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Sensitivity Analysis of Generalized Variational Inequalities

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Dafermos studied the sensitivity properties of the solutions of a variational inequality with regard to continuity and differentiability of such solutions with respect to a parameter λ. In the present paper we extend this analysis for a generalized variational inequality of the type introduced by Noor of which the variational inequality of Dafermos is a particular case. Our results are such that they automatically extend the regularity properties of solutions with respect to a parameter λ when the variational inequality is treated on a Hilbert space.

1. INTRODUCTION

Dafermos [1] studied the sensitivity property of solutions, of a particular kind of variational inequality, on a parameter which takes values on an open subset of Euclidean space R^k. Noor [4] has investigated a most general class of nonlinear variational inequalities with regard to the existence of their solutions. The type of variational inequalities treated by Noor occurs frequently in a framework for studying many unrelated free and moving boundary value problems arising in contact problems in elastostatics, fluid flow through porous media, and lubrication problems. The purpose of our present paper is to analyze the sensitivity property of solutions of the type of variational inequalities just mentioned on a parameter. A special feature which is more pronounced in our treatment is that our analysis carries through in a setting which can be described as well in an infinite dimensional space (in particular on a Hilbert space).

We now introduce the parametric form of the variational inequality [4] as follows. Let Λ be an open subset of R^k in which the parameter λ takes values and assume

$$\{\mathscr{K}_\lambda : \lambda \in \Lambda\} \tag{1.1}$$

is a family of closed convex subsets of R^n.

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We now introduce the parametric form of the variational inequality [4] as follows. Find

$$x \in \mathscr{K}_\lambda: a(u, \lambda, v-u) + b(u, v) - b(u, u) \geqslant \langle A(u, \lambda), v-u \rangle, \quad \forall v \in \mathscr{K}_\lambda, \tag{1.2}$$

where, $a(\cdot, \lambda, \cdot)$ is a coercive continuous bilinear form on R^n; that is, there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$a(v, \lambda, v) \geqslant \alpha \|v\|^2, \qquad \text{for all} \quad v \in R^n \tag{1.3}$$

and

$$a(u, \lambda, v) \leqslant \beta \|u\| \, \|v\| \qquad \text{for all } u, v \in R^n, \tag{1.4}$$

where both α and β are independent of λ. Also let $b(u, v)$: $R^n \times R^n \to R$ satisfy the following properties:

(i) $b(\cdot, \cdot)$ is linear in the first variable.

(ii) $b(\cdot, \cdot)$ is bounded; that is, there exists constant $\gamma > 0$ such that

$$|b(u, v)| \leqslant \gamma \|u\| \, \|v\|, \qquad \text{for all} \quad u, v \in R^n. \tag{1.5}$$

(iii) $b(\cdot, \cdot)$ is either convex or linear in the second argument.

(iv) For every $u, v, \omega \in R^n$

$$|b(u, v) - b(u, \omega)| \leqslant b(u, v - \omega) \tag{1.6}$$

$$b(u, v \pm \omega) \leqslant b(u, v) + b(u, \omega). \tag{1.7}$$

$A(\lambda, \cdot)$ is antimonotone, i.e., for all $u, v \in \mathscr{K}_\lambda$,

$$\langle A(\lambda, u) - A(\lambda, v), u - v \rangle \leqslant 0 \tag{1.8}$$

and $A(\lambda, \cdot)$ is Lipschitz continuous, i.e.,

$$\|A(\lambda, u) - A(\lambda, v)\| \leqslant \xi \|u - v\|, \tag{1.9}$$

for all $u, v \in \mathscr{K}_\lambda$, and ξ is independent of λ.

The sensitivity problem connected with (1.2) can be summarised in the following terms. Assuming for some $\bar{\lambda} \in \Lambda$, (1.2) admits a solution $\bar{x}$, one would like to investigate conditions under which, for each λ in a neighbourhood of $\bar{\lambda}$, (1.2) has a unique solution $x(\lambda)$ near $\bar{x}$ and the function $x(\lambda)$ is continuous, Lipschitz continuous, or differentiable.

Remark. To keep the treatment more simple we have restricted the above formulation on a finite dimensional space. As hinted earlier, and more so because of the results of [4], such formulations are more general in nature and carry over to a base space which happens to be a Hilbert space.

2. Local Uniqueness and Continuity

Consider the family of variational inequalities (1.2), where λ takes values in an open neighbourhood Λ of λ in R^k, and $\mathscr{K}_\lambda$ is a closed convex set in R^n. We assume $a(u, \lambda, u)$ is defined on $X \times \Lambda \times X$, where X is the closure of a ball in R^n centerd at $\bar{x}$ and satisfies the coerciveness condition (1.3) and boundedness condition (1.4), and $b(u, v)$ satisfies the properties (1.5)–(1.7). $A(\lambda, \cdot)$ is antimonotone and Lipschitz continuous as depicted in (1.8) and (1.9). We have the following lemmas which can be proved by the techniques of Noor [4].

Lemma 2.1. *Let ρ be a number such that $0<\rho<2(\alpha-\gamma-\xi)/(\beta^2-(\gamma+\xi)^2)$ and $\rho<1/(\gamma+\xi)$. Then there exists a θ with $0<\theta<1$ such that*

$$\|\phi(\lambda, u_1)-\phi(\lambda, u_2)\leqslant \theta\|u_1-u_2\|$$

for all $u_1, u_2 \in X$, where ϕ is defined as, given $u \in X$,

$$\langle \phi(\lambda, u), v\rangle = (u, v) - \rho a(u, \lambda, v) - \rho b(u, v) + \rho\langle A(\lambda, u), v\rangle \qquad \textit{for all} \quad v \in X.$$

β is the boundedness constant of the bilinear form $a(u, v)$.

Lemma 2.2. *We define a map $G(u, \lambda) = P_{\mathscr{K}_\lambda}\phi(\lambda, u)$, where $P_{\mathscr{K}_\lambda}$ is the projection for the closed convex set $\mathscr{K}_\lambda$, for each $\lambda \in \Lambda$. Then a point $u \in \mathscr{K}_\lambda$ is a solution of the variational inequality* (1.2) *if and only if u is a fixed point of the map $G(u, \lambda)$, $u \in \mathscr{K}_\lambda$, $\lambda \in \Lambda$.*

We define $G^*(u, \lambda) = P_{\mathscr{K}_\lambda \cap X}\phi(\lambda, u)$, for $(X, \lambda) \in X \times \Lambda$, because we are interested in solutions of (1.2) that lie in the interior of X. $G(u, \lambda)$ is a contraction with respect to u, uniformly in $\lambda \in \Lambda$, which follows as a consequence of Lemma 2.2 since $P_{\mathscr{K}_\lambda \cap X}$ is a nonexpansive map.

Lemma 2.3.

$$\|G^*(u, \lambda) - G^*(v, \lambda)\| \leqslant \theta\|u - v\|, \qquad \textit{for all} \quad u, v \in X, \lambda \in \Lambda, 0 \leqslant \theta < 1. \tag{2.1}$$

From the Banach fixed point theorem it is now apparent that for every $\lambda \in \Lambda$, $G^*(u, \lambda)$ has a unique fixed point $u(\lambda)$.

We now show the following: (i) $u(\lambda)$ depends continuously upon λ, (ii) for λ near $\bar{\lambda}$, $u(\lambda)$ is in fact a fixed point of $G(x, \lambda)$, i.e., a solution of the variational inequality (1.2).

LEMMA 2.4. *Assume that $a(\bar{u}, \lambda, v)$ is continuous (or Lipschitz continuous in λ at $\bar{\lambda}$ for each $v \in X$, $A(\lambda, \bar{u})$ is continuous in λ at $\bar{\lambda}$, and for fixed $\bar{v} \in X$, the map*

$$\lambda \to P_{\mathscr{K}_\lambda \cap X} \phi(\lambda, \bar{v})$$

is continuous (or Lipschitz continuous) in λ at $\bar{\lambda}$, then $u(\lambda)$ is continuous (or Lipschitz continuous) at $\lambda = \bar{\lambda}$.

Proof. Fix $\lambda \in \Lambda$, then using triangle inequality and (2.1)

$$\begin{aligned}\|u(\lambda) - u(\bar{\lambda})\| &= \|G^*(u(\lambda), \lambda) - G^*(u(\bar{\lambda}), \bar{\lambda})\| \\ &\leqslant \|G^*(u(\lambda), \lambda) - G^*(u(\bar{\lambda}), \lambda)\| \\ &\quad + \|G^*(u(\bar{\lambda}), \lambda) - G^*(u(\bar{\lambda}), \bar{\lambda})\| \\ &\leqslant \theta\|u(\lambda) - u(\bar{\lambda})\| + \|G^*(u(\bar{\lambda}), \lambda) - G^*(u(\bar{\lambda}), \bar{\lambda})\|. \end{aligned} \tag{2.2}$$

Also using the nonexpansiveness of the projection map we have the inequalities

$$\begin{aligned}&\|G^*(u(\bar{\lambda}), \lambda) - G^*(u(\bar{\lambda}), \bar{\lambda})\| \\ &\quad \leqslant \|P_{\mathscr{K}_\lambda \cap X} \phi(\lambda, u(\bar{\lambda})) - P_{\mathscr{K}_{\bar{\lambda}} \cap X} \phi(\bar{\lambda}, u(\bar{\lambda}))\| \\ &\quad \leqslant \|P_{\mathscr{K}_\lambda \cap X} \phi(\lambda, u(\bar{\lambda})) - P_{\mathscr{K}_\lambda \cap X} \phi(\bar{\lambda}, u(\bar{\lambda}))\| \\ &\qquad + \|P_{\mathscr{K}_\lambda \cap X} \phi(\bar{\lambda}, u(\bar{\lambda})) - P_{\mathscr{K}_{\bar{\lambda}} \cap X} \phi(\bar{\lambda}, u(\bar{\lambda}))\| \\ &\quad \leqslant \|\phi(\lambda, u(\bar{\lambda})) - \phi(\bar{\lambda}, u(\bar{\lambda}))\| + \|P_{\mathscr{K}_\lambda \cap X} \phi(\bar{\lambda}, u(\bar{\lambda})) \\ &\qquad - P_{\mathscr{K}_{\bar{\lambda}} \cap X} \phi(\bar{\lambda}, u(\bar{\lambda}))\|. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) and putting $u(\bar{\lambda}) = \bar{u}$ we get

$$\begin{aligned}\|u(\lambda) - \bar{u}\| &\leqslant (1/1 - \theta)\|\phi(\lambda, \bar{u}) - \phi(\bar{\lambda}, \bar{u})\| \\ &\quad + (1/1 - \theta)\|P_{\mathscr{K}_\lambda \cap X} \phi(\bar{\lambda}, \bar{u}) - P_{\mathscr{K}_{\bar{\lambda}} \cap X} \phi(\bar{\lambda}, \bar{u})\|. \end{aligned} \tag{2.4}$$

Now because of assumptions on $a(\bar{u}, \bar{\lambda}, \cdot)$ and $A(\lambda, \bar{u})$ it is easy to see that $\|\phi(\lambda, \bar{u}) - \phi(\bar{\lambda}, \bar{u})\| \to 0$ as $\lambda \to \bar{\lambda}$; also since $\lambda \to P_{\mathscr{K}_\lambda \cap X} \phi(\bar{\lambda}, u(\bar{\lambda}))$ is continuous by assumption, the second norm in the r.h.s. of (2.3) goes to zero as $\lambda \to \bar{\lambda}$ and hence the proof is complete.

The following results follow by similar arguments as given in Lemma 2.4.

LEMMA 2.5. *Under the assumptions of Lemma 2.4, there exists a neighbourhood $\mathscr{L} \subset \Lambda$ of $\bar{\lambda}$ such that for $\lambda \in \mathscr{L}$, $u(\lambda)$ is the unique solution of the parametric variational inequality* (1.2) *in the interior of X.*

THEOREM 2.1. *Consider the parametric variational inequality* (1.2) *which admits a solution $\bar{u}$ at $\bar{\lambda}$. Assume the conditions* (1.2) *to* (1.9). *Assume also that the conditions of Lemma 2.4 also hold at $\lambda = \bar{\lambda}$. Then there exists a neighbourhood $\mathscr{L} \subset \Lambda$ of $\bar{\lambda}$ such that for every $\lambda \in \mathscr{L}$, the variational inequality* (1.2) *admits a unique solution $u(\lambda)$ in the interior of X, $u(\bar{\lambda}) = \bar{u}$, and $u(\lambda)$ is continuous* (*or Lipschitz continuous*) *at $\lambda = \bar{\lambda}$.*

Remark 2.1. The situation as discussed in [1] is analogous in the present case with regard to the sufficient condition [1, Proposition 2.1] which guarantees the continuity of the map $\lambda \to P_{\mathscr{K}_\lambda \cap X} \bar{Y}$, for fixed $\bar{Y} \in X$. Also as a potential application, Remark 2.2 of [1] remains valid for the feasible set $\mathscr{K}_\lambda$ defined locally by

$$\mathscr{K}_\lambda = \{x \in X \mid g_i(x, \lambda) = 0,\ i = 1, 2, ..., s,\ g_j(x, \lambda) \geqslant 0,\ j = s+1, ..., m\}.$$

We state the following theorem which gives an extension of the differentiability property of $u(\lambda)$ as solution of (1.2). The proof follows as in [1], by applying Lemma 2.3 and implicit function theorem.

THEOREM 2.2. *Consider the parametric variational inequality* (1.2). *Assume that the conditions* (1.2) *to* (1.9) *are satisfied. Suppose $\phi(\lambda, u)$ is continuously differentiable on $X \times \Lambda$. Assume that the map*

$$(v, \lambda) \to P_{\mathscr{K}_\lambda \cap X}\, v$$

is continuously differentiable on some neighborhood of the point $(\bar{v}, \bar{\lambda})$, where $\bar{v} = \phi(\bar{\lambda}, \bar{u})$. Then the function $u(\lambda)$, as defined through Theorem 2.1, is continuously differentiable in some neighbourhood $\mathscr{L}$ of $\bar{\lambda}$.

Remark 2.2. The gradient formula (implicit) for $u(\lambda)$ can be given as

$$\nabla_\lambda u(\lambda) = [I - \nabla_u G^*]^{-1}\, \nabla_\lambda G^*,$$

where G^* is defined as in Lemma 2.1. Also we observe that in case $a(u, \lambda, v)$ is continuously differentiable on $X \times \Lambda \times X$ and $A(\lambda, u)$ is weakly continuously differentiable on $\Lambda \times X$ then it would imply that $\phi(\lambda, u)$ is weakly continuously differentiable on $\Lambda \times X$. Since our treatment is on finite dimensional space this would be sufficient for the strongly continuous differentiability of $\phi(\lambda, u)$ on $\Lambda \times X$, and this specific assumption would be satisfied in Theorem 2.2.

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