho'_1\rho'_2,\quad \rho_3=\rho'_3.$$

Hence,

$$[4,4,4]=\langle\rho_0(z)=i\bar{z},\quad \rho_1(z)=\bar{z},\quad \rho_2(z)=-i\bar{z}+1+i,\quad \rho_3(z)=1/\bar{z}\rangle,$$

$$[4,4,4]^+=\langle\sigma_1(z)=iz,\quad \sigma_2(z)=iz+1-i,\quad \sigma_3(z)=1+i-i/z\rangle.$$

Similarly, $[3, 6, 3]$ and $[6, 3, 6]$ are subgroups of index 4 and 6, respectively, in $[6, 3, 3]$. In terms of generators ρ'_ν (say) of $[6, 3, 3]$, the generating reflections for $[3, 6, 3]$ and $[6, 3, 6]$ are

$$\rho_0 = \rho'_0, \qquad \rho_1 = \rho'_1 \rho'_0 \rho'_1, \qquad \rho_2 = \rho'_2, \qquad \rho_3 = \rho'_3,$$

and

$$\rho_0 = \rho'_1, \qquad \rho_1 = \rho'_0, \qquad \rho_2 = \rho'_2 \rho'_1 \rho'_0 \rho'_1 \rho'_2, \qquad \rho_3 = \rho'_3,$$

respectively. Hence,

$$[3, 6, 3] = \langle \rho_0(z) = \bar{z}, \qquad \rho_1(z) = \omega \bar{z}, \qquad \rho_2(z) = 1 - \bar{z}, \qquad \rho_3(z) = 1/\bar{z} \rangle,$$

$$[3, 6, 3]^+ = \langle \sigma_1(z) = \omega^2 z, \qquad \sigma_2(z) = \omega(1 - z), \qquad \sigma_3(z) = 1 - 1/z \rangle,$$

$$[6, 3, 6] = \langle \rho_0(z) = -\omega^2 \bar{z}, \qquad \rho_1(z) = \bar{z}, \qquad \rho_2(z) = 1 - \omega^2 + \omega^2 \bar{z}, \qquad \rho_3(z) = 1/\bar{z} \rangle,$$

$$[6, 3, 6]^+ = \langle \sigma_1(z) = -\omega^2 z, \qquad \sigma_2(z) = 1 - \omega + \omega z, \qquad \sigma_3(z) = 1 - \omega^2 + \omega^2/z \rangle.$$

Below we will represent the Möbius transformations σ_i by matrices, the correspondence

$$\frac{az + b}{cz + d} \leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

being one-to-one up to scalar multiplication.

7. The type $\{4, 4, 3\}$

In matrix notation the generators σ_1, σ_2 and σ_3 of $[4, 4, 3]^+$ can be expressed as

$$\sigma_1 = \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} -i & i \\ 0 & 1 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{9}$$

Theorem 7.1. $[4, 4, 3]^+ \cong PL_2^{\langle i \rangle}(\mathbb{Z}[i]) \cong PSL_2(\mathbb{Z}[i]) \rtimes C_2$.

Proof. We first note that the matrices (9) as elements of $L_2^{\langle i \rangle}(\mathbb{Z}[i])$ generate that group, using Lemma 5.1(c) with σ_1, $\sigma_1^{-1} \sigma_2 \sigma_3$, $\sigma_1 \sigma_2 \sigma_1^2$ and $\sigma_2 \sigma_1^{-1}$. The centre of the group is $C_2^{\langle i \rangle}(\mathbb{Z}[i]) = \langle iI \rangle$. However, the correspondence of Möbius transformations and matrices gives us

$$[4, 4, 3]^+ \cong C \cdot L_2^{\langle i \rangle}(\mathbb{Z}[i])/C \cong L_2^{\langle i \rangle}(\mathbb{Z}[i])/C_2^{\langle i \rangle}(\mathbb{Z}[i]) = PL_2^{\langle i \rangle}(\mathbb{Z}[i]),$$

where $C = \{\lambda I \mid \lambda \in \mathbb{C}^*\}$ is the centre of $GL_2(\mathbb{C})$; here the second isomorphism holds because of $C \cap L_2^{\langle i \rangle}(\mathbb{Z}[i]) = C_2^{\langle i \rangle}(\mathbb{Z}[i])$. Similarly, since $C_2^{\langle i \rangle}(\mathbb{Z}[i]) \cap SL_2(\mathbb{Z}[i]) = C_2(\mathbb{Z}[i])$, we have

$$PSL_2(\mathbb{Z}[i]) = SL_2(\mathbb{Z}[i])/C_2(\mathbb{Z}[i]) \cong C_2^{\langle i \rangle}(\mathbb{Z}[i]) \cdot SL_2(\mathbb{Z}[i])/C_2^{\langle i \rangle}(\mathbb{Z}[i]) =: U.$$

Now, the subgroup $C_2^{\langle i \rangle}(\mathbb{Z}[i]) \cdot SL_2(\mathbb{Z}[i])$ consists precisely of all matrices in $L_2^{\langle i \rangle}(\mathbb{Z}[i])$ with determinant ± 1, i.e. equals $L_2^{\langle -1 \rangle}(\mathbb{Z}[i])$. It follows that U together with the involution

$$\sigma_2 \sigma_3 C_2^{\langle i \rangle}(\mathbb{Z}[i]) = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} C_2^{\langle i \rangle}(\mathbb{Z}[i])$$

generate $PL_2^{\langle i \rangle}(\mathbb{Z}[i])$, which then must be isomorphic to $U \rtimes C_2$. $\square$

Lemma 7.2. *For each integer $m \geqslant 3$ and for each subgroup H of $C_2^{\langle i \rangle}(\mathbb{Z}_m[i])$ containing $\langle iI \rangle$, there is a chiral or a directly regular polytope of type $\{4, 4, 3\}$ with the rotation group isomorphic to $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$.*

Proof. By Lemma 5.4 we have an epimorphism

$$\varphi_{m,H} : [4,4,3]^+ \cong PL_2^{\langle i \rangle}(\mathbb{Z}[i]) \to L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H.$$

The images of the generators under $\varphi_{m,H}$ satisfy the same relations as σ_v (but now the relations do not suffice to define the group). There is little possibility of confusion if we denote by σ_v the image of σ_v under $\varphi_{m,H}$ (since we can think of the 'new' σ_v's as the 'old' σ_v's mod m). Since $m \geqslant 3$, the elements $\sigma_1, \sigma_2, \sigma_3$ have again orders $4, 4, 3$, respectively. Then, subject to the intersection property, $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H \cong \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is the rotation group of a chiral or a directly regular polytope of type $\{4, 4, 3\}$; see Section 1.

For the intersection property we need to check $\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{1\} = \langle \sigma_2 \rangle \cap \langle \sigma_3 \rangle$ and $U := \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle \subseteq \langle \sigma_2 \rangle$. The former equalities are trivial. For the latter, note first that $\langle \sigma_2, \sigma_3 \rangle \cong [4, 3]^+ \cong S_4$. In fact, since $\langle \sigma_2, \sigma_3 \rangle$ is a quotient of $[4, 3]^+$ and σ_2, σ_3 have orders $4, 3$, respectively, we must have $|\langle \sigma_2, \sigma_3 \rangle| = 12$ or 24; but the first case is impossible, since S_4 has no normal subgroup of order 2. Now, to find U note that $|U|$ must divide 24, and must be divisible by $4 = |\sigma_2|$. The orders 8, 12 and 24 are easily disproved, so that $|U| = 4$ and hence $U = \langle \sigma_2 \rangle$. This completes the proof. $\square$

We now proceed to identify the facets of the polytopes from Lemma 7.2. Since the facet type is $\{4, 4\}$ and each facet is either chiral or regular, we see that the facets must be isomorphic to toroidal maps $\{4, 4\}_{(b,c)}$. From $[4, 4]^+ \cong \langle \sigma_1, \sigma_2 \rangle$ we obtain $[4, 4]_{(b,c)}^+$ (see the presentation (4)) by the addition of

$$(\sigma_1^{-1} \sigma_2)^b (\sigma_1 \sigma_2^{-1})^c = 1$$

to the defining relations for $[4, 4]^+$. To find the possible values for b and c we consider

$$(\sigma_1^{-1} \sigma_2)^b (\sigma_1 \sigma_2^{-1})^c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^b \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}^c = \begin{bmatrix} 1 & -(b+ci) \\ 0 & 1 \end{bmatrix}.$$

This matrix is the identity in $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ if and only if $b + ci = 0$ in $\mathbb{Z}_m[i]$.

Theorem 7.3. *For each integer $m \geqslant 3$ and for each conjugation invariant subgroup H of $C_2^{\langle i \rangle}(\mathbb{Z}_m[i])$ containing $\langle iI \rangle$ there is a directly regular polytope $\mathscr{P}$ in $\langle \{4,4\}_{(m,0)}, \{4,3\} \rangle$ such that the rotation group of $\mathscr{P}$ is isomorphic to $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$.*

Proof. From Lemma 7.2 we know that there exists a polytope of type $\{4,4,3\}$ with the rotation group $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$. As remarked in Section 1, the polytope is regular if and only if there exists an involutory automorphism ρ of the group such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2 \sigma_2$ and $\rho(\sigma_3) = \sigma_3$. Or, using the corresponding matrices,

$$\rho: \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -i & i \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} i & -i \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, ρ is induced by conjugation of $\mathbb{Z}_m[i]$. Note for this that H is conjugation invariant.

To identify the facets we note that $b + ic = 0$ in $\mathbb{Z}_m[i]$ if and only if either $b = c = m$ or $b = m$, $c = 0$ (or the other way around). We can rule out the possibility $b = c = m$ since the order of

$$\sigma_1^{-1} \sigma_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

is m, whereas in $[4,4]_{(m,m)}^+$ it should be $2m$.

Remark. In Theorem 7.3, if H is not conjugation invariant, then in general $\mathscr{P}$ will only be chiral. In fact, the right and left Petrie polygons will generally have different lengths; see Section 12.

To recognize the groups explicitly we use the following lemma.

Lemma 7.4. *Let $m = 2^e p_1^{e_1} \cdot \ldots \cdot p_k^{e_k} (> 2)$ be the prime decomposition of m in $\mathbb{Z}$.*

(a) The equation $x^2 = i$ is solvable in $\mathbb{Z}_m[i]$ if and only if $e = 0$ and $p_j \not\equiv -3 \pmod 8$ for each j.

(b) Let $x \in \mathbb{Z}_m[i]$ be such that $x^2 = i$. Then $\langle x \rangle = \langle \pm 1, \pm i, \pm x, \pm xi \rangle$ is conjugation invariant in $\mathbb{Z}_m[i]$ if and only if either $p_j \equiv \pm 1 \pmod 8$ for all j or $p_j \equiv 1, 3 \pmod 8$ for all j.

Proof. The equation $x^2 = i$ has a solution in $\mathbb{Z}_m[i]$ if and only if $(u+v)(u-v) \equiv 0 \pmod m$, $2uv \equiv 1 \pmod m$ has a solution. Here the second equation forces m to be odd. Now, these equations are solvable $\mod m$ if and only if they are solvable $\mod p_j^{e_j}$ for each j; i.e. if and only if $u \equiv v$, $u^2 \equiv 2^{-1} \pmod{p_j^{e_j}}$ or $u \equiv -v$, $u^2 \equiv -2^{-1} \pmod{p_j^{e_j}}$ are solvable. Hence, we have a solution $\mod m$ if and only if 2 or -2 is a quadratic residue $\mod p_j$ for each j, i.e. if and only if $p_j \not\equiv -3 \pmod 8$ for each j.

This proves (a). Note for part (b) that for conjugation invariance we must have $\bar{x} = -ix$ or $\bar{x} = ix$, that is, $x = u + ui$ or $x = u - ui$ for some $u \in \mathbb{Z}_m$, but then 2 or -2 is a quadratic residue mod m, respectively. $\square$

Corollary 7.5. *Let $m = 2^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ be the prime decomposition of m, $m > 2$. There exist directly regular polytopes in $\langle \{4, 4\}_{(m, 0)}, \{4, 3\} \rangle$ whose rotation groups are isomorphic to*

(a) $PSL_2(\mathbb{Z}_m[i])$ *if $e = 0$ and $p_j \not\equiv -3 \pmod 8$ for each j;*

(b) $\hat{P}SL_2(\mathbb{Z}_m[i])$ *if $e = 0$ and either $p_j \equiv \pm 1 \pmod 8$ for each j or $p_j \equiv 1, 3 \pmod 8$ for each j;*

(c) $PSL_2(\mathbb{Z}_m[i]) \bowtie C_2$ *and $\hat{P}SL_2(\mathbb{Z}_m[i]) \bowtie C_2$, if $e \neq 0$ or $p_j \equiv -3 \pmod 8$ for at least one j.*

Proof. Let $C \leqslant C_2(\mathbb{Z}_m[i])$, $\langle iI \rangle \leqslant H \leqslant C_2^{\langle i \rangle}(\mathbb{Z}_m[i])$, $C \leqslant H$, and $\chi : SL_2(\mathbb{Z}_m[i])/C \to L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ be the canonical homomorphism. We choose C and H such that $SL_2(\mathbb{Z}_m[i]) \cap H = C$, so that χ is injective. Note that χ is surjective if and only if there exists an element in H with determinant i; in particular, $x^2 = i$ must be solvable in $\mathbb{Z}_m[i]$.

Let $C = C_2(\mathbb{Z}_m[i])$ and $H = C_2^{\langle i \rangle}(\mathbb{Z}_m[i])$. Then H is conjugation invariant. By Theorem 7.3 and Lemma 7.4, if $e = 0$ and $p_j \not\equiv -3 \pmod 8$ for each j, we have a polytope with rotation group $PSL_2(\mathbb{Z}_m[i])$. If $e \neq 0$ of $p_j \equiv -3 \pmod 8$ for some j, then χ is not surjective and has image $L_2^{\langle -1 \rangle}(\mathbb{Z}_m[i])/H$. The element

$$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} \cdot H (= \sigma_2 \sigma_3 H)$$

is an involution in $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ not in this image. It follows that the rotation group $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ of the polytope is isomorphic to $PSL_2(\mathbb{Z}_m[i]) \bowtie C_2$.

Let $C = \{ \pm I \}$. If $e = 0$ and either $p_j \equiv \pm 1 \pmod 8$ for each j or $p_j \equiv 1, 3 \pmod 8$ for each j, let $x \in \mathbb{Z}_m[i]$ be such that $x^2 = i$ and $\langle x \rangle$ is conjugation invariant. Then $H = \langle xI \rangle = \{ \pm I, \pm iI, \pm xI, \pm xiI \}$ is conjugation invariant, and the rotation group of the polytope is $\hat{P}SL_2(\mathbb{Z}_m[i])$. If $e \neq 0$ or $p_j \equiv -3 \pmod 8$ for some j, choose $H = \langle iI \rangle$. Again, χ is not surjective, and the rotation group of the polytope becomes $\hat{P}SL_2(\mathbb{Z}_m[i]) \bowtie C_2$. $\square$

Remarks. (a) The groups $PSL_2(\mathbb{Z}_m[i])$ and $\hat{P}SL_2(\mathbb{Z}_m[i])$ occurring in Corollary 7.5 coincide precisely for $m = 2^e p^\lambda$ where $e = 0, 1$ and $p \equiv -1 \pmod 4$, or $m = 4$. In fact, precisely for these m the equation $x^2 = 1$ has only two solutions in $\mathbb{Z}_m[i]$, namely ± 1.

(b) If $m = p \equiv -1 \pmod 4$, then $\mathbb{Z}_p[i] \cong GF(p^2)$ and thus $PSL_2(\mathbb{Z}_p[i]) \cong PSL_2(p^2)$.

Theorem 7.6. *Let $m = 2^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ (> 2) be the prime decomposition of m, and assume that $e = 0, 1$ and $p_j \equiv 1 \pmod 4$ for each $j = 1, \ldots, k$. Let $i \in \mathbb{Z}_m$ be such that $i^2 \equiv -1 \pmod m$, and let b, c be the unique pair of positive integers such that $m = b^2 + c^2$, $(b, c) = 1$ and $b \equiv -ic \pmod m$. Then for each subgroup H of $C_2^{\langle i \rangle}(\mathbb{Z}_m)$ containing $\langle iI \rangle$*

there exists a chiral polytope in $\langle\{4,4\}_{(b,c)},\{4,3\}\rangle$ with the group isomorphic to $L_2^{\langle i\rangle}(\mathbb{Z}_m)/H$.

Proof. First note that our conditions on m are precisely those that guarantee the existence of i (cf. [15, p. 50]). Also note that our assumptions on m imply the existence and uniqueness of b, c (cf. [14, p. 117]).

By Lemma 5.5(a) there is the canonical epimorphism

$$\Phi_{m,H}: PL_2^{\langle i\rangle}(\mathbb{Z}[i]) \to L_2^{\langle i\rangle}(\mathbb{Z}_m)/H.$$

As in the proof of Lemma 7.2 we have the existence of a chiral or a directly regular polytope of type $\{4,4,3\}$ with cubical vertex-figures and with the rotation group isomorphic to $L_2^{\langle i\rangle}(\mathbb{Z}_m)/H \cong \langle\sigma_1, \sigma_2, \sigma_3\rangle$, where σ_v is the image of the matrix σ_v in (9) under $\Phi_m \varphi_m$.

To identify the facets as $\{4,4\}_{(b,c)}$ note that, by the remarks preceding Theorem 7.3 and by our choice of b, c, the required relation $(\sigma_1^{-1}\sigma_2)^b(\sigma_1\sigma_2^{-1})^c = 1$ holds in $\langle\sigma_1, \sigma_2\rangle$. However, the 'translation' $\sigma_1^{-1}\sigma_2$ (and $\sigma_1\sigma_2^{-1}$) is easily seen to have order m, so that the facets must in fact be $\{4,4\}_{(b,c)}$. Finally, since the facets are chiral, the polytope must also be chiral. This completes the proof. $\square$

Note that the integers b and c in Theorem 7.6 depend on the choice of i. The number of solutions of $x^2 \equiv -1 \,(\mathrm{mod}\,m)$ is exactly 2^k (cf. [14, p. 116]). It follows that the number of solutions of $m = b^2 + c^2$ with b, $c > 0$ and $(b,c) = 1$ is exactly 2^k. Here the pairs b, c and c, b are counted as distinct solutions, corresponding to i and $-i$. Hence, Theorem 7.6 gives us (at least) 2^k polytopes with groups $L_2^{\langle i\rangle}(\mathbb{Z}_m)/H$, $\langle iI\rangle \leqslant H \leqslant C_2^{\langle i\rangle}(\mathbb{Z}_m)$. As we shall see below, the groups for different choices of i are isomorphic. However, the polytopes are isomorphic (as abstract polytopes) only if i is replaced by $-i$, corresponding to switching b and c. In fact, the polytopes for i and $-i$ are the two enantiomorphic forms of the same underlying chiral polytope. To see this, note that changing the generators $\sigma_1, \sigma_2, \sigma_3$ of (9) to $\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3$ gives precisely the generators of (9) for $-i$.

For example, if $m = 65 = 5\cdot13$, then $m = 1^2 + 8^2 = 4^2 + 7^2$, while the pair 1, 8 belongs to $i = 8$, the pair 4, 7 belongs to $i = 18$ ($\not\equiv -8$). Hence, one gets different kinds of facets $\{4,4\}_{(1,8)}$ and $\{4,4\}_{(4,7)}$ for the same m. If i is replaced by $-i$, we can also obtain facets $\{4,4\}_{(8,1)}$ and $\{4,4\}_{(7,4)}$, respectively.

Corollary 7.7. *Let $m = 2^e p_1^{e_1} \cdots p_k^{e_k}$ be the prime decomposition of $m > 2$, and let $e = 0, 1$ and $p_j \equiv 1 \,(\mathrm{mod}\,4)$ for each $j = 1, \ldots, k$. Let b, c be positive integers such that $m = b^2 + c^2$ and $(b,c) = 1$. There exist chiral polytopes in $\langle\{4,4\}_{(b,c)},\{4,3\}\rangle$ whose groups are isomorphic to*
  (a) $PSL_2(\mathbb{Z}_m)$ *and* $\hat{P}SL_2(\mathbb{Z}_m)$ *if* $p_j \equiv 1 \,(\mathrm{mod}\,8)$ *for each* j;
  (b) $PSL_2(\mathbb{Z}_m)\rtimes C_2$ *and* $\hat{P}SL_2(\mathbb{Z}_m)\rtimes C_2$, *if* $p_j \equiv -3 \,(\mathrm{mod}\,8)$ *for at least one* j.

Proof. By our assumptions on b, c there exists a unique $i \in \mathbb{Z}_m$ with the property that $i^2 \equiv -1 \pmod{m}$ and $b \equiv -ic \pmod{m}$. Let $C \leqslant C_2(\mathbb{Z}_m)$, $\langle iI \rangle \leqslant H \leqslant C_2^{\langle i \rangle}(\mathbb{Z}_m)$, $C \leqslant H$, and $\chi : SL_2(\mathbb{Z}_m)/C \to L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$. Again we choose C and H such that χ becomes injective, i.e. $SL_2(\mathbb{Z}_m) \cap H = C$. As in Corollary 7.5, for χ to be surjective we need an element in H of determinant i.

Now, $x^2 \equiv i \pmod{m}$ has a solution if and only if $x^2 \equiv i \pmod{p_j^{e_j}}$ has a solution for each j. This is satisfied if and only if $x^8 \equiv 1 \pmod{p_j^{e_j}}$ has exactly eight distinct solutions. On the other hand, this congruence has exactly $(8, p_j - 1)$ solutions (cf. [14, p. 47]), so that we must have $p_j \equiv 1 \pmod 8$ for each j.

Let $C = C_2(\mathbb{Z}_m)$ and $H = C_2^{\langle i \rangle}(\mathbb{Z}_m)$. By Theorem 7.6, if $p_j \equiv 1 \pmod 8$ for each j, we have a polytope with group $PSL_2(\mathbb{Z}_m)$. If $p_j \equiv -3 \pmod 8$ for some j, then χ is not surjective and has image $L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/H$. Again,

$$\begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} \cdot H$$

is an involution not in this image, so that the group of the polytope is $PSL_2(\mathbb{Z}_m) \rtimes C_2$.

Let $C = \{\pm I\}$. If $p_j \equiv 1 \pmod 8$ for each j, choose x such that $x^2 \equiv i \pmod{m}$, and let $H = \langle xI \rangle$. Then the group of the polytope is $\hat{P}SL_2(\mathbb{Z}_m)$. If $p_j \equiv -3 \pmod 8$ for some j, let $H = \langle iI \rangle$. Then the group of the polytope is $\hat{P}SL_2(\mathbb{Z}_m) \rtimes C_2$. $\square$

Remark. The groups $PSL_2(\mathbb{Z}_m)$ and $\hat{P}SL_2(\mathbb{Z}_m)$ coincide if and only if $m = 2^e p^\lambda$ where $e = 0, 1$ and $\lambda \geqslant 1$, or $m = 4$. Hence, when $m > 2$, they coincide if and only if there exist primitive roots $\bmod\, m$. Furthermore, the image of the canonical homomorphism $PSL_2(\mathbb{Z}_m) \to PGL_2(\mathbb{Z}_m)$ is a subgroup in $PGL_2(\mathbb{Z}_m)$ with elements represented by matrices whose determinants are quadratic residues $\bmod\, m$. The index of this subgroup is 2 if there exist primitive roots and $\bmod\, m$. Hence, when $m = 4$ or $m = 2^e p^\lambda$ with $e = 0, 1$ and $\lambda \geqslant 1$, then $PSL_2(\mathbb{Z}_m) \rtimes C_2 \cong \hat{P}SL_2(\mathbb{Z}_m) \rtimes C_2 \cong PGL_2(\mathbb{Z}_m)$.

To summarize the above construction of polytopes, for every positive integer $m \geqslant 3$ we have an epimorphism

$$PL_2^{\langle i \rangle}(\mathbb{Z}[i]) \xrightarrow{\varphi_{m,H}} L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H, \tag{10}$$

with $\langle iI \rangle = \varphi_{m,H}(C_2^{\langle i \rangle}(\mathbb{Z}[i])) \leqslant H \leqslant C_2^{\langle i \rangle}(\mathbb{Z}_m[i])$, Furthermore, whenever $x^2 + 1 \equiv 0 \pmod{m}$ is solvable in $\mathbb{Z}_m$, we have a commutative diagram of homomorphisms

$$L_2^{\langle i \rangle}(\mathbb{Z}[i]) \xrightarrow{\varphi_m} L_2^{\langle i \rangle}(\mathbb{Z}_m[i]) \tag{11}$$

$$L_2^{\langle i \rangle}(\mathbb{Z}_m)$$

This induces an epimorphism

$$PL_2^{\langle i \rangle}(\mathbb{Z}[i]) \xrightarrow{\Phi_{m,H}} L_2^{\langle i \rangle}(\mathbb{Z}_m)/H, \tag{12}$$

with $\langle iI \rangle = \Phi_{m,H}(C_2^{\langle i \rangle}(\mathbb{Z}[i])) \leqslant H \leqslant C_2^{\langle i \rangle}(\mathbb{Z}_m)$.

The above epimorphisms and an appropriate choice of generators were used to construct chiral and directly regular polytopes of type $\{4,4,3\}$ whose rotation groups are precisely the groups in the diagrams (10) and (12). More precisely, the chiral polytopes $\mathscr{P}$ of Theorem 7.6 in the class $\langle \{4,4\}_{(b,c)}, \{4,3\} \rangle$ are covered by suitable directly regular polytopes $\mathscr{L}$ (say) of Theorem 7.3 in the class $\langle \{4,4\}_{(m,0)}, \{4,3\} \rangle$, with $m = b^2 + c^2$ (provided that H is conjugation invariant). For example, since Φ_m is surjective and $\Phi_m(C_2^{\langle i \rangle}(\mathbb{Z}_m[i])) \subset C_2^{\langle i \rangle}(\mathbb{Z}_m)$, the map Φ_m induces an epimorphism $PL_2^{\langle i \rangle}(\mathbb{Z}_m[i]) \to PL_2^{\langle i \rangle}(\mathbb{Z}_m)$ and thus a corresponding projection of $\mathscr{L}$ onto $\mathscr{P}$.

8. The type $\{4,4,4\}$

In matrix notation the generators σ_j of $[4,4,4]^+$ are represented by

$$\sigma_1 = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} i & 1-i \\ 0 & 1 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1+i & -i \\ 1 & 0 \end{bmatrix}. \tag{13}$$

Considered as elements in $L_2^{\langle i \rangle}(\mathbb{Z}[i])$ the matrices in (13) generate a subgroup L (say). The centre of L is $\langle iI \rangle$. Then the correspondence of Möbius transformations and matrices gives us

$$[4,4,4]^+ \cong C \cdot L/C \cong L/L \cap C = L/\langle iI \rangle =: \Lambda,$$

where $C = \{\lambda I \mid \lambda \in \mathbb{C}^*\}$. We use the notation $\sigma_1, \sigma_2, \sigma_3$ for both the generators of L and their images in Λ. As explained in Section 6, $\Lambda = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is a subgroup of index 3 in $[4,4,3]^+ \cong PL_2^{\langle i \rangle}(\mathbb{Z}[i])$. Using the methods of Section 7 we construct polytopes, directly regular and chiral, in this case by considering the restrictions of the maps in (10) and (12) to this subgroup. Let $\tilde{\varphi}_{m,H} : \Lambda \to L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ and $\tilde{\Phi}_{m,H} : \Lambda \to L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$ be the restrictions of $\varphi_{m,H}$ and $\Phi_{m,H}$, respectively, to Λ. Note that H in $\tilde{\varphi}_{m,H}$ and H in $\tilde{\Phi}_{m,H}$ represent different subgroups. We shall abuse notation and use the same letter, but this should cause no confusion.

Lemma 8.1. *Let* $R = \mathbb{Z}_m[i]$, $\chi = \tilde{\varphi}_{m,H}$, *or* $R = \mathbb{Z}_m$, $\chi = \tilde{\Phi}_{m,H}$.
  (a) *If* m *is odd, then* χ *is an epimorphism.*
  (b) *If* m *is even, then* $\chi(\Lambda)$ *is a subgroup of index* 3 *in* $L_2^{\langle i \rangle}(R)/H$.

Proof. Consider the σ_j's as elements of $L_2^{\langle i \rangle}(R)$. Since

$$\sigma_1^{-1}\sigma_2 = \begin{bmatrix} 1 & -1-i \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \sigma_1\sigma_2^{-1} = \begin{bmatrix} 1 & -1+i \\ 0 & 1 \end{bmatrix},$$

it follows that

$$\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

If m is odd, 2 is invertible in $\mathbb{Z}_m$ and hence

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \chi(\Lambda).$$

Also

$$\begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} = \sigma_1\sigma_2^{-1} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \chi(\Lambda).$$

Furthermore,

$$\sigma_1^2\sigma_2\sigma_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and as in the proof of Lemma 5.4, we use Lemma 5.3 to show that the natural homomorphism $L \to L_2^{\langle i \rangle}(R)$ is surjective. Then χ is surjective.

Let m be even. To prove that $\chi(\Lambda)$ is a subgroup of $L_2^{\langle i \rangle}(R)/H$ of index 3, consider the ring homomorphism $f: R \to \mathbb{Z}_2$ given by $u + iv \mapsto u + v$ or $u \to u$ if $R = \mathbb{Z}_m[i]$ or $R = \mathbb{Z}_m$, respectively. This induces a homomorphism $\tilde{f}: L_2^{\langle i \rangle}(R)/H \to SL_2(\mathbb{Z}_2)$. By arguments similar to those used in the proof of Lemma 5.4, $\tilde{f}$ is surjective. Now, $\tilde{f}$ maps $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ onto

$$\langle \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \rangle = \left\langle I, I, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong C_2,$$

which has index 3 in $SL_2(\mathbb{Z}_2)$. It follows that $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and $L_2^{\langle i \rangle}(R)/H$ cannot coincide, and hence the index is 3. This completes the proof. □

There is little possibility of confusion if we denote by σ_ν the image of σ_ν under $\tilde{\varphi}_{m,H}$ or $\tilde{\Phi}_{m,H}$ (as we did in the above proof).

Theorem 8.2. *Let $m > 2$ be an integer and let H be a conjugation invariant subgroup of $C_2^{\langle i \rangle}(\mathbb{Z}_m[i])$ containing $\langle iI \rangle$. Then there exists a directly regular polytope $\mathscr{P}$ such that*

(a) if m is odd, $\mathscr{P}$ is self-dual, $\mathscr{P}$ is in $\langle \{4,4\}_{(m,0)}, \{4,4\}_{(m,0)} \rangle$ and $A^+(\mathscr{P}) \cong L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$;

(b) if m is even, $\mathscr{P}$ is in $\langle \{4,4\}_{(m/2,m/2)}, \{4,4\}_{(m,0)} \rangle$ and $A^+(\mathscr{P})$ is a subgroup of $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ of index 3.

Proof. Modulo the intersection property (which we prove later) the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ of $L_2^{\langle i \rangle}(\mathbb{Z}_m[i])/H$ (of index 1 or 3) is the rotation group of a chiral or a directly regular polytope $\mathscr{P}$. Define the group automorphism ρ of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ by

$$\rho\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \begin{bmatrix} \bar{\alpha} & \bar{\beta}i \\ -\bar{\gamma}i & \bar{\delta} \end{bmatrix} = \sigma_1 \cdot \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{bmatrix} \cdot \sigma_1^{-1}.$$

Then $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2 \sigma_2$ and $\rho(\sigma_3) = \sigma_3$ and hence $\mathscr{P}$ must be regular.

To identify the facets we consider

$$(\sigma_1^{-1}\sigma_2)^b(\sigma_1\sigma_2^{-1})^c = \begin{bmatrix} 1 & (-b+ci)(1+i) \\ 0 & 1 \end{bmatrix}. \tag{14}$$

Hence, we must consider $(-b+ci)(1+i)=0$ in $\mathbb{Z}_m[i]$.

If m is odd, $1+i$ is invertible since $(1+i)(1-i)=2$ and 2 is invertible. Then the equation is equivalent to $-b+ci=0$ in $\mathbb{Z}_m[i]$, and the facets are either $\{4,4\}_{(m,m)}$ or $\{4,4\}_{(m,0)}$. We can rule out the possibility of $\{4,4\}_{(m,m)}$, since the order of

$$\sigma_1^{-1}\sigma_2 = \begin{bmatrix} 1 & -(1+i) \\ 0 & 1 \end{bmatrix}$$

is m, and so the facets must be $\{4,4\}_{(m,0)}$. Note that conjugation by the element

$$\begin{bmatrix} 1 & -i \\ 1 & -1 \end{bmatrix}$$

interchanges σ_1 with σ_3^{-1} and σ_2 with σ_2^{-1}, and hence the polytope is self-dual. To complete the proof of (a) we use Lemma 8.1.

Let m be even. Since $(-b+ci)(1+i)=0$ implies $2(-b+ci)=0$, the facets could be $\{4,4\}_{(m/2,m/2)}$, $\{4,4\}_{(m,m)}$, $\{4,4\}_{(m/2,0)}$ or $\{4,4\}_{(m,0)}$. However, the order of $\sigma_1^{-1}\sigma_2$ is m, so that we are left only with $\{4,4\}_{(m/2,m/2)}$ or $\{4,4\}_{(m,0)}$. To rule out the possibility of $\{4,4\}_{(m,0)}$, we consider the transformation

$$\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

of order $m/2$. This transformation shifts a Petrie polygon of a facet two steps along itself. Here a Petrie polygon is a zig-zag along the edges of the map such that each two consecutive edges, but no three, belong to a face (cf. [6]). Hence, the length of the Petrie polygon is m, so that the facets are maps $\{4,4\}_{(m/2,m/2)}$.

Note that for even m the above argument for self-duality does not extend, since

$$\begin{bmatrix} 1 & -i \\ 1 & -1 \end{bmatrix}$$

is not invertible over $\mathbb{Z}_m[i]$. To identify the vertex-figures we must consider

$$(\sigma_2^{-1}\sigma_3)^d(\sigma_2\sigma_3^{-1})^a = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^d \begin{bmatrix} 1+i & -i \\ i & 1-i \end{bmatrix}^a$$

$$= \begin{bmatrix} 1+d & -d \\ d & 1-d \end{bmatrix} \begin{bmatrix} 1+ai & -ai \\ ai & 1-ai \end{bmatrix}$$

$$= \begin{bmatrix} 1+(d+ai) & -(d+ai) \\ d+ai & 1-(d+ai) \end{bmatrix}. \tag{15}$$

It follows that the vertex-figures are $\{4,4\}_{(m,0)}$ or $\{4,4\}_{(m,m)}$, but $\sigma_2^{-1}\sigma_3$ is of order m, and hence the vertex figures are maps $\{4,4\}_{(m,0)}$. To complete the proof of (b) we use Lemma 8.1.

Concluding we need to check the intersection property. It suffices to prove $\langle\sigma_1,\sigma_2\rangle\cap\langle\sigma_2,\sigma_3\rangle=\langle\sigma_2\rangle$. First note that

$$T_1 := \langle\sigma_1^{-1}\sigma_2,\sigma_1\sigma_2^{-1}\rangle = \left\{ \begin{bmatrix} 1 & \alpha(1+i) \\ 0 & 1 \end{bmatrix} \middle| \alpha\in\mathbb{Z}_m[i] \right\}$$

and

$$T_2 := \langle\sigma_2^{-1}\sigma_3,\sigma_2\sigma_3^{-1}\rangle = \left\{ \begin{bmatrix} 1+\beta & -\beta \\ \beta & 1-\beta \end{bmatrix} \middle| \beta\in\mathbb{Z}_m[i] \right\}$$

are the 'translation subgroups' of $\langle\sigma_1,\sigma_2\rangle$ and $\langle\sigma_2,\sigma_3\rangle$, respectively. Also, $\langle\sigma_1,\sigma_2\rangle=\langle\sigma_2\rangle\cdot T_1$ and $\langle\sigma_2,\sigma_3\rangle=\langle\sigma_2\rangle\cdot T_2$. Now, let $\sigma\in\langle\sigma_1,\sigma_2\rangle\cap\langle\sigma_2,\sigma_3\rangle$, say $\sigma=\sigma_2^j\tau_1=\sigma_2^k\tau_2$ with $\tau_1\in T_1$, $\tau_2\in T_2$. Then $\tau_2\tau_1^{-1}=\sigma_2^{j-k}$, but for some α,β in $\mathbb{Z}_m[i]$ we have

$$\tau_2\tau_1^{-1} = \begin{bmatrix} 1+\beta & \alpha(1+i)(1+\beta)-\beta \\ \beta & \beta\alpha(1+i)+1-\beta \end{bmatrix},$$

so that a comparison with the elements in $\langle\sigma_2\rangle$ shows that $\beta=0$. It follows that $\tau_2=1$, and hence $\sigma=\sigma_2^k\in\langle\sigma_2\rangle$. This completes the proof of the theorem. □

The proof of Corollary 7.5 implies the following consequence of Theorem 8.2. See also the remark following Corollary 7.5.

Corollary 8.3. *Let* $m=p_1^{e_1}\cdot\cdots\cdot p_k^{e_k}$ *be the prime decomposition of* m *such that* $p_j\neq 2$ *for each* j. *There exist self-dual directly regular polytopes in* $\langle\{4,4\}_{(m,0)},\{4,4\}_{(m,0)}\rangle$ *whose rotation groups are isomorphic to*

(a) $PSL_2(\mathbb{Z}_m[i])$ *if* $p_j\not\equiv -3 \pmod 8$ *for each* j;

(b) $PSL_2(\mathbb{Z}_m[i])$ *if either* $p_j\equiv\pm 1 \pmod 8$ *for each* j *or* $p_j\equiv -3 \pmod 8$ *for each* j.

(c) $PSL_2(\mathbb{Z}_m[i])\rtimes C_2$ *and* $\hat{P}SL_2(\mathbb{Z}_m[i])\rtimes C_2$, *if* $p_j\equiv -3 \pmod 8$ *for at leat one* j.

Theorem 8.4. *Let* $m=2^e p_1^{e_1}\cdot\cdots\cdot p_k^{e_k}$ *be the prime decomposition of* $m>2$, *and assume* $e=0,1$ *and* $p_j\equiv 1 \pmod 4$ *for each* $j=1,\dots,k$. *Let* $i\in\mathbb{Z}_m$ *be such that* $i^2\equiv -1 \pmod m$, *and let* b,c *be the unique pair of positive integers* b,c *such that* $m=b^2+c^2$, $(b,c)=1$ *and*

$b \equiv -ic \pmod{m}$. *Let H be a subgroup of $C_2^{\langle i \rangle}(\mathbb{Z}_m)$ containing $\langle iI \rangle$. Then there exists a chiral polytope $\mathscr{P}$ such that*

(a) *if m is odd, $\mathscr{P}$ is self-dual, $\mathscr{P}$ is in $\langle \{4,4\}_{(c,b)}, \{4,4\}_{(b,c)} \rangle$, and $A(\mathscr{P}) \cong L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$;*

(b) *if m is even, $\mathscr{P}$ is in $\langle \{4,4\}_{(a,d)}, \{4,4\}_{(b,c)} \rangle$ with $a = (c-b)/2$ [1] and $d = (c+b)/2$, and $A(\mathscr{P})$ is a subgroup of $L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$ of index 3.*

Proof. Let $\mu: \Lambda \to L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$ be the restriction of $\Phi_{m,H}$ in (12) to Λ. Again we write $\sigma_1, \sigma_2, \sigma_3$ for the images of the generators of Λ under μ. Modulo the intersection property, the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ of $L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$ (of index 1 or 3) is the rotation group of a chiral or directly regular polytope $\mathscr{P}$. As we shall see below, the facets of $\mathscr{P}$ are chiral, so $\mathscr{P}$ must also be chiral.

To identify the facets consider (as in (14))

$$(\sigma_1^{-1}\sigma_2)^k(\sigma_1\sigma_2^{-1})^l = \begin{bmatrix} 1 & (-k+li)(1+i) \\ 0 & 1 \end{bmatrix}.$$

This leads to the equation $(l+ik)(-1+i) = (-k+li)(1+i) = 0$ over $\mathbb{Z}_m$. By our choice of b and c, $l = b$ and $k = c$ is one possible solution.

If m is odd, then the 'translation'

$$\sigma_1^{-1}\sigma_2 = \begin{bmatrix} 1 & -(1+i) \\ 0 & 1 \end{bmatrix}$$

has order m, so that the facets must in fact be isomorphic to $\{4,4\}_{(c,b)}$. As in the proof of the previous theorem, the matrix

$$\begin{bmatrix} 1 & -i \\ 1 & -1 \end{bmatrix}$$

can be used to show that $\mathscr{P}$ is self-dual. Note that conjugation by this matrix leads to the relation $(\sigma_2^{-1}\sigma_3)^b(\sigma_2\sigma_3^{-1})^c = 1$, implying that the vertex-figures are maps $\{4,4\}_{(b,c)}$. Finally, $A(\mathscr{P}) = A^+(\mathscr{P}) \cong L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$, by Lemma 8.1.

Let m be even. First note that $d + ia \equiv 0 \pmod{m}$. It follows that $(\sigma_1^{-1}\sigma_2)^a(\sigma_1\sigma_2^{-1})^d = 1$. This time $\sigma_1^{-1}\sigma_2$ has order $m/2 = a^2 + d^2$, so that the facets are maps $\{4,4\}_{(a,d)}$. To find the vertex-figures we can use (15) to show that $(\sigma_2^{-1}\sigma_3)^b(\sigma_2\sigma_3^{-1})^c = 1$. Again the order of $\sigma_2^{-1}\sigma_3$ is m, so that the vertex-figures are maps $\{4,4\}_{(b,c)}$. By Lemma 8.1 $A^+(\mathscr{P})$ is of index 3 in $L_2^{\langle i \rangle}(\mathbb{Z}_m)/H$.

Finally, the proof of the intersection property carries over from the proof of Theorem 8.2, but now with $\alpha, \beta \in \mathbb{Z}_m$. This completes the proof. □

Corollary 8.5. *Let $m = p_1^{e_1} \cdots p_k^{e_k}$ be the prime decomposition of m such that $p_j \equiv 1 \pmod{4}$ for each $j = 1, \ldots, k$. Let b, c be positive integers such that $m = b^2 + c^2$ and*

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[1] Note that if $a < 0$ then $\{4,4\}_{(a,d)} = \{4,4\}_{(d,-a)}$.

$(b,c) = 1$. *There exist self-dual chiral polytopes in* $\langle \{4,4\}_{(c,b)}, \{4,4\}_{(b,c)} \rangle$ *whose groups are isomorphic to*

  (a) $PSL_2(\mathbb{Z}_m)$ *and* $\hat{P}SL_2(\mathbb{Z}_m)$, *if* $p_j \equiv 1 \pmod 8$ *for each* j;

  (b) $PSL_2(\mathbb{Z}_m) \bowtie C_2$ *and* $\hat{P}SL_2(\mathbb{Z}_m) \bowtie C_2$, *if* $p_j \equiv -3 \pmod 8$ *for at least one* j.

Proof. Follows from Corollaries 7.7 and 8.3. □

Concluding we remark that the results of this section could also be derived from the results of the previous section by employing suitable mixing operations in the sense of [20, Section 6] to the (rotation) groups. However, this does not lead to shorter proofs.

9. The type $\{6,3,3\}$

The generators σ_1, σ_2 and σ_3 of $[6,3,3]^+$ can be represented by the following matrices in $GL_2(\mathbb{Z}[\omega])$:

$$\sigma_1 = \begin{bmatrix} \omega^2 & 0 \\ 0 & -\omega \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} \omega & -\omega \\ 0 & \omega^2 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{16}$$

Theorem 9.1. $[6,3,3]^+ \cong PL_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) = PSL_2(\mathbb{Z}[\omega]) \bowtie C_2$.

Proof. Use Lemma 5.2(b) with $A = \sigma_2 \sigma_3 \sigma_1^2$, $B = \sigma_2^{-1} \sigma_1^2$, $C = \sigma_2 \sigma_1^2 \sigma_2$ and

$$\sigma_1^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

to show that the matrices in (16) considered as elements of $L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega])$ generate that group. The centres of $SL_2(\mathbb{Z}[\omega])$ and $L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega])$ are $\{\pm I\}$. Then

$$[6,3,3]^+ \cong C \cdot L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega])/C \cong L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega])/\{\pm I\} = PL_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]),$$

with $C = \{\lambda I \mid \lambda \in \mathbb{C}^*\}$; note for this that $C \cap L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) = \{\pm I\}$. Furthermore, $PL_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) \cong PSL_2(\mathbb{Z}[\omega]) \bowtie C_2$ with $C_2 = \langle \sigma_1^3 \{\pm I\} \rangle$. □

For later reference we need the following number theoretical lemmas. From now on, in prime decompositions we distinguish the primes 2 and 3.

Lemma 9.2. *Let* $m = 2^d 3^e p_1^{e_1} \cdots p_k^{e_k}$ *be the prime decomposition of* m. *Then* $x^2 + x + 1 \equiv 0 \pmod m$ *is solvable if and only if* $d = 0$, $e = 0, 1$ *and* $p_j \equiv 1 \pmod 3$ *for each* $j = 1, \dots, k$.

Proof. We first notice that for solvability m must be odd, for $x^2 + x + 1$ is odd and hence is not congruent to 0 $\pmod m$ if m is even. Then, 4 is invertible, and hence $x^2 + x + 1 \equiv 0 \pmod m$ is solvable if and only if $y^2 = (2x+1)^2 \equiv -3 \pmod m$ is solvable.

Now, $y^2 \equiv -3 \pmod{m}$ is equivalent to the system $y^2 \equiv -3 \pmod{3^e}$, $y^2 \equiv -3 \pmod{p_1^{e_1}}, \ldots, y^2 \equiv -3 \pmod{p_k^{e_k}}$. However, $y^2 \equiv -3 \pmod{3^e}$ is solvable if and only if $e = 0, 1$. Also $y^2 \equiv -3 \pmod{p^\lambda}$ is solvable if and only if $y^2 \equiv -3 \pmod{p}$ is solvable whenever $p > 3$ is prime. However, -3 is a quadratic residue mod p if and only if $p \equiv 1 \pmod{3}$. $\square$

Since -1 is a quadratic residue mod m if and only if $m = 2^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ with $e = 0, 1$ and $p_j \equiv 1 \pmod{4}$ for each $j = 1, \ldots, k$, the following lemma is obvious.

Lemma 9.3. *Let* $m = 3^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ *be the prime decomposition of* m *such that* $e = 0$, 1 *and* $p_j \equiv 1 \pmod{3}$ *for each* $j = 1, \ldots, k$. *Then* -1 *is a quadratic residue mod* m *if and only if* $e = 0$ *and* $p_j \equiv 1 \pmod{12}$ *for each* $j = 1, \ldots, k$.

Lemma 9.4. *Let* $m = 2^d 3^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ *be the prime decomposition of* m.

(a) *Then* $x^2 = -1$ *has a solution in* $\mathbb{Z}_m[\omega]$ *if and only if* $d = 0, 1$, $e = 0$ *and* $p_j \not\equiv -5 \pmod{12}$ *for each* $j = 1, \ldots, k$.

(b) *Let* $x \in \mathbb{Z}_m[\omega]$ *be such that* $x^2 = -1$. *Then* $\langle x \rangle = \{\pm 1, \pm x\}$ *is conjugation invariant in* $\mathbb{Z}_m[\omega]$ *if and only if either* $p_j \equiv 1, 5 \pmod{12}$ *for each* j *or* $p_j \equiv \pm 1 \pmod{12}$ *for each* j.

Proof. Let $x = u + v\omega$. Then $x^2 = -1$ in $\mathbb{Z}_m[\omega]$ if and only if $u^2 - v^2 \equiv -1 \pmod{m}$, $v(2u - v) \equiv 0 \pmod{m}$; i.e. if and only if $u^2 - v^2 \equiv -1 \pmod{q}$, $v(2u - v) \equiv 0 \pmod{q}$ for $q = 2^d$, 3^e, $p_j^{e_j}$ for all j. Note that $v(2u - v) \equiv 0 \pmod{q}$ implies $v \equiv 0$ or $v \equiv 2u \pmod{q}$ if $q \neq 2^d$; then there exists a solution u, v modulo q if $e = 0$ or $p_j \not\equiv -5 \pmod{12}$. For $q = 2^d$ there exists a solution only if $d = 0, 1$. This proves (a). Note for (b) that for conjugation invariance we must have $\bar{x} = x$ or $\bar{x} = -x$, i.e. $x = u$ or $x = u(1 + 2\omega)$ for some $u \in \mathbb{Z}_m$. However, then -1 or 3 is a quadratic residue mod m, respectively. $\square$

Lemma 9.5. *Let* $m > 1$ *be an integer such that*

$$x^2 + x + 1 \equiv 0 \pmod{m} \tag{17}$$

is solvable. Let ω *be a solution. Then there exists a unique pair* b, c *of positive integers satisfying*

$$m = b^2 + bc + c^2, \quad (b, c) = 1, \quad c \equiv \omega b \pmod{m}. \tag{18}$$

Proof. The proof is analogous to that of [14, p. 117] for the equation $x^2 + 1 \equiv 0 \pmod{m}$. As we do not know of any explicit reference, we give a proof here. Only for the purpose of this proof we change the notation of the ring $\mathbb{Z}[\omega]$ of Eisenstein integers to $\mathbb{Z}[\rho]$ where $\rho = e^{2\pi i/3}$.

First note that in $\mathbb{Z}[\rho]$ we have

$$x^2 + xy + y^2 = (x - y\rho)(x - y\bar{\rho}) = (x - y\rho)(x - y\rho^2).$$

If z is a unit of $\mathbb{Z}[\rho]$ and $z(x - y\rho) = \tilde{x} - \tilde{y}\rho$, then $x^2 + xy + y^2 = \tilde{x}^2 + \tilde{x}\tilde{y} + \tilde{y}^2$. Hence, by multiplying $x - y\rho$ by a unit if need be, we can achieve $x, y \geq 0$.

By (17) and Lemma 9.2 we have $m = 3^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ with $e = 0, 1$ and $p_j \equiv 1 \pmod 3$ for each j. We proceed by induction on m. The case $m = 3$ is trivial.

It is well-known that if $m = p$ is a prime with $p \equiv 1 \pmod 3$, there exists a unique pair of positive integers u, v such that

$$p = u^2 + uv + v^2, \quad (u, v) = 1, \quad v \equiv \omega u \pmod p$$

(cf. [15, p. 95]). This proves the lemma for $m = p$. Assume (18) holds for $m = p^\lambda$. Let $m = p^{\lambda + 1}$ and $p \neq 3$; then $\omega \not\equiv \pm 1 \pmod p$. Since $\omega^2 + \omega + 1 \equiv 0 \pmod{p^\lambda}$, there exist x, y such that $p^\lambda = x^2 + xy + y^2$, $(x, y) = 1$ and $y \equiv \omega x \pmod{p^\lambda}$. Then $p^{\lambda + 1} = (x^2 + xy + y^2)(u^2 + uv + v^2) = r^2 + rs + s^2$ where $r = xu - yv$ and $s = xv + yu + yv$. First we show that $(r, s) = 1$, and that we can assume $s \equiv \omega r \pmod{p^{\lambda + 1}}$. However, $(r, s) \neq 1$ implies $p \mid (r, s)$ and hence $0 \equiv r = xu - yv = xu(1 - \omega^2) \not\equiv 0 \pmod p$; hence, $(r, s) = 1$. Further, since $(r, p) = 1$ there exists γ such that $r\gamma \equiv s \pmod{p^{\lambda + 1}}$. It follows that $0 \equiv r^2 + rs + s^2 \equiv r^2(1 + \gamma + \gamma^2) \pmod{p^{\lambda + 1}}$. Hence, $\gamma \equiv \omega, \omega^2 \pmod{p^{\lambda + 1}}$. If $\gamma \equiv \omega \pmod{p^{\lambda + 1}}$, we are done. If $\gamma \equiv \omega^2 \pmod{p^{\lambda + 1}}$, we can exchange r and s. Finally, it was remarked above that by changing $s - r\rho$ in $p^{\lambda + 1} = r^2 + rs + s^2 = (s - r\rho)(s - r\rho^2)$ by a unit of $\mathbb{Z}[\rho]$ to $\tilde{s} - \tilde{r}\rho$ (if need be), we can also achieve $r, s > 0$. Note for this that we still have $(\tilde{r}, \tilde{s}) = 1$ and $\tilde{s} \equiv \omega\tilde{r} \pmod{p^{\lambda + 1}}$; the latter can be easily seen by employing the ring homomorphism $\mathbb{Z}[\rho] \to \mathbb{Z}_m$ given by $a + b\rho \to a + b\omega$. This completes the existence proof for prime powers.

Now let $m = a \cdot b$, $a, b > 1$, $(a, b) = 1$. Assume

$$a = u^2 + uv + v^2, \quad u, v > 0, \quad (u, v) = 1, \quad v \equiv \omega u \pmod a,$$

$$b = x^2 + xy + y^2, \quad x, y > 0, \quad (x, y) = 1, \quad y \equiv \omega x \pmod b.$$

Then $m = ab = r^2 + rs + s^2$ with $r = xu - yv$, $s = xv + yu + yv$. Again, if we can prove $(r, s) = 1$ and $s \equiv \omega r \pmod m$, then a similar argument to that above shows that we can also achieve $r, s > 0$.

If $(r, s) \neq 1$, then $p \mid (r, s)$ for some prime p, and we write $xu - yv = p\alpha$ and $xv + yu + yv = p\beta$ with positive integers α, β. Then $xuv = p\alpha v + yv^2 = p\beta u - yu^2 - yuv$ and hence $y(u^2 + uv + v^2) = p(\beta u - \alpha v)$. Hence, $p \mid y$ or $p \mid u^2 + uv + v^2 = a$. However, $p \mid y$ implies $p \mid xu = r + yv$ and $p \mid xv = s - yu - yv$, and hence $p \mid u, v$; note here that $p \nmid x$. It follows that necessarily $p \mid a$. In a similar fashion, one proves $p \mid b$, contradicting $(a, b) = 1$. Hence, $(r, s) = 1$. Further, since $s = xv + yu + yv$, we have

$$s \equiv x\omega u + yu + y\omega u \equiv \omega xu - \omega^2 yu \equiv \omega(xu - yv) \equiv \omega r \pmod a,$$

$$s \equiv xv + \omega xu + \omega xv \equiv \omega xu - \omega^2 xv \equiv \omega(xu - yv) \equiv \omega r \pmod b.$$

However, $(a, b) = 1$, so that $s \equiv \omega r \pmod m$. This completes the existence proof.

Finally, to prove uniqueness assume there are two pairs b, c and $\tilde{b}, \tilde{c}$ satisfying (18). Then $m^2 = (b^2 + bc + c^2)(\tilde{b}^2 + \tilde{b}\tilde{c} + \tilde{c}^2) = r^2 + rs + s^2$ with $r = b\tilde{c} - c\tilde{b}$, $s = b\tilde{b} + c\tilde{c} + c\tilde{b}$, but $s \equiv b\tilde{b}(\omega^2 + \omega + 1) \equiv 0 \pmod m$ and s is positive. It follows that $s = m$ and $r = 0$. Let

$t := b/\tilde{b} = c/\tilde{c}$. Then $m = b^2 + bc + c^2 = t^2(\tilde{b}^2 + \tilde{b}\tilde{c} + \tilde{c}^2) = t^2 m$, so that $t = 1$. Hence, $b = \tilde{b}$ and $c = \tilde{c}$. This completes the proof. $\square$

For every positive integer m we have an epimorphism (Lemma 5.4)

$$PL_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) \xrightarrow{\psi_{m,K}} L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K, \tag{19}$$

with $\{\pm I\} \leqslant K \leqslant C_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$. Furthermore, whenever $m = 3^e p_1^{e_1} \cdots \cdot p_k^{e_k}$ where $e = 0, 1$ and $p_j \equiv 1 \pmod 3$ for each $j = 1, \ldots, k$, we have a commutative diagram of epimorphisms

$$L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) \xrightarrow{\psi_m} L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$$

$$\searrow \qquad \swarrow \Psi_m$$

$$L_2^{\langle -1 \rangle}(\mathbb{Z}_m)$$

This induces an epimorphism

$$PL_2^{\langle -1 \rangle}(\mathbb{Z}[i]) \xrightarrow{\psi_{m,H}} L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K, \tag{20}$$

with $\{\pm I\} \leqslant K \leqslant C_2^{\langle -1 \rangle}(\mathbb{Z}_m)$.

We will use the images of the generators (16) under the epimorphisms $\psi_{m,K}$ and $\Psi_{m,K}$ to construct chiral and regular polytopes of type $\{6,3,3\}$.

Again let us denote by σ_v the image of σ_v under $\psi_{m,K}$ or $\Psi_{m,K}$. Using the homomorphisms (19) and (20) and a proof similar to the proof of Lemma 7.2 we have the following lemma.

Lemma 9.6. *Let R be $\mathbb{Z}_m[\omega]$ with $m \geqslant 2$ or $\mathbb{Z}_m$ with $m > 3$ and m as in Lemma 9.2. Then for each subgroup K of $C_2^{\langle -1 \rangle}(R)$ containing $\{\pm I\}$, there is a chiral or a directly regular abstract polytope of type $\{6,3,3\}$ with the rotation group isomorphic to $\langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong L_2^{\langle -1 \rangle}(R)/K$.*

Proof. The conditions on m imply that $\sigma_1, \sigma_2, \sigma_3$ are of order 6, 3, 3, respectively. The intersection property for the group is easily checked; see [19, p. 91].

We proceed to identify the facets of the polytopes of Lemma 9.6. Since each facet is either chiral or regular of type $\{6,3\}$, it must be isomorphic to a toroidal map $\{6,3\}_{(b,c)}$. Recall from Section 2 that $\{6,3\}_{(b,c)}$ is regular if and only if $bc(b - c) = 0$. We obtain $\{6,3\}_{(b,c)}^+$ (see the presentation (6)) by the addition of

$$(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^b(\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2)^c = 1$$

to the defining relations for $[6,3]^+$. In terms of the matrices (16) the left-hand side becomes

$$\begin{bmatrix} 1 & -\omega^2 \\ 0 & 1 \end{bmatrix}^b \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}^c = \begin{bmatrix} 1 & (c - b\omega)\omega \\ 0 & 1 \end{bmatrix}.$$

Let R and K be as in Lemma 9.6. Since ω is invertible in R, the above matrix is the identity in $L_2^{\langle -1 \rangle}(R)/K$ if and only if $c - b\omega = 0$ in R. □

Theorem 9.7. *For each integer $m \geq 2$ and each conjugation invariant subgroup K of $C_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$ containing $\{\pm I\}$, there is a directly regular polytope $\mathscr{P}$ in $\langle\{6,3\}_{(m,0)}, \{3,3\}\rangle$ such that the rotation group of $\mathscr{P}$ is isomorphic to $L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K$.*

Proof. From Lemma 9.6 we have the existence of the polytope $\mathscr{P}$ with rotation group generated by the matrices (16) taken modulo m. The polytope is regular, since here exists an involutory automorphism ρ such that $\rho(\sigma_1) = \sigma_1^{-1}$, $\rho(\sigma_2) = \sigma_1^2 \sigma_2$ and $\rho(\sigma_3) = \sigma_3$. In fact, ρ is induced by conjugation in $\mathbb{Z}_m[\omega]$, i.e. by $\alpha + \beta\omega \mapsto \alpha + \beta\bar{\omega} = \alpha + \beta\omega^2$. Note that K is conjugation invariant.

To identify the facets note that $c - b\omega = 0$ n $\mathbb{Z}_m[\omega]$ is and only if either $b = c = m$ or $b = m, c = 0$ (or vice versa). We can rule out the possibility of $b = c = m$ since the order of $\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$ is m, and not $3m$ as it is for $\{6,3\}_{(m,m)}$. □

Corollary 9.8. *Let $m = 2^d 3^e p_1^{e_1} \cdots p_k^{e_k}$ be the prime decomposition of m. There exist directly regular polytopes in $\langle\{6,3\}_{(m,0)}, \{3,3\}\rangle$ whose rotation groups are isomorphic to*

(a) $PSL_2(\mathbb{Z}_m[\omega])$ *if $d = 0, 1$, $e = 0$ and $p_j \not\equiv -5 \pmod{12}$ for each j;*

(b) $\hat{P}SL_2(\mathbb{Z}_m[\omega])$ *if $d = 0, 1$, $e = 0$ and either $p_j \equiv 1, 5 \pmod{12}$ for each j or $p_j \equiv \pm 1 \pmod{12}$ for each j;*

(c) $PSL_2(\mathbb{Z}_m[\omega]) \rtimes C_2$ *and $\hat{P}SL_2(\mathbb{Z}_m[\omega]) \rtimes C_2$, if $d \geq 2$, or $e \geq 1$, or $p_j \equiv -5 \pmod{12}$ for at least one j.*

Proof. Let $C \leq C_2(\mathbb{Z}_m[\omega])$, $\{\pm I\} \leq K \leq C_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$, $C \leq K$ and

$$\chi: SL_2(\mathbb{Z}_m[\omega])/C \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K$$

be the canonical homomorphism. Again, C and K will be such that $SL_2(\mathbb{Z}_m[\omega]) \cap K = C$, implying that χ is injective. Note that χ is surjective if and only if there exists an element in K with determinant -1.

Let $C = C_2(\mathbb{Z}_m[\omega])$ and $K = C_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$. Then K is conjugation invariant. By Theorem 9.7 and Lemma 9.4 if $d = 0, 1$, $e = 0$ and $p_j \not\equiv -5 \pmod{12}$ for each j, we have a polytope with rotation group $PSL_2(\mathbb{Z}_m[\omega])$. Otherwis χ is not surjective and the rotation group of the polytope is $PSL_2(\mathbb{Z}_m[\omega]) \rtimes C_2$, with $C_2 = \langle \sigma_1^3 K \rangle$.

Let $C = \{\pm I\}$. Under the conditions of (b) for m, let x be a solution of $x^2 = -1$ in $\mathbb{Z}_m[\omega]$ such that $\langle x \rangle$ is conjugation invariant, and let $K = \langle xI \rangle$. Then the rotation group is $\hat{P}SL_2(\mathbb{Z}_m[\omega])$. If m is as in (c), choose $K = \{\pm I\}$, so that the rotation group is $\hat{P}SL_2(\mathbb{Z}_m[\omega]) \rtimes C_2$.

Remarks. (a) The groups $PSL_2(\mathbb{Z}_m[\omega])$ and $\hat{P}SL_2(\mathbb{Z}_m[\omega])$ of Corollary 9.8 coincide precisely for $m = 2^d 3^e$ with $d = 0, 1$, or $m = 2^d p^\lambda$ with $d = 0, 1$, $\lambda \geq 1$, and $p_j \equiv 5, -1 \pmod{12}$.

(b) If $m = p \equiv -1 \pmod 3$, then $\mathbb{Z}_p[\omega] \cong GF(p^2)$ and $PSL_2(\mathbb{Z}_m[\omega]) \cong PSL_2(p^2)$.

Theorem 9.9. *Let $m = 3^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ be the prime decomposition of m, and assume $e = 0, 1$ and $p_j \equiv 1 \pmod 3$ for each $j = 1, \ldots, k$. Let $\omega \in \mathbb{Z}_m$ be such that $\omega^2 + \omega + 1 \equiv 0 \pmod m$, and let b, c be the unique pair of positive integers such that $m = b^2 + bc + c^2$, $(b, c) = 1$ and $c \equiv \omega b \pmod m$. Then for each subgroup K of $C_2^{\langle -1 \rangle}(\mathbb{Z}_m)$ containing $\{\pm I\}$ there exists a chiral polytope in $\langle \{6, 3\}_{(b,c)}, \{3, 3\} \rangle$ with group isomorphic to $L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$.*

Proof. First recall Lemma 9.5. Since, by Lemma 9.2, the equation $x^2 + x + 1 \equiv 0 \pmod m$ is solvable, we have (see Lemma 5.5(b)) the canonical epimorphism $\Psi_{m,K}: PL_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$. We use Lemma 9.6 to construct a polytope $\mathscr{P}$ of type $\{6, 3, 3\}$ from $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$.

To identify the facets as $\{6, 3\}_{(b,c)}$ we use the remarks preceding Theorem 9.7. With our choice of b, c the required relation $(\sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1^{-1})^b (\sigma_1^{-1} \sigma_2^{-1} \sigma_1 \sigma_2)^c = 1$ holds in $\langle \sigma_1, \sigma_2 \rangle$. Since the 'translation' $\sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1^{-1}$ has order m, the facets must be maps $\{6, 3\}_{(b,c)}$. Since the facets are chiral, the polytope must be chiral as well. □

Corollary 9.10. *Let $m = 3^e p_1^{e_1} \cdot \cdots \cdot p_k^{e_k}$ be the prime decomposition of m, and assume that $e = 0, 1$ and $p_j \equiv 1 \pmod 3$ for each $j = 1, \ldots, k$. Let b, c be positive integers such that $m = b^2 + bc + c^2$, $(b, c) = 1$. There exist chiral polytopes in $\langle \{6, 3\}_{(b,c)}, \{3, 3\} \rangle$ whose groups are isomorphic to*

    (a) $PSL_2(\mathbb{Z}_m)$ *and* $\widehat{PSL_2}(\mathbb{Z}_m)$, *if $e = 0$ and $p_j \equiv 1 \pmod{12}$ for each $j = 1, \ldots, k$;*

    (b) $PSL_2(\mathbb{Z}_m) \bowtie C_2$ *and* $\widehat{PSL_2}(\mathbb{Z}_m) \bowtie C_1$, *if $e = 1$ or $p_j \equiv -5 \pmod{12}$ for at least one $j = 1, \ldots, k$.*

Proof. By our assumptions on b, c there exists a unique $\omega \in \mathbb{Z}_m$ such that $\omega^2 + \omega + 1 \equiv 0 \pmod m$, $c \equiv \omega b \pmod m$. Let $C \leqslant C_2(\mathbb{Z}_m)$, $\{\pm I\} \leqslant K \leqslant C_2^{\langle -1 \rangle}(\mathbb{Z}_m)$, $C \leqslant K$ and $\chi: SL_2(\mathbb{Z}_m)/C \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ be the canonical homomorphism. Again, C and K will be such that $SL_2(\mathbb{Z}_m) \cap K = C$, implying that χ is injective. Note that χ is surjective if and only if there exists an element in K with determinant -1.

Let $C = C_2(\mathbb{Z}_m)$ and $K = C_2^{\langle -1 \rangle}(\mathbb{Z}_m)$. By Theorem 9.9 and Lemma 9.3 if $e = 0$ and $p_j \equiv 1 \pmod{12}$ for each j, we have a polytope with group $PSL_2(\mathbb{Z}_m)$. Otherwise, the group of the polytope is $PSL_2(\mathbb{Z}_m) \bowtie C_2$ where $C_2 = \langle \sigma_1^3 K \rangle$.

Let $C = \{\pm I\}$. Under the conditions in (a) on m choose $K = \langle xI \rangle$ with $x^2 \equiv -1 \pmod m$. Then the group of the polytope is $\widehat{PSL_2}(\mathbb{Z}_m)$; otherwise, let $K = \{\pm I\}$, then the group is $\widehat{PSL_2}(\mathbb{Z}_m) \bowtie C_2$. □

10. The type $\{3, 6, 3\}$

We recall from Section 6 that $[3, 6, 3]^+$ is a subgroup of index 4 in $[6, 3, 3]^+$. In matrix notation the generators of $[3, 6, 3]^+$ are represented by

$$\sigma_1 = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} -\omega^2 & \omega^2 \\ 0 & \omega \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}. \qquad (21)$$

Considered as elements in $L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])$ the matrices in (21) generate a subgroup L (say). The centre of L is $\{\pm I\}$. Then the correspondence of Möbius transformations and matrices gives us

$$[3,6,3]^+ \cong C \cdot L/C \cong L/L \cap C = L/\{\pm I\} =: \Lambda,$$

where $C = \{\lambda I \mid \lambda \in \mathbb{C}^*\}$. Again we use the notation $\sigma_1, \sigma_2, \sigma_3$ for the generators of Λ. Let $\tilde{\psi}_{m,K} \colon \Lambda \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K$ and $\tilde{\Psi}_{m,K} \colon \Lambda \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ be the restrictions of $\psi_{m,K}$ and $\Psi_{m,K}$, respectively, to Λ. Again, we shall abuse notation by using K to denote two different groups.

Lemma 10.1. *Let* $R = \mathbb{Z}_m[\omega]$, $\chi = \tilde{\psi}_{m,K}$, *or* $R = \mathbb{Z}_m$, $\chi = \tilde{\Psi}_{m,K}$.
  (a) *If* $m \equiv \pm 1 \pmod 3$, *then* χ *is an epimorphism.*
  (b) *If* $m \equiv 0 \pmod 3$, *then* $\chi(\Lambda)$ *is a subgroup of index 4 in* $L_2^{\langle -1 \rangle}(R)/K$.

Proof. Consider the σ_j's as elements of $L_2^{\langle -1 \rangle}(R)$. Let $m \equiv \pm 1 \pmod 3$ so that 3 is invertible $\bmod\, m$. Then a suitable power of

$$\sigma_1^{-1}\sigma_2^3\sigma_1^{-1}\sigma_2 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \text{ gives } \sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \chi(\Lambda).$$

Multiplication of σ by

$$\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1} = \begin{bmatrix} 1 & \omega-1 \\ 0 & 1 \end{bmatrix} \text{ shows } \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \in \chi(\Lambda).$$

Also

$$\sigma^{-1}\sigma_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \chi(\Lambda).$$

Now, as in the proof of Lemma 5.4, we use Lemma 5.3 to prove that the homomorphism $L \to L_2^{\langle -1 \rangle}(R)$ is surjective. Then χ is surjective.

Let $m \equiv 0 \pmod 3$. Consider the ring homomorphism $f \colon R \to \mathbb{Z}_3$ given by $u + v\omega \mapsto u + v$ or $u \mapsto u$ if $R = \mathbb{Z}_m[\omega]$ or $R = \mathbb{Z}_m$, respectively. The induced homomorphism $\tilde{f} \colon L_2^{\langle -1 \rangle}(R)/K \to L_2^{\langle -1 \rangle}(\mathbb{Z}_3)/\{\pm I\}$ is surjective; see the proof of Lemma 5.4. Now, $\tilde{f}$ maps $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ onto

$$\langle \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \rangle \cong \left\langle I, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle \cong S_3,$$

which has index 4 in $L_2^{\langle -1 \rangle}(\mathbb{Z}_3)/\{\pm I\}$. It follows that $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ has index 4 in $L_2^{\langle -1 \rangle}(R)/K$. □

Theorem 10.2. *Let* $m \geqslant 2$ *be an integer and let* K *be a conjugation invariant subgroup of* $C_2^{\langle -1 \rangle}(\mathbb{Z}[\omega])$ *containing* $\{\pm I\}$. *Then there exists a directly regular polytope* $\mathscr{P}$ *such that*

(a) *if* $m \equiv \pm 1 \pmod 3$, $\mathscr{P}$ *is self-dual,* $\mathscr{P}$ *is in* $\langle \{3,6\}_{(m,0)}, \{6,3\}_{(m,0)} \rangle$ *and* $A^+(\mathscr{P}) \cong L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K$;

(b) *if* $m \equiv 0 \pmod 3$, $\mathscr{P}$ *is in* $\langle \{3,6\}_{(m/3,m/3)}, \{6,3\}_{(m,0)} \rangle$ *and* $A^+(\mathscr{P})$ *is a subgroup of* $L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K$ *of index* 4.

Proof. Modulo the intersection property (which we prove later) the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ of $L_2^{\langle -1 \rangle}(\mathbb{Z}_m[\omega])/K$ (of index 1 or 4) is the rotation group of a chiral or a directly regular polytope $\mathscr{P}$. The polytope $\mathscr{P}$ is indeed regular, since conjugation of $\mathbb{Z}_m[\omega]$ induces an involutory group automorphism of $A^+(\mathscr{P})$ that maps $\sigma_1, \sigma_2, \sigma_3$ onto $\sigma_1^{-1}, \sigma_1^2 \sigma_2, \sigma_3$, respectively.

To identify the facets of $\mathscr{P}$, we must (see the presentation (5)) consider

$$(\sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2)^b (\sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1})^c = \begin{bmatrix} 1 & \bar{\omega}(1-\omega)(b-c\omega) \\ 0 & 1 \end{bmatrix}, \tag{22}$$

which is the identity if and only if $(1-\omega)(b-c\omega) = 0$ in $\mathbb{Z}_m[\omega]$. Let $m \equiv \pm 1 \pmod 3$, so that 3 is invertible mod m. Then multiplication by $1 - \bar{\omega}$ shows that $(1-\omega)(b-c\omega) = 0$ if and only if $b - c\omega = 0$. Hence, the facets are maps $\{3,6\}_{(m,m)}$ or $\{3,6\}_{(m,0)}$. Since the order of

$$\sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} = \begin{bmatrix} 1 & \omega - 1 \\ 0 & 1 \end{bmatrix}$$

is m (and not $3m$), the facets must be maps $\{3,6\}_{(m,0)}$. The polytope is self-dual, since conjugation by the matrix

$$\begin{bmatrix} -1 & -\omega \\ \omega & 1 \end{bmatrix}$$

induces an involutory group automorphism of $A^+(\mathscr{P})$ interchanging σ_1 with σ_3^{-1} and σ_2 with σ_2^{-1}. It follows that the vertex-figures are maps $\{6,3\}_{(m,0)}$. To complete the proof of (a) we use Lemma 10.1.

Let $m \equiv 0 \pmod 3$. From $0 = (1-\omega)(b-c\omega) = b - c - \omega(2c+b)$ we see that the facets must be maps $\{3,6\}_{(m/3,m/3)}$ or $\{3,6\}_{(m,0)}$. However, the transformation

$$\sigma_1 \sigma_2^{-2} \sigma_1^{-1} \sigma_2^2 = \begin{bmatrix} 1 & 3\omega^2 \\ 0 & 1 \end{bmatrix}$$

has order $m/3$, so that the facets are maps $\{3,6\}_{(m/3,m/3)}$. Note here that this transformation shifts a Petrie 2-chain of $\{3,6\}$ two steps along itself; here a Petrie 2-chain is a zig-zag along the edges of $\{3,6\}$ which leaves at each vertex two faces to the right or left, in an alternating fashion.

To identify the vertex-figures $\{6,3\}_{(d,e)}$ consider

$$(\sigma_3^{-1}\sigma_2\sigma_3\sigma_2^{-1})^d(\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_3)^e = \begin{bmatrix} 1+\omega & 1 \\ -\omega^2 & 1-\omega \end{bmatrix}^d \begin{bmatrix} 1-\omega^2 & -\omega \\ 1 & 1+\omega^2 \end{bmatrix}^e$$

$$= \begin{bmatrix} 1+d\omega & d \\ -d\omega^2 & 1-d\omega \end{bmatrix} \begin{bmatrix} 1-e\omega^2 & -e\omega \\ e & 1+e\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(d-e\omega)\omega & d-e\omega \\ -(d-e\omega)\omega^2 & 1-(d-e\omega)\omega \end{bmatrix}. \tag{23}$$

For this to be the identity we must have $d-e\omega=0$ in $\mathbb{Z}_m[\omega]$. Hence, the vertex-figures are maps $\{6,3\}_{(m,0)}$ or $\{6,3\}_{(m,m)}$. Since $\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_3$ has order m, they must in fact be maps $\{6,3\}_{(m,0)}$. Now part (b) of the theorem follows from Lemma 10.1(b).

Finally, the proof of the intersection property $\langle\sigma_1,\sigma_2\rangle\cap\langle\sigma_2,\sigma_3\rangle = \langle\sigma_2\rangle$ is similar to that in the proof of Theorem 8.2. Here the 'translation subgroups' of $\langle\sigma_1,\sigma_2\rangle$ and $\langle\sigma_2,\sigma_3\rangle$ are given by

$$T_1 = \langle\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2, \sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\rangle = \left\{\begin{bmatrix} 1 & (\omega^2-1)\alpha \\ 0 & 1 \end{bmatrix} \middle| \alpha\in\mathbb{Z}_m[\omega]\right\}$$

and

$$T_2 = \langle\sigma_3^{-1}\sigma_2\sigma_3\sigma_2^{-1}, \sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_3\rangle = \left\{\begin{bmatrix} 1+\beta\omega & \beta \\ -\beta\omega^2 & 1-\beta\omega \end{bmatrix} \middle| \beta\in\mathbb{Z}_m[\omega]\right\},$$

respectively, so that

$$T_1 T_2 = \left\{\begin{bmatrix} 1+\beta\omega-\alpha\beta(1+2\omega) & \beta-\alpha(2+\omega)+\alpha\beta(\omega-1) \\ -\beta\omega^2 & 1-\beta\omega \end{bmatrix} \middle| \alpha,\beta\in\mathbb{Z}_m[\omega]\right\}.$$

Now the proof of the intersection property follows as in the proof of Theorem 8.2.   □

The next corollary is an immediate consequence of the proof of Corollary 9.8.

Corollary 10.3. *Let* $m=2^d p_1^{e_1}\cdot\cdots\cdot p_k^{e_k}$ *be the prime decomposition of* m, *and assume that* $p_j\geqslant 5$ *for* $j=1,\ldots,k$. *Then there exist self-dual directly regular polytopes in* $\langle\{3,6\}_{(m,0)}, \{6,3\}_{(m,0)}\rangle$ *whose rotation groups are isomorphic to*

(a) $PSL_2(\mathbb{Z}_m[\omega])$ *if* $d=0,1$ *and* $p_j\not\equiv-5 \pmod{12}$ *for each* j;

(b) $\hat{P}SL_2(\mathbb{Z}_m[\omega])$ *if* $d=0,1$, *and either* $p_j\equiv 1,5\pmod{12}$ *for each* j *or* $p_j\equiv\pm 1\pmod{12}$ *for each* j;

(c) $PSL_2(\mathbb{Z}_m[\omega])\bowtie C_2$ *and* $\hat{P}SL_2(\mathbb{Z}_m[\omega])\bowtie C_2$, *if* $d\geqslant 2$ *or* $p_j\equiv-5\pmod{12}$ *for at least one* j.

Theorem 10.4. *Let* $m=3^e p_1^{e_1}\cdot\cdots\cdot p_k^{e_k}$ *be the prime decomposition of* m, *and assume that* $e=0,1$ *and* $p_j\equiv 1\pmod 3$ *for each* $j=1,\ldots,k$. *Let* $\omega\in\mathbb{Z}_m$ *be such that* $\omega^2+\omega+1\equiv 0\pmod m$, *and let* b,c *be the unique pair of positive integers such that* $m=b^2+bc+c^2$, $(b,c)=1$ *and* $c\equiv\omega b\pmod m$. *Let* K *be a subgroup of* $C_2^{(s-1)}(\mathbb{Z}_m)$ *containing* $\{\pm I\}$. *Then there exists a chiral polytope* $\mathscr{P}$ *such that*

(a) *if* $m \equiv 1 \pmod 3$, $\mathscr{P}$ *is self-dual*, $\mathscr{P}$ *is in* $\langle \{3,6\}_{(c,b)}, \{6,3\}_{(c,b)} \rangle$ *and* $A(\mathscr{P}) \cong L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$;

(b) *if* $m \equiv 0 \pmod 3$, $\mathscr{P}$ *is in* $\langle \{3,6\}_{(a,d)}, \{6,3\}_{(c,b)} \rangle$ *with* $a = (c-b)/3$ *and* $d = (2b+c)/3$, *and* $A(\mathscr{P})$ *is a subgroup of* $L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ *of index* 4.

Proof. Again we write $\sigma_1, \sigma_2, \sigma_3$ for the images of the generators of A under the homomorphism $\mu: A \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$, the restriction of $\Psi_{m,K}$ in (20) to A. Modulo the intersection property, the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ of $L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ (of index 1 or 4) is the rotation group of a chiral or directly regular polytope $\mathscr{P}$.

To find the structure of the facets consider (as in (22))

$$(\sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2)^4 (\sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1})^l = \begin{bmatrix} 1 & \bar{\omega}(1-\omega)(k-l\omega) \\ 0 & 1 \end{bmatrix}.$$

This leads to the equation $(1-\omega)(k-l\omega) = 0$ over $\mathbb{Z}_m$. By our choice of b and c, $k = c$ and $l = b$ is one possible solution.

If $m \equiv 1 \pmod 3$, then $(\omega - 1)(\omega^2 - 1) = 3$ shows that $\omega - 1$ is invertible modulo m, so that the order of

$$\sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} = \begin{bmatrix} 1 & \omega - 1 \\ 0 & 1 \end{bmatrix}$$

is m. It follows that the facets are maps $\{3,6\}_{(c,b)}$. As in the proof of Theorem 10.2 conjugation by

$$\begin{bmatrix} -1 & -\omega \\ \omega & 1 \end{bmatrix}$$

gives a group automorphism interchanging σ_1 with σ_3^{-1} and σ_2 with σ_2^{-1}. Hence, $\mathscr{P}$ is self-dual. The vertex-figures are maps $\{6,3\}_{(c,b)}$, as can be seen either by self-duality arguments or by computations as in the next case. To complete the proof of (a) we use Lemma 10.1(a).

Let $m \equiv 0 \pmod 3$. First note that $c \equiv \omega b$ implies that $a, d \in \mathbb{Z}$ and $a \equiv \omega d$. Now the order of $\sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1}$ is $m/3 = a^2 + ad + d^2$, so that the facets are maps $\{3,6\}_{(a,d)}$. To find the structure of the vertex-figure we need to consider an equation like (23) over $\mathbb{Z}_m$. This shows that the relation $(\sigma_3^{-1} \sigma_2 \sigma_3 \sigma_2^{-1})^c (\sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_3)^b = 1$ holds in $A(\mathscr{P})$. However, the order of

$$\sigma_3^{-1} \sigma_2 \sigma_3 \sigma_2^{-1} = \begin{bmatrix} 1+\omega & 1 \\ -\omega^2 & 1-\omega \end{bmatrix}$$

is m, so that the vertex-figures are maps $\{6,3\}_{(c,b)}$. By Lemma 10.1(b) the index of $A(\mathscr{P})$ in $L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ is 4.

Finally, the proof of the intersection property carries over from the proof of Theorem 10.2, but now with $\alpha, \beta \in \mathbb{Z}_m$. This completes the proof. $\square$

Corollary 10.5. *Let* $m = p_1^{e_1} \cdots p_k^{e_k}$ *be the prime decomposition of* m, *and let* $p_j \equiv 1 \pmod 3$ *for each* $j = 1, \ldots, k$. *Let* b, c *be positive integers such that* $m = b^2 + bc + c^2$, $(b, c) = 1$. *There exist self-dual chiral polytopes in* $\langle \{3, 6\}_{(c, b)}, \{6, 3\}_{(c, b)} \rangle$ *whose rotation groups are isomorphic to*

(a) $PSL_2(\mathbb{Z}_m)$ *and* $\hat{PSL}_2(\mathbb{Z}_m)$, *if* $p_j \equiv 1 \pmod{12}$ *for each* $j = 1, \ldots, k$;

(b) $PSL_2(\mathbb{Z}_m) \rtimes C_2$ *and* $\hat{PSL}_2(\mathbb{Z}_m) \rtimes C_2$, *if* $p_j \equiv -5 \pmod{12}$ *for at least one* $j = 1, \ldots, k$.

Proof. Follows from that of Corollary 9.10.

Note that the results of this and the next section could also be derived from the results of the previous section by employing suitable mixing operations in the sense of [20] to the (rotation) groups. As in Section 8 this does not lead to shorter proofs.

11. The type {6, 3, 6}

Recall from Section 6 that $[6, 3, 6]^+$ is a subgroup of index 6 in $[6, 3, 3]^+$. Hence, to construct the polytopes of type $\{6, 3, 6\}$ we proceed as in Section 10. We first represent the generators of $[6, 3, 6]^+$ as

$$\sigma_1 = \begin{bmatrix} -\omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} \omega^2 & \omega - \omega^2 \\ 0 & \omega \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} \omega^2 - \omega & \omega \\ \omega^2 & 0 \end{bmatrix}. \qquad (24)$$

Considered as elements in $L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega])$ the matrices in (24) generate a subgroup L (say) whose centre is $\{\pm I\}$. Then

$$[6, 3, 6]^+ \cong C \cdot L / C \cong L / L \cap C = L / \{\pm I\} =: \Lambda,$$

with $C = \{\lambda I \mid \lambda \in \mathbb{C}^*\}$. Again we write $\sigma_1, \sigma_2, \sigma_3$ for the generators of Λ. Let $\tilde{\tilde{\psi}}_{m, K}$: $\Lambda \to L_2^{\langle -1 \rangle}(\mathbb{Z}[\omega]) / K$ and $\tilde{\tilde{\Psi}}_{m, K}: \Lambda \to L_2^{\langle -1 \rangle}(\mathbb{Z}_m) / K$ be the restrictions of $\psi_{m, K}$ and $\Psi_{m, K}$, respectively, to Λ.

Lemma 11.1. *Let* $R = \mathbb{Z}_m[\omega]$, $\chi = \tilde{\tilde{\psi}}_{m, K}$, *or* $R = \mathbb{Z}_m$, $\chi = \tilde{\tilde{\Psi}}_{m, K}$.

(a) *If* $m \equiv \pm 1 \pmod 3$, *then* χ *is an epimorphism.*

(b) *If* $m \equiv 0 \pmod 3$, *then* $\chi(\Lambda)$ *is a subgroup of index 6 in* $L_2^{\langle -1 \rangle}(R) / K$.

Proof. The proof is similar to that of Lemma 10.1. For (a), first note that

$$\sigma_1^2 \sigma_2^{-1} \sigma_1^{-2} \sigma_2 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \qquad \text{implies} \quad \sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \chi(\Lambda)$$

and

$$\sigma \sigma_1^2 \sigma_2^{-1} = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \in \chi(\Lambda).$$

Also

$$\sigma_2\sigma_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \chi(\Lambda),$$

and since

$$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{for } \alpha \in R,$$

we can use arguments such as in the proof of Lemma 5.4 to complete the proof of (a). For (b), note that over $\mathbb{Z}_3$ the matrices of (24) generate a subgroup of $L_2^{\langle -1\rangle}(\mathbb{Z}_3)/\{\pm I\}$ which is isomorphic to $C_2 \times C_2$ and thus has index 6. As in the proof of Lemma 10.1 it follows that $\chi(\Lambda)$ has index 6 in $L_2^{\langle -1\rangle}(R)/K$.

Theorem 11.2. *Let $m \geqslant 3$ be an integer and let K be a conjugation invariant subgroup of $C_2^{\langle -1\rangle}(\mathbb{Z}[\omega])$ containing $\{\pm I\}$. Then there exists a self-dual directly regular polytope $\mathscr{P}$ such that*

(a) *if $m \equiv \pm 1 \pmod 3$, then $\mathscr{P}$ is in $\langle\{6,3\}_{(m,0)}, \{3,6\}_{(m,0)}\rangle$ and $A^+(\mathscr{P}) \cong L_2^{\langle -1\rangle}(\mathbb{Z}_m[\omega])/K$;*

(b) *if $m \equiv 0 \pmod 3$, then $\mathscr{P}$ is in $\langle\{6,3\}_{(m/3,m/3)}, \{3,6\}_{(m/3,m/3)}\rangle$ and $A^+(\mathscr{P})$ is a subgroup of $L_2^{\langle -1\rangle}(\mathbb{Z}_m[\omega])/K$ of index 6.*

Proof. Modulo the intersection property, the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ of $L_2^{\langle -1\rangle}(\mathbb{Z}_m[\omega])/K$ (of index 1 or 6) is the rotation group of a chiral or a directly regular polytope $\mathscr{P}$ of type $\{6,3,6\}$; note for this that $m \geqslant 3$. Define the involutory group automorphism ρ of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ by

$$\rho\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \begin{bmatrix} \bar{\alpha} & -\bar{\beta}\omega^2 \\ -\bar{\gamma}\omega & \bar{\delta} \end{bmatrix} = \sigma_1 \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{bmatrix} \sigma_1^{-1},$$

where conjugation of $\mathbb{Z}_m[\omega]$ is given by $\overline{u+v\omega} = u+v\omega^2$. Then ρ maps $\sigma_1, \sigma_2, \sigma_3$ onto $\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3$, respectively, and hence $\mathscr{P}$ is indeed regular. Furthermore, conjugation by the matrix

$$\begin{bmatrix} 1 & \omega^2 \\ 1 & -1 \end{bmatrix}$$

induces an involutory group automorphism of $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ which interchanges σ_1 with σ_3^{-1} and σ_2 with σ_2^{-1}. It follows that $\mathscr{P}$ must be self-dual.

To identify the facets of $\mathscr{P}$ consider

$$(\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1})^k(\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2)^l = \begin{bmatrix} 1 & (\omega-1)(-k+l\omega) \\ 0 & 1 \end{bmatrix}. \tag{25}$$

This is the identity if and only if $(\omega - 1)(k - l\omega) = 0$ in $\mathbb{Z}_m[\omega]$. Also

$$\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1} = \begin{bmatrix} 1 & 1-\omega \\ 0 & 1 \end{bmatrix}$$

has order m.

Now, if $m \equiv \pm 1 \pmod 3$, then $(\omega - 1)(\omega^2 - 1) = 3$ shows that $\omega - 1$ is invertible in $\mathbb{Z}_m[\omega]$, so that the facets must be maps $\{6, 3\}_{(m, 0)}$. If $m \equiv 0 \pmod 3$, then $\frac{1}{3}m(\omega - 1)^2 = \frac{1}{3}m(\omega^2 - 2\omega + 1) = -m\omega = 0$ and so we must have facets $\{6, 3\}_{(m/3, m/3)}$. Together with Lemma 11.1 this proves (a) and (b).

Finally, for the proof of the intersection property note that in the group of the vertex-figure we have

$$(\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_3)^r(\sigma_3\sigma_2\sigma_3^{-1}\sigma_2^{-1})^s =$$

$$\begin{bmatrix} -r(\omega + 2) + s(\omega - 1) - 1 & r(\omega + 2) - s(\omega - 1) \\ -r(\omega + 2) + s(\omega - 1) & r(\omega + 2) - s(\omega - 1) - 1 \end{bmatrix}. \qquad (26)$$

This can be used to complete the proof as in Theorem 10.2. □

Corollary 11.3. *Corollary 10.3 remains true if the class* $\langle\{3,6\}_{(m,0)}, \{6,3\}_{(m,0)}\rangle$ *is replaced by the class* $\langle\{6,3\}_{(m,0)}, \{3,6\}_{(m,0)}\rangle$.

Proof. See the proof of Corollary 9.8. □

Theorem 11.4. *Let* $m = 3^e p_1^{e_1} \cdots p_k^{e_k}$ *be the prime decomposition of* $m \geq 4$, *and assume that* $e = 0, 1$ *and* $p_j \equiv 1 \pmod 3$ *for each* $j = 1, \dots, k$. *Let* $\omega \in \mathbb{Z}_m$ *be such that* $\omega^2 + \omega + 1 \equiv 0 \pmod m$, *and let* b, c *be the unique pair of positive integers such that* $m = b^2 + bc + c^2$, $(b, c) = 1$ *and* $c \equiv \omega b \pmod m$. *Let* K *be a subgroup of* $C_2^{\langle -1 \rangle}(\mathbb{Z}_m)$ *containing* $\{\pm I\}$. *Then there exists a self-dual chiral polytope* $\mathcal{P}$ *such that*

(a) *if* $m \equiv 1 \pmod 3$, *then* $\mathcal{P}$ *is in* $\langle\{6,3\}_{(c,b)}, \{3,6\}_{(c,b)}\rangle$ *and* $A(\mathcal{P}) \cong L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$;

(b) *if* $m \equiv 0 \pmod 3$, *then* $\mathcal{P}$ *is in* $\langle\{6,3\}_{(c,b)}, \{3,6\}_{(a,d)}\rangle$ *with* $a = (c - b)/3$ *and* $d = (2b + c)/3$, *and* $A(\mathcal{P})$ *is a subgroup of* $L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ *of index 6.*

Proof. Modulo the intersection property, the subgroup $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ of $L_2^{\langle -1 \rangle}(\mathbb{Z}_m)/K$ (of index 1 or 6) is the rotation group of a chiral or regular polytope $\mathcal{P}$ of type $\{6, 3, 6\}$; note that $m \geq 4$. By the same arguments as in the proof of Theorem 11.2, $\mathcal{P}$ is self-dual.

To find the facets of $\mathcal{P}$ we consider (25) over $\mathbb{Z}_m$. A similar analysis as in the proof of Theorem 10.4 shows that the facets are maps $\{6, 3\}_{(c,b)}$ if $m \equiv 1 \pmod 3$, or $\{6, 3\}_{(a,d)}$ if $m \equiv 0 \pmod 3$. Since the facets are chiral, $\mathcal{P}$ is chiral too. The self-duality of $\mathcal{P}$ and considerations involving (26) over $\mathbb{Z}_m$ imply that the vertex-figures are maps $\{3, 6\}_{(c,b)}$ and $\{3, 6\}_{(a,d)}$, respectively. Then (a) and (b) follow from Lemma 11.1. Finally, the proof of the intersection property is similar to that of Theorem 11.2. □

Corollary 11.5. *Corollary* 10.5 *remains true if the class* $\langle \{3,6\}_{(c,b)}, \{6,3\}_{(c,b)} \rangle$ *is replaced by the class* $\langle \{6,3\}_{(c,b)}, \{3,6\}_{(c,b)} \rangle$.

12. Petrie polygons

For a regular or a chiral polytope it is sometimes useful to know the length of its Petrie polygons. For the definition of the Petrie polygon for a regular polytope, we refer to [25, pp. 315–316]. This definition naturally extends to chiral polytopes, but in this case (since there are two orbits on the flags of the polytope) there are two 'kinds' of Petrie polygons: left- and right-handed. The right-handed Petrie polygon is shifted one step along itself by $\sigma_1 \sigma_3$, and the left-handed one by $\sigma_1^{-1} \sigma_3$. We proceed to find the orders of these transformations. For example, for the polytopes of type $\{4,4,3\}$ in Section 7 we can do the following. Note that

$$\sigma_1 \sigma_3 = \begin{bmatrix} -i & i \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} - i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\sigma_1 \sigma_3 = A - iB$. We may allow $(\sigma_1 \sigma_3)^k = a_k(A - iB) + b_k I$, for some a_k and b_k. Then

$$(\sigma_1 \sigma_3)^{k+1} = (b_k - ia_k)(A - iB) + ia_k I.$$

Hence, $b_{k+1} = ia_k$ and $a_{k+1} = b_k - ia_k$, and we have the following recursive formula:

$$a_0 = 0, \qquad a_1 = 1, \qquad a_{k+1} + ia_k - ia_{k-1} = 0.$$

In conclusion, if $\mathscr{P}$ is a polytope described in Theorem 7.3 or 7.6, the order of $\sigma_1 \sigma_3$ is the smallest integer k such that $a_k = 0$ in $\mathbb{Z}_m[i]$ or $\mathbb{Z}_m$, respectively, and $a_{k-1} I \in H$. Here the condition on $a_{k-1} I$ is automatically satisfied if $H = C_2^{\langle -1 \rangle}(\mathbb{Z}_m[i])$ or $H = C_2^{\langle -1 \rangle}(\mathbb{Z}_m)$, respectively.

To find the order of

$$\sigma_1^{-1} \sigma_3 = \begin{bmatrix} i & -i \\ 1 & 0 \end{bmatrix},$$

we see that it is sufficient to replace i by $-i$ in the above recursion formula. With

$$C = \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = B,$$

$\sigma_1^{-1} \sigma_3 = C + iB$ and $\sigma_1^{-1} \sigma_3 = c_k(C + iB) + d_k I$,

this leads to the recursion

$$c_0 = 0, \qquad c_1 = 1, \qquad c_{k+1} - ic_k + ic_{k-1} = 0.$$

Now, for $\mathbb{Z}_m[i]$ the change $i \mapsto -i$ correspond to conjugation in $\mathbb{Z}_m[i]$, so that the two recursions are conjugate. Also, by assumption H is invariant under conjugation. It follows that the orders of $\sigma_1^{-1}\sigma_3$ and $\sigma_1\sigma_3$ are the same, in agreement with the fact that the polytope of Theorem 7.3 is regular. However, this is no longer true in the chiral case of Theorem 7.6.

Similar remarks extend to the polytopes in Sections 8–11.

Acknowledgement

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