Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 255 (2008) 374–448

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems

Pascal Auscher [a], Andreas Axelsson [b,*], Steve Hofmann [c]

[a] *Université de Paris-Sud, UMR du CNRS 8628, 91405 Orsay Cedex, France*
[b] *Matematiska institutionen, Stockholms universitet, 106 91 Stockholm, Sweden*
[c] *Mathematics Department, University of Missouri, Columbia, MO 65211, USA*

Received 6 February 2008; accepted 14 February 2008

Available online 18 April 2008

Communicated by C. Kenig

Abstract

We prove that Neumann, Dirichlet and regularity problems for divergence form elliptic equations in the half-space are well posed in L_2 for small complex L_∞ perturbations of a coefficient matrix which is either real symmetric, of block form or constant. All matrices are assumed to be independent of the transversal coordinate. We solve the Neumann, Dirichlet and regularity problems through a new boundary operator method which makes use of operators in the functional calculus of an underlaying first order Dirac type operator. We establish quadratic estimates for this Dirac operator, which implies that the associated Hardy projection operators are bounded and depend continuously on the coefficient matrix. We also prove that certain transmission problems for k-forms are well posed for small perturbations of block matrices.

Keywords: Neumann problem; Dirichlet problem; Elliptic equation; Non-symmetric coefficients; Dirac operator; Functional calculus; Quadratic estimates; Perturbation theory; Carleson measure

Contents

* Corresponding author.
E-mail addresses: pascal.auscher@math.u-psud.fr (P. Auscher), andax@math.su.se (A. Axelsson), hofmann@math.missouri.edu (S. Hofmann).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.02.007

1. Introduction

In this paper we prove that the Neumann, Dirichlet and regularity problems are well posed in $L_2(\mathbf{R}^n)$ for divergence form second order elliptic equations

$$\operatorname{div}_{t,x} A(x)\nabla_{t,x} U(t,x) = 0 \tag{1.1}$$

on the half-space $\mathbf{R}^{n+1}_+ := \{(t,x) \in \mathbf{R} \times \mathbf{R}^n;\ t > 0\}$, $n \geqslant 1$, when A is a small complex L_∞ perturbation of either a block matrix, a constant matrix or a real symmetric matrix. Furthermore, the matrix $A = (a_{ij}(x))_{i,j=0}^n \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{n+1}))$ is assumed to be t-independent with complex coefficients and accretive, with quantitative bounds $\|A\|_\infty$ and κ_A, where $\kappa_A > 0$ is the largest constant such that

$$\operatorname{Re}\big(A(x)v, v\big) \geqslant \kappa_A |v|^2, \quad \text{for all } v \in \mathbf{C}^{n+1},\ x \in \mathbf{R}^n.$$

We shall approach Eq. (1.1) from a first order point of view, rewriting it as the first order system

$$\begin{cases} \operatorname{div}_{t,x} A(x) F(t,x) = 0, \\ \operatorname{curl}_{t,x} F(t,x) = 0, \end{cases} \tag{1.2}$$

where $F(t,x) = \nabla_{t,x} U(t,x)$. Recall that a vector field $F = F_0 e_0 + F_1 e_1 + \cdots + F_n e_n$ can be written in this way as a gradient if and only if $\operatorname{curl}_{t,x} F = 0$, by which we understand that $\partial_j F_i = \partial_i F_j$, for all $i, j = 0, \dots, n$. We write $\{e_0, e_1, \dots, e_n\}$ for the standard basis for $\mathbf{R}^{n+1}$ with e_0 upward pointing into $\mathbf{R}^{n+1}_+$, and write $t = x_0$ for the vertical coordinate. For the vertical derivative, we write $\partial_0 = \partial_t$. Denote also by $F_\parallel := F_1 e_1 + \cdots + F_n e_n$, the tangential part of F, and write $\operatorname{curl}_x F_\parallel = 0$ if $\partial_j F_i = \partial_i F_j$, for all $i, j = 1, \dots, n$.

In the formulation of the boundary value problems below, we assume that $A = A(x)$ is a given coefficient matrix with properties as above. Furthermore, by saying that $F_t(x) = F(t,x)$

satisfies (1.2) we shall mean that $F_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{n+1}))$ and that for each fixed $t > 0$, we have $\operatorname{div}_x(AF)_\parallel = -(A(x)\partial_0 F)_0$, $\nabla_x F_0 = \partial_0 F_\parallel$ and $\operatorname{curl}_x F_\parallel = 0$, where the derivatives on the left-hand sides are taken in the sense of distributions. If this holds for F, then in particular we can write $F = \nabla_{t,x} U$, with $U \in W^1_{2,\mathrm{loc}}(\mathbf{R}^{n+1}_+)$, and we see that U satisfies (1.1) in the sense that

$$\iint_{\mathbf{R}^{n+1}_+} \big(A(x)\nabla_{t,x}U(t,x), \nabla_{t,x}\varphi(t,x)\big)\, dt\, dx = 0, \quad \text{for all } \varphi \in C_0^\infty\big(\mathbf{R}^{n+1}_+\big).$$

Neumann problem (Neu-A). Given a function $\phi(x) \in L_2(\mathbf{R}^n; \mathbf{C})$, find a vector field $F_t(x) = F(t,x)$ in $\mathbf{R}^{n+1}_+$ such that $F_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{n+1}))$ and F satisfies (1.2) for $t > 0$, and furthermore $\lim_{t\to\infty} F_t = 0$ and $\lim_{t\to 0} F_t = f$ in L_2 norm, where the conormal part of f satisfies the boundary condition

$$e_0 \cdot (Af) = \sum_{j=0}^{n} A_{0j} f_j = \phi, \quad \text{on } \mathbf{R}^n = \partial \mathbf{R}^{n+1}_+.$$

For U, this means that the conormal derivative $\frac{\partial U}{\partial \nu_A}(0,x) = \phi(x)$ in $L_2(\mathbf{R}^n)$.

Regularity problem (Reg-A). Given a function $\psi : \mathbf{R}^n \to \mathbf{C}$ with tangential gradient $\nabla_x \psi \in L_2(\mathbf{R}^n; \mathbf{C}^n)$, find a vector field $F_t(x) = F(t,x)$ in $\mathbf{R}^{n+1}_+$ such that $F_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{n+1}))$ and F satisfies (1.2) for $t > 0$, and furthermore $\lim_{t\to\infty} F_t = 0$ and $\lim_{t\to 0} F_t = f$ in L_2 norm, where the tangential part of f satisfies the boundary condition

$$f_\parallel = f_1 e_1 + \cdots + f_n e_n = \nabla_x \psi, \quad \text{on } \mathbf{R}^n = \partial \mathbf{R}^{n+1}_+.$$

For U, this means that $\nabla_x U(x,0) = \nabla_x \psi(x)$, i.e. $U(x,0) = \psi(x)$ in $\dot W^1_2(\mathbf{R}^n)$.

Dirichlet problem (Dir-A). Given a function $u(x) \in L_2(\mathbf{R}^n; \mathbf{C})$, find a function $U_t(x) = U(t,x)$ in $\mathbf{R}^{n+1}_+$ such that $U_t \in C^2(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}))$, $\nabla_{t,x} U_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{n+1}))$ and $\nabla_{t,x} U_t$ satisfies (1.2) for $t > 0$, and furthermore $\lim_{t\to\infty} U_t = 0$, $\lim_{t\to\infty} \nabla_{t,x} U_t = 0$ and $\lim_{t\to 0} U_t = u$ in L_2 norm.

We shall also use first order methods based on (1.2) to solve the Dirichlet problem. However, here we use a different relation between (1.1) and (1.2) which we now describe. Assume $F(t,x) = \sum_{k=0}^n F_i(t,x) e_i$ is a vector field satisfying (1.2). Applying ∂_t to the first equation and using that $\operatorname{curl}_{t,x} F = 0$ yields

$$0 = \partial_t \big(\operatorname{div}_{t,x} A(x) F\big) = \operatorname{div}_{t,x} A(x)(\partial_t F) = \operatorname{div}_{t,x} A(x)(\nabla_{t,x} F_0),$$

since the coefficients are assumed to be t-independent. Thus the normal component $U := F_0$ satisfies (1.1). Note that when $A = I$, the functions $F_1, \ldots, F_n$ are conjugates to U in the sense of Stein and Weiss [25]. From this we see that solvability of (Dir-A) is a direct consequence of solvability of the following auxiliary Neumann problem.

Neumann problem (Neu$^\perp$-A). Given a function $\phi(x) \in L_2(\mathbf{R}^n; \mathbf{C})$, find a vector field $F_t(x) = F(t,x)$ in $\mathbf{R}^{n+1}_+$ such that $F_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{n+1}))$ and F satisfies (1.2) for $t > 0$, and furthermore $\lim_{t\to\infty} F_t = 0$ and $\lim_{t\to 0} F_t = f$ in L_2 norm, where the normal part of f is $e_0 \cdot f = f_0 = \phi$.

The main result of this paper is the following L_∞ perturbation result for the boundary value problems.

Theorem 1.1. *Let $A_0(x) = ((a_0)_{ij}(x))_{i,j=0}^n$ be a t-independent, complex, accretive coefficient matrix function. Furthermore assume that A_0 has one of the following extra properties.*

(b) *A_0 is a block matrix, i.e. $(a_0)_{0i}(x) = (a_0)_{i0}(x) = 0$ for all $1 \leqslant i \leqslant n$ and all $x \in \mathbf{R}^n$.*
(c) *A_0 is a constant coefficient matrix, i.e. $A_0(x) = A_0(y)$ for all $x, y \in \mathbf{R}^n$.*
(s) *A_0 is a real symmetric matrix, i.e. $(a_0)_{ij}(x) = (a_0)_{ji}(x) \in \mathbf{R}$ for all $0 \leqslant i, j \leqslant n$ and all $x \in \mathbf{R}^n$.*

Then there exists $\varepsilon > 0$ depending only on $\|A_0\|_\infty$, the accretivity constant κ_{A_0} and the dimension n, such that if $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{n+1}))$ is t-independent and satisfies $\|A - A_0\|_\infty < \varepsilon$, then Neumann and Regularity problems (Neu-A), (Neu$^\perp$-A) *and* (Reg-A) *above have a unique solution $F(t,x)$ with the required properties for every boundary function $g(x)$, being $\phi(x)$ and $\nabla_x \psi(x)$, respectively. Furthermore Dirichlet problem* (Dir-A) *above has a unique solution $U(t,x)$ with the required properties for every boundary function $u(x)$.*

The solutions depend continuously on the data with the following equivalences of norms. If we define the triple bar norm $|||G_t|||^2 := \int_0^\infty \|G_t\|_2^2 t^{-1}\,dt$ and the non-tangential maximal function

$$\widetilde{N}_*(F)(x) := \sup_{t>0}\left(\fint_{|s-t|<c_0 t} \fint_{|y-x|<c_1 t} \left|F(s,y)\right|^2 ds\,dy\right)^{1/2},$$

where $\fint_E := |E|^{-1}\int_E$ and $c_0 \in (0,1)$, $c_1 > 0$ are constants, then for Neumann and Regularity problems we have

$$\|g\|_2 \approx \|f\|_2 \approx \sup_{t>0}\|F_t\|_2 \approx |||t\partial_t F_t||| \approx \left\|\widetilde{N}_*(F)\right\|_2,$$

and for Dirichlet problem we have

$$\|u\|_2 \approx \sup_{t>0}\|U_t\|_2 \approx |||t\partial_t U_t||| \approx |||t\nabla_x U_t||| \approx \left\|\widetilde{N}_*(U)\right\|_2.$$

Moreover, the solution operators S_A, being $S_A(g) = F$ or $S_A(u) = U$, respectively, depend Lipschitz continuously on A, i.e. there exists $C < \infty$ such that

$$\|S_{A_2} - S_{A_1}\|_{L_2(\mathbf{R}^n)\to\mathcal{X}} \leqslant C\|A_2 - A_1\|_{L_\infty(\mathbf{R}^n)}$$

when $\|A_i - A_0\|_\infty < \varepsilon$, $i = 1, 2$, where $\|F\|_\mathcal{X}$ or $\|U\|_\mathcal{X}$ denotes any of the norms above.

Throughout this paper, we use the notation $X \approx Y$ and $X \lesssim Y$ to mean that there exists a constant $C > 0$ so that $X/C \leqslant Y \leqslant CX$ and $X \leqslant CY$, respectively. The value of C varies from one usage to the next, but then is always fixed.

Let us now review the history of works on these boundary value problems, starting with the case of matrices of the form $\tilde{A}$ which we now describe. By standard arguments, Theorem 1.1 also shows well-posedness of the corresponding boundary value problems on the region Ω above a Lipschitz graph $\Sigma = \{(t,x);\ t = g(x)\}$, where $g : \mathbf{R}^n \to \mathbf{R}$ is a Lipschitz function. Indeed, if the function $U(t,x)$ satisfies $\operatorname{div}_{t,x} A(x)\nabla_{t,x} U = 0$ in Ω then $\widetilde{U}(t,x) := U(t+g(x),x)$ satisfies $\operatorname{div}_{t,x} \tilde{A}(x)\nabla_{t,x}\widetilde{U} = 0$ in $\mathbf{R}^{n+1}_+$, where

$$\tilde{A}(x) := \begin{bmatrix} 1 & -(\nabla_x g(x))^t \\ 0 & I \end{bmatrix} A(x) \begin{bmatrix} 1 & 0 \\ -\nabla_x g(x) & I \end{bmatrix}.$$

Thus Theorem 1.1 gives conditions on A for which Neumann problem (conormal derivative $e_0 \cdot \tilde{A}\nabla_{t,x}\widetilde{U} = (e_0 - \nabla_x g)\cdot A\nabla_{t,x}U$ given), Regularity problem (tangential gradient $\nabla_x \widetilde{U} = (\partial_t U)\nabla_x g + \nabla_x U$ given), and Dirichlet problem ($\widetilde{U} = U$ given) are well posed. Note that A is real symmetric if and only if $\tilde{A}$ is, but that $\tilde{A}$ being constant or of block form does not imply the same for A. For the Laplace equation $A = I$ in Ω, solvability of (Neu-$\tilde{I}$) and (Reg-$\tilde{I}$) was first proved by Jerison and Kenig [16], and solvability of (Dir-$\tilde{I}$) was first proved by Dahlberg [12]. Later Verchota [26] showed that these boundary value problems are solvable with the layer potential integral equation method.

For general real symmetric matrices A, not being of the "Jacobian type" $\tilde{I}$ above, the well-posedness of (Dir-A) was first proved by Jerison and Kenig [17], and (Neu-A) and (Reg-A) by Kenig and Pipher [20]. These results make use of the Rellich estimate technique. For Neumann and Regularity problems, this integration by parts technique yields an equivalence

$$\left\| \frac{\partial U}{\partial \nu_A} \right\|_{L_2(\mathbf{R}^n)} \approx \|\nabla_x U\|_{L_2(\mathbf{R}^n)}, \tag{1.3}$$

which is seen to be equivalent with the first estimate $\|f\| \approx \|g\|$ in the theorem above, and shows that the boundary trace f splits into two parts of comparable size.

Turning to the unperturbed case where $A = A_0$ satisfies (b), then (1.3) is still valid, but the proof is far deeper than Rellich estimates. In fact, it is equivalent with the Kato square root estimate proved by Auscher, Hofmann, Lacey, M^{c}Intosh and Tchamitchian in [4]. (For the non-divergence form case $a_{00} \neq 1$, see [10].) For details concerning this equivalence between the Kato problem and the boundedness and invertibility of the Dirichlet-to-Neumann map $\nabla_x U \mapsto \frac{\partial U}{\partial \nu}$ we refer to Kenig [19, Remark 2.5.6], where also many further references in the field can be found.

We now consider what is previously known in the case when A does not satisfy (b), (c) or (s). Here (Dir-A) has been showed to be well posed by Fabes, Jerison and Kenig [14] for small perturbations of (c), using the method of multilinear expansions. More recently, the boundary value problems have been studied in the L_p setting and for real but non-symmetric matrices in the plane, i.e. $n = 1$. Here Kenig, Koch, Pipher and Toro [22] have obtained solvability of the Dirichlet problem for sufficiently large p, and Kenig and Rule [21] have shown solvability of the Neumann and regularity problems for sufficiently small dual exponent p'.

In the perturbed case $A \approx A_0$ when A_0 satisfies (c) or (s), the well-posedness of (Neu-A), (Reg-A) and (Dir-A) is also proved in [2] by Alfonseca, Auscher, Axelsson, Hofmann and Kim. With the further assumption of pointwise resolvent kernel bounds, perturbation of case (b) is also implicit in [2]. It is worth comparing the present methods to those of [2]. In [2], due to the presence of kernel bounds, the solvability of the boundary value problems is meant in the sense of non-tangential maximal estimates at the boundary and this follows from the use of layer potentials. The first main result in [2] in the unperturbed case (s), is the proof via singular integral operator theory of boundedness and invertibility of layer potentials. The second main result in [2] is the stability of the simultaneous occurrence of both boundedness and invertibility, which hold in the unperturbed cases (c), (s) and (b). Solvability then follows.

Here, we setup a different resolution algorithm (forcing us to introduce some substantial material), which consists in solving the first order system (1.2) instead of (1.1), also by a boundary operator method, but acting on the gradient of solutions involving a generalised Cauchy operator E_A, the goal being to establish boundedness of E_A and invertibility of related operators $E_A \pm N_A$. Boundedness of $E_A = \text{sgn}(T_A)$ is obtained via quadratic estimates of an underlaying first order differential operator T_A, and the deep fact is here that those quadratic estimates alone are stable under perturbations. Stability of invertibility is then easy. The perturbation argument requires sophisticated harmonic analysis techniques inspired by the strategy of [2]. In particular, the latter uses extensively the technology of the solution of the Kato problem for second order operators in [4], whereas we utilise here the work of Axelsson, Keith and M^{c}Intosh [10], which adapts and extends this technology to first order operators of Dirac type. Indeed, we note that our Dirac type operators T_A are of the form Π_A of [10] in the case of block matrices (b) of Theorem 1.1. But T_A has a more complicated structure when A is not a block matrix and we understand how to prove boundedness of E_A at the moment only in the cases specified by Theorem 1.1. We also note that the present paper, like [10], makes no use of kernel bounds and only needs L_2 off-diagonal bounds for the operators, which always holds.

The boundary operator method for first order Dirac type operators, used here to solve second order boundary value problems, was developed in the thesis of Axelsson [6], which has been published as the four papers [5,7–9]. It covers operators on Lipschitz domains as described above and in Example 1.5. The result in [10] pursued the program initiated by Auscher, M^{c}Intosh and Nahmod in [3], consisting of connecting the Kato problem and the functional calculus of first order differential operators of Dirac type. As said, it thus applies to the boundary value problems for operators of case (b). What is new here is the setup for full matrices encompassing the above. We prove also a sort of meta-theorem (see Theorem 1.3) which roughly says that the set of matrices for which the needed quadratic estimates on T_A hold, is open.

We also show that non-tangential maximal estimates hold for our solutions. By uniqueness in the class of solutions of (1.1) with non-tangential maximal estimates, this implies that our solutions are the same as those in [2] for perturbations of the real symmetric and constant cases. The non-tangential maximal estimate here also yields an indirect proof of non-tangential limits of solutions of (1.2) which hold for the solutions of (1.1) in [2]. We do not know how to prove this fact directly in the framework of this article. Note also that we prove here that the non-tangential maximal functions have comparable L_2-norms for different values of the parameters c_0 and c_1, and that the slightly different non-tangential maximal function used in [2] therefore has comparable norm.

Before turning to the method of proof for Theorem 1.1, we would like to stress the importance of the final result that the solution operators $g \mapsto F$ and $u \mapsto U$ depend Lipschitz continuously on L_∞ changes of the matrix A around A_0. This is an important motivation for considering complex A, as the authors do not know any proof of this perturbation result which does not make use of boundedness of the operators in a complex neighbourhood of A_0. We also remark that we in fact prove that $A \mapsto S_A$ is holomorphic, from which we deduce Lipschitz continuity as a corollary.

In this paper, we shall mainly focus on the boundary value problems (Neu-A) and (Reg-A). The reason is that, assuming the Cauchy operator E_A is bounded, we prove in Section 2.5 that well-posedness follows as

$$(\text{Reg-}A^*) \iff (\text{Neu}^\perp\text{-}A) \implies (\text{Dir-}A).$$

That (Reg-A^*) implies (Dir-A) has been proved by Kenig and Pipher [20, Theorem 5.4] in the case of real matrices A.

1.1. Operators and vector fields

We now explain the basic ideas of the method we use for the proof of Theorem 1.1. The appropriate Hilbert space on the boundary $\mathbf{R}^n$ is

$$\hat{\mathcal{H}}^1 := \{ f \in L_2(\mathbf{R}^n; \mathbf{C}^{n+1});\ \operatorname{curl}_x(f_\parallel) = 0 \}.$$

The condition on f means that its tangential part is curl-free. Indeed, the trace $f(x)$ of a vector field $F(t,x)$ solving (1.2) belongs to $\hat{\mathcal{H}}^1$ due to the second equation in (1.2). The basic picture, building on ideas from [7], is that the Hilbert space splits into two different pairs of complementary subspaces as

$$\hat{\mathcal{H}}^1 = E_A^+\hat{\mathcal{H}}^1 \oplus E_A^-\hat{\mathcal{H}}^1 = N_A^+\hat{\mathcal{H}}^1 \oplus N_A^-\hat{\mathcal{H}}^1. \tag{1.4}$$

We first discuss the splitting into the Hardy type subspaces $E_A^\pm\hat{\mathcal{H}}^1$, consisting of L_2 boundary traces of vector fields $F^\pm$ solving (1.2) in $\mathbf{R}^{n+1}_\pm$, respectively. Our main work in this paper is to establish boundedness of the projection operators $E_A^\pm$ for certain A. These projections can be written $E_A^\pm = \frac{1}{2}(I \pm E_A)$, where E_A for simple A is a singular integral operator of Cauchy type. However, in the general case E_A may fail to be a singular integral operator. To handle the projections $E_A^\pm$ we make use of functional calculus of closed Hilbert space operators, and show that $E_A^\pm = \chi_\pm(T_A)$ are the spectral projections of an underlaying bisectorial operator T_A in $\hat{\mathcal{H}}^1$. The functions $\chi_\pm(z)$ are the characteristic functions for the right and left complex half-planes. To find T_A, assume $F(t,x)$ satisfies (1.2) in $\mathbf{R}^{n+1}_+$ and solve for the vertical derivative

$$\partial_t F_0 = -a_{00}^{-1}\left(\sum_{i=1}^n a_{0i}\partial_i F_0 + \partial_i (AF)_i\right),$$
$$\partial_t F_i = \partial_i F_0, \quad i = 1, \ldots, n.$$

The right-hand side defines an operator $-T_A$ in $\hat{\mathcal{H}}^1$ which on $F(t,x)$, for fixed $t > 0$, satisfies

$$\partial_t F + T_A F = 0.$$

Concretely, if we identify $f = f_0 e_0 + f_\parallel$ with $(f_0, f_\parallel)^t$, where $f_\parallel$ is a tangential curl-free vector field, then

$$T_A f = \begin{bmatrix} A_{00}^{-1}((A_{0\parallel}, \nabla_x) + \operatorname{div}_x A_{\parallel 0}) & A_{00}^{-1}\operatorname{div}_x A_{\parallel\parallel} \\ -\nabla_x & 0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_\parallel \end{bmatrix}, \tag{1.5}$$

where

$$\mathsf{D}(T_A) = \{ f = (f_0, f_\|)^t \in \hat{\mathcal{H}}^1;\ \nabla_x f_0 \in L_2,\ \operatorname{div}_x (Af)_\| \in L_2 \} \quad \text{and}$$

$$A = \begin{bmatrix} A_{00} & A_{0\|} \\ A_{\|0} & A_{\|\|} \end{bmatrix}.$$

If $F(t,x)$ is a vector field in $\mathbf{R}^{n+1}_+$ satisfying (1.2), then using this operator T_A, we can reproduce F provided we know the full trace $f = F|_{\mathbf{R}^n}$, through a Cauchy type reproducing formula $F(t,x) = (e^{-t|T_A|} f)(x)$. However, in (Neu-$A$) and (Reg-$A$) only "half" of the trace f is known since the boundary conditions for f are $e_0 \cdot Af = \phi$ and $e_0 \wedge f = e_0 \wedge \nabla \psi$, respectively.

We now turn to the second splitting in (1.4), which is used to split the boundary trace f into the regularity and Neumann data. We define the A-tangential and normal subspaces of $\hat{\mathcal{H}}^1$ to be the null spaces of these two operators:

$$N_A^+ \hat{\mathcal{H}}^1 := \{ f \in \hat{\mathcal{H}}^1;\ e_0 \cdot Af = 0 \},$$
$$N_A^- \hat{\mathcal{H}}^1 := \{ f \in \hat{\mathcal{H}}^1;\ e_0 \wedge f = 0 \}.$$

In contrast with the Hardy subspaces, it is straightforward to show that we have a topological splitting $\hat{\mathcal{H}}^1 = N_A^+ \hat{\mathcal{H}}^1 \oplus N_A^- \hat{\mathcal{H}}^1$, and therefore that the corresponding pair of projections $N_A^\pm$ are bounded. We can now reformulate (Neu-A) and (Reg-A) as follows. Neumann problem (Neu-A) being well posed means that the restricted projection

$$N_A^- : E_A^+ \hat{\mathcal{H}}^1 \to N_A^- \hat{\mathcal{H}}^1$$

is an isomorphism, since $N_A^- \hat{\mathcal{H}}^1$ is a complement of the null space of $e_0 \cdot A(\cdot)$. Similarly Regularity problem (Reg-A) being well posed means that the restricted projection

$$N_A^+ : E_A^+ \hat{\mathcal{H}}^1 \to N_A^+ \hat{\mathcal{H}}^1$$

is an isomorphism, since $N_A^+ \hat{\mathcal{H}}^1$ is a complement of the null space of $e_0 \wedge (\cdot)$. Note that what is important here is which subspace $N_A^\pm$ projects along, not what subspace they project onto.

We shall also find it convenient to use the operators $E_A := E_A^+ - E_A^- = \operatorname{sgn}(T_A)$ and $N_A := N_A^+ - N_A^-$. These are reflection operators, i.e. $E_A^2 = I$ and $N_A^2 = I$, and we have $E_A^\pm = \frac{1}{2}(I \pm E_A)$ and $N_A^\pm = \frac{1}{2}(I \pm N_A)$.

Example 1.2. Let $n = 1$ and $A = I$. Then the space $\hat{\mathcal{H}}^1$ is simply $L_2(\mathbf{R}; \mathbf{C}^2)$ and the fundamental operator T_A becomes

$$T := T_I = \begin{bmatrix} 0 & \frac{d}{dx} \\ -\frac{d}{dx} & 0 \end{bmatrix} \approx \begin{bmatrix} 0 & i\xi \\ -i\xi & 0 \end{bmatrix},$$

if $\approx$ denotes conjugation with Fourier transform. Furthermore

$$E := E_I = \operatorname{sgn}(T) = \begin{bmatrix} 0 & iH \\ -iH & 0 \end{bmatrix}, \qquad N := N_I = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$Hf(x) := \frac{i}{\pi}\text{p.v.}\int \frac{f(y)}{x-y}\,dy.$$

Note that the operator E is contained in the Borel functional calculus of the self-adjoint operator T, and it follows that $\|E\| = 1$. On the other hand the operator N is outside the Borel functional calculus of T. Indeed, the operators $b(T)$ in the Borel functional calculus of T all commute, but we have the anticommutation relation

$$EN + NE = 0. \tag{1.6}$$

A consequence of this equation is that the boundary value problems (Neu-I) and (Reg-I) are well posed, or equivalently that $N^- : E^+L_2 \to N^-L_2$ and $N^+ : E^+L_2 \to N^+L_2$ are isomorphisms. Consider for example (Reg-I) and assume we want to solve $N^+f = g$ for $f \in E^+L_2$. Applying $4E^+$ to the equation gives

$$4E^+g = (I+E)(I+N)f = f + Ef + Nf + ENf = f + f + Nf - Nf = 2f,$$

so $f = 2E^+g$ and it follows that $N^+ : E^+L_2 \to N^+L_2$ is an isomorphism. Having solved for $f \in E^+L_2$, we can find the solution $F(t,x) = C_t^+ f(x)$ in $\mathbf{R}^2_+$ by using the Cauchy extension $C_t^+ := e^{-t|T|}E^+$ for $t > 0$. As a convolution operator, the Fourier multiplier C_t^+ has the expression

$$\begin{aligned} C_t^+(u_0e_0 + u_1e_1) &= \left(\frac{1}{2\pi}\int_{\mathbf{R}} \frac{tu_0(y) - (x-y)u_1(y)}{t^2 + (x-y)^2}\,dy\right)e_0 \\ &\quad + \left(\frac{1}{2\pi}\int_{\mathbf{R}} \frac{(x-y)u_0(y) + tu_1(y)}{t^2 + (x-y)^2}\,dy\right)e_1, \end{aligned}$$

and in particular

$$F(t,x) = C_t^+ f(x) = 2C_t^+(g_1e_1) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{(-(x-y)e_0 + te_1)g_1(y)}{t^2 + (x-y)^2}\,dy.$$

For a more general A, even if A is real symmetric, the operator T_A is not self-adjoint and proving boundedness of $E_A^{\pm}$ is a highly non-trivial problem. Also Eq. (1.6) fails for general A, and therefore such explicit formulae for the solution $F(t,x)$ as in Example 1.2 are not available. However, to show well-posedness of (Neu-A) and (Reg-A) it suffices to show that $I \pm \frac{1}{2}(E_AN_A + N_AE_A) = \frac{1}{2}(E_A \pm N_A)^2$ are invertible, as explained in [7]. To summarise, in order to solve (Neu-A) and (Reg-A) we need that:

(i) the Hardy projections $E_A^{\pm}$ are bounded, so that we have a topological splitting $\hat{\mathcal{H}}^1 = E_A^+\hat{\mathcal{H}}^1 \oplus E_A^-\hat{\mathcal{H}}^1$, and
(ii) the restricted projections $N_A^+ : E_A^+\hat{\mathcal{H}}^1 \to N_A^+\hat{\mathcal{H}}^1$ and $N_A^- : E_A^+\hat{\mathcal{H}}^1 \to N_A^-\hat{\mathcal{H}}^1$ are isomorphisms.

To prove Theorem 1.1, we shall prove the following.

(i′) That T_A satisfies quadratic estimates for all A such that $\|A - A_0\|_\infty < \varepsilon$. From this it will follow that $E_A^\pm$ are bounded for all such A and that $A \mapsto E_A^\pm$ is continuous (in fact holomorphic).

(ii′) That (ii) holds for the unperturbed A_0. From (i′) and since clearly $N_A^\pm$ are bounded and depend continuously on A, it then follows from a continuity argument that (ii) holds for all A in a neighbourhood of A_0.

We emphasise that boundedness of the Hardy projections $E_A^\pm$ alone does not show that (Neu-A) and (Reg-A) are well posed. In our framework, the boundary value problems being well posed means that the Hardy space $E_A^+\hat{\mathcal{H}}^1$, which exists as a closed subspace when the Hardy projections are bounded, is transversal to the A-tangential and normal subspaces $N_A^\pm\hat{\mathcal{H}}^1$. A concrete example showing that (Neu-A) and (Reg-A) may fail to be well posed, even though E_A is bounded, is furnished by the matrix

$$A(x) = \begin{bmatrix} 1 & k\,\mathrm{sgn}(x) \\ -k\,\mathrm{sgn}(x) & 1 \end{bmatrix},$$

with parameter $k \in \mathbf{R}$. The corresponding elliptic equation (1.1) in $\mathbf{R}^2_+$ was studied by Kenig, Koch, Pipher and Toro in [22], where they showed that (Dir-A) fails to be well posed for certain values of k. Moreover, that (Neu-A) and (Reg-A) also fails for some k, is shown by Kenig and Rule [21]. On the other hand, $\|E_A\| = 1$ for all $k \in \mathbf{R}$ since according to (1.5)

$$T_A = \begin{bmatrix} k(\mathrm{sgn}(x)\frac{d}{dx} - \frac{d}{dx}\,\mathrm{sgn}(x)) & \frac{d}{dx} \\ -\frac{d}{dx} & 0 \end{bmatrix}$$

is self-adjoint, and therefore $E_A^\pm$ are orthogonal projections.

1.2. Embedding in a Dirac equation

Unfortunately there is a technical problem in applying harmonic analysis to the operator T_A in order to prove (i′): the space $\hat{\mathcal{H}}^1$ is defined through the non-local condition $\mathrm{curl}_x(f_\parallel) = 0$. This prevents us from using multiplication operators, for example when localising with a cut-off $f \mapsto \eta f$, as these does not preserve $\hat{\mathcal{H}}^1$. To avoid this problem we embed $\hat{\mathcal{H}}^1 \subset \mathcal{H} := L_2(\mathbf{R}^n; \bigwedge_{\mathbf{C}} \mathbf{R}^{n+1})$, where

$$\bigwedge\nolimits_{\mathbf{C}} \mathbf{R}^{n+1} = \bigwedge\nolimits^0 \oplus \bigwedge\nolimits^1 \oplus \bigwedge\nolimits^2 \oplus \cdots \oplus \bigwedge\nolimits^{n+1}$$

is the full complex exterior algebra of $\mathbf{R}^{n+1}$, which in particular contains the vectors $\bigwedge^1 = \mathbf{C}^{n+1}$ and the scalars $\bigwedge^0 = \mathbf{C}$ along with all k-vectors $\bigwedge^k$. (We identify k-vectors with the dual k-forms in Euclidean space.) In this way we obtain closure, i.e. all operators, including multiplication operators preserve $\mathcal{H}$. Furthermore we embed Eq. (1.2) in a Dirac type equation

$$\big(d_{t,x} + \widetilde{B}(x)^{-1} d^*_{t,x} B(x)\big) F(t,x) = 0, \tag{1.7}$$

where $B, \widetilde{B} \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge_{\mathbf{C}} \mathbf{R}^{n+1}))$ are accretive and $B|_{\bigwedge^1} = A$. Here $d_{t,x}$ and $d^*_{t,x}$ are the exterior and interior derivative operators, as defined in Section 2. In particular, if $F(t,x) : \mathbf{R}^{n+1}_+ \to \bigwedge^1$ is a vector field satisfying (1.2) then

$$d^*_{t,x}(BF) = -\operatorname{div}_{t,x}(AF) = 0,$$
$$d_{t,x}F = \operatorname{curl}_{t,x} F = 0.$$

Thus (1.7) follows from (1.2) for any choice of the auxiliary function $\widetilde{B}$. In the same way as in Section 1.1 we shall solve for the vertical derivative in (1.7) and obtain an operator T_B acting in $\mathcal{H}$, such that $T_A = T_B|_{\hat{\mathcal{H}}^1}$. For applications to the Neumann and regularity problems it suffices to consider $B = I \oplus A \oplus I \oplus \cdots \oplus I$. The stability result for the operator T_A we prove in this paper is the following.

Theorem 1.3. *Let A_0 be a t-independent, complex, accretive coefficient matrix function such that T_{B_0} has quadratic estimates in $\mathcal{H}$, where $B_0 = I \oplus A_0 \oplus I \oplus \cdots \oplus I$. Then there exists $\varepsilon_0 > 0$ depending only on $\|A_0\|_\infty$, κ_{A_0} and n, such that if $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{n+1}))$ is t-independent and satisfies $\|A - A_0\|_\infty < \varepsilon_0$, then T_B has quadratic estimates in $\mathcal{H}$, where $B = I \oplus A \oplus I \oplus \cdots \oplus I$. In particular T_A has quadratic estimates in $\hat{\mathcal{H}}^1$.*

Thus $\hat{\mathcal{H}}^1$ splits into Hardy subspaces, the spectral subspaces of T_A, i.e. each $f \in \hat{\mathcal{H}}^1$ can be uniquely written $f = f^+ + f^-$, where $f^\pm = F^\pm|_{\mathbf{R}^n}$ and $F^\pm(t,x) = e^{\mp t|T_A|} E^\pm_A f(x)$ satisfy (1.2) in $\mathbf{R}^{n+1}_\pm$. Moreover, we have equivalence of the norms

$$\|f\|_2 \approx \|f^+\|_2 + \|f^-\|_2$$

and $\|f^\pm\|_2 \approx \sup_{t>0} \|F^\pm_{\pm t}\|_2 \approx |||t\partial_t F^\pm_{\pm t}||| \approx \|\widetilde{N}_*(F^\pm)\|_2$.

With a more general choice

$$B = B^0 \oplus B^1 \oplus B^2 \oplus \cdots \oplus B^{n+1},$$

where $B^k \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$, we also obtain new perturbation results concerning boundary value problems and more generally transmission problems for k-vector fields. For k-vector fields we consider the function space

$$\hat{\mathcal{H}}^k_B := \left\{ f \in L_2\Big(\mathbf{R}^n; \mathcal{L}\big(\textstyle\bigwedge^k\big)\Big);\ d_x(f_\|) = 0 = d^*_x\big(\big(B^k f\big)_\perp\big)\right\} \subset \mathcal{H},$$

where $f_\|$ and $f_\perp$ denote the tangential and normal parts of f. This is the appropriate function space since traces of k-vector fields $F(t,x)$ in $\mathbf{R}^{n+1}_\pm$ satisfying

$$\begin{cases} d^*_{t,x}\big(B^k(x)F(t,x)\big) = 0, \\ d_{t,x}F(t,x) = 0, \end{cases} \tag{1.8}$$

belong to $\hat{\mathcal{H}}^k_B$. Note that for 1-vectors, i.e. vectors, the condition $d^*_x((B^1 f)_\perp) = 0$ is void and $\hat{\mathcal{H}}^1_B = \hat{\mathcal{H}}^1$. For k-vector fields, we consider the following.

Transmission problem (Tr-$B^k\alpha^\pm$). Let $B^k = B^k(x) \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ be accretive and let $\alpha^\pm \in \mathbf{C}$ be given jump parameters. Given a k-vector field $g \in \hat{\mathcal{H}}^k_B$, find k-vector fields $F^+_t(x) = F^+(t,x)$ in $\mathbf{R}^{n+1}_+$ and $F^-_t(x) = F^-(t,x)$ in $\mathbf{R}^{n+1}_-$ such that $F^\pm_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \bigwedge^k))$ and $F^\pm$ satisfies (1.8) for $\pm t > 0$, and furthermore $\lim_{t\to\pm\infty} F^\pm_t = 0$ and $\lim_{t\to 0^\pm} F^\pm_t = f^\pm$ in L_2 norm, where the traces $f^\pm$ satisfy the jump conditions

$$\begin{cases} e_0 \wedge (\alpha^- f^+ - \alpha^+ f^-) = e_0 \wedge g, \\ e_0 \lrcorner (B^k(\alpha^+ f^+ - \alpha^- f^-)) = e_0 \lrcorner (B^k g). \end{cases}$$

The second main result of this paper is the following L_∞ perturbation result of Transmission problem (Tr-$B^k\alpha^\pm$).

Theorem 1.4. *Let $B^k_0 = B^k_0(x) \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ be t-independent, accretive and possibly complex, and assume that B^k_0 is a block matrix, i.e.*

$$(B^k_0)_{st} = 0, \quad \text{whenever } 0 \in (s \setminus t) \cup (t \setminus s)$$

and $s, t \subset \{0, 1, \dots, n\}$ has lengths $|s| = |t| = k$. Then there exist $\varepsilon > 0$ and $C < \infty$ depending only on $\|B^k_0\|_\infty$, the accretivity constant $\kappa_{B^k_0}$ and dimension n, such that if $B^k = B^k(x) \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ is t-independent and satisfies

$$\|B^k - B^k_0\|_{L_\infty(\mathbf{R}^n)} < \min(\varepsilon, C|(\alpha^+/\alpha^-)^2 + 1|), \tag{1.9}$$

then Transmission problem (Tr-$B^k\alpha^\pm$) *above is well posed in the sense that for every boundary function $g \in \hat{\mathcal{H}}^k_B$, there exist unique k-vector fields $F^\pm(t,x)$ with properties as in* (Tr-$B^k\alpha^\pm$). *The solution $F^\pm_t$ depends continuously on g with equivalences of norms $\|g\|_2 \approx \|f^+ + f^-\|_2 \approx \|f^+\|_2 + \|f^-\|_2$ and*

$$\|f^\pm\|_2 \approx \sup_{t>0} \|F^\pm_{\pm t}\|_2 \approx |||t\partial_t F^\pm_{\pm t}|||.$$

This perturbation theorem for transmission problems for k-vectors has two important corollaries. On one hand it specialises when $k = 1$ to a generalisation of Theorem 1.1(b), giving perturbation results for transmission problems across $\mathbf{R}^n$ for the divergence form equation (1.1). The details of this Neumann-regularity transmission problem is stated as (Tr-$A\alpha^\pm$) in Section 4.3.

On the other hand it specialises when either $\alpha^+ = 0$ or $\alpha^- = 0$ to a generalisation of Theorem 1.1(b), giving perturbation results for boundary-value problems for k-vectors. Our result for these boundary-value problems (Nor-B^k) and (Tan-B^k) is given as Corollary 4.17 in Section 4.3.

Example 1.5. In the case $\widetilde{B} = B$, operators of the form $d_{t,x} + B^{-1}d^*_{t,x}B$ appear naturally when pulling back the unperturbed Hodge–Dirac operator $d + d^*$ with a change of variables. As above, consider the region Ω above a Lipschitz graph $\Sigma = \{(t,x);\ t = g(x)\}$. We define the pullback of the field $F : \Omega \to \bigwedge \mathbf{R}^{n+1}$ to be the field

$$(\rho^* F)(t,x) := \underline{\rho}^{\mathrm{T}}(x) F(\rho(t,x))$$

in $\mathbf{R}^{n+1}_+$, where $\rho(t,x) = (t + g(x), x)$ is the parametrisation of Ω, having differential

$$\underline{\rho}(x)|_{\bigwedge^1} = \begin{bmatrix} 1 & \nabla g(x) \\ 0 & I \end{bmatrix}$$

acting on vector fields and extended naturally to $\bigwedge \mathbf{R}^{n+1}$, and $\underline{\rho}^{\mathrm{T}}(x)$ denotes the transposed matrix. From the well-known fact that $d_{t,x}$ commutes with ρ^*, we get the intertwining relation

$$\big(d_{t,x} + G d^*_{t,x} G^{-1}\big)(\rho^* F) = \rho^*\big(\big(d_{t,x} + d^*_{t,x}\big)F\big), \tag{1.10}$$

where $G(x) = (g_{ij}(x)) = \underline{\rho}^{\mathrm{T}}(x)\underline{\rho}(x)$ is the metric for the parametrisation, being real symmetric. Solving for the vertical derivative in the equation $(d_{t,x} + d^*_{t,x})F = 0$ in Ω, gives us an operator $\mathbf{D}_\Sigma$ in $L_2(\Sigma; \bigwedge \mathbf{R}^{n+1})$, which is similar to the operator $(e_0 - \nabla g(x))^{-1}(d_x + d^*_x)$ in $\mathcal{H}$. From (1.10) it follows that

$$T_{G^{-1}}\big(\underline{\rho}(x)^{\mathrm{T}} f(x)\big) = \underline{\rho}(x)^{\mathrm{T}}(\mathbf{D}_\Sigma f)(x).$$

It is known that the operator $\mathbf{D}_\Sigma$ satisfies quadratic estimates, and therefore so does $T_{G^{-1}}$. For references and further discussion of $\mathbf{D}_\Sigma$, see [10, Consequence 3.6]. From this we get the bounded Clifford–Cauchy singular integral operator

$$\begin{aligned} &E_{G^{-1}}\big(\underline{\rho}(x)^{\mathrm{T}} f(x)\big) \\ &\quad = \underline{\rho}(x)^{\mathrm{T}} \frac{2}{\sigma_n} \text{p.v.} \int_{\mathbf{R}^n} \frac{(g(x)e_0 + x) - (g(y)e_0 + y)}{(|y-x|^2 + (g(y) - g(x))^2)^{(n+1)/2}} \big(e_0 - \nabla g(y)\big) f(y)\, dy, \end{aligned}$$

where σ_n is the area of the unit n-sphere in $\mathbf{R}^{n+1}$ and $E_{G^{-1}} = \operatorname{sgn}(T_{G^{-1}})$. For this reason, we shall refer to $E_B = \operatorname{sgn}(T_B)$ as generalised Cauchy integral operators and $E^{\pm}_B = \chi_\pm(T_B)$ as generalised Hardy projection operators, also when $B \neq G^{-1}$ and $\widetilde{B} \neq B$.

1.3. Outline of the paper

In Section 2 we explain how we use the exterior algebra $\bigwedge_{\mathbf{C}} \mathbf{R}^{n+1}$ and the exterior and interior derivative operators d and d^*. In Section 2.1 we introduce the Dirac type operator T_B which extends T_A to the full exterior algebra as well as projection operators $N^{\pm}_B$ which extend the A-tangential and normal projections $N^{\pm}_A$ from above. Section 2.2 is concerned with the spectral properties of T_B, where we prove that T_B is a bisectorial operator and that the resolvents $(\lambda I - T_B)^{-1}$ has L_2 off-diagonal estimates. Section 2.3 surveys the theory of functional calculus of bisectorial operators like T_B. In Section 2.5, Lemmas 2.49 and 2.55 characterise the classes of solutions F_t and U_t, respectively, to the boundary value problems, and are used in particular to prove uniqueness.

In Section 3 we prove (i′) quadratic estimates and (ii′) invertibility in the unperturbed case $B = B_0$. The most involved case is when B_0 is real symmetric. In order to prove the quadratic estimates we use the results from Section 2.4 that T_{B_0} leaves certain subspaces $\hat{\mathcal{H}}^k_{B_0}$ and $\check{\mathcal{H}}_{B_0}$ invariant and that therefore it suffices to establish quadratic estimates in each subspace separately.

Section 4 contains the core harmonic analysis results of the paper, where we establish quadratic estimates for T_B when $B \approx B_0$ through a perturbation argument based on Eqs. (4.3)–(4.8). Section 4.1 treats the case when B_0 is a block matrix and makes use of techniques from the solution of the Kato square root problem [4]. In Lemma 4.10 we construct a new set of test functions f_Q^w which also can be used in the Carleson measure estimate in [10] to simplify the proof there. Section 4.2 treats matrices of the form $B = I \oplus A \oplus I \oplus \cdots \oplus I$. The key technique is Lemma 4.14 where we compare B_0 with a corresponding block matrix $\hat{B}_0$ for which the results from Section 4.1 applies. The reason why this approximation $B_0 \approx \hat{B}_0$ works is that the normal vector component $F^{1,0}$ has additional regularity by Lemma 2.12 when F satisfy the Dirac type equation (1.7).

The paper ends with Section 4.3, where we bring the results together and prove Theorems 1.4, 1.1 and 1.3.

2. Operator theory and algebra

In this section we develop the operator theoretic framework we use to prove the perturbation theorems stated in the introduction. In particular we introduce our basic operator T_B, along with perturbations of the normal and tangential projection operators N^- and N^+. These all act in the Hilbert space $\mathcal{H} := L_2(\mathbf{R}^n; \bigwedge_{\mathbf{C}} \mathbf{R}^{n+1})$ on the boundary $\mathbf{R}^n = \partial\mathbf{R}^{n+1}_+ = \partial\mathbf{R}^{n+1}_-$. In $\mathbf{R}^{n+1}$ we write the standard ON basis as $\{e_0, e_1, \ldots, e_n\}$, where e_0 denote the vertical direction and $e_1, \ldots, e_n$ span the horizontal hyperplane $\mathbf{R}^n$. We write the corresponding coordinates as $x_0, x_1, \ldots, x_n$ and we also use the notation $t = x_0$. The corresponding partial derivatives we write as $\partial_i = \frac{\partial}{\partial x_i}$ and $\partial_t = \partial_0 = \frac{\partial}{\partial t}$. Our functions $f \in \mathcal{H}$ take values in the full complex exterior algebra over $\mathbf{R}^{n+1}$

$$\bigwedge = \bigwedge_{\mathbf{C}} \mathbf{R}^{n+1} = \bigwedge^0 \oplus \bigwedge^1 \oplus \cdots \oplus \bigwedge^{n+1}.$$

This is a 2^{n+1}-dimensional linear space with $n+2$ pairwise orthogonal subspaces $\bigwedge^k$ of dimensions $\binom{n+1}{k}$. With the notation $e_s := e_{s_1} \wedge \ldots \wedge e_{s_k}$ if $s = \{s_1, \ldots, s_k\} \subset \{0, 1, \ldots, n\}$ and $s_1 < s_2 < \cdots < s_k$, the space $\bigwedge^k$ of homogeneous *k-vectors* is the linear span of $\{e_s;\ |s| = k\}$. In particular, identifying $e_\emptyset$ with 1 and the singleton set $\{j\}$ with j, we have $\bigwedge^0 = \mathbf{C}$ and $\bigwedge^1 = \mathbf{C}^{n+1}$. A general element in $\bigwedge$ is called a *multivector* and is a direct sum of k-vectors of different degrees k.

Definition 2.1. Introduce the sesqui-linear scalar product

$$(f, g) = \left(\sum_s f_s e_s, \sum_t g_t e_t\right) = \sum_s f_s \overline{g}_s,$$

on $\bigwedge$ and the bilinear scalar product $f \cdot g = \sum_s f_s g_s$. Define the counting function $\sigma(s,t) := \#\{(s_i, t_j);\ s_i > t_j\}$, where $s = \{s_i\}$, $t = \{t_j\} \subset \{0, 1, \ldots, n\}$.

(i) The *exterior product* $f \wedge g$ is the complex bilinear product for which

$$e_s \wedge e_t = (-1)^{\sigma(s,t)} e_{s \cup t} \text{ if } s \cap t = \emptyset \quad \text{and} \quad \text{zero otherwise.}$$

(ii) The (left) *interior product* $f \lrcorner g$ is the complex bilinear product for which $(e_s \lrcorner e_t, e_u) = (e_t, e_s \wedge e_u)$ for all $s, t, u \subset \{0, 1, \ldots, n\}$. Explicitly we have

$$e_s \lrcorner e_t = (-1)^{\sigma(s,t\setminus s)}\, e_{t\setminus s} \text{ if } s \subset t \quad \text{and} \quad \text{zero otherwise.}$$

Example 2.2. The most common situation is when forming a product between a vector $a = \sum_{j=0}^{n} a_j e_j \in \bigwedge^1$ and a k-vector

$$f = \sum_{0 \leqslant s_1 < \cdots < s_k \leqslant n} f_{s_1,\ldots,s_k} e_{s_1} \wedge \ldots \wedge e_{s_k}.$$

In this case the interior product $a \lrcorner f$ is a $(k-1)$-vector, whereas the exterior product $a \wedge f$ yields a $(k+1)$-vector. Note also that we embed all different spaces $\bigwedge^k$ of homogeneous k-vectors as pairwise orthogonal subspaces in the 2^{n+1}-dimensional linear space $\bigwedge$. Thus we may add the two products to obtain (the Clifford product)

$$a \lrcorner f + a \wedge f \in \textstyle\bigwedge^{k-1} \oplus \bigwedge^{k+1} \subset \bigwedge.$$

In the special case when $f = b$ is also a vector, i.e. $k = 1$, we have

$$a \lrcorner b = \sum_{j=0}^{n} a_j b_j \in \textstyle\bigwedge^0,$$

$$a \wedge b = \sum_{0 \leqslant i < j \leqslant n} (a_i b_j - a_j b_i) e_i \wedge e_j \in \textstyle\bigwedge^2,$$

where we write $b_{\{j\}} = b_j$. We see that $a \lrcorner b$ coincide with the bilinear scalar product $a \cdot b$. Furthermore, in three dimensions $n+1 = 3$, the exterior product $a \wedge b \in \bigwedge^2$ can be identified with the vector product $a \times b \in \bigwedge^1$ by using the Hodge star identifications $e_{\{1,2\}} \approx e_0$, $-e_{\{0,2\}} \approx e_1$ and $e_{\{0,1\}} \approx e_2$.

The following anticommutativity, associativity and derivation properties of these products summarise the fundamental algebra we shall need in this paper.

Lemma 2.3. *If $a, b \in \bigwedge^1$ are vectors and $f, g, h \in \bigwedge$, then*

$$a \wedge b = -b \wedge a, \qquad a \wedge a = 0,$$

$$f \wedge (g \wedge h) = (f \wedge g) \wedge h, \qquad f \lrcorner (g \lrcorner h) = (g \wedge f) \lrcorner h,$$

$$a \lrcorner (b \wedge f) = (a \cdot b) f - b \wedge (a \lrcorner f).$$

We shall also frequently use that if $a \in \bigwedge^1$ is a real vector, then $(a \lrcorner f, g) = (f, a \wedge g)$.

Proof. That $a \wedge b = -b \wedge a$ and $a \wedge a = 0$ is readily seen from Example 2.2. These and the associativity $f \wedge (g \wedge h) = (f \wedge g) \wedge h$ are well-known properties of the exterior product. To see how $f \lrcorner (g \lrcorner h) = (g \wedge f) \lrcorner h$ follows, note first that by linearity it suffices to consider the case when f, g and h all are real, i.e. have real coefficients in the standard basis $\{e_s\}$. Under this assumption we pair with an arbitrary $w \in \bigwedge$ and use that left interior and exterior multiplication are adjoint operations. We get

$$\big(f \lrcorner (g \lrcorner h), w\big) = (g \lrcorner h, f \wedge w) = \big(h, g \wedge (f \wedge w)\big)$$
$$= \big(h, (g \wedge f) \wedge w\big) = \big((g \wedge f) \lrcorner h, w\big).$$

For the derivation identity, by linearity it suffices to prove

$$e_i \lrcorner (e_j \wedge e_s) + e_j \wedge (e_i \lrcorner e_s) = \begin{cases} e_s, & i = j, \\ 0, & i \neq j, \end{cases}$$

for all $s \subset \{0, 1, \ldots, n\}$. This is straightforward to verify from Definition 2.1. □

Definition 2.4. For a multivector $f \in \bigwedge$, we write $\mu f := e_0 \wedge f$, $\mu^* f := e_0 \lrcorner f$ and $mf := e_0 \wedge f + e_0 \lrcorner f = (\mu + \mu^*) f$. We call

$$f_\perp = N^- f := \mu\mu^* f = m\mu^* f = \mu m f,$$
$$f_\parallel = N^+ f := \mu^*\mu f = m\mu f = \mu^* m f,$$

the *normal* and *tangential* parts of f, respectively.

Concretely, if $f = \sum_{\{s_1,\ldots,s_k\}} f_{s_1,\ldots,s_k} e_{s_1} \wedge \ldots \wedge e_{s_k}$, then its normal part is

$$f_\perp = \sum_{\{s_1,\ldots,s_k\} \ni 0} f_{s_1,\ldots,s_k} e_{s_1} \wedge \ldots \wedge e_{s_k}$$

and its tangential part is

$$f_\parallel = \sum_{\{s_1,\ldots,s_k\} \not\ni 0} f_{s_1,\ldots,s_k} e_{s_1} \wedge \ldots \wedge e_{s_k}.$$

In particular e_s is normal if $0 \in s$ and tangential if $0 \notin s$. Note that both the subspace of normal multivectors $N^- \bigwedge$ and the subspace of tangential multivectors $N^+ \bigwedge$ have dimension 2^n, although the subspaces $N^\pm \bigwedge^k$ have different dimensions in general.

Throughout this paper, upper case letters denote $\bigwedge$-valued functions $F(t, x) = F_t(x)$ in the domain $\mathbf{R}^{n+1}_+$ or $\mathbf{R}^{n+1}_-$ whereas lower case letters will denote $\bigwedge$-valued functions f on the boundary $\mathbf{R}^n$. We use the sesqui-linear scalar product $(f, g) := \int_{\mathbf{R}^n} (f(x), g(x))\, dx$ on the Hilbert space $\mathcal{H}$.

Definition 2.5. Using the nabla symbols $\nabla_x = \nabla = \sum_{j=1}^n e_j \partial_j$ in $\mathbf{R}^n$ and $\nabla_{t,x} = \sum_{j=0}^n e_j \partial_j$ in $\mathbf{R}^{n+1}$, we define the operators of *exterior* and *interior* derivation as

$$d_x f = df = \nabla \wedge f := \sum_{j=1}^{n} e_j \wedge \partial_j f, \qquad d_x^* f = d^* f = -\nabla \lrcorner f := -\sum_{j=1}^{n} e_j \lrcorner \partial_j f,$$

$$d_{t,x} F = \nabla_{t,x} \wedge F := \sum_{j=0}^{n} e_j \wedge \partial_j F, \qquad d_{t,x}^* F = -\nabla_{t,x} \lrcorner F := -\sum_{j=0}^{n} e_j \lrcorner \partial_j F.$$

We shall also find it convenient to use the operators $\underline{d} := imd$ and $\underline{d}^* = -id^*m = imd^*$.

Remark 2.6. Clearly, in the splitting $\bigwedge = \bigwedge^0 \oplus \bigwedge^1 \oplus \cdots \oplus \bigwedge^{n+1}$, the operators d and d^* map $d : L_2(\mathbf{R}^n; \bigwedge^k) \to L_2(\mathbf{R}^n; \bigwedge^{k+1})$ and $d^* : L_2(\mathbf{R}^n; \bigwedge^k) \to L_2(\mathbf{R}^n; \bigwedge^{k-1})$ as unbounded operators. Moreover, if we further decompose the space of homogeneous k-vectors into its normal and tangential subspaces as

$$\bigwedge = \bigwedge_{\parallel}^0 \oplus \big(\bigwedge_{\perp}^1 \oplus \bigwedge_{\parallel}^1\big) \oplus \big(\bigwedge_{\perp}^2 \oplus \bigwedge_{\parallel}^2\big) \oplus \cdots \oplus \big(\bigwedge_{\perp}^n \oplus \bigwedge_{\parallel}^n\big) \oplus \bigwedge_{\perp}^{n+1},$$

where $\bigwedge_{\perp}^k := N^- \bigwedge^k$ and $\bigwedge_{\parallel}^k := N^+ \bigwedge^k$, then we see that the operators $\underline{d}$ and $\underline{d}^*$ map

$$\underline{d} : L_2\big(\mathbf{R}^n; \bigwedge_{\perp}^k\big) \to L_2\big(\mathbf{R}^n; \bigwedge_{\parallel}^k\big), \qquad \underline{d} : L_2\big(\mathbf{R}^n; \bigwedge_{\parallel}^k\big) \to L_2\big(\mathbf{R}^n; \bigwedge_{\perp}^{k+2}\big),$$
$$\underline{d}^* : L_2\big(\mathbf{R}^n; \bigwedge_{\parallel}^k\big) \to L_2\big(\mathbf{R}^n; \bigwedge_{\perp}^k\big), \qquad \underline{d}^* : L_2\big(\mathbf{R}^n; \bigwedge_{\perp}^k\big) \to L_2\big(\mathbf{R}^n; \bigwedge_{\parallel}^{k-2}\big).$$

Lemma 2.7. *We have, on appropriate domains,* $(d_{t,x})^2 = (d_{t,x}^*)^2 = d^2 = (d^*)^2 = \mu^2 = (\mu^*)^2 = 0$ *and the anti-commutation relations*

$$\{m, d_{t,x}\} = \partial_t = \big\{m, -d_{t,x}^*\big\}, \qquad m^2 = \{\mu, \mu^*\} = I,$$
$$\{d, d^*\} = -\Delta = -\sum_1^n \partial_k^2, \qquad \{d, \mu\} = \{d, \mu^*\} = 0,$$

where $\{A, B\} = AB + BA$ *denotes the anticommutator.*

Proof. The proofs are straightforward, using Lemma 2.3. Let us prove that $\partial_t = \{m, -d_{t,x}^*\}$. For a function $F(t, x)$ we have

$$\nabla_{x,t} \lrcorner (mF) = \sum_0^n e_i \lrcorner (m\partial_i F) = \sum_0^n e_i \lrcorner (e_0 \lrcorner \partial_i F) + \sum_0^n e_i \lrcorner (e_0 \wedge \partial_i F).$$

Using the anticommutativity and associative properties, the first sum is

$$\sum_0^n (e_0 \wedge e_i) \lrcorner \partial_i F = -\sum_0^n (e_i \wedge e_0) \lrcorner \partial_i F = -e_0 \lrcorner \left(\sum_0^n e_i \lrcorner \partial_i F\right),$$

whereas using the derivation property, the second sum is

$$\sum_0^n e_i \lrcorner (e_0 \wedge \partial_i F) = \partial_0 F - e_0 \wedge \left(\sum_0^n e_i \lrcorner \partial_i F \right).$$

Adding up we obtain $\nabla_{x,t} \lrcorner (mF) = \partial_t F - m(\nabla_{x,t} \lrcorner F)$. $\square$

2.1. The basic operators

Definition 2.8. Throughout this paper we denote by $B \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ a bounded, accretive and complex matrix function acting on $f \in \mathcal{H}$ as $f(x) \mapsto B(x)f(x)$, with quantitative bounds $\|B\|_\infty$ and $\kappa_B > 0$, where κ_B is the largest constant such that

$$\operatorname{Re}\big(B(x)w, w\big) \geqslant \kappa_B |w|^2, \quad \text{for all } w \in \textstyle\bigwedge,\ x \in \mathbf{R}^n.$$

We shall also assume that B is of the form $B = B^0 \oplus B^1 \oplus B^2 \oplus \cdots \oplus B^{n+1}$, where $B^k \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$, so that B preserve the space of k-vectors. For the matrix part B^1 acting on vectors, we use the alternative notation $A = B^1$. We also define $\widetilde{B} := mBm$. It is not true in general that $\widetilde{B}$ preserve k-vectors.

Remark 2.9. It would be more optimal to replace the quantitative bounds $\|B\|_\infty$ and $\kappa_B > 0$ with K_B and k_B, where K_B and k_B are the optimal constants such that

$$k_B \|f\|^2 \leqslant \big|(Bf, f)\big| \leqslant K_B \|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

We consider $F(t, x)$ satisfying the Dirac type equation

$$\big(md_{t,x} + B^{-1} m d^*_{t,x} B\big) F(t, x) = 0 \tag{2.1}$$

or equivalently

$$\big(d_{t,x} + \widetilde{B}^{-1} d^*_{t,x} B\big) F(t, x) = 0. \tag{2.2}$$

In order to solve for the vertical derivative $\partial_t F$, we note that

$$d_{t,x} F = dF + \mu \partial_t F, \qquad d^*_{t,x} F = d^* F - \mu^* \partial_t F.$$

Inserted in (2.1) this yields

$$\partial_t F + \tfrac{1}{N^+ - B^{-1} N^- B} \big(md + B^{-1} m d^* B\big) F = 0.$$

Definition 2.10. Write $M_B := N^+ - B^{-1} N^- B$ and define the unbounded operator

$$T_B := M_B^{-1}\big(md + B^{-1} m d^* B\big) = -i M_B^{-1}\big(\underline{d} + B^{-1} \underline{d}^* B\big) = \tfrac{1}{\mu - \widetilde{B}^{-1} \mu^* B}\big(d + \widetilde{B}^{-1} d^* B\big)$$

in $\mathcal{H}$ with domain $\mathsf{D}(T_B) := \mathsf{D}(d) \cap B^{-1} \mathsf{D}(d^*)$.

A rather surprising fact is that the most obvious choice for $\widetilde{B}$, namely $\widetilde{B} = B$ is not the best, but rather $\widetilde{B} = mBm$. For example, this is the only choice for which a Rellich type formula, as in Proposition 3.8, holds on all $\mathcal{H}$, in the case of Hermitean coefficients B.

Note that T_B is closely related to operators of the form $\Pi_B = \Gamma + B^{-1}\Gamma^* B$, where Γ denotes a first order, homogeneous partial differential operator with constant coefficients such that $\Gamma^2 = 0$, which were studied in [10]. Unfortunately, the factor M_B^{-1} does not commute with Π_B for general B. However, it has other useful commutation properties.

Lemma 2.11. *The operator M_B is an isomorphism and*

$$\big(B^{-1}N^-B\big)M_B^{-1} = M_B^{-1}N^-, \qquad N^-M_B^{-1} = M_B^{-1}\big(B^{-1}N^-B\big),$$
$$\big(B^{-1}N^+B\big)M_B^{-1} = M_B^{-1}N^+, \qquad N^+M_B^{-1} = M_B^{-1}\big(B^{-1}N^+B\big).$$

Proof. Note that $M_B = B^{-1}(BN^+ - N^-B)$, where the last factor is the diagonal matrix $(-B_{\perp\perp}) \oplus B_{\|\|}$ in the splitting $\mathcal{H} = N^-\mathcal{H} \oplus N^+\mathcal{H}$, if $B_{\perp\perp}$ and $B_{\|\|}$ denote the diagonal blocks of B, and thus $BN^+ - N^-B$ commutes with $N^\pm$. Furthermore, the diagonal blocks are accretive, so $BN^+ - N^-B$ and thus M_B is invertible. This proves the two equations to the right. To obtain the equations to the left, replace B by B^{-1} and note that

$$-N^- + B^{-1}N^+B = N^+ - B^{-1}N^-B. \qquad \square$$

Let us comment on the terminology "Dirac type equation" for (2.1). Normally this denotes a first-order differential operator, like $d_{t,x} + d_{t,x}^*$, whose square acts componentwise as the Laplace operator. In our situation, the following holds.

Lemma 2.12. *If $F(t,x)$ satisfies* (2.1)*, then $d_{t,x}^* B d_{t,x}(mF) = 0$. In particular it follows that*

$$\operatorname{div}_{t,x} A\nabla_{t,x} F^{1,0} = 0$$

if $F = F^0 + (F^{1,0}e_0 + F^{1,\|}) + F^2 + \cdots + F^{n+1}$, i.e. $F^{1,0}$ is the normal component of the vector part of F.

Proof. We use the anticommutation relations $md_{t,x} = \partial_t - d_{t,x}m$ and $md_{t,x}^* = -\partial_t - d_{t,x}^* m$ from Lemma 2.7, which shows that (2.1) is equivalent with

$$\big(d_{t,x}m + B^{-1}d_{t,x}^* mB\big)F(t,x) = 0,$$

since $\partial_t - B^{-1}\partial_t B = 0$. Applying $d_{t,x}^* B$ to this equation shows that $d_{t,x}^* B d_{t,x}(mF) = 0$, and evaluating the scalar $\bigwedge^0$ part shows that $\operatorname{div}_{t,x} A\nabla_{t,x} F^{1,0} = 0$ since $d_{t,x}^* B d_{t,x}$ preserves k-vectors. $\square$

The following notions are central in our operator theoretic framework.

Definition 2.13. Let $\mathcal{H}$ be a Hilbert space.

(i) A (*topological*) *splitting* of $\mathcal{H}$ is a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$. In particular, we have $\|f_1 + f_2\| \approx \|f_1\| + \|f_2\|$ if $f_i \in \mathcal{H}_i$.
(ii) Two bounded operators R^+ and R^- in $\mathcal{H}$ are called *complementary projections* if $R^+ + R^- = I$, $(R^\pm)^2 = R^\pm$ and $R^\pm R^\mp = 0$.
(iii) A bounded operator R in $\mathcal{H}$ is called a *reflection operator* if $R^2 = I$.

We note the following connection between these concepts.

Lemma 2.14. *There is a one-to-one correspondence*

$$R = R^+ - R^- \longleftrightarrow R^\pm = \tfrac{1}{2}(I \pm R) \longleftrightarrow \mathcal{H} = R^+\mathcal{H} \oplus R^-\mathcal{H}$$

between reflection operators in $\mathcal{H}$, complementary projections in $\mathcal{H}$ and topological splittings of $\mathcal{H}$. We write $R^\pm\mathcal{H} = \mathsf{R}(R^\pm)$ for the range of the projection $R^\pm$.

In Definition 2.4 we introduced the complementary projections $N^\pm$ associated with the splitting of $\mathcal{H}$ into the subspaces of tangential and normal multivector fields. The corresponding reflection operator is

$$N := N^+ - N^- = \mu^*\mu - \mu\mu^* = (\mu^* - \mu)m = m(\mu - \mu^*).$$

We also introduce B-perturbed versions of the tangential and normal subspaces $N^-\mathcal{H}$ and $N^+\mathcal{H}$ as

$$B^{-1}N^+\mathcal{H} := \{B^{-1}f;\ f \in N^+\mathcal{H}\},$$
$$B^{-1}N^-\mathcal{H} := \{B^{-1}f;\ f \in N^-\mathcal{H}\}.$$

In Definition 2.10 we encountered one of the complementary projections $B^{-1}N^+B$ and $B^{-1}N^-B$ associated with the splitting

$$\mathcal{H} = B^{-1}N^+\mathcal{H} \oplus B^{-1}N^-\mathcal{H}.$$

However, more important will be the following complementary projections.

Definition 2.15. Let $\hat{N}_B^+$ and $\hat{N}_B^-$ be the complementary projections associated with the splitting

$$\mathcal{H} = B^{-1}N^+\mathcal{H} \oplus N^-\mathcal{H}.$$

We sometimes use the shorter notation $N_B^+ := \hat{N}_B^+$ and $N_B^- := \hat{N}_B^-$. Also let $\check{N}_B^+$ and $\check{N}_B^-$ be the complementary projections associated with the splitting

$$\mathcal{H} = N^+\mathcal{H} \oplus B^{-1}N^-\mathcal{H}.$$

Let $N_B = \hat{N}_B := \hat{N}_B^+ - \hat{N}_B^-$ and $\check{N}_B := \check{N}_B^+ - \check{N}_B^-$ be the associated reflection operators.

With the notation $\mu_B^* := B^{-1}\mu^* B$, these operators are

$$\hat{N}_B^+ = \mu_B^*\big(\mu + \mu_B^*\big)^{-1} = \big(\mu + \mu_B^*\big)^{-1}\mu,$$
$$\hat{N}_B^- = \mu\big(\mu + \mu_B^*\big)^{-1} = \big(\mu + \mu_B^*\big)^{-1}\mu_B^*,$$
$$\hat{N}_B = \big(\mu_B^* - \mu\big)\big(\mu + \mu_B^*\big)^{-1} = \big(\mu + \mu_B^*\big)^{-1}\big(\mu - \mu_B^*\big),$$

and similarly for $\check{N}_B^{\pm}$ and $\check{N}_B$. We shall prove in Section 2.2 that all these operators are bounded. It is mainly the operators $\hat{N}_B^+$, $\hat{N}_B^-$ and $\hat{N}_B$ that we shall use.

Definition 2.16. Let $\langle f, g\rangle_B := ((BN^+ - N^-B)f, g)$, for $f, g \in \mathcal{H}$. As $BN^+ - N^-B = BM_B$ is invertible, $\langle\cdot,\cdot\rangle_B$ is a duality, i.e. there exists $C < \infty$ such that

$$\big|\langle f, g\rangle_B\big| \leqslant C\|f\|\|g\|,$$
$$\|f\| \leqslant C \sup_{g\neq 0}\big|\langle f, g\rangle_B\big|/\|g\|, \qquad \|g\| \leqslant C \sup_{f\neq 0}\big|\langle f, g\rangle_B\big|/\|f\|.$$

We write T' for the adjoint of an operator T with respect to this duality, i.e. if $\langle Tf, g\rangle_B = \langle f, T'g\rangle_B$ for all $f, g \in \mathcal{H}$.

Proposition 2.17. *We have adjoint operators $(\hat{N}_B)' = \check{N}_{B^*}$, $(\check{N}_B)' = \hat{N}_{B^*}$, $N' = N$ and $(T_B)' = -T_{B^*}$.*

Proof. To prove that $(\hat{N}_B)' = \check{N}_{B^*}$, recall that $\hat{N}_B$ is the reflection operator for the splitting $\mathcal{H} = B^{-1}N^+\mathcal{H} \oplus N^-\mathcal{H}$ and $\check{N}_{B^*}$ is the reflection operator for the splitting $\mathcal{H} = N^+\mathcal{H} \oplus (B^*)^{-1}N^-\mathcal{H}$. Thus we need to prove that

$$\big\langle B^{-1}N^+f_1 - N^-f_2, N^+g_1 + (B^*)^{-1}N^-g_2\big\rangle_B$$
$$= \big\langle B^{-1}N^+f_1 + N^-f_2, N^+g_1 - (B^*)^{-1}N^-g_2\big\rangle_B,$$

for all f_i, g_i, i.e. that $\langle B^{-1}N^+f_1, (B^*)^{-1}N^-g_2\rangle_B = 0 = \langle N^-f_2, N^+g_1\rangle_B$. Use Lemma 2.11 to obtain

$$\big\langle B^{-1}N^+f_1, (B^*)^{-1}N^-g_2\big\rangle_B = \big(\big(BN^+ - N^-B\big)B^{-1}N^+f_1, (B^*)^{-1}N^-g_2\big)$$
$$= \big(N^-M_B B^{-1}N^+f_1, g_2\big) = \big(M_B B^{-1}N^-N^+f_1, g_2\big) = 0.$$

A similar calculation shows that $\langle N^-f_2, N^+g_1\rangle_B = 0$. The proof of $(\check{N}_B)' = \hat{N}_{B^*}$ is similar. For the unperturbed reflection operator N, the duality $N' = N$ follows directly from the fact the $BN^+ - N^-B$ is diagonal in the splitting $\mathcal{H} = N^+\mathcal{H} \oplus N^-\mathcal{H}$.

To prove $(T_B)' = -T_{B^*}$, we calculate

$$\langle T_B f, g\rangle_B = \big(\big(BN^+ - N^-B\big)M_B^{-1}\big(md + B^{-1}md^*B\big)f, g\big)$$
$$= \big((Bmd + md^*B)f, g\big) = \big(f, (B^*dm + d^*mB^*)g\big)$$

$$= -\big(f, \big(N^+B^* - B^*N^-\big)\big(N^+B^* - B^*N^-\big)^{-1}(B^*md + md^*B^*)g\big)$$
$$= -\big\langle f, \big(N^+ - (B^*)^{-1}N^-B^*\big)^{-1}\big(md + (B^*)^{-1}md^*B^*\big)g\big\rangle_B = \langle f, -T_{B^*}g\rangle_B,$$

where we have used that $(B^*)^{-1}N^+B^* - N^- = N^+ - (B^*)^{-1}N^-B^*$. □

Remark 2.18. Our main use of these dualities is for proving surjectivity. Recall the following standard technique for proving invertibility of a bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$. Assume we have at our disposal two pairs of dual spaces $\langle \mathcal{H}_1, \mathcal{K}_1\rangle_1$ and $\langle \mathcal{H}_2, \mathcal{K}_2\rangle_2$ and that the adjoint operator is $T' : \mathcal{K}_2 \to \mathcal{K}_1$. If we can prove *a priori estimates*

$$\|Tf\| \gtrsim \|f\|, \quad \text{for all } f \in \mathcal{H}_1,$$

then T is injective and has closed range. If furthermore T' is injective, in particular if it satisfies a priori estimates, then T is surjective and therefore an isomorphism.

Remark 2.19. We shall also need to restrict dualities to subspaces. Let $\langle \mathcal{H}, \mathcal{K}\rangle$ be a duality, i.e. let $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{K} \to \mathbf{C}$ satisfy the estimates in Definition 2.16. If $\mathcal{H}_1 \subset \mathcal{H}$ is a subspace, then a subspace $\mathcal{K}_1 \subset \mathcal{K}$ is such that $\langle \mathcal{H}_1, \mathcal{K}_1\rangle$ satisfies estimates as in Definition 2.16 if and only if $\mathcal{K}_1$ is a complementary subspace to the annihilator $\{g \in \mathcal{K};\ \langle f, g\rangle = 0, \text{ for all } f \in \mathcal{H}_1\}$, in the sense of Definition 2.13(i).

In particular, if $R^\pm$ are complementary projections in $\mathcal{H}$, the adjoint operators $(R^\pm)'$ are also complementary projections and the duality $\langle \mathcal{H}, \mathcal{H}\rangle$ restricts to a duality $\langle R^+\mathcal{H}, (R^+)'\mathcal{H}\rangle$, since the annihilator of $R^+\mathcal{H}$ is $\mathsf{N}((R^+)')$, which is complementary to $(R^+)'\mathcal{H}$.

2.2. *Hodge decompositions and resolvent estimates*

In this section, we estimate the spectrum of the operator T_B. For this we make use of Hodge type decompositions of $\mathcal{H}$ as explained below.

Definition 2.20. By a *nilpotent operator* Γ in a Hilbert space $\mathcal{H}$, we mean a closed, densely defined operator such that $\overline{\mathsf{R}(\Gamma)} \subset \mathsf{N}(\Gamma)$. In particular $\Gamma^2 f = 0$ if $f \in \mathsf{D}(\Gamma)$. We say that a nilpotent operator is *exact* if $\overline{\mathsf{R}(\Gamma)} = \mathsf{N}(\Gamma)$.

If $\widetilde{\Gamma}$ is another nilpotent operator, then we say that Γ and $\widetilde{\Gamma}$ are *transversal* if there is a constant $c = c(\Gamma, \widetilde{\Gamma}) < 1$ such that

$$\big|(f, g)\big| \leqslant c\|f\|\|g\|, \quad f \in \overline{\mathsf{R}(\Gamma)},\ g \in \overline{\mathsf{R}(\widetilde{\Gamma})},$$

or equivalently if $\|f + g\| \approx \|f\| + \|g\|$ for all $f \in \overline{\mathsf{R}(\Gamma)}$ and $g \in \overline{\mathsf{R}(\widetilde{\Gamma})}$. Note that any nilpotent operator Γ is transversal to its adjoint Γ^* with $c = 0$, since $\overline{\mathsf{R}(\Gamma)} \subset \mathsf{N}(\Gamma) = \overline{\mathsf{R}(\Gamma^*)}^\perp$.

Below we collect the properties of Hodge type splittings which we need in this paper. This generalises results from [10, Proposition 2.2] and [9, Proposition 3.11].

Lemma 2.21. *Let Γ and $\widetilde{\Gamma}$ be two nilpotent operators in a Hilbert space $\mathcal{H}$ which are exact and transversal with constant $c(\Gamma, \widetilde{\Gamma})$, and assume also that the adjoints Γ^* and $\widetilde{\Gamma}^*$ are transversal. Let B be a bounded, accretive multiplication operator and assume that*

$$\max\bigl(c(\Gamma, \widetilde{\Gamma}), c(\Gamma^*, \widetilde{\Gamma}^*)\bigr) < \kappa_B/\|B\|_\infty.$$

Define operators $\Pi := \Gamma + \widetilde{\Gamma}$, $\widetilde{\Gamma}_B := B^{-1}\widetilde{\Gamma}B$ and $\Pi_B := \Gamma + \widetilde{\Gamma}_B$ with domains $\mathsf{D}(\Pi) := \mathsf{D}(\Gamma) \cap \mathsf{D}(\widetilde{\Gamma})$, $\mathsf{D}(\widetilde{\Gamma}_B) := B^{-1}\mathsf{D}(\widetilde{\Gamma})$ and $\mathsf{D}(\Pi_B) := \mathsf{D}(\Gamma) \cap \mathsf{D}(\widetilde{\Gamma}_B)$, respectively.

(i) *We have a topological splitting*

$$\mathcal{H} = \mathsf{N}(\Gamma) \oplus \mathsf{N}(\widetilde{\Gamma}_B),$$

so that $\|f_1 + f_2\| \approx \|f_1\| + \|f_2\|$, $f_1 \in \mathsf{N}(\Gamma)$, $f_2 \in \mathsf{N}(\widetilde{\Gamma}_B)$. The operator Π_B is a closed operator with dense domain and range. Furthermore $\widetilde{\Gamma}_B$ is an exact nilpotent operator. The complementary Hodge type projections associated with the splitting are

$$\mathbf{P}_B^1 := \Gamma\Pi_B^{-1} = \Pi_B^{-1}\widetilde{\Gamma}_B = \Gamma\Pi_B^{-2}\widetilde{\Gamma}_B,$$
$$\mathbf{P}_B^2 := \widetilde{\Gamma}_B\Pi_B^{-1} = \Pi_B^{-1}\Gamma = \widetilde{\Gamma}_B\Pi_B^{-2}\Gamma,$$

where we identify a bounded, densely defined operator with its bounded extension to $\mathcal{H}$. In the special case when $B = I$ we write $\mathbf{P}^1$ and $\mathbf{P}^2$ for these projections.

(ii) *Let B_1 and B_2 be two bounded, accretive operators in $\mathcal{H}$, with $\kappa_{B_i}/\|B_i\|_\infty > \max(c(\Gamma, \widetilde{\Gamma}), c(\Gamma^*, \widetilde{\Gamma}^*))$, $i = 1, 2$. Then there exists $C < \infty$, depending only on $\|B_i\|_\infty$, κ_{B_i}, $c(\Gamma, \widetilde{\Gamma})$ and $c(\Gamma^*, \widetilde{\Gamma}^*)$, such that*

$$\bigl\|\bigl(\Gamma + B_1^{-1}\widetilde{\Gamma}B_2\bigr)f\bigr\| \geqslant C^{-1}\lambda\|f\|, \quad \textit{for all } f \in \mathsf{D}(\Gamma) \cap B_2^{-1}\mathsf{D}(\widetilde{\Gamma}),$$

if $\|\Pi f\| \geqslant \lambda\|f\|$ for all $f \in \mathsf{D}(\Pi)$.

Proof. Since $\widetilde{\Gamma}_B$ is conjugated to $\widetilde{\Gamma}$, it is an exact nilpotent operator. To prove (i), it suffices to prove the estimate

$$\|f\| + \|g\| \lesssim \|f + g\|, \quad f \in \mathsf{N}(\Gamma),\ g \in \mathsf{N}(\widetilde{\Gamma}_B). \tag{2.3}$$

Indeed, this shows that $\mathsf{N}(\Gamma) \oplus \mathsf{N}(\widetilde{\Gamma}_B) \subset \mathcal{H}$. Furthermore, replacing $(\Gamma, \widetilde{\Gamma}, B)$ with $(\widetilde{\Gamma}^*, \Gamma^*, B^*)$ shows that $\mathsf{N}(\widetilde{\Gamma}^*) \oplus \mathsf{N}(\Gamma^*_{B^*}) \subset \mathcal{H}$. Conjugating with B^* then shows that $\mathsf{N}((\widetilde{\Gamma}_B)^*) \oplus \mathsf{N}(\Gamma^*) \subset \mathcal{H}$. Therefore a duality argument proves the splitting $\mathcal{H} = \mathsf{N}(\Gamma) \oplus \mathsf{N}(\widetilde{\Gamma}_B)$. From this it follows that

$$\mathsf{D}(\Pi_B) = \bigl(\mathsf{D}(\widetilde{\Gamma}_B) \cap \mathsf{N}(\Gamma)\bigr) \oplus \bigl(\mathsf{D}(\Gamma) \cap \mathsf{N}(\widetilde{\Gamma}_B)\bigr)$$

and $\mathsf{R}(\Pi_B) = \mathsf{R}(\Gamma) \oplus \mathsf{R}(\widetilde{\Gamma}_B)$ are dense and that Π_B is closed, as well as the boundedness of the associated projections.

To prove (2.3), we use that $Bg \in \mathsf{N}(\widetilde{\Gamma})$ and thus $|(f, Bg)| \leqslant c(\Gamma, \widetilde{\Gamma})\|f\|\|Bg\|$, and estimate:

$$\begin{aligned}\|f\|^2 &\leqslant \kappa_B^{-1}\operatorname{Re}(Bf,f) \leqslant \kappa_B^{-1}\operatorname{Re}\big(\big(B(f+g),f\big)-(Bg,f)\big)\\ &\leqslant \kappa_B^{-1}\|B\|_\infty\|f+g\|\|f\|+\kappa_B^{-1}c(\Gamma,\widetilde{\Gamma})\|Bg\|\|f\|\\ &\leqslant \kappa_B^{-1}\|B\|_\infty\|f+g\|\|f\|+\kappa_B^{-1}c(\Gamma,\widetilde{\Gamma})\|B\|_\infty\big(\|f+g\|+\|f\|\big)\|f\|\\ &\leqslant \kappa_B^{-1}\|B\|_\infty\big(1+c(\Gamma,\widetilde{\Gamma})\big)\|f+g\|\|f\|+\kappa_B^{-1}\|B\|_\infty c(\Gamma,\widetilde{\Gamma})\|f\|^2.\end{aligned}$$

Solving for $\|f\|^2$, this shows that $\|f\| \lesssim \|f+g\|$ provided $\kappa_B^{-1}\|B\|_\infty c(\Gamma,\widetilde{\Gamma}) < 1$, which proves (2.3).

To prove (ii) we factorise

$$\Gamma + B_1^{-1}\widetilde{\Gamma}B_2 = \big(\mathbf{P}^1 + B_1^{-1}\mathbf{P}^2\big)\Pi\big(\mathbf{P}^2+\mathbf{P}^1B_2\big).$$

Here $(\mathbf{P}^1 + B_1^{-1}\mathbf{P}^2)^{-1} = \mathbf{P}^1_{B_1} + B_1\mathbf{P}^2_{B_1}$ and $(\mathbf{P}^2+\mathbf{P}^1B_2)^{-1} = \mathbf{P}^1_{B_2}B_2^{-1} + \mathbf{P}^2_{B_2}$. Indeed, we have

$$\begin{aligned}\big(\mathbf{P}^1 + B_1^{-1}\mathbf{P}^2\big)\big(\mathbf{P}^1_{B_1} + B_1\mathbf{P}^2_{B_1}\big) &= \mathbf{P}^1\mathbf{P}^1_{B_1} + \mathbf{P}^1B_1\mathbf{P}^2_{B_1} + B_1^{-1}\mathbf{P}^2\mathbf{P}^1_{B_1} + B_1^{-1}\mathbf{P}^2B_1\mathbf{P}^2_{B_1}\\ &= \mathbf{P}^1\mathbf{P}^1_{B_1} + B_1^{-1}\mathbf{P}^2B_1\mathbf{P}^2_{B_1}\\ &= \big(\mathbf{P}^1+\mathbf{P}^2\big)\mathbf{P}^1_{B_1} + B_1^{-1}\big(\mathbf{P}^1+\mathbf{P}^2\big)B_1\mathbf{P}^2_{B_1}\\ &= \mathbf{P}^1_{B_1} + \mathbf{P}^2_{B_1} = I,\end{aligned}$$

$$\begin{aligned}\big(\mathbf{P}^2+\mathbf{P}^1B_2\big)\big(\mathbf{P}^1_{B_2}B_2^{-1}+\mathbf{P}^2_{B_2}\big) &= \mathbf{P}^2\mathbf{P}^1_{B_2}B_2^{-1} + \mathbf{P}^2\mathbf{P}^2_{B_2} + \mathbf{P}^1B_2\mathbf{P}^1_{B_2}B_2^{-1} + \mathbf{P}^1B_2\mathbf{P}^2_{B_2}\\ &= \mathbf{P}^2\mathbf{P}^2_{B_2} + \mathbf{P}^1B_2\mathbf{P}^1_{B_2}B_2^{-1}\\ &= \mathbf{P}^2\big(\mathbf{P}^1_{B_2}+\mathbf{P}^2_{B_2}\big) + \mathbf{P}^1B_2\big(\mathbf{P}^1_{B_2}+\mathbf{P}^2_{B_2}\big)B_2^{-1} = \mathbf{P}^2+\mathbf{P}^1 = I.\end{aligned}$$

A similar calculation shows that $\mathbf{P}^1_{B_1} + B_1\mathbf{P}^2_{B_1}$ and $\mathbf{P}^1_{B_2}B_2^{-1} + \mathbf{P}^2_{B_2}$ also are left inverses. Thus $\mathbf{P}^1 + B_1^{-1}\mathbf{P}^2$ and $\mathbf{P}^2+\mathbf{P}^1B_2$ are isomorphisms and (ii) follows. □

We can now prove the following perturbed normal and tangential splittings of $\mathcal{H}$.

$$\begin{aligned}\mathcal{H} &= \hat{N}_B^+\mathcal{H}\oplus\hat{N}_B^-\mathcal{H} = B^{-1}N^+\mathcal{H}\oplus N^-\mathcal{H},\\ \mathcal{H} &= \check{N}_B^+\mathcal{H}\oplus\check{N}_B^-\mathcal{H} = N^+\mathcal{H}\oplus B^{-1}N^-\mathcal{H}.\end{aligned}$$

To see this, let first $\Gamma = \mu$ and $\widetilde{\Gamma} = \Gamma^* = \mu^*$ in Lemma 2.21(i). It follows that $\mathsf{N}(\Gamma) = N^-\mathcal{H}$, $\mathsf{N}(\Gamma_B^*) = B^{-1}N^+\mathcal{H}$, $\mathbf{P}^1_B = \hat{N}_B^-$ and $\mathbf{P}^2_B = \hat{N}_B^+$. On the other hand, choosing $\Gamma = \mu^*$ and $\widetilde{\Gamma} = \Gamma^* = \mu$, we see that $\mathsf{N}(\Gamma) = N^+\mathcal{H}$, $\mathsf{N}(\Gamma_B^*) = B^{-1}N^-\mathcal{H}$, $\mathbf{P}^1_B = \check{N}_B^+$ and $\mathbf{P}^2_B = \check{N}_B^-$. Lemma 2.21(i) thus shows that all the oblique normal and tangential projections $\hat{N}_B^\pm$ and $\check{N}_B^\pm$ from Definition 2.15 are bounded operators on $\mathcal{H}$, i.e. we have the stated splittings.

Next we apply Lemma 2.21(ii) to prove resolvent bounds for the operator T_B from Definition 2.10. Define closed and open sectors and double sectors in the complex plane by

$$\begin{aligned}S_{\omega+} &:= \big\{z\in\mathbf{C};\ |\arg z|\leqslant\omega\big\}\cup\{0\}, & S_\omega &:= S_{\omega+}\cup(-S_{\omega+}),\\ S^o_{\nu+} &:= \big\{z\in\mathbf{C};\ z\neq 0,\ |\arg z|<\nu\big\}, & S^o_\nu &:= S^o_{\nu+}\cup\big(-S^o_{\nu+}\big).\end{aligned}$$

Proposition 2.22. *The operator T_B is a closed operator in $\mathcal{H}$ with dense domain and range, for any accretive, complex matrix function $B \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$. Furthermore, T_B is a bisectorial operator with $\sigma(T_B) \subset S_\omega$, where*

$$\omega := \arccos\bigl(\kappa_B/\bigl(2\|B\|_\infty\bigr)\bigr) \in [\pi/3, \pi/2)$$

and if $\omega < \nu < \pi/2$, then there exists $C < \infty$ depending only on ν, κ_B and $\|B\|_\infty$ such that

$$\bigl\|(\lambda - T_B)^{-1}\bigr\| \leqslant C/|\lambda|, \quad \textit{for all } \lambda \notin S_\nu.$$

Proof. It is a consequence of Lemma 2.21(i) that the operator $\Pi_B = \underline{d} + B^{-1}\underline{d}^* B$ is a closed operator with dense domain and range. Since $T_B = -iM_B^{-1}\Pi_B$ with M_B an isomorphism, it follows that T_B also is closed with dense domain and range.

To prove the resolvent estimate, write $\lambda = 1/(i\tau)$ where $\tau \in S^o_{\pi/2-\nu}$. We first prove that $\|u\| \leqslant C\|f\|$ if $(I - i\tau T_B)u = f$, uniformly for $\tau \in S^o_{\pi/2-\nu}$. Multiply the equation with $i(\mu - \widetilde{B}^{-1}\mu^* B)$ to obtain

$$\bigl(\Gamma + \widetilde{B}^{-1}\widetilde{\Gamma}B\bigr)u = i\bigl(\mu - \widetilde{B}^{-1}\mu^* B\bigr)f,$$

where $\Gamma := i\underline{\mu} + \tau d$ and $\widetilde{\Gamma} := -i\mu^* + \tau d^*$ are nilpotent by Lemma 2.7. It suffices to prove $\|u\| \lesssim \|(\Gamma + \widetilde{B}^{-1}\widetilde{\Gamma}B)u\|$.

(i) By orthogonality we have

$$\begin{aligned}\bigl\|(\Gamma + \Gamma^*)u\bigr\|^2 = \|\Gamma u\|^2 + \|\Gamma^* u\|^2 = \|\mu u\|^2 + |\tau|^2\|du\|^2 + 2\,\mathrm{Re}(i\mu u, \tau du)\\ + \|\mu^* u\|^2 + |\tau|^2\|d^* u\|^2 + 2\,\mathrm{Re}(-i\mu^* u, \bar{\tau} d^* u),\end{aligned}$$

where

$$\begin{aligned}\mathrm{Re}(i\mu u, \tau du) + \mathrm{Re}(-i\mu^* u, \bar{\tau} d^* u) &= \mathrm{Re}(i\bar{\tau} d^* \mu u, u) + \mathrm{Re}(u, i\bar{\tau}\mu d^* u)\\ &= \mathrm{Re}\bigl(i\bar{\tau}\{d^*, \mu\}u, u\bigr) = 0,\end{aligned}$$

by Lemma 2.7. Thus $\|(\Gamma + \Gamma^*)u\|^2 = \|(\mu + \mu^*)u\|^2 + |\tau|^2\|(d + d^*)u\|^2 \geqslant \|mu\|^2 = \|u\|^2$. In particular Γ is exact.

(ii) Next we prove that Γ and $\widetilde{\Gamma}$ are transversal, with a bound $c < 1$ uniformly for all $\tau \in S^o_{\pi/2-\nu}$. By exactness, it suffices to bound (f, g) for $f = \Gamma u \in \mathsf{R}(\Gamma)$ and $\widetilde{\Gamma} g = 0$. Furthermore, using the orthogonal Hodge splitting $\mathcal{H} = \mathsf{N}(\Gamma) \oplus \mathsf{N}(\Gamma^*)$, we may assume that $\Gamma^* u = 0$. We get

$$\begin{aligned}(f, g) = (\Gamma u, g) = (u, \Gamma^* g) = \bigl(u, (-i\mu^* + \bar{\tau} d^*)g\bigr)\\ = \bigl(u, -i\mu^* g + i(\bar{\tau}/\tau)\mu^* g\bigr) = 2(\tau - \bar{\tau})/(2i\bar{\tau})(u, \mu^* g),\end{aligned}$$

and thus $|(f, g)| \leqslant 2|\sin(\arg\tau)|\|u\|\|g\| \leqslant c\|f\|\|g\|$ by (i), where $c < \kappa_B/\|B\|_\infty$ since $\pi/2 - \nu < \arcsin(\kappa_B/(2\|B\|_\infty))$. A similar argument shows that Γ^* and $\widetilde{\Gamma}^*$ are transversal with the same constant $c < \kappa_B/\|B\|_\infty$ uniformly for $\tau \in S^o_{\pi/2-\nu}$.

(iii) To apply Lemma 2.21(ii) it now suffices to prove that $\|(\Gamma + \widetilde{\Gamma})u\| \geqslant C^{-1}\|u\|$ uniformly for all $\tau \in S^o_{\pi/2-\nu}$. From Lemma 2.21(i) with $B = I$ we have $\|(\Gamma + \widetilde{\Gamma})u\| \approx \|\Gamma u\| + \|\widetilde{\Gamma} u\|$.

Using the Hodge splitting $\mathcal{H} = \mathsf{N}(\Gamma) \oplus \mathsf{N}(\widetilde{\Gamma})$ it suffices to prove $\|\Gamma u\| \gtrsim \|u\|$ for $u \in \mathsf{N}(\widetilde{\Gamma})$, and $\|\widetilde{\Gamma} v\| \gtrsim \|v\|$ for $v \in \mathsf{N}(\Gamma)$. To prove for example the first estimate, write $\mathsf{N}(\widetilde{\Gamma}) \ni u = u_1 + u_2 \in \mathsf{N}(\Gamma) \oplus \mathsf{N}(\Gamma^*)$. Then

$$\begin{aligned}\|\Gamma u\|^2 = \|\Gamma u_2\|^2 \geqslant \|u_2\|^2 = \|u - u_1\|^2 \\ \geqslant \|u\|^2 + \|u_1\|^2 - 2c\|u\|\|u_1\| \geqslant \big(1 - c^2\big)\|u\|^2,\end{aligned}$$

where we have used (i) in the second step and (ii) in the fourth step.

This proves that $\|u\| \leqslant C\|f\|$ if $(I - i\tau T_B)u = f$. Since $(I - i\tau T_B)' = I - i\bar{\tau} T_{B^*}$ by Proposition 2.17, a duality argument shows that $I - i\tau T_B$ is onto, and the proof is complete. □

From the uniform boundedness of the resolvents $R_t^B := (I + itT_B)^{-1}$ for $t \in \mathbf{R}$ and the boundedness of the Hodge projections $\mathbf{P}_B^1 := \underline{d}\Pi_B^{-1}$ and $\mathbf{P}_B^2 := \underline{d}_B^*\Pi_B^{-1}$, where $\Pi_B = \underline{d} + \underline{d}_B^*$, $\underline{d} = imd$ and $\underline{d}_B^* = B^{-1}\underline{d}^* B$, we can now deduce boundedness of operators related to $P_t^B = \frac{1}{2}(R_{-t}^B + R_t^B)$ and $Q_t^B = \frac{1}{2i}(R_{-t}^B - R_t^B)$.

Corollary 2.23. *The following families of operators are all uniformly bounded for $t > 0$.*

$$\begin{aligned}
R_t^B &:= (I + itT_B)^{-1}, & t\underline{d}P_t^B &= i\mathbf{P}_B^1 M_B Q_t^B,\\
P_t^B &:= \big(I + t^2T_B^2\big)^{-1}, & t\underline{d}_B^* P_t^B &= i\mathbf{P}_B^2 M_B Q_t^B,\\
Q_t^B &:= tT_B\big(I + t^2T_B^2\big)^{-1}, & P_t^B M_B^{-1} t\underline{d} &= iQ_t^B\mathbf{P}_B^2,\\
t^2T_B^2P_t^B &= tT_BQ_t^B = I - P_t^B, & P_t^B M_B^{-1} t\underline{d}_B^* &= iQ_t^B\mathbf{P}_B^1,\\
t\underline{d}Q_t^B &= i\mathbf{P}_B^1 M_B\big(I - P_t^B\big), & t\underline{d}P_t^B M_B^{-1} t\underline{d} &= \mathbf{P}_B^1 M_B\big(P_t^B - I\big)\mathbf{P}_B^2,\\
t\underline{d}_B^* Q_t^B &= i\mathbf{P}_B^2 M_B\big(I - P_t^B\big), & t\underline{d}P_t^B M_B^{-1} t\underline{d}_B^* &= \mathbf{P}_B^1 M_B\big(P_t^B - I\big)\mathbf{P}_B^1,\\
Q_t^B M_B^{-1} t\underline{d} &= i\big(I - P_t^B\big)\mathbf{P}_B^2, & t\underline{d}_B^* P_t^B M_B^{-1} t\underline{d} &= \mathbf{P}_B^2 M_B\big(P_t^B - I\big)\mathbf{P}_B^2,\\
Q_t^B M_B^{-1} t\underline{d}_B^* &= i\big(I - P_t^B\big)\mathbf{P}_B^1, & t\underline{d}_B^* P_t^B M_B^{-1} t\underline{d}_B^* &= \mathbf{P}_B^2 M_B\big(P_t^B - I\big)\mathbf{P}_B^1.
\end{aligned}$$

These families of operators are not only bounded, but have L_2 off-diagonal bounds in the following sense.

Definition 2.24. Let $(U_t)_{t>0}$ be a family of operators on $\mathcal{H}$, and let $M \geqslant 0$. We say that $(U_t)_{t>0}$ has L_2 *off-diagonal bounds* (with exponent M) if there exists $C_M < \infty$ such that

$$\|U_t f\|_{L_2(E)} \leqslant C_M\big\langle \operatorname{dist}(E, F)/t\big\rangle^{-M}\|f\|$$

whenever $E, F \subset \mathbf{R}^n$ and $\operatorname{supp} f \subset F$. Here $\langle x\rangle := 1 + |x|$, and $\operatorname{dist}(E, F) := \inf\{|x - y|\colon x \in E,\ y \in F\}$. We write $\|U_t\|_{\mathrm{off},M}$ for the smallest constant C_M. The exact value of M is normally not important and we write $\|U_t\|_{\mathrm{off}}$, where it is understood that M is chosen sufficiently large but fixed.

Proposition 2.25. *All the operator families from Corollary* 2.23 *has L_2 off-diagonal bounds for all exponents $M \geqslant 0$.*

Proof. First consider the resolvents $R_t^B = (I + itT_B)^{-1}$. As we already have proved uniform bounds for R_t^B, it suffices to prove

$$\big\|(I + itT_B)^{-1} f\big\|_{L_2(E)} \leqslant C_M\big(|t|/\operatorname{dist}(E,F)\big)^M \|f\|$$

for $|t| \leqslant \operatorname{dist}(E,F)$. We prove this by induction on M as in [10, Proposition 5.2]. Let $\eta : \mathbf{R}^n \to [0,1]$ be a bump function such that $\eta|_E = 1$, $\operatorname{supp}\eta \subset \widetilde{E} := \{x \in \mathbf{R}^n;\ \operatorname{dist}(x,E) \leqslant \operatorname{dist}(x,F)\}$ and $\|\nabla\eta\|_\infty \lesssim 1/\operatorname{dist}(E,F) \approx 1/\operatorname{dist}(\widetilde{E},F)$. Since the commutator is

$$\big[\eta I, R_t^B\big] = tR_t^B M_B^{-1}\big([\eta I, \underline{d}] + B^{-1}[\eta I, \underline{d}^*]B\big)R_t^B,$$

where $\|[\eta I, \underline{d}]\|, \|[\eta I, \underline{d}^*]\| \lesssim \|\nabla\eta\|_\infty$, we get

$$\big\|R_t^B f\big\|_{L_2(E)} \leqslant \big\|\eta R_t^B f\big\| = \big\|\big[\eta I, R_t^B\big] f\big\| \lesssim |t|\|\nabla\eta\|_\infty \big\|R_t^B f\big\|_{L_2(\widetilde{E})},$$

where we used that $\eta f = 0$. By induction, this proves the off-diagonal bounds for R_t^B. From this, off-diagonal bounds for P_t^B, Q_t^B and $I - P_t^B$ also follows immediately.

Next we consider $t\underline{d}P_t^B$ and use Lemma 2.21(i) to obtain

$$\begin{aligned}
\big\|t\underline{d}P_t^B f\big\|_{L_2(E)} &\leqslant \big\|\eta t\underline{d}P_t^B f\big\| \leqslant \big\|[\eta I, t\underline{d}]P_t^B f\big\| + \big\|t\underline{d}\eta P_t^B f\big\| \\
&\lesssim |t|\|\nabla\eta\|_\infty \big\|P_t^B f\big\|_{L_2(\widetilde{E})} + \big\|tT_B\eta P_t^B f\big\| \\
&\lesssim |t|\|\nabla\eta\|_\infty \big\|P_t^B f\big\|_{L_2(\widetilde{E})} + \big\|[\eta, tT_B]P_t^B f\big\| + \big\|\eta tT_B P_t^B f\big\| \\
&\lesssim |t|\|\nabla\eta\|_\infty \big\|P_t^B f\big\|_{L_2(\widetilde{E})} + \big\|Q_t^B f\big\|_{L_2(\widetilde{E})}. \qquad (2.4)
\end{aligned}$$

This and the corresponding calculation with $\underline{d}$ replaced by $\underline{d}_B^*$ proves the off-diagonal bounds for $t\underline{d}P_t^B$ and $t\underline{d}_B^* P_t^B$. From this the result for $P_t^B M_B^{-1} t\underline{d}$ and $P_t^B M_B^{-1} t\underline{d}_B^*$ follows immediately with a duality argument. Indeed, $(M_B^{-1}\underline{d})' = M_{B^*}^{-1}\underline{d}_{B^*}^*$ and $(M_B^{-1}\underline{d}_B^*)' = M_{B^*}^{-1}\underline{d}$ is proved similarly to $T_B' = -T_{B^*}$ in Proposition 2.17.

The proof for $t\underline{d}Q_t^B$, $t\underline{d}_B^* Q_t^B$, $Q_t^B M_B^{-1} t\underline{d}$ and $Q_t^B M_B^{-1} t\underline{d}_B^*$ is similar, replacing P_t and Q_t with Q_t and $I - P_t$. Finally, the last four estimates follows from a computation like (2.4), for example replacing P_t and Q_t with $P_t^B M_B^{-1} t\underline{d}$ and $Q_t^B M_B^{-1} t\underline{d}$ proves the estimate for $t\underline{d}P_t^B M_B^{-1} t\underline{d}$. □

We finish this section with a lemma to be used in Section 4. This lemma is proved with an argument similar to that in [15, Lemma 2.3]. For completeness, we include a short proof.

Lemma 2.26. *Assume that $(U_t)_{t>0}$ and $(V_t)_{t>0}$ both have L_2 off-diagonal bounds with exponent M. Then $(U_t V_t)_{t>0}$ has L_2 off-diagonal bounds with exponent M and*

$$\|U_t V_t\|_{\mathrm{off},M} \leqslant 2^{M+1}\|U_t\|_{\mathrm{off},M}\|V_t\|_{\mathrm{off},M}.$$

Proof. By Definition 2.24 we need to prove that

$$\|U_t V_t f\|_{L_2(E)} \lesssim \big\langle \operatorname{dist}(E,F)/t\big\rangle^{-M}\|f\|$$

whenever $\operatorname{supp} f \subset F$. To this end let $G := \{x \in \mathbf{R}^n;\ \operatorname{dist}(x, E) \leqslant \rho/2\}$, where $\rho := \operatorname{dist}(E, F)$. We get

$$\begin{aligned}\|U_t V_t f\|_{L_2(E)} &\leqslant \big\|U_t(\chi_G V_t f)\big\|_{L_2(E)} + \big\|U_t(\chi_{\mathbf{R}^n \setminus G} V_t f)\big\|_{L_2(E)} \\ &\leqslant C_0^{U_t} \|V_t f\|_{L_2(G)} + C_M^{U_t} \langle \rho/2t\rangle^{-M} \|V_t f\|_{L_2(\mathbf{R}^n \setminus G)} \\ &\leqslant C_0^{U_t} C_M^{V_t} \langle \rho/2t\rangle^{-M} \|f\| + C_M^{U_t} \langle \rho/2t\rangle^{-M} C_0^{V_t} \|f\| \leqslant 2^{M+1} C_M^{U_t} C_M^{V_t} \langle \rho/t\rangle^{-M} \|f\|.\end{aligned}$$

□

2.3. Quadratic estimates: generalities

In Proposition 2.22 we proved the spectral estimate $\sigma(T_B) \subset S_\omega$ for some angle $\omega < \pi/2$ with bounds on the resolvent outside S_ω. In this section we survey some general facts about the functional calculus of the operator T_B. For a further background and discussion of these matters we refer to [1,10].

Definition 2.27. For $\omega < \nu < \pi/2$, we define the following classes of holomorphic functions $f \in H(S_\nu^o)$ on the open double sector S_ν^o:

$$\begin{aligned}\Psi\big(S_\nu^o\big) &:= \big\{\psi \in H\big(S_\nu^o\big);\ \big|\psi(z)\big| \leqslant C \min\big(|z|^s, |z|^{-s}\big),\ z \in S_\nu^o,\ \text{for some } s > 0,\ C < \infty\big\}, \\ H_\infty\big(S_\nu^o\big) &:= \big\{b \in H\big(S_\nu^o\big);\ \big|b(z)\big| \leqslant C,\ z \in S_\nu^o,\ \text{for some } C < \infty\big\}, \\ F\big(S_\nu^o\big) &:= \big\{w \in H\big(S_\nu^o\big);\ \big|w(z)\big| \leqslant C \max\big(|z|^s, |z|^{-s}\big),\ z \in S_\nu^o,\ \text{for some } s < \infty,\ C < \infty\big\}.\end{aligned}$$

Thus $\Psi(S_\nu^o) \subset H_\infty(S_\nu^o) \subset F(S_\nu^o) \subset H(S_\nu^o)$.

For $\psi \in \Psi(S_\nu^o)$, we define a bounded operator $\psi(T_B)$ through the Dunford functional calculus

$$\psi(T_B) := \frac{1}{2\pi i} \int_\gamma \psi(\lambda)(\lambda I - T_B)^{-1}\, d\lambda, \tag{2.5}$$

where γ is the unbounded contour $\{\pm r e^{\pm i\theta};\ r > 0\}$, $\omega < \theta < \nu$, parametrised counterclockwise around S_ω. The decay estimate on ψ and the resolvent bounds of Proposition 2.22 guarantee that $\|\psi(T_B)\| < \infty$.

For general $w \in F(S_\nu^o)$ we define

$$w(T_B) := \big(Q^B\big)^{-k} \big(q^k w\big)(T_B),$$

where k is an integer larger than s if $|w(z)| \leqslant C \max(|z|^s, |z|^{-s})$, and $q(z) := z(1 + z^2)^{-1}$ and $Q^B := q(T_B)$. This yields a closed, densely defined operator $w(T_B)$ in $\mathcal{H}$. Furthermore, we have

$$\begin{aligned}\overline{\lambda_1 w_1(T_B) + \lambda_2 w_2(T_B)} &= (\lambda_1 w_1 + \lambda_2 w_2)(T_B), \\ \overline{w_1(T_B) w_2(T_B)} &= (w_1 w_2)(T_B),\end{aligned} \tag{2.6}$$

for all w_1 and $w_2 \in F(S_\nu^o)$. Here $\overline{T} = S$ means that the graph $\mathsf{G}(T)$ is dense in the graph $\mathsf{G}(S)$.

The functional calculus $w \mapsto w(T_B)$ have the following convergence properties as proved in [1].

Lemma 2.28. *If $b_k \in H_\infty(S_\nu^o)$ is a sequence uniformly bounded on S_ν^o which converges to b uniformly on compact subsets, and if $b_k(T_B)$ are uniformly bounded operators, then*

$$b_k(T_B)f \to b(T_B)f, \quad \textit{for all } f \in \mathcal{H},$$

and $\|b(T_B)\| \leqslant \limsup_k \|b_k(T_B)\|$.

Definition 2.29. The following operators in the functional calculus are of special importance to us.

(1) $q_t(z) = q(tz) := tz(1+t^2z^2)^{-1} \in \Psi(S_\nu^o)$, which give the operator Q_t^B.
(2) $|z|^s := (z^2)^{s/2} \in F(S_\nu^o)$, which give the operator $|T_B|^s$. Note that $|z|$ does not denote absolute value here, but $z \mapsto |z|$ is holomorphic on S_ν^o.
(3) $e^{-t|z|} \in H_\infty(S_\nu^o)$, which give the operator $e^{-t|T_B|}$.
(4) The characteristic functions

$$\chi^\pm(z) = \begin{cases} 1 & \text{if } \pm\operatorname{Re} z > 0, \\ 0 & \text{if } \pm\operatorname{Re} z < 0 \end{cases}$$

which give the generalised *Hardy projections* $E_B^\pm := \chi^\pm(T_B)$.
(5) The signum function

$$\operatorname{sgn}(z) = \chi^+(z) - \chi^-(z)$$

which give the generalised *Cauchy integral* $E_B := \operatorname{sgn}(T_B)$.

The main work in this paper is to prove the boundedness the projections $E_B^\pm$. As in Lemma 2.14, if these are bounded then they correspond to a splitting

$$\mathcal{H} = E_B^+\mathcal{H} \oplus E_B^-\mathcal{H}$$

of $\mathcal{H}$ into the *Hardy subspaces* $E_B^\pm\mathcal{H}$ associated with Eq. (2.1). That the projections are bounded is also equivalent with having a bounded reflection operator E_B.

Definition 2.30. For a function $F(t,x)$ defined in $\mathbf{R}_\pm^{n+1}$ we write

$$|||F|||_\pm := \left(\int_0^\infty \big\|F(\pm t, x)\big\|^2 \frac{dt}{t} \right)^{1/2}$$

and for short $|||F|||_+ =: |||F|||$. When $F(t,x) = (\Theta_t f)(x)$ for some family of operators $(\Theta_t)_{t>0}$, we use the notation

$$|||\Theta_t|||_{\mathrm{op}} := \sup_{\|f\|=1} |||\Theta_t f|||.$$

Our main goal in this paper will be to prove quadratic estimates of the form

$$\int_0^\infty \|Q_t^B f\|^2 \frac{dt}{t} \approx \|f\|^2, \tag{2.7}$$

for certain coefficients B. We recall the following two basic results concerning quadratic estimates which are proved by Schur estimates. For details we refer to [1].

Proposition 2.31. *Let $\psi \in \Psi(S_\nu^o)$ be non-vanishing on both $S_{\nu+}^o$ and $S_{\nu-}^o$, and define $\psi_t(z) := \psi(tz)$. Then there exists $0 < C < \infty$ such that*

$$C^{-1}\big\|\big|Q_t^B f\big\|\big| \leqslant \big\|\big|\psi_t(T_B)f\big\|\big| \leqslant C\big\|\big|Q_t^B f\big\|\big|.$$

Proposition 2.32. *If T_B satisfies quadratic estimates* (2.7)*, then T_B has bounded $H_\infty(S_\nu^o)$ functional calculus, i.e.*

$$\big\|b(T_B)\big\| \lesssim \|b\|_\infty, \quad \textit{for all } b \in H_\infty\big(S_\nu^o\big).$$

Thus $H_\infty(S_\nu^o) \ni b \mapsto b(T_B) \in \mathcal{L}(\mathcal{H})$ is a continuous homomorphism.

Before proving quadratic estimates for T_B for certain B in Sections 3 and 4, we introduce a dense subspace on which the operator $b(T_B)$ is defined for any $b \in H_\infty(S_\nu^o)$.

Definition 2.33. Let $\mathcal{V}_B$ be the dense linear subspace

$$\mathcal{V}_B := \bigcup_{s>0}\big(\mathsf{D}\big(|T_B|^s\big) \cap \mathsf{R}\big(|T_B|^s\big)\big) \subset \mathcal{H}.$$

We see that $\mathsf{D}(|T_B|^s) \cap \mathsf{R}(|T_B|^s)$ increases when s decreases. The density of $\mathsf{D}(|T_B|) \cap \mathsf{R}(|T_B|)$, and therefore of $\mathcal{V}_B$, follows from the fact that

$$2\int_\alpha^\beta \big(Q_t^B\big)^2 f \frac{dt}{t} = \big(P_\alpha^B - P_\beta^B\big) f \to f,$$

as $(\alpha, \beta) \to (0, \infty)$, for all $f \in \mathcal{H}$.

Moreover, if $b \in H_\infty(S_\nu^o)$ and $f \in \mathcal{V}_B$ then $b(T_B)f \in \mathcal{V}_B \subset \mathcal{H}$. To see this, write

$$b(T_B)f = (b\psi)(T_B)\big(\psi(T_B)^{-1} f\big),$$

where $\psi(z)^{-1} := (1 + |z|^s)/|z|^{s/2}$ if $f \in \mathsf{D}(|T_B|^s) \cap \mathsf{R}(|T_B|^s)$. Then $\psi(T_B)^{-1} f \in \mathcal{H}$ and $(b\psi)(T_B)$ is bounded since $b\psi \in \Psi(S_\nu^o)$. Furthermore, if $s' < s/2$ then $|T_B|^{s'}(\psi(T_B)^{-1} f) \in \mathcal{H}$ and $|T_B|^{-s'}(\psi(T_B)^{-1} f) \in \mathcal{H}$, so $b(T_B)f \in \mathsf{D}(|T_B|^{s'}) \cap \mathsf{R}(|T_B|^{s'})$.

Lemma 2.34. *We have an algebraic splitting*

$$\mathcal{V}_B = E_B^+ \mathcal{V}_B + E_B^- \mathcal{V}_B,$$

$E_B^+\mathcal{V}_B \cap E_B^-\mathcal{V}_B = \{0\}$, *and* $f \in E_B^\pm\mathcal{V}_B$ *is in one-to-one correspondence with* $F(t,x) = F_t(x) := (e^{\mp t|T_B|}f)(x)$ *in* $\mathbf{R}_\pm^{n+1}$, *and*

$$\lim_{t\to 0^\pm} F_t = f, \qquad \lim_{t\to\pm\infty} F_t = 0,$$

$$F_t \in \mathsf{D}(T_B) = \mathsf{D}(d) \cap \mathsf{D}(d^*B), \qquad \partial_t F_t = -T_B F_t \in L_2(\mathbf{R}^n), \quad \pm t > 0.$$

Proof. That each $f \in \mathcal{V}_B$ can be uniquely written $f = f^+ + f^-$, where $f^\pm \in E_B^\pm\mathcal{V}_B$, follows from (2.6). To verify the properties of F_t, it suffices to consider the case $f \in E_B^+\mathcal{V}_B$ as the case $f \in E_B^-\mathcal{V}_B$ is similar. Since $ze^{-t|z|} \in \Psi(S_\nu^0)$ it follows that $F_t \in \mathsf{D}(T_B)$. Moreover, since $\frac{1}{h}(e^{-(t+h)|z|} - e^{-t|z|}) \to -|z|e^{-t|z|}$ uniformly on S_ν^0, it follows from Lemma 2.28 that $\partial_t F_t = -|T_B|F_t$ and since $F_t \in E_B^+\mathcal{V}_B$ we have $|T_B|F_t = T_B F_t$.

To prove the limits, assume that $f \in \mathsf{D}(|T_B|^s) \cap \mathsf{R}(|T_B|^s)$ for some $0 < s < 1$. Writing $f = |T_B|^{-s}u$, we see that

$$F_t - f = t^s\psi(tT_B)u, \quad \text{where } \psi(z) = \big(e^{-|z|} - 1\big)/|z|^s.$$

Similarly with $f = |T_B|^s v$, we get

$$F_t = t^{-s}\psi(tT_B)v, \quad \text{where } \psi(z) = |z|^s e^{-|z|}.$$

Since in both cases $\psi(tT_B)$ are uniformly bounded in t, using a direct norm estimate in (2.5), it follows that both limits are 0 as $t \to 0$ and $t \to \infty$, respectively. In particular, $f = \lim_{t\to 0} F_t$ is uniquely determined by F. □

We now further discuss the quadratic estimates (2.7). First note the following consequence of the duality $T_B' = -T_{B^*}$ from Proposition 2.17. Again we refer to [1] for further details.

Lemma 2.35. *If* $|||Q_t^B f||| \lesssim \|f\|$ *for all* $f \in \mathcal{H}$, *then* $|||Q_t^{B^*} f||| \gtrsim \|f\|$ *for all* $f \in \mathcal{H}$.

In Section 3.3 we shall use the following Hardy space reduction of the quadratic estimate. This is a technique due to Coifman, Jones and Semmes [11], and adapted to the setting of functional calculus by McIntosh and Qian [23, Theorem 5.2].

Proposition 2.36. *Assume that we have reverse quadratic estimates in* $E_{B^*}^+\mathcal{V}_{B^*}$ *and* $E_{B^*}^-\mathcal{V}_{B^*}$, *i.e.*

$$\|g\| \lesssim |||t\partial_t G_t|||_\pm, \quad g \in E_{B^*}^\pm\mathcal{V}_{B^*},$$

where $G_t = e^{\mp t|T_{B^*}|}g$. *Then*

$$\big|\big|\big|Q_t^B f\big|\big|\big| \lesssim \|f\|, \quad f \in \mathcal{H}.$$

In particular, if we have reverse quadratic estimates in both Hardy spaces for both operators T_B *and* T_{B^*}, *then* $|||Q_t^B f||| \approx \|f\| \approx |||Q_t^{B^*} f|||$, $f \in \mathcal{H}$, *and thus* T_B *and* T_{B^*} *have bounded* $H_\infty(S_\nu^o)$ *functional calculus for all* $\omega < \nu < \pi/2$.

Proof. Assume first that

$$\|g\| \lesssim \|\!|\!| t\partial_t G_t |\!|\!|, \quad g \in E^+_{B^*}\mathcal{V}_{B^*},$$

and write $\psi_t(z) := tze^{-t|z|}$ so that $t\partial_t G_t = -\psi_t(T_{B^*})g$ by Lemma 2.34. Let $f \in \mathcal{H}$ and define $q_t^{\mp}(z) := \chi^{\mp} q_t(z)$ so that $q_t^{\mp}(T_B) = E_B^{\mp} Q_t^B$. To prove that

$$\big|\!\big|\!\big| q_t^-(T_B) f \big|\!\big|\!\big| \lesssim \|f\|, \quad f \in \mathcal{H},$$

it suffices to bound $\int_\alpha^\beta \|q_t^-(T_B)f\|^2 \frac{dt}{t}$ uniformly for $\alpha > 0$ and $\beta < \infty$. To this end, define the auxiliary functions

$$h_t := \left(\int_\alpha^\beta \big\| q_t^-(T_B) f \big\|^2 \frac{dt}{t} \right)^{-1/2} \big(N^+ B^* - B^* N^-\big)^{-1} q_t^-(T_B) f,$$

$$g := -\int_\alpha^\beta q_t^+(T_{B^*}) h_t \frac{dt}{t},$$

so that $\int_\alpha^\beta \|h_t\|^2 \frac{dt}{t} \leqslant C$ and $g \in E^+_{B^*}\mathcal{V}_{B^*}$, and calculate

$$\begin{aligned}
\left(\int_\alpha^\beta \big\| q_t^-(T_B) f \big\|^2 \frac{dt}{t} \right)^{1/2} &= \int_\alpha^\beta \big\langle q_t^-(T_B) f, h_t \big\rangle_B \frac{dt}{t} = \langle f, g \rangle_B \\
&\lesssim \|f\| \, \|g\| \lesssim \|f\| \big|\!\big|\!\big| \psi_s(T_{B^*}) g \big|\!\big|\!\big| \\
&\lesssim \|f\| \left(\int_0^\infty \left(\int_\alpha^\beta \big\| \big(\psi_s q_t^+\big)(T_{B^*}) \big\| \, \|h_t\| \frac{dt}{t} \right)^2 \frac{ds}{s} \right)^{1/2} \\
&\lesssim \|f\| \left(\int_\alpha^\beta \|h_t\|^2 \frac{dt}{t} \right)^{1/2} \lesssim \|f\|.
\end{aligned}$$

In the second equality we have used that $q_t^-(T_B)' = q_t^-(-T_{B^*}) = -q_t^+(T_{B^*})$, in the second estimate we used the hypothesis and the second last estimate is a Schur estimate. We here use that $\|(\psi_s q_t^+)(T_{B^*})\| \lesssim \eta(t/s)$, where $\eta(x) := \min(x^s, x^{-s})$ for some $s > 0$.

With a similar argument $\|\!|\!| q_t^+(T_B) f |\!|\!|\| \lesssim \|f\|$, $f \in \mathcal{H}$, follows from the reverse quadratic estimate for $g \in E^-_{B^*}\mathcal{V}_{B^*}$. If both reverse estimates holds for B^*, then

$$\big|\!\big|\!\big| Q_t^B f \big|\!\big|\!\big| \leqslant \big|\!\big|\!\big| q_t^-(T_B) f \big|\!\big|\!\big| + \big|\!\big|\!\big| q_t^+(T_B) f \big|\!\big|\!\big| \lesssim \|f\|, \quad f \in \mathcal{H},$$

and if the same holds for B and B^* interchanged, then Lemma 2.35 proves that $\|\!|\!| Q_t^B f |\!|\!|\| \approx \|f\|$, $f \in \mathcal{H}$, and by Proposition 2.32 this proves that T_B has bounded $H_\infty(S^o_\nu)$ functional calculus, and similarly for T_{B^*}. $\square$

We end this section with a discussion of the holomorphic perturbation theory for the functional calculus of T_B.

Definition 2.37. Let $z \mapsto U_z \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an operator-valued function defined on an open subset $D \subset \mathbf{C}$ of the complex plane. We say that U_z is *holomorphic* if for all $z \in D$ there exists an operator $U_z' \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that

$$\left\| \frac{1}{w}(U_{z+w} - U_z) - U_z' \right\|_{\mathcal{X} \to \mathcal{Y}} \to 0, \quad w \to 0.$$

Lemma 2.38. *Let $z \mapsto U_z \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be an operator-valued function defined on an open subset $D \subset \mathbf{C}$ of the complex plane. Then the following are equivalent.*

- (i) *$z \mapsto U_z$ is holomorphic.*
- (ii) *The scalar function $h(z) = (U_z f, g)$ is a holomorphic function for all $f \in \widetilde{\mathcal{X}}$ and $g \in \widetilde{\mathcal{Y}}^*$, where $\widetilde{\mathcal{X}} \subset \mathcal{X}$ and $\widetilde{\mathcal{Y}}^* \subset \mathcal{Y}^*$ are dense, and $\|U_z\|$ is locally bounded.*

In particular, if $z \mapsto U_z^k$ are holomorphic on D for $k = 1, 2, \ldots$ and

$$\left(U_z^k f, g\right) = h_k(z) \to h(z) = (U_z f, g), \quad \textit{for all } z \in D,\ f \in \widetilde{\mathcal{X}},\ g \in \widetilde{\mathcal{Y}}^*,$$

and $\sup_{z \in K, k \geqslant 1} \|U_z^k\| < \infty$ *for each compact subset $K \subset D$, then $z \mapsto U_z$ is holomorphic on D.*

Proof. For the equivalence between (i) and (ii), see Kato [18, Theorem III 3.12]. To prove the convergence result, it suffices to show that $h(z)$ is holomorphic on D. That this is true follows from an application of the dominated convergence theorem in the Cauchy integral formula for $h_k(z)$. □

Below, we shall assume that $z \mapsto B_z$ is a given holomorphic matrix-valued function defined on an open subset $D \subset \mathbf{C}$ such that B_z is a multiplication operator as in Definition 2.8 for each $z \in D$, and that $\omega_D := \sup_{z \in D} \arccos(\kappa_{B_z}/(2\|B_z\|_\infty)) < \pi/2$. Let $\omega_D < \nu < \pi/2$.

Lemma 2.39. *For $\tau \in S^0_{\pi/2-\nu}$, the operator-valued function $D \ni z \mapsto (I - i\tau T_{B_z})^{-1}$ is holomorphic.*

Proof. Similar to the proof of Proposition 2.22 we have

$$(I - i\tau T_{B_z})^{-1} = (\widetilde{B}_z \Gamma + \widetilde{\Gamma} B_z)^{-1} i(\widetilde{B}_z \mu - \mu^* B_z),$$

where $\Gamma = i\mu + \tau d$ and $\widetilde{\Gamma} = -i\mu + \tau d^*$. It is clear that the multiplication operator $\widetilde{B}_z \mu - \mu^* B_z$ depends holomorphically on z, so it suffices to show that $z \mapsto (\widetilde{B}_z \Gamma + \widetilde{\Gamma} B_z)^{-1}$ is holomorphic. Let $z \in D$, write $B := B_z$ and $B_w := B_{z+w}$ and calculate

$$\begin{aligned} &(\widetilde{B}_w \Gamma + \widetilde{\Gamma} B_w)^{-1} - (\widetilde{B} \Gamma + \widetilde{\Gamma} B)^{-1} \\ &\quad = -(\widetilde{B}_w \Gamma + \widetilde{\Gamma} B_w)^{-1} (\widetilde{B}_w - \widetilde{B}) \big(\Gamma (\widetilde{B} \Gamma + \widetilde{\Gamma} B)^{-1}\big) \\ &\qquad - \big((\widetilde{B}_w \Gamma + \widetilde{\Gamma} B_w)^{-1} \widetilde{\Gamma}\big)(B_w - B)(\widetilde{B} \Gamma + \widetilde{\Gamma} B)^{-1}. \end{aligned} \tag{2.8}$$

The first and last factors in both terms on the right-hand side are uniformly bounded by Lemma 2.21(i) so we deduce continuity of $w \mapsto (\widetilde{B}_w \Gamma + \widetilde{\Gamma} B_w)^{-1}$. Furthermore, multiplying Eq. (2.8) from the right with $\widetilde{\Gamma}$, the first term on the right vanishes as

$$\Gamma(\widetilde{B}\Gamma + \widetilde{\Gamma}B)^{-1}\widetilde{\Gamma} = (\widetilde{B}\Gamma + \widetilde{\Gamma}B)^{-1}\widetilde{\Gamma}^2 = 0,$$

and we deduce continuity of $w \mapsto (\widetilde{B}_w \Gamma + \widetilde{\Gamma} B_w)^{-1}\widetilde{\Gamma}$. Thus, dividing Eq. (2.8) by w and letting $w \to 0$ we see that the limit exists and equals

$$-(\widetilde{B}\Gamma + \widetilde{\Gamma}B)^{-1}\widetilde{B}'\big(\Gamma(\widetilde{B}\Gamma + \widetilde{\Gamma}B)^{-1}\big) - \big((\widetilde{B}\Gamma + \widetilde{\Gamma}B)^{-1}\widetilde{\Gamma}\big)B'(\widetilde{B}\Gamma + \widetilde{\Gamma}B)^{-1}. \qquad \square$$

Lemma 2.40. *If $\psi \in \Psi(S_\nu^o)$, then $D \ni z \mapsto \psi(T_{B_z})$ is holomorphic.*

Proof. Let γ be the unbounded contour $\{\pm re^{\pm i\theta};\ r > 0\}$, $\omega_D < \theta < \nu$, parametrised counter-clockwise around S_{ω_D}. By inspection of the proof of Lemma 2.39 we have

$$\sup_{\lambda \in \gamma} |\lambda| \left\| \frac{1}{w}\big((\lambda - T_{B_{z+w}})^{-1} - (\lambda - T_{B_z})^{-1}\big) - \partial_z(\lambda - T_{B_z})^{-1} \right\| \to 0,$$

as $w \to 0$. Thus

$$\frac{1}{w}\big(\psi(T_{B_{z+w}}) - \psi(T_{B_z})\big) \to \frac{1}{2\pi i}\int_\gamma \psi(\lambda)\partial_z(\lambda - T_{B_z})^{-1}\,d\lambda,$$

since $\int_\gamma |\psi(\lambda)||\frac{d\lambda}{\lambda}| < \infty$. $\square$

Lemma 2.41. *Assume that T_{B_z} satisfy quadratic estimates $|||Q_t^{B_z} f||| \approx \|f\|$, $f \in \mathcal{H}$, locally uniformly for $z \in D$. If $b \in H_\infty(S_\nu^o)$, $\psi \in \Psi(S_\nu^o)$ and $0 < \alpha < \beta < \infty$, then the following operators depend holomorphically on $z \in D$.*

(i) $u(x) \mapsto (b(T_{B_z})u)(x) : \mathcal{H} \to \mathcal{H}$;
(ii) $u(x) \mapsto v(t,x) = (b(tT_{B_z})u)(x) : \mathcal{H} \to L_2(\mathbf{R}^n \times (\alpha,\beta); \bigwedge)$;
(iii) $u(x) \mapsto v(t,x) = (\psi_t(T_{B_z})u)(x) : \mathcal{H} \to L_2(\mathbf{R}^{n+1}_+, \frac{dtdx}{t}; \bigwedge)$.

Proof. (i) Take a uniformly bounded sequence $\psi_k(z) \in \Psi(S_\nu^o)$ which converges to $b(z) \in H_\infty(S_\nu^o)$ uniformly on compact subsets of S_ν^o. Lemma 2.38 then applies with $U_z^k = \psi_k(T_{B_z})$ and $U_z = b(T_{B_z})$, using Lemmas 2.40 and 2.28.

(ii) It suffices by Lemma 2.38 to show that $h(z) = \int_\alpha^\beta h_t(z)\,dt$ is holomorphic, where $h_t(z) = (b(tT_{B_z})f, G_t)$, for all $f(x) \in \mathcal{H}$ and $G(t,x) \in C_0^\infty(\mathbf{R}^n \times (\alpha,\beta); \bigwedge)$. That $h_t(z)$ is holomorphic for each t is clear from (i), and for $h(z)$ this follows from an application of the Fubini theorem to the Cauchy integral formula for $h_t(z)$.

(iii) Consider the truncations $U_z^k : u(x) \mapsto v(t,x) = \chi_k(t)(\psi_t(T_{B_z})u)(x)$, where χ_k denotes the characteristic function of the interval $(1/k, k)$. It is clear from (ii) that U_z^k is holomorphic, and letting $k \to \infty$ we deduce from Lemma 2.38 that $\psi_t(T_{B_z})$ is holomorphic. $\square$

Proposition 2.42. *Assume that T_B satisfies quadratic estimates $|||Q_t^B f||| \approx \|f\|$, $f \in \mathcal{H}$, locally uniformly for all B such that $\|B - B_0\|_\infty < \varepsilon$, and let $\psi \in \Psi(S_\nu^o)$. Then we have the Lipschitz estimates*

$$\big\|b(T_{B_2}) - b(T_{B_1})\big\| \leqslant C\|b\|_\infty\|B_2 - B_1\|_\infty, \quad b \in H_\infty\big(S_\nu^o\big),$$
$$\big|\big|\big|\psi_t(T_{B_2}) - \psi_t(T_{B_1})\big|\big|\big| \leqslant C\|B_2 - B_1\|_\infty,$$

when $\|B_i - B_0\| < \varepsilon/2$, $i = 1, 2$.

Proof. Let $B(z) := B_1 + z(B_2 - B_1)/\|B_2 - B_1\|_\infty$, so that $B(z)$ is holomorphic in a neighbourhood of the interval $[0, \|B_2 - B_1\|_\infty]$. In this neighbourhood, we have bounds $\|b(T_{B(z)})\| \lesssim \|b\|_\infty$ by Proposition 2.32 and holomorphic dependence on z by Lemma 2.41(i). Schwarz' lemma now applies and proves that $\|\frac{d}{dz} b(T_{B(z)})\| \lesssim \|b\|_\infty$ for all $z \in [0, \|B_2 - B_1\|_\infty]$. This shows that

$$\big\|b(T_{B_2}) - b(T_{B_1})\big\| \leqslant \int_0^{\|B_2 - B_1\|} \left\|\frac{d}{dt} b(T_{B(t)})\right\| dt \leqslant C\|b\|_\infty\|B_2 - B_1\|_\infty.$$

The proof of the Lipschitz estimate for $\psi_t(T_B)$ similarly follows from Lemma 2.41(iii). □

2.4. Decoupling of the Dirac equation

In Section 2.1 we introduced the Dirac type equation

$$\big(md_{t,x} + B^{-1}md_{t,x}^* B\big)F = 0 \tag{2.9}$$

satisfied by functions $F(t, x) : \mathbf{R}_\pm^{n+1} \to \bigwedge$. Of particular interest is when both terms vanish, i.e. when

$$\begin{cases} d_{t,x}F = 0, \\ d_{t,x}^*(BF) = 0, \end{cases} \quad \text{or equivalently} \quad \text{when} \quad \begin{cases} dF = -\mu\partial_t F, \\ d^*(BF) = \mu^* B\partial_t F. \end{cases} \tag{2.10}$$

Consider a solution $F(t, x) = e^{\mp t|T_B|} f$ to (2.9) in $\mathbf{R}_\pm^{n+1}$ as in Lemma 2.34, where $f, T_B f \in E_B^\pm \mathcal{V}_B$. Using $\partial_t F = -T_B F$ and Lemma 2.11, we get

$$\begin{aligned}(md_{t,x}F)|_{t=0} &= m(d - \mu T_B)f = \big(md - N^+ M_B^{-1}\big(md + B^{-1}md^* B\big)\big)f \\ &= M_B^{-1}\big(\big(M_B - B^{-1}N^+ B\big)md - B^{-1}N^+ md^* B\big)f \\ &= M_B^{-1}\big(-N^- md - B^{-1}N^+ md^* B\big)f = M_B^{-1}\big(d\mu + B^{-1}d^*\mu^* B\big)f.\end{aligned}$$

Thus we see that $d_{t,x}F = 0 = d_{t,x}^*(BF)$ at $t = 0$ if and only if $d\mu f = 0 = d^*\mu^* Bf$.

We may also rewrite Eq. (2.9) as

$$\big(d_{t,x}m + B^{-1}d_{t,x}^* mB\big)F(t, x) = 0, \tag{2.11}$$

by using $md_{t,x} = \partial_t - d_{t,x}m$ and $md^*_{t,x} = -\partial_t - d^*_{t,x}m$ from Lemma 2.7. Consider the case when both terms vanish, i.e. when

$$\begin{cases} d_{t,x}(mF) = 0, \\ d^*_{t,x}(mBF) = 0, \end{cases} \quad \text{or equivalently} \quad \text{when} \begin{cases} dF = \mu^*\partial_t F, \\ d^*(BF) = -\mu B\partial_t F. \end{cases} \tag{2.12}$$

For $F(t,x) = e^{\mp t|T_B|}f$ solving (2.9) we also have

$$\begin{aligned} d_{t,x}(mF)|_{t=0} &= \partial_t F|_{t=0} - md_{t,x}F|_{t=0} \\ &= M_B^{-1}\big(dm + B^{-1}d^*mB\big)f - M_B^{-1}\big(d\mu + B^{-1}d^*\mu^*B\big)f \\ &= M_B^{-1}\big(d\mu^* + B^{-1}d^*\mu B\big)f. \end{aligned}$$

Thus we see that $d_{t,x}(mF) = 0 = d^*_{t,x}(mBF)$ at $t = 0$ if and only if $d\mu^*f = 0 = d^*\mu Bf$.

Definition 2.43. Introduce the closed, densely defined operators

$$\hat{T}_B := -M_B^{-1}\big(d\mu^* + B^{-1}d^*\mu B\big), \qquad \check{T}_B := -M_B^{-1}\big(d\mu + B^{-1}d^*\mu^*B\big),$$

with domains

$$\begin{aligned} \mathsf{D}(\hat{T}_B) &:= \big\{f \in \mathcal{H};\ \mu^*f \in \mathsf{D}(d),\ \mu Bf \in \mathsf{D}(d^*)\big\}, \\ \mathsf{D}(\check{T}_B) &:= \big\{f \in \mathcal{H};\ \mu f \in \mathsf{D}(d),\ \mu^*Bf \in \mathsf{D}(d^*)\big\}, \end{aligned}$$

respectively, and define the closed subspaces

$$\begin{aligned} \hat{\mathcal{H}}_B &:= \big\{f \in \mathcal{H};\ d(\mu f) = 0 = d^*(\mu^*Bf)\big\}, \\ \check{\mathcal{H}}_B &:= \big\{f \in \mathcal{H};\ d(\mu^*f) = 0 = d^*(\mu Bf)\big\}. \end{aligned}$$

Proposition 2.44. *We have a topological splitting of $\mathcal{H}$ into closed subspaces*

$$\mathcal{H} = \hat{\mathcal{H}}_B \oplus \check{\mathcal{H}}_B.$$

Furthermore we have $T_B = \hat{T}_B + \check{T}_B$ with $\mathsf{D}(T_B) = \mathsf{D}(\hat{T}_B) \cap \mathsf{D}(\check{T}_B)$, and

$$\overline{\mathsf{R}(\hat{T}_B)} = \hat{\mathcal{H}}_B = \mathsf{N}(\check{T}_B), \qquad \overline{\mathsf{R}(\check{T}_B)} = \check{\mathcal{H}}_B = \mathsf{N}(\hat{T}_B).$$

Thus, if we identify $\hat{T}_B$ with its restriction to $\hat{\mathcal{H}}_B$ and $\check{T}_B$ with its restriction to $\check{\mathcal{H}}_B$, then these are bisectorial operators with spectral and resolvent estimates as in Proposition 2.22.

Proof. Clearly $\mathsf{D}(\hat{T}_B) \cap \mathsf{D}(\check{T}_B) = \mathsf{D}(d) \cap B^{-1}\mathsf{D}(d^*) = \mathsf{D}(T_B)$ and $\hat{T}_B + \check{T}_B = T_B$. Furthermore Lemma 2.21(i) shows that $\mathsf{N}(\check{T}_B) = \hat{\mathcal{H}}_B$ and $\mathsf{N}(\hat{T}_B) = \check{\mathcal{H}}_B$.

(1) To show $\mathsf{R}(\hat{T}_B) \subset \hat{\mathcal{H}}_B$, let $f \in \mathsf{D}(\hat{T}_B)$ and use Lemma 2.11 to get

$$\begin{aligned}
d\mu(\hat{T}_B f) &= -d\mu M_B^{-1}\big(d\mu^* + B^{-1}d^*\mu B\big)f \\
&= -dmN^+M_B^{-1}\big(d\mu^* + B^{-1}d^*\mu B\big)f \\
&= -dmM_B^{-1}B^{-1}N^+B\big(d\mu^* + B^{-1}d^*\mu B\big)f \\
&= -dmM_B^{-1}\big(B^{-1}N^+B - N^-\big)d\mu^* f \\
&= -dmd\mu^* f = md^2\mu^* f = 0,
\end{aligned}$$

and similarly

$$\begin{aligned}
d^*\mu^* B(\hat{T}_B f) &= -d^*mB\big(B^{-1}N^-B\big)M_B^{-1}\big(d\mu^* + B^{-1}d^*\mu B\big)f \\
&= -d^*mBM_B^{-1}N^-\big(d\mu^* + B^{-1}d^*\mu B\big)f \\
&= -d^*mBM_B^{-1}\big(N^- - B^{-1}N^+B\big)B^{-1}d^*\mu Bf \\
&= d^*md^*\mu Bf = -m(d^*)^2\mu Bf = 0.
\end{aligned}$$

A similar calculation shows that $\mathsf{R}(\check{T}_B) \subset \check{\mathcal{H}}_B$.

(2) Next we show that for all $f \in \mathsf{D}(T_B)$, we have

$$\|T_B f\| \approx \|\hat{T}_B f\| + \|\check{T}_B f\|.$$

Using that M_B and B are isomorphisms, Lemma 2.21(i) with $\Gamma = d$ and $\widetilde{\Gamma} = d^*$ and orthogonality $\|\mu g\|^2 + \|\mu^* g\|^2 = \|g\|^2$, we obtain

$$\begin{aligned}
\|\hat{T}_B f\| + \|\check{T}_B f\| &\approx \big(\|\mu^* df\| + \|\mu d^* Bf\|\big) + \big(\|\mu df\| + \|\mu^* d^* Bf\|\big) \\
&\approx \|df\| + \|d^* Bf\| \approx \|T_B f\|.
\end{aligned}$$

(3) Clearly $\hat{\mathcal{H}}_B \cap \check{\mathcal{H}}_B = \{0\}$ and $\mathsf{R}(T_B) \subset \mathsf{R}(\hat{T}_B) + \mathsf{R}(\check{T}_B)$. Taking closures, using that T_B has dense range in $\mathcal{H}$ and using (2) yields

$$\mathcal{H} = \overline{\mathsf{R}(\hat{T}_B)} \oplus \overline{\mathsf{R}(\check{T}_B)}.$$

Thus from (1) it follows that $\overline{\mathsf{R}(\hat{T}_B)} = \hat{\mathcal{H}}_B$ and $\overline{\mathsf{R}(\check{T}_B)} = \check{\mathcal{H}}_B$ and that $\mathcal{H} = \hat{\mathcal{H}}_B \oplus \check{\mathcal{H}}_B$. □

It follows that T_B is diagonal in the splitting $\mathcal{H} = \hat{\mathcal{H}}_B \oplus \check{\mathcal{H}}_B$. We shall now further decompose the subspace $\hat{\mathcal{H}}_B$ into the subspaces of homogeneous k-vector field $\bigwedge^k$, which also are preserved by T_B. The same decomposition can be made for $\check{\mathcal{H}}_B$, but this is not useful since T_B does not preserve these subspaces.

Definition 2.45. Let $\mathcal{H}^k := L_2(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ and $\hat{\mathcal{H}}_B^k := \mathcal{H}^k \cap \hat{\mathcal{H}}_B$.

Lemma 2.46. *The operator T_B preserve all subspaces $\hat{\mathcal{H}}_B^0, \hat{\mathcal{H}}_B^1, \hat{\mathcal{H}}_B^2, \ldots, \hat{\mathcal{H}}_B^{n+1}$ and $\check{\mathcal{H}}_B$ (but not the subspaces $\mathcal{H}^k$) and we have a splitting*

$$\hat{\mathcal{H}}_B = \hat{\mathcal{H}}_B^0 \oplus \hat{\mathcal{H}}_B^1 \oplus \hat{\mathcal{H}}_B^2 \oplus \cdots \oplus \hat{\mathcal{H}}_B^{n+1}.$$

Furthermore, for the operators $\hat{N}_B$, $\check{N}_B$ *and* N, *we have mapping properties*

$$\hat{N}_B : \hat{\mathcal{H}}^k_B \to \hat{\mathcal{H}}^k_B, \qquad \check{N}_B : \check{\mathcal{H}}_B \to \check{\mathcal{H}}_B \quad \text{and} \quad N : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1.$$

Proof. As $d\mu^*$, $d^*\mu$, B and $N^\pm$ all preserve $\mathcal{H}^k$ it follows that $T_B(\hat{\mathcal{H}}^k_B) = \hat{T}_B(\hat{\mathcal{H}}^k_B) \subset \hat{\mathcal{H}}^k_B$. To show the splitting of $\hat{\mathcal{H}}_B$ it suffices to prove

$$\hat{\mathcal{H}}_B \subset \hat{\mathcal{H}}^0_B \oplus \hat{\mathcal{H}}^1_B \oplus \hat{\mathcal{H}}^2_B \oplus \cdots \oplus \hat{\mathcal{H}}^{n+1}_B.$$

To this end, let $f \in \hat{\mathcal{H}}_B$ and write f_k for the $\bigwedge^k$ part of f. Since $d\mu f = 0 = d^*\mu^* B f$ we get

$$0 = (d\mu f)_{k+2} = d\big((\mu f)_{k+1}\big) = d\mu(f_k),$$
$$0 = (d^*\mu^* B f)_{k-2} = d^*\big((\mu^* B f)_{k-1}\big) = d^*\mu^*\big((Bf)_k\big) = d^*\mu^* B(f_k),$$

for all k. Thus $f_k \in \hat{\mathcal{H}}^k_B$ and $f \in \bigoplus_k \hat{\mathcal{H}}^k_B$.

To prove the mapping properties for $\hat{N}_B$, note that if $f = f_1 + f_2$ in the splitting $\mathcal{H} = B^{-1}N^+\mathcal{H} \oplus N^-\mathcal{H}$, then $f \in \hat{\mathcal{H}}_B$ if and only if $\mu f_2 \in \mathsf{D}(d)$ and $\mu B f_1 \in \mathsf{D}(d^*)$, according to Definition 2.43. Clearly $f_1 - f_2 = \hat{N}_B(f) \in \hat{\mathcal{H}}_B$ if $f \in \hat{\mathcal{H}}_B$. Since $\hat{N}_B$ preserves $\mathcal{H}^k$, the desired mapping property follows. The proofs for $\check{N}_B$ and N are similar. □

Lemma 2.47. *With* $\langle\cdot,\cdot\rangle_B$ *denoting the duality from Definition* 2.16, *we have dual operators*

$$(\hat{T}_B)' = -\hat{T}_{B^*}, \qquad (\check{T}_B)' = -\check{T}_{B^*},$$

and restricted dualities $\langle\hat{\mathcal{H}}_B, \hat{\mathcal{H}}_{B^*}\rangle_B$, $\langle\check{\mathcal{H}}_B, \check{\mathcal{H}}_{B^*}\rangle_B$ *and* $\langle\hat{\mathcal{H}}^k_B, \hat{\mathcal{H}}^k_{B^*}\rangle_B$ *for all* k. *In the case* $k = 1$, *we shall write the duality as* $\langle\hat{\mathcal{H}}^1, \hat{\mathcal{H}}^1\rangle_A$.

Proof. The proofs of $\hat{T}'_B = -\hat{T}_{B^*}$ and $\check{T}'_B = -\check{T}_{B^*}$ are similar to that of $T'_B = -T_{B^*}$ in Proposition 2.17. From this we get that the annihilator of $\hat{\mathcal{H}}_B = \overline{\mathsf{R}(\hat{T}_B)}$ is $\mathsf{N}(\hat{T}_{B^*}) = \check{\mathcal{H}}_{B^*}$ which is a complement of $\hat{\mathcal{H}}_{B^*}$. Thus we see from Remark 2.19 that $\langle\hat{\mathcal{H}}_B, \hat{\mathcal{H}}_{B^*}\rangle_B$ is a duality. The proof of the duality $\langle\check{\mathcal{H}}_B, \check{\mathcal{H}}_{B^*}\rangle_B$ is similar. Also, since $BN^+ - N^-B$ in the definition of $\langle\cdot,\cdot\rangle_B$ preserves $\mathcal{H}^k$ we also have dualities $\langle\hat{\mathcal{H}}^k_B, \hat{\mathcal{H}}^k_{B^*}\rangle_B$ for all k. □

Remark 2.48.

- The subspace of vector fields with a curl-free tangential part

$$\hat{\mathcal{H}}^1_B = \big\{ f \in L_2\big(\mathbf{R}^n; \textstyle\bigwedge^1\big);\ d(e_0 \wedge f) = 0 \big\}$$

  is independent of B and coincides with the space $\hat{\mathcal{H}}^1$ from Section 1.1. Furthermore, the operator T_A there coincides with $T_B|_{\hat{\mathcal{H}}^1} = \hat{T}_B|_{\hat{\mathcal{H}}^1}$.
- The dense subspace $\mathcal{V}_B \subset \mathcal{H}$ splits $\mathcal{V}_B = \hat{\mathcal{V}}_B \oplus \check{\mathcal{V}}_B$ with Proposition 2.44, where $\hat{\mathcal{V}}_B := \mathcal{V}_B \cap \hat{\mathcal{H}}_B \subset \hat{\mathcal{H}}_B$ and $\check{\mathcal{V}}_B := \mathcal{V}_B \cap \check{\mathcal{H}}_B \subset \check{\mathcal{H}}_B$ are dense subspaces, and we can further decompose $\hat{\mathcal{V}}_B$ into homogeneous k-vector fields, $\hat{\mathcal{V}}_B = \hat{\mathcal{V}}^0_B \oplus \hat{\mathcal{V}}^1_B \oplus \cdots \oplus \hat{\mathcal{V}}^{n+1}_B$, where $\hat{\mathcal{V}}^k_B := \mathcal{V}_B \cap \hat{\mathcal{H}}^k_B$.

- Furthermore, all these dense subspaces $\hat{\mathcal{V}}^k_B$, $\hat{\mathcal{V}}_B$, $\check{\mathcal{V}}_B$ of $\hat{\mathcal{H}}^k_B$, $\hat{\mathcal{H}}_B$, $\check{\mathcal{H}}_B$ splits algebraically into Hardy spaces similar to Lemma 2.34, e.g. $\hat{\mathcal{V}}^k_B = E^+_B\hat{\mathcal{V}}^k_B + E^-_B\hat{\mathcal{V}}^k_B$. Here $f \in E^\pm_B\hat{\mathcal{V}}_B$ if and only if $F(t,x)$ satisfies (2.10), and $f \in E^\pm_B\check{\mathcal{V}}_B$ if and only if $F(t,x)$ satisfies (2.12).

2.5. Operator equations and estimates for solutions

Our objective in this section is to set up our boundary operator method for solving the boundary value problems (Neu-A), (Reg-A), (Neu$^\perp$-A) and (Dir-A) as well as the transmission problem (Tr-$B^k\alpha^\pm$). Under the assumption that T_B has quadratic estimates, which is made throughout this section, we first show that solutions $F(t,x)$ are determined by their traces f through the reproducing Cauchy type formula $F_t = e^{-t|T_B|}f$. Recall from Lemma 2.46 that both operators E_B and N_B preserve all subspaces $\hat{\mathcal{H}}^k_B$. We shall write $E_{B^k} = E_B|_{\hat{\mathcal{H}}^k_B} = \operatorname{sgn}(T_B|_{\hat{\mathcal{H}}^k_B})$ and $N_{B^k} = N_B|_{\hat{\mathcal{H}}^k_B}$ for the restrictions. In particular, we write $A = B^1$ when $k = 1$.

Lemma 2.49. *Assume that T_B satisfies quadratic estimates, let $f \in \mathcal{H}^k$ and let $(0,\infty) \ni t \mapsto F_t(x) = F(t,x) \in \mathcal{H}^k$ be a family of functions. Then the following are equivalent.*

(i) $f \in E^+_B\hat{\mathcal{H}}^k_B$ *and* $F_t = e^{-t|T_B|}f$.
(ii) $F_t \in C^1(\mathbf{R}_+;\mathcal{H}^k)$ *and satisfies the equations*

$$\begin{cases} d^*_{t,x}\big(B(x)F(t,x)\big) = 0,\\ d_{t,x}F(t,x) = 0,\end{cases}$$

and have L_2 limits $\lim_{t\to0^+}F_t = f$ *and* $\lim_{t\to\infty}F_t = 0$.

In fact, such F_t belong to $C^j(\mathbf{R}_+;\mathcal{H}^k)$ for all $j \geqslant 1$ and are in one-to-one correspondence with the trace $f \in E^+_B\hat{\mathcal{H}}^k_B$, and we have equivalences of norms

$$\|f\| \approx \sup_{t>0}\|F_t\| \approx |||t\partial_tF_t|||.$$

The corresponding reproducing formula $F_t = e^{t|T_B|}f$, $f = F|_{\mathbf{R}^n} \in E^-_B\hat{\mathcal{H}}^k_B$ is also valid for F solving the equations in $\mathbf{R}^{n+1}_-$, and the corresponding estimates hold.

Proof. (i) implies (ii). As in the proof of Lemma 2.34, from Lemma 2.28 it follows that $\lim_{t\to0}F_t = f$ and $\lim_{t\to\infty}F_t = 0$, and also that $\partial_t^jF_t = (-|T_A|)^je^{-t|T_A|}f$. Therefore $F_t \in C^j(\mathbf{R}_+;\mathcal{H}^k)$ for all j. For $j = 1$, we get $0 = (\partial_t + |T_B|)F_t = (\partial_t + T_B)F_t$, since $F_t \in E^+_B\hat{\mathcal{H}}^k_B$. As explained in Section 2.4 this is equivalent with the two equations $d_{t,x}F = 0$ and $d^*_{t,x}(BF) = 0$.

(ii) implies (i). The two equations can be written $(-\mu^*\partial_t + d^*)BF = 0$ and $(\mu\partial_t + d)F = 0$. Applying μ^* and μ, respectively, to these equations and using nilpotence, we obtain $\mu^*d^*BF = 0$ and $\mu dF = 0$. Therefore $F_t \in \hat{\mathcal{H}}^k_B$, and since $\hat{\mathcal{H}}^k_B$ is closed we also have $f \in \hat{\mathcal{H}}^k_B$.

Next we write $F_t = F_t^+ + F_t^-$, where $F_t^\pm := E^\pm_BF_t \in E^\pm_B\hat{\mathcal{H}}^k_B$ and similarly $f = f^+ + f^-$. We rewrite the equations satisfied by F_t as $\partial_tF_t + T_BF_t = 0$. Applying the projections $E^\pm_B$ to this equation yields

$$\partial_tF_t^+ + |T_B|F_t^+ = 0, \qquad \partial_tF_t^- - |T_B|F_t^- = 0,$$

since $T_B F_t^\pm = \pm|T_B|F_t^\pm$. Fix $t > 0$. Then it follows that $e^{(t-s)|T_B|}F_s^-$ is constant for $s \in (t,\infty)$ and that $e^{(s-t)|T_B|}F_s^+$ is constant for $s \in (0,t)$. Taking limits, this shows that

$$F_t^- = \lim_{s\to t^+} e^{(t-s)|T_B|}F_s^- = \lim_{s\to\infty} e^{(t-s)|T_B|}F_s^- = 0 \quad \text{and}$$
$$F_t^+ = \lim_{s\to t^-} e^{(s-t)|T_B|}F_s^+ = \lim_{s\to 0} e^{(s-t)|T_B|}F_s^+ = e^{-t|T_B|}f^+.$$

Therefore $f = f^+ \in E_B^+\hat{\mathcal{H}}_B^k$ and $F_t = e^{-t|T_B|}f$.

To prove the norm estimates we note that $\|f\| = \lim_{t\to 0}\|F_t\|$. By Proposition 2.32, $e^{-t|T_B|}$ are uniformly bounded and thus $\sup_{t>0}\|F_t\| \lesssim \|f\|$. Furthermore, using Proposition 2.31 with $\psi(z) = ze^{-|z|}$ shows that $|||t\partial_t F_t||| \approx \|f\|$. □

Remark 2.50. In proving that (ii) implies (i), it suffices to assume that $\|F_t\|$ grows at most polynomially when $t\to\infty$. Indeed, from the equation $F_t^- = e^{-(s-t)|T_B|}F_s^-$ for $s > t$, it then follows that

$$\left\|T_B^k F_t^-\right\| = (s-t)^{-k}\left\|\big((s-t)T_B\big)^k e^{-(s-t)|T_B|}F_s^-\right\| \leqslant C(s-t)^{-k}\|F_s\| \to 0,$$

when $s\to\infty$. Since T_B is injective, this shows that $F_t^- = 0$ and therefore that $F_t = e^{-t|T_B|}f \in E_B^+\hat{\mathcal{H}}_B^k$ as before.

We now proceed by showing how, given data g in (Tr-$B^k\alpha^\pm$), we can solve for the trace $f = F|_{\mathbf{R}^n}$ by using the boundary operators E_B and N_B.

Lemma 2.51. *Assume that T_B satisfies quadratic estimates, so that the Hardy projections $E_B^\pm$ are bounded by Proposition* 2.32*, let $\alpha^\pm \in \mathbf{C}$ be given jump parameters and define the associated spectral point $\lambda := (\alpha^+ + \alpha^-)/(\alpha^+ - \alpha^-)$. Then*

$$\lambda - E_{B^k}N_{B^k} : \hat{\mathcal{H}}_B^k \to \hat{\mathcal{H}}_B^k$$

is an isomorphism if and only if the transmission problem (Tr-$B^k\alpha^\pm$) *is well posed.*

Proof. If we identify the k-vector fields $F^\pm(t,x)$ in Transmission problem (Tr-$B^k\alpha^\pm$) with the boundary traces $f^\pm$ and write $f = f^+ + f^-$ using Lemma 2.49, then we see that the transmission problem is equivalent with the system of equations

$$\begin{cases} N_{B^k}^+\big(\alpha^- E_{B^k}^+ - \alpha^+ E_{B^k}^-\big) f = N_{B^k}^+ g, \\ N_{B^k}^-\big(\alpha^+ E_{B^k}^+ - \alpha^- E_{B^k}^-\big) f = N_{B^k}^- g. \end{cases}$$

Using $E_{B^k}^\pm = \frac{1}{2}(I \pm E_{B^k})$ and adding up the equations, we see that the system is equivalent with the equation

$$(\lambda - E_{B^k}N_{B^k})f = \frac{2}{\alpha^+ - \alpha^-}E_{B^k}g.$$

This proves the lemma. □

Next we consider $k = 1$ and the boundary value problems (Neu-A), (Reg-A) and (Neu$^\perp$-A). By Lemma 2.49, we have the following.

(1) (Neu-A) is well posed if and only if the restricted projection $N_A^- : E_A^+\hat{\mathcal{H}}^1 \to N_A^-\hat{\mathcal{H}}^1$ is an isomorphism.
(2) (Reg-A) is well posed if and only if the restricted projection $N_A^+ : E_A^+\hat{\mathcal{H}}^1 \to N_A^+\hat{\mathcal{H}}^1$ is an isomorphism, or equivalently if and only if $N^+ : E_A^+\hat{\mathcal{H}}^1 \to N^+\hat{\mathcal{H}}^1$ is an isomorphism. Note that both N_A^+ and N^+ project along $N^-\mathcal{H}^1$.
(3) (Neu$^\perp$-A) is well posed if and only if the restricted projection $N^- : E_A^+\hat{\mathcal{H}}^1 \to N^-\hat{\mathcal{H}}^1$ is an isomorphism.

Proposition 2.52. *Assume that T_A satisfies quadratic estimates. Then* (Reg-A) *is well posed if and only if* (Neu$^\perp$-A^*) *is well posed.*

Proof. We need to show that if $N^+ : E_A^+\hat{\mathcal{H}}^1 \to N^+\hat{\mathcal{H}}^1$ is an isomorphism, then so is $N^- : E_{A^*}^+\hat{\mathcal{H}}^1 \to N^-\hat{\mathcal{H}}^1$. The proof uses two facts. First that we have adjoint operators $(E_A)' = -E_{A^*}$ and $N' = N$ according to Proposition 2.17 and Lemma 2.47. As in Remark 2.19, this shows that we have dual spaces $\langle E_A^+\hat{\mathcal{H}}^1, E_{A^*}^-\hat{\mathcal{H}}^1\rangle_A$ and $\langle N^+\hat{\mathcal{H}}^1, N^+\hat{\mathcal{H}}^1\rangle_A$, and we see that

$$\langle N^+ f, g\rangle_A = \langle f, g\rangle_A = \langle f, E_{A^*}^- g\rangle_A,$$

for all $f \in E_A^+\hat{\mathcal{H}}^1$ and $g \in N^+\hat{\mathcal{H}}^1$. Therefore, the restricted projections $N^+ : E_A^+\hat{\mathcal{H}}^1 \to N^+\hat{\mathcal{H}}^1$ and $E_{A^*}^- : N^+\hat{\mathcal{H}}^1 \to E_{A^*}^-\hat{\mathcal{H}}^1$ are adjoint.

Secondly, if $R_1^\pm$ and $R_2^\pm$ are two pairs of complementary projections in a Hilbert space $\mathcal{H}$, as in Definition 2.13, then $R_1^- : R_2^+\mathcal{H} \to R_1^-\mathcal{H}$ has a priori estimates, as in Remark 2.18, if and only if $R_2^- : R_1^+\mathcal{H} \to R_2^-\mathcal{H}$ has a priori estimates. Indeed, both statements are seen to be equivalent with that the subspaces $R_1^+\mathcal{H}$ and $R_2^+\mathcal{H}$ are transversal, i.e. that the estimate $\|f_1 + f_2\| \approx \|f_1\| + \|f_2\|$ holds for all $f_1 \in R_1^+\mathcal{H}$ and $f_2 \in R_2^+\mathcal{H}$. To see this, assume that $\|R_1^- f_2\| \gtrsim \|f_2\|$ holds for all $f_2 \in R_2^+\mathcal{H}$. Then $\|f_2\| \lesssim \|R_1^-(f_1 + f_2)\| \lesssim \|f_1 + f_2\|$ for all $f_i \in R_i^+\mathcal{H}$, which proves transversality. Conversely, assume that $R_1^+\mathcal{H}$ and $R_2^+\mathcal{H}$ are transversal. Then $f_2 - R_1^- f_2 = R_1^+ f_2 =: f_1 \in R_1^+\mathcal{H}$ for all $f_2 \in R_2^+\mathcal{H}$. Therefore $\|R_1^- f_2\| = \|f_2 - f_1\| \approx \|f_2\| + \|f_1\| \geqslant \|f_2\|$. The same argument can be used to show that transversality also holds if and only if $\|R_2^- f_1\| \gtrsim \|f_1\|$ holds for all $f_1 \in R_1^+\mathcal{H}$.

To prove the proposition, assume that $N^+ : E_A^+\hat{\mathcal{H}}^1 \to N^+\hat{\mathcal{H}}^1$ is an isomorphism. It follows that the adjoint operator $E_{A^*}^- : N^+\hat{\mathcal{H}}^1 \to E_{A^*}^-\hat{\mathcal{H}}^1$ also is an isomorphism. Using the second fact above twice, shows that

$$E_A^- : N^-\hat{\mathcal{H}}^1 \to E_A^-\hat{\mathcal{H}}^1 \quad \text{and} \quad N^- : E_{A^*}^+\hat{\mathcal{H}}^1 \to N^-\hat{\mathcal{H}}^1$$

have a priori estimates. As these are adjoint operators as well, both must in fact be isomorphisms. In particular we have shown that (Neu$^\perp$-A^*) is well posed. The proof of the converse implication is similar. □

When perturbing A, it is preferable not to have operators defined on spaces like $E_A^+\hat{\mathcal{H}}$, which varies with A. We have the following.

Lemma 2.53. *Assume that T_A satisfies quadratic estimates.*

(1) *If $I - E_A N_A : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$ is an isomorphism, then Neumann problem* (Neu-A) *is well posed.*
(2) *If $I + E_A N_A : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$ is an isomorphism, or if $I + E_A N : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$ is an isomorphism, then Regularity problem* (Reg-A) *is well posed.*
(3) *If $I - E_A N : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$ is an isomorphism, then Neumann problem* (Neu$^\perp$-A) *is well posed.*

Proof. Assume for example that $I + E_A N_A$ is an isomorphism. We need to prove that $N_A^+ : E_A^+\hat{\mathcal{H}}^1 \to N_A^+\hat{\mathcal{H}}^1$ is an isomorphism. Note that if $N_A^+ f = g$ where $f \in E_A^+\hat{\mathcal{H}}^1$, then

$$g = N_A^+ f = \frac{1}{2}(I + N_A)f = \frac{1}{2}(E_A + N_A)f = \frac{1}{2}E_A(I + E_A N_A)f.$$

If $g \in N_A^+\hat{\mathcal{H}}^1$, let $f := 2(E_A + N_A)^{-1}g$. Then it follows that $0 = N_A^- g = \frac{1}{2}(E_A + N_A)(E_A^- f)$, since $N_A^-(E_A + N_A) = \frac{1}{2}(E_A + N_A - I - N_A E_A) = (E_A + N_A)E_A^-$, so $E_A^- f = 0$ and therefore $f \in E_A^+\hat{\mathcal{H}}^1$.

A similar calculation proves well-posedness of the other boundary value problems. □

Remark 2.54. More generally, letting $k = 1$ and $(\alpha^+, \alpha^-) = (1, 0)$, i.e. $\lambda = 1$, in Lemma 2.51, we see that $I - E_A N_A$ is an isomorphism if and only if the restricted projections

$$N_A^+ : E_A^-\hat{\mathcal{H}}^1 \to N_A^+\hat{\mathcal{H}}^1 \quad \text{and} \quad N_A^- : E_A^+\hat{\mathcal{H}}^1 \to N_A^-\hat{\mathcal{H}}^1$$

are isomorphisms. Similarly, if $(\alpha^+, \alpha^-) = (0, 1)$, i.e. $\lambda = -1$ in Lemma 2.51, we see that $I + E_A N_A$ is an isomorphism if and only if the restricted projections $N_A^+ : E_A^+\hat{\mathcal{H}}^1 \to N_A^+\hat{\mathcal{H}}^1$ and $N_A^- : E_A^-\hat{\mathcal{H}}^1 \to N_A^-\hat{\mathcal{H}}^1$ are isomorphisms.

We next turn to the Dirichlet problem (Dir-A), where we aim to prove an analogue of Lemma 2.49 which characterises the solution U_t as a Poisson integral of the boundary trace u. As discussed in the introduction, we shall use (Neu$^\perp$-A) to construct the solution U_t. Given Dirichlet data $u \in L_2(\mathbf{R}^n; \mathbf{C})$, we form $ue_0 \in N^-\hat{\mathcal{H}}^1$. It then follows from Lemma 2.12 that the vector field F_t solving (Neu$^\perp$-A) with data ue_0, has a normal component $U := F_0$ which satisfies the second order equation (1.1). We now define the *Poisson integral* of u to be

$$\mathcal{P}_t(u) := (F_t, e_0), \quad \text{when } F_t = e^{-t|T_A|}f \text{ and } (f, e_0) = u.$$

Lemma 2.55. *Assume that T_A satisfies quadratic estimates and that Neumann problem* (Neu$^\perp$-A) *is well posed. Let $u \in L_2(\mathbf{R}^n; \mathbf{C})$ and let $(0, \infty) \ni t \mapsto U_t(x) = U(t, x) \in L_2(\mathbf{R}^n; \mathbf{C})$ be a family of functions. Then the following are equivalent.*

(i) *$U_t = \mathcal{P}_t u$ for all $t > 0$.*
(ii) *$U_t \in C^2(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}))$, $\nabla_{t,x} U_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{n+1}))$ and U satisfies the equation*

$$\mathrm{div}_{t,x} A(x)\nabla_{t,x} U(t,x) = 0,$$

and we have L_2 limits $\lim_{t\to 0^+} U_t = u$, $\lim_{t\to\infty} U_t = 0$ *and* $\lim_{t\to\infty} \nabla_{t,x} U_t = 0$.

If this holds, then $U_t, \nabla_{t,x} U_t \in C^j(\mathbf{R}_+; L_2(\mathbf{R}^n))$ for all $j \geqslant 1$. Furthermore, U_t is in one-to-one correspondence with the trace u, and we have equivalences of norms

$$\|u\| \approx \sup_{t>0} \|U_t\| \approx |||t\partial_t U_t|||.$$

Proof. (i) implies (ii). Assume that $U_t = (F_t, e_0)$, where $F_t = e^{-t|T_A|} f$ and $(f, e_0) = u$. As in the proof of Lemma 2.34, from Lemma 2.28 it follows that $F_t \to f$, and therefore $U_t \to u$, and also that $\partial_t^j F_t = (-|T_A|)^j e^{-t|T_A|} f$. Therefore $F_t \in C^j(\mathbf{R}_+; L_2(\mathbf{R}^n))$ for all j, and so does U_t. For $j = 1$, we see that F satisfies the Dirac type equation since $|T_A| F_t = T_A F_t$, and Lemma 2.12 thus shows that U_t satisfies the second order equation. Furthermore, we note from the expression (1.5) for T_A, that $\nabla_x U_t = (\partial_t F_t)_\parallel$. Thus $\nabla_x U_t \in C^j(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}))$ for all j, and yet another application of Lemma 2.28 shows that $U_t = o(1)$ and $\nabla_{t,x} U_t = o(1/t)$ when $t \to \infty$.

(ii) implies (i). Assume U_t has the stated properties and boundary trace u. Consider the family of vector fields $G_t := \nabla_{t,x} U_t$. Since these satisfy Lemma 2.49(ii) for $t \geqslant s > 0$ with boundary trace G_s, we obtain that $G_{s+t} = e^{-t|T_A|} G_s$ for all $s, t > 0$. For the normal components, this means that

$$\partial_0 U_{s+t} = \mathcal{P}_t(\partial_0 U_s),$$

or equivalently that $\partial_s(U_{s+t} - \mathcal{P}_t(U_s)) = 0$. Since $\lim_{s\to\infty} U_s = 0$, we must have $U_{s+t} = \mathcal{P}_t(U_s)$ for all $s, t > 0$. Letting $s \to 0$, we conclude that $U_t = \mathcal{P}_t(u)$.

The equivalence of norms $\|u\| \approx \sup_{t>0} \|U_t\|$ follows from the uniform boundedness of the operators $\mathcal{P}_t$. For the equivalence $\|u\| \approx |||t\partial_t U_t|||$ we use that (Neu$^\perp$-A) is well posed and the corresponding square function estimate for F_t from Lemma 2.49 to get $\|u\| \approx \|f\| \approx |||t\partial_t F_t||| \approx |||t\partial_t U_t|||$, since for all $t > 0$ we have $\|\partial_t U_t\| = \|N^-(\partial_t F_t)\| \approx \|\partial_t F_t\|$. □

We end this section with the proof of the non-tangential estimate $\|\widetilde{N}_*(F)\| \approx \|f\|$ in Theorem 1.1.

Proposition 2.56. *Assume that T_A satisfies quadratic estimates. Let $F_t = e^{-t|T_A|} f$, where $f \in E_A^+ \hat{\mathcal{H}}^1$. Then $\|f\| \approx \|\widetilde{N}_*(F)\|$, where the non-tangential maximal function is*

$$\widetilde{N}_*(F)(x) := \sup_{t>0} \biggl(\fint\!\!\fint_{D(t,x)} |F(s,y)|^2 \, ds\, dy \biggr)^{1/2},$$

and $D(t,x) := \{(s,y) \in \mathbf{R}^{n+1}_+;\ |s-t| < c_0 t,\ |y-x| < c_1 t\}$, *for given constants* $c_0 \in (0,1)$ *and* $c_1 > 0$.

The proof uses the following lemma.

Lemma 2.57. *Let $f \in \hat{\mathcal{H}}^1$ and define $H_t = (1 + itT_B)^{-1} f \in \hat{\mathcal{H}}^1$. Write $H_t = H_t^{1,0} e_0 + H_t^{1,\parallel}$ and $f = f^{1,0} e_0 + f^{1,\parallel}$.*

(i) *The normal component* $H_t^{1,0}$ *satisfies the second order divergence form equation*

$$[1 \quad it\,\mathrm{div}]\begin{bmatrix} a_{\perp\perp} & a_{\perp\|} \\ a_{\|\perp} & a_{\|\|} \end{bmatrix}\begin{bmatrix} 1 \\ it\nabla \end{bmatrix} H_t^{1,0} = [1 \quad it\,\mathrm{div}]\begin{bmatrix} a_{\perp\perp} f^{1,0} \\ -a_{\|\|} f^{1,\|} \end{bmatrix},$$

where we identify normal vectors ue_0 *with scalars* u*, and the tangential component* $H_t^{1,\|}$ *satisfies*

$$H_t^{1,\|} = f^{1,\|} + it\nabla H_t^{1,0}.$$

(ii) *There exist* $p < 2$ *and* $q > 2$ *such that for any fixed* $r_0 < \infty$ *we have*

$$\left(\fint_{B(x,r_0 t)} |H_t^{1,0}|^q \right)^{1/q} + \left(\fint_{B(x,r_0 t)} |H_t^{1,\|}|^p \right)^{1/p} \lesssim M\big(|f|^p\big)^{1/p}(x),$$

for all $x \in \mathbf{R}^n$ *and* $t > 0$*. Here* $M(f)(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|\,dy$ *denotes the Hardy–Littlewood maximal function.*

Proof. (i) Multiplying the equation $(1 + itT_B)H_t = f$ with M_B we get

$$\big(m(\mu + itd) + B^{-1}\big(m(-\mu^* + itd^*)\big)B\big)H_t = M_B f.$$

Similar to the proof of Lemma 2.12 we can now use the anticommutation relations from Lemma 2.7 to rewrite this equation as

$$\big(-(\mu + itd)m + B^{-1}\big(-(-\mu^* + itd^*)m\big)B\big)H_t = M_B f,$$

since $\{m, \mu + itd\} = I$, $\{m, -\mu^* + itd^*\} = -I$ and $I - B^{-1}IB = 0$. Then apply $(-\mu^* + itd^*)B$ to obtain

$$(\mu^* - itd^*)B(\mu + itd)(mH_t) = (-\mu^* + itd^*)\big(BN^+ - N^- B\big)f,$$

since $-\mu^* + itd^*$ is nilpotent. Evaluating the scalar part of this equation, we get the desired identity.

To find the identity for $H_t^{1,\|}$, we use the expression for $T_A = \hat{T}_B|_{\hat{\mathcal{H}}^1}$ from Definition 2.43. Multiplying the equation $(I + it\hat{T}_A)H_t = f$ by $AN^+ - N^- A$ yields

$$a_{\|\|} H_t^{1,\|} - a_{00} H_t^{1,0} e_0 - it\big(A\nabla H_t^{1,0} + d^*\mu A H_t\big) = a_{\|\|} f^{1,\|} - a_{00} f_t^{1,0} e_0.$$

Evaluating the tangential part of this equation gives the desired identity.

(ii) By rescaling, we see from (i) that it suffices to show that

$$\left(\int_{B(x,r_0)} \big|u(y)\big|^q\,dy \right)^{1/q} + \left(\int_{B(x,r_0)} \big|\nabla u(y)\big|^p\,dy \right)^{1/p} \lesssim M\big(|g|^p\big)^{1/p}(x), \tag{2.13}$$

for all $x \in \mathbf{R}^n$ and all u and g satisfying an equation

$$[\,1 \quad i\,\mathrm{div}\,]\begin{bmatrix} a'_{\perp\perp} & a'_{\perp\|} \\ a'_{\|\perp} & a'_{\|\|} \end{bmatrix}\begin{bmatrix} 1 \\ i\nabla \end{bmatrix} u = [\,1 \quad i\,\mathrm{div}\,]\begin{bmatrix} a'_{\perp\perp} g^{1,0} \\ -a'_{\|\|} g^{1,\|} \end{bmatrix},$$

where A' is a matrix with same norm and accretivity constant as for A. Indeed, by rescaling we see that $u(x) = H_t^{1,0}(tx)$, $g(x) = f(tx)$ and $A'(x) = A(tx)$ satisfies this hypothesis. To prove (2.13), we use that the maps $g \mapsto u$ and $g \mapsto \nabla u$ have $L_p(\mathbf{R}^n) \to L_q(\mathbf{R}^n)$ and $L_p(\mathbf{R}^n) \to L_p(\mathbf{R}^n)$ off-diagonal bounds, respectively, with exponent M for any $M > 0$, i.e. there exists $C_M < \infty$ such that

$$\|u\|_{L_q(E)} + \|\nabla u\|_{L_p(E)} \leqslant C_M \big\langle \mathrm{dist}(E, F)\big\rangle^{-M} \|g\|_p \tag{2.14}$$

whenever $E, F \subset \mathbf{R}^n$ and $\operatorname{supp} g \subset F$. To see this, let $L := [\,1\ \ i\,\mathrm{div}\,]A'[\,1\ \ i\nabla\,]^t$. Note that $L : W_2^1(\mathbf{R}^n) \to W_2^{-1}(\mathbf{R}^n)$ is an isomorphism and that $L : W_p^1(\mathbf{R}^n) \to W_p^{-1}(\mathbf{R}^n)$ is bounded. Then by the stability result of Šneĭberg [24], it follows that there exists $\varepsilon > 0$ such that $L : W_p^1(\mathbf{R}^n) \to W_p^{-1}(\mathbf{R}^n)$ is an isomorphism when $|p - 2| < \varepsilon$. We then fix $p_0 \in (2 - \varepsilon, 2)$ and use Sobolev's embedding theorem to see that

$$\|u\|_{q_0} + \|\nabla u\|_{p_0} \lesssim \|u\|_{W^1_{p_0}} \lesssim \|Lu\|_{W^{-1}_{p_0}} \lesssim \|g\|_{p_0}.$$

By choosing p_0 close to 2, we may assume that $q_0 > 2$. Thus we have bounded maps $g \mapsto u : L_{p_0}(F) \to L_{q_0}(E)$ and $g \mapsto \nabla u : L_{p_0}(F) \to L_{p_0}(E)$, with norms $\leqslant C$. Also, by Proposition 2.25, the norms of $g \mapsto u : L_2(F) \to L_2(E)$ and $g \mapsto \nabla u : L_2(F) \to L_2(E)$ are bounded by $\langle \mathrm{dist}(E, F)\rangle^{-M_0}$. Interpolation now proves (2.14) for some $p_0 < p < 2$, $2 < q < q_0$ and $M = M_0(1 - p_0/p)/(1 - p_0/2)$.

Finally we show how (2.14) implies (2.13). Let $E = F_0 := B(x, r_0)$ and for $k \geqslant 1$ let $F_k := B(x, 2^k r_0) \setminus B(x, 2^{k-1} r_0)$. This gives

$$\begin{aligned} \Bigg(\int\limits_{B(x,r_0)} \big|u(y)\big|^q\,dy\Bigg)^{1/q} + \Bigg(\int\limits_{B(x,r_0)} \big|\nabla u(y)\big|^p\,dy\Bigg)^{1/p} &\lesssim \sum_{k=0}^{\infty} 2^{-Mk}\Bigg(\int\limits_{F_k} \big|g(y)\big|^p\,dy\Bigg)^{1/p} \\ &\lesssim \sum_{k=0}^{\infty} 2^{(n/p - M)k}\Bigg(\fint\limits_{B(x,2^k r_0)} \big|g(y)\big|^p\,dy\Bigg)^{1/p} \\ &\lesssim \big(M\big(|g|^p\big)(x)\big)^{1/p}, \end{aligned}$$

provided we chose $M > n/p$. □

Proof of Proposition 2.56. To prove that $\|\widetilde{N}_*(F)\| \gtrsim \|f\|$, we calculate

$$\begin{aligned} \big\|\widetilde{N}_*(F)\big\|^2 &\gtrsim \sup_{t>0} \int\limits_{\mathbf{R}^n} \fint\limits_{|y-x|<c_1 t} \fint\limits_{|s-t|<c_0 t} \big|F(s, y)\big|^2\,ds\,dy\,dx \\ &= \sup_{t>0} \fint\limits_{|s-t|<c_0 t} \|F_s\|^2\,ds \gtrsim \sup_{t>0} \|F_{(1+c_0)t}\|^2 \approx \|f\|^2. \end{aligned}$$

We have here used that $F_{(1+c_0)t} = e^{-((1+c_0)t-s)|T_A|} F_s$, which by Proposition 2.32 shows that $\|F_{(1+c_0)t}\| \lesssim \|F_s\|$ when $|s-t| < c_0 t$.

To prove $\|\widetilde{N}_*(F)\| \lesssim \|f\|$, we note that since $\operatorname{curl}_{t,x} F = 0$, we can write $F = \nabla_{t,x} U$ for some scalar potential U, and we see that U solves the second order equation (1.1). With notation $\widetilde{D}(t,x) = \{(s,y);\ |s-t| < \tilde{c}_0 t,\ |y-x| < \tilde{c}_1 t\}$, for constants $c_0 < \tilde{c}_0 < 1$ and $c_1 < \tilde{c}_1 < \infty$, and $\overline{U} := \fint\fint_{\widetilde{D}(t,x)} U(s,y)\, ds\, dy$, we have

$$\Bigg(\fint\fint_{D(t,x)} \big|t\nabla U(s,y)\big|^2\, ds\, dy \Bigg)^{1/2} \lesssim \Bigg(\fint\fint_{\widetilde{D}(t,x)} \big|U(s,y) - \overline{U}\big|^2\, ds\, dy \Bigg)^{1/2}$$
$$\lesssim \Bigg(\fint\fint_{\widetilde{D}(t,x)} \big|t\nabla U(s,y)\big|^p\, ds\, dy \Bigg)^{1/p},$$

with $2(n+1)/(n+3) < p < 2$. The first estimate uses Caccioppoli's inequality and the second estimate uses Poincaré's inequality. Thus it suffices to bound the L_2 norm of

$$\sup_{t>0} \Bigg(\fint\fint_{\widetilde{D}(t,x)} \big|F(s,y)\big|^p\, ds\, dy \Bigg)^{1/p}.$$

To this end, we write $F_s = H_s + \psi_s(T_A) f$, where $H_s := (I + isT_A)^{-1} f$ and $\psi(z) := e^{-|z|} - (1+iz)^{-1}$. Using the quadratic estimates, the second term has estimates

$$\int_{\mathbf{R}^n} \sup_{t>0} \Bigg(\fint\fint_{\widetilde{D}(t,x)} \big|\psi_s(T_A) f(y)\big|^p\, ds\, dy \Bigg)^{2/p} dx$$
$$\lesssim \int_{\mathbf{R}^n} \sup_{t>0} \Bigg(\int_{|s-t|<\tilde{c}_0 t} \int_{|y-x|<\tilde{c}_1 t} \big|\psi_s(T_A) f(y)\big|^2 \frac{dy\, ds}{t^{n+1}} \Bigg) dx$$
$$\lesssim \int_0^\infty \int_{\mathbf{R}^n} \int_{|y-x|<\tilde{c}_1 s/(1-\tilde{c}_0)} \big|\psi_s(T_A) f(y)\big|^2 s^{-(n+1)}\, dy\, dx\, ds$$
$$\lesssim \int_0^\infty \big\|\psi_s(T_A) f\big\|^2 \frac{ds}{s} \lesssim \|f\|^2.$$

For the first term, we use Lemma 2.57(ii) and obtain

$$\int_{\mathbf{R}^n} \sup_{t>0} \Bigg(\fint\fint_{\widetilde{D}(t,x)} \big|H_s(y)\big|^p\, ds\, dy \Bigg)^{2/p} dx$$
$$\lesssim \int_{\mathbf{R}^n} \sup_{t>0} \sup_{|s-t|<\tilde{c}_0 t} \Bigg(\fint_{B(x,\tilde{c}_1 s/(1-\tilde{c}_0))} \big|H_s(y)\big|^p\, dy \Bigg)^{2/p} dx$$

$$\lesssim \left\| M\left(|f|^p\right)\right\|_{2/p}^{2/p} \lesssim \left\| |f|^p \right\|_{2/p}^{2/p} = \|f\|_2^2,$$

using the boundedness of the Hardy–Littlewood maximal function on $L_{2/p}(\mathbf{R}^n)$. This completes the proof. □

3. Invertibility of unperturbed operators

In this section we prove Theorems 1.4 and 1.1 for the unperturbed problem, i.e. for B_0^k and A_0, respectively. We do this by verifying the hypothesis in Lemmas 2.51 and 2.53, i.e. we prove that $T_{B_0^k}$ satisfies quadratic estimates and $\lambda - E_{B_0^k} N_{B_0^k}$ is an isomorphism, and that T_{A_0} satisfies quadratic estimates and $I \pm E_{A_0} N_{A_0}$ are isomorphisms, respectively.

3.1. Block coefficients

In this section we assume that $B = B_0 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ have properties as in Definition 2.8 with the extra property that it is a block matrix, i.e.

$$B = \begin{bmatrix} B_{\perp\perp} & 0 \\ 0 & B_{\|\|} \end{bmatrix}$$

in the splitting $\mathcal{H} = N^-\mathcal{H} \oplus N^+\mathcal{H}$. Note that B being of this form is equivalent with the commutation relations $N^\pm B = BN^\pm$.

Lemma 3.1. *Let B be a block matrix as above. Then*

$$T_B = \Gamma + B^{-1}\Gamma^* B,$$

where $\Gamma = Nmd = -iN\underline{d}$ is a nilpotent first order, homogeneous partial differential operator with constant coefficients.

Proof. Since $N^\pm B = BN^\pm$, it follows that $M_B = N^+ - B^{-1}N^-B = N^+ - N^- = N$ and

$$T_B = N\left(md + B^{-1} md^* B\right) = \Gamma + B^{-1}\Gamma^* B,$$

since $N^2 = I$ and $NB^{-1} = B^{-1}N$. The operator Γ is nilpotent since $\Gamma^2 = NmdNmd = NmNdmd = -NmNmd^2 = 0$. □

Remark 3.2. Note that if $\Pi_B = \Gamma + \Gamma_B^*$, where $\Gamma_B^* = B^{-1}\Gamma^* B$ and Γ is nilpotent, is an operator of the form considered in [10], then Π_B intertwines Γ and Γ_B^* in the sense that $\Pi_B \Gamma u = \Gamma_B^* \Pi_B u$ for all $u \in \mathsf{D}(\Gamma_B^* \Pi_B)$ and $\Pi_B \Gamma_B^* u = \Gamma \Pi_B u$ for all $u \in \mathsf{D}(\Gamma \Pi_B)$. Thus Π_B^2 commutes with both Γ and Γ_B^* on appropriate domains. In particular, if $P_t^B = (1 + t^2\Pi_B^2)^{-1}$ and $Q_t^B = t\Pi_B(1 + t^2\Pi_B^2)^{-1}$, then we find that $\Gamma P_t^B u = P_t^B \Gamma u$, $\Gamma_B^* Q_t^B u = Q_t^B \Gamma u$ for all $u \in \mathsf{D}(\Gamma)$ and $\Gamma_B^* P_t^B u = P_t^B \Gamma_B^* u$, $\Gamma Q_t^B u = Q_t^B \Gamma_B^* u$ for all $u \in \mathsf{D}(\Gamma_B^*)$.

Theorem 3.3. *Let B be a block matrix as above. Then*

(i) *T_B satisfies quadratic estimates, and*

(ii) $E_BN_B + N_BE_B = 0$ *and* $N_B = N$. *In particular* $\lambda - E_BN_B$ *is an isomorphism whenever* $\lambda \notin \{i, -i\}$ *with*

$$(\lambda - E_BN_B)^{-1} = \frac{1}{\lambda^2+1}(\lambda - N_BE_B).$$

Proof. For operators of the form $\Gamma + B^{-1}\Gamma^*B$, quadratic estimates were proved in [10], with essentially the same methods as we use here in Section 4.1.

To prove (ii), note that since B is a block matrix, it follows that $B^{-1}N^{\pm}\mathcal{H} = N^{\pm}\mathcal{H}$. Thus the projections $N_B^{\pm}$ associated with the splitting $\mathcal{H} = N^-\mathcal{H} \oplus B^{-1}N^+\mathcal{H}$ are $N^{\pm}$ and the associated reflection operator is $N_B = N$.

To prove invertibility of $\lambda - E_BN_B$, we note that

$$T_BN_B = \big(\Gamma + B^{-1}\Gamma^*B\big)N = -N\big(\Gamma + B^{-1}\Gamma^*B\big) = -N_BT_B,$$

since N commutes with d, d^* and B, and anticommutes with m. Thus

$$E_BN_B = \operatorname{sgn}(T_B)N_B = N_B \operatorname{sgn}(N_BT_BN_B) = N_B \operatorname{sgn}(-T_B) = -N_BE_B,$$

since $\operatorname{sgn}(z)$ is odd. Using this anticommutation formula we obtain

$$(\lambda - N_BE_B)(\lambda - E_BN_B) = \lambda^2 + 1 - \lambda(E_BN_B + N_BE_B) = \lambda^2 + 1,$$

and similarly $(\lambda - E_BN_B)(\lambda - N_BE_B) = \lambda^2 + 1$, from which the stated formula for the inverse follows. □

3.2. *Constant coefficients*

We here collect results in the case when $B(x) = B \in \mathcal{L}(\bigwedge)$ is a constant accretive matrix. In this case we make use of the Fourier transform

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := \frac{1}{2\pi i}\int_{\mathbf{R}^n} u(x)e^{-i(x,\xi)}\,dx,$$

acting componentwise. If we let

$$\mu_\xi f(\xi) := \xi \wedge f(\xi), \qquad \mu_\xi^* f(\xi) := \xi \lrcorner f(\xi),$$

then T_B, conjugated with $\mathcal{F}$, is the multiplication operator

$$M_\xi f(\xi) := M_B^{-1}\big(im\mu_\xi - iB^{-1}m\mu_\xi^*B\big)f(\xi), \quad \xi \in \mathbf{R}^n.$$

Lemma 3.4. *For all* $t \in \mathbf{R}$ *and* $\xi \in \mathbf{R}^n$ *we have*

$$\big|(it + M_\xi)^{-1}\big| \approx \big(t^2 + |\xi|^2\big)^{-1/2}.$$

Proof. Let $u = (it + M_\xi)f$. It suffices to prove that $\|u\|^2 \gtrsim (t^2 + |\xi|^2)\|f\|^2$. With $\widetilde{B} := mBm$, it follows from the definition of M_ξ that

$$mM_B u = \big(\Gamma + \widetilde{B}^{-1}\Gamma^* B\big)f,$$

where $\Gamma = i(t\mu + \mu_\xi)$. Using Lemma 2.7 we get

$$\big\|(\Gamma + \Gamma^*)g\big\|^2 = \|\Gamma g\|^2 + \|\Gamma^* g\|^2 = \big((\Gamma^*\Gamma + \Gamma^*\Gamma)g, g\big) = \big(t^2 + |\xi|^2\big)\|g\|^2.$$

Therefore our estimate follows from Lemma 2.21(ii). □

Proposition 3.5. *If $B(x) = B \in \mathcal{L}(\bigwedge)$ is a constant, accretive matrix, then T_B satisfies quadratic estimates.*

Proof. Using the lemma, we obtain the estimate

$$\left|\frac{tM_\xi}{1 + t^2 M_\xi^2}\right| \leqslant t|M_\xi|\big|(i - M_{t\xi})^{-1}\big|\big|(i + M_{t\xi})^{-1}\big| \lesssim \frac{t|\xi|}{1 + t^2|\xi|^2}.$$

Thus using Plancherel's formula we obtain

$$\begin{aligned}\int_0^\infty \left\|\frac{tT_B}{1 + t^2 T_B^2}u\right\|^2 \frac{dt}{t} &\approx \int_0^\infty \left\|\frac{tM_\xi}{1 + t^2 M_\xi^2}\hat{u}\right\|^2 \frac{dt}{t} \\ &\lesssim \int_{\mathbf{R}^n}\left(\int_0^\infty \left(\frac{t|\xi|}{1 + t^2|\xi|^2}\right)^2 \frac{dt}{t}\right)\big|\hat{u}(\xi)\big|^2\, d\xi \approx \|u\|^2,\end{aligned}$$

where the last step follows from a change of variables $s = t|\xi|$. □

Next we prove that Neumann and Regularity problems are well posed in the case of a complex, constant, accretive matrix

$$A = \begin{bmatrix} a_{00} & a_{0\|} \\ a_{\|0} & a_{\|\|}\end{bmatrix}.$$

For this it suffices to consider $B = I \oplus A \oplus I \oplus I \oplus \cdots \oplus I$ and the action of $T_A = T_B|_{\hat{\mathcal{H}}^1}$ on the invariant subspace $\hat{\mathcal{H}}^1$. Recall that $f \in \hat{\mathcal{H}}^1$ means that f is a vector field $f : \mathbf{R}^n \to \bigwedge^1 = \mathbf{C}^{n+1}$ such that $df_\| = 0$. On the Fourier transform side $f \in \hat{\mathcal{H}}^1$ is seen to correspond to a vector field $\hat{f} : \mathbf{R}^n \to \bigwedge^1$ such that $\xi \wedge \hat{f}_\| = 0$, i.e. $\hat{f}$ is such that its tangential part $f_\|$ is a radial vector field. Thus the space

$$\mathcal{F}\big(\hat{\mathcal{H}}^1\big) = \big\{\hat{f} \in L_2\big(\mathbf{R}^n; \mathbf{C}^{n+1}\big);\ \xi \wedge \hat{f}_\| = 0\big\}$$

can be identified with $L_2(\mathbf{R}^n;\mathbf{C}^2)$ if we use $\{e_0, \xi/|\xi|\}$ as basis for the two-dimensional space $\hat{\mathcal{H}}^1_\xi$ to which $\hat{f}(\xi)$ belongs. Furthermore, the operator T_A is conjugated to the multiplication operator

$$M_\xi = M_A^{-1}\big(i\mu^*\mu_\xi - iA^{-1}\mu\mu_\xi^* A\big) : \mathcal{F}\big(\hat{\mathcal{H}}^1\big) \to \mathcal{F}\big(\hat{\mathcal{H}}^1\big),$$

under the Fourier transform, where $M_A := N^+ - A^{-1}N^-A$. We see from the expression (1.5) for T_A that, at a fixed point $\xi \in S^{n-1}$, the matrix for M_ξ in the basis $\{e_0, \xi\}$ is

$$\begin{bmatrix} \frac{i}{a_{00}}(a_{0\|} + a_{\|0}, \xi) & \frac{i}{a_{00}}(a_{\|\|}\xi, \xi) \\ -i & 0 \end{bmatrix}.$$

Theorem 3.6. *Let $A(x) = A \in \mathcal{L}(\bigwedge)$ be a constant, complex, accretive matrix. Then*

$$I \pm E_A N_A : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$$

are isomorphisms. In particular, Neumann and Regularity problems (Neu-A) *and* (Reg-A) *are well posed.*

Proof. From Remark 2.54 we see that it suffices to prove that all four restricted projections

$$N_A^\pm : E_A^\pm \hat{\mathcal{H}}^1 \to N_A^\pm \hat{\mathcal{H}}^1$$

are isomorphisms, or equivalently that the constant multiplication operators $N_A^\pm$ are isomorphisms on $\mathcal{F}(\hat{\mathcal{H}}^1)$. For the projections $\chi_\pm(M_\xi)$ conjugated to the Hardy projection operators $E_A^\pm$, we observe that $M_{t\xi} = tM_\xi$ and hence $\chi_\pm(tM_\xi) = \chi_\pm(M_\xi)$ for all $t > 0$. Thus it suffices to verify that

$$N_A^\pm : \chi_\pm(M_\xi)\hat{\mathcal{H}}^1_\xi \to N_A^\pm \hat{\mathcal{H}}^1_\xi$$

are isomorphisms for each $\xi \in S^{n-1}$. For such fixed ξ, using the basis $\{e_0, \xi\}$ from above, $N_A^- \hat{\mathcal{H}}^1_\xi = \{\hat{f} \in \hat{\mathcal{H}}^1_\xi;\ e_0 \wedge \hat{f} = 0\}$ is spanned by $[\,1\ \ 0\,]^t$ and $N_A^+ \hat{\mathcal{H}}^1_\xi = \{\hat{f} \in \hat{\mathcal{H}}^1_\xi;\ (A\hat{f}, e_0) = 0\}$ is spanned by $[\,(a_{0\|}, \xi)\ \ -a_{00}\,]^t$. Indeed $(A(ze_0 + w\xi), e_0) = za_{00} + w(a_{0\|}, \xi)$.

If we call e_ξ^+ and e_ξ^- the two eigenvectors of M_ξ, it follows that $\chi_\pm(M_\xi)\hat{\mathcal{H}}^1_\xi$ are spanned by these, as these subspaces are one-dimensional. It suffices to show that $[\,1\ \ 0\,]^t$ and $[\,(a_{0\|}, \xi)\ \ -a_{00}\,]^t$ are not eigenvectors of M_ξ. We have

$$M_\xi \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \begin{bmatrix} \frac{1}{a_{00}}(a_{0\|} + a_{\|0}, \xi) \\ -1 \end{bmatrix},$$

$$M_\xi \begin{bmatrix} (a_{0\|}, \xi) \\ -a_{00} \end{bmatrix} = i \begin{bmatrix} \frac{1}{a_{00}}(a_{0\|} + a_{\|0}, \xi)(a_{0\|}, \xi) - (a_{\|\|}\xi, \xi) \\ -(a_{0\|}, \xi) \end{bmatrix}.$$

Clearly the normal vector is not an eigenvector. To prove that the second is not an eigenvector, note that the cross product is

$$-(a_{0\|}, \xi)^2 + (a_{0\|} + a_{\|0}, \xi)(a_{0\|}, \xi) - a_{00}(a_{\|\|}\xi, \xi) = (a_{\|0}, \xi)(a_{0\|}, \xi) - a_{00}(a_{\|\|}\xi, \xi).$$

The right-hand side is non-zero since

$$\left(\begin{bmatrix} a_{00} & a_{0\|} \\ a_{\|0} & a_{\|\|} \end{bmatrix}\begin{bmatrix} z \\ w\xi \end{bmatrix}, \begin{bmatrix} z \\ w\xi \end{bmatrix}\right) = \left(\begin{bmatrix} a_{00} & (a_{0\|},\xi) \\ (a_{\|0},\xi) & (a_{\|\|}\xi,\xi) \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}\right)$$

is a non-degenerate quadratic form as A is accretive. □

Remark 3.7. We note that the method above also can be used to show that $I \pm E_A N : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$, with the unperturbed operator N, are isomorphisms when A is constant. Here we also need to observe that the tangential vector $[\,0\ \ 1\,]^{\mathrm{t}}$ is not an eigenvector to M_ξ.

3.3. Real symmetric coefficients

In this section, we assume that $B^* = B$. We first prove a Rellich type estimate.

Proposition 3.8. *Assume that $B^* = B$ and that $f \in E_B^+ \mathcal{V}_B$ or $f \in E_B^- \mathcal{V}_B$. Then*

$$(Bf, f) = 2\,\mathrm{Re}\big(e_0 \lrcorner (Bf), e_0 \lrcorner f\big) = 2\,\mathrm{Re}\big(e_0 \wedge (Bf), e_0 \wedge f\big).$$

In particular $\|f\| \approx \|\hat{N}_B^- f\| \approx \|\check{N}_B^- f\| \approx \|\check{N}_B^+ f\| \approx \|\hat{N}_B^+ f\|$.

Proof. It suffices to consider $f \in E_B^+ \mathcal{V}_B$ as the case $f \in E_B^- \mathcal{V}_B$ is treated similarly. We use Lemma 2.34 and write $F_t := e^{-t|T_B|} f$. Hence $(0,\infty) \ni t \mapsto F_t \in \mathcal{H}$ is differentiable, $\lim_{t\to 0} F_t = f$ and $\lim_{t\to\infty} F_t = 0$. Furthermore $F_t \in \mathsf{D}(d_x)$ and $BF_t \in \mathsf{D}(d_x^*)$ for all $t \in (0,\infty)$. We note the formulae

$$d_{t,x}^* m G_t + m d_{t,x}^* G_t = -\partial_t G_t,$$
$$B m d_{t,x} F_t = -m d_{t,x}^* B F_t.$$

The first identity, which we apply with $G_t = BF_t$, follows from Lemma 2.7, whereas the second is equivalent to $\partial_t F_t + T_B F_t = 0$, and follows from Lemma 2.34. We get

$$\begin{aligned}(Bf, f) &= -\int_0^\infty (\partial_t BF_t, F_t) + (BF_t, \partial_t F_t) = -2\,\mathrm{Re}\int_0^\infty (\partial_t BF_t, F_t)\\ &= 2\,\mathrm{Re}\int_0^\infty \big(d_{t,x}^* m BF_t + m d_{t,x}^* BF_t, F_t\big) = 2\,\mathrm{Re}\int_0^\infty \big(d_{t,x}^* m BF_t - B m d_{t,x} F_t, F_t\big)\\ &= 2\,\mathrm{Re}\int_0^\infty \big(d_{t,x}^* m BF_t, F_t\big) - (m BF_t, d_{t,x} F_t)\\ &= 2\,\mathrm{Re}\int_0^\infty \big((d^* - \mu^*\partial_t) m BF_t, F_t\big) - \big(m BF_t, (d + \mu\partial_t) F_t\big)\end{aligned}$$

$$= -2\,\mathrm{Re} \int_0^\infty (\partial_t m B F_t, \mu F_t) + (m B F_t, \partial_t \mu F_t)$$

$$= 2\,\mathrm{Re}(m B f, \mu f) = 2\,\mathrm{Re}\big(e_0 \wedge (Bf), e_0 \wedge f\big).$$

Note that all integrals are convergent since $\|F_t\| \lesssim \min(1, t^{-s})$ and since $\partial_t F_t = -T_B F_t$ with $\|T_B F_t\| \lesssim \min(t^{s-1}, t^{-1})$ if $f \in \mathsf{D}(|T_B|^s) \cap \mathsf{R}(|T_B|^{-s})$. This follows as in the proof of Lemma 2.34. Furthermore, we note that

$$(e_0 \wedge Bf, e_0 \wedge f) = \big(Bf, e_0 \lrcorner (e_0 \wedge f)\big)$$
$$= \big(Bf, f - e_0 \wedge (e_0 \lrcorner f)\big) = (Bf, f) - \big(e_0 \lrcorner (Bf), e_0 \lrcorner f\big).$$

Together with the calculation above this proves that $(Bf, f) = 2\,\mathrm{Re}(e_0 \lrcorner (Bf), e_0 \lrcorner f)$.

To prove that $\|f\| \approx \|\hat{N}_B^+ f\| \approx \|\hat{N}_B^- f\|$, it suffices to show that $\|f\| \lesssim \|\hat{N}_B^\pm f\|$ since $\hat{N}_B^\pm$ are bounded. From the Rellich type identities above we have $\|f\|^2 \lesssim \|e_0 \lrcorner (Bf)\| \|e_0 \lrcorner f\| \lesssim \|\hat{N}_B^- f\| \|f\|$ which proves $\|f\| \lesssim \|\hat{N}_B^- f\|$, and using the other identity we obtain $\|f\|^2 \lesssim \|e_0 \wedge (Bf)\| \|e_0 \wedge f\| \lesssim \|f\| \|\hat{N}_B^+ f\|$. The proof of $\|f\| \approx \|\check{N}_B^+ f\| \approx \|\check{N}_B^- f\|$ is similar. □

We note that Proposition 3.8 also proves that the norms of the two components of $f \in E_B^\pm \mathcal{V}_B$ in the unperturbed splitting $\mathcal{H} = N^- \mathcal{H} \oplus N^+ \mathcal{H}$ are comparable.

Corollary 3.9. *Assume that $B^* = B$ and that $f \in E_B^+ \mathcal{V}_B$ or $f \in E_B^- \mathcal{V}_B$, and decompose B in the splitting $\mathcal{H} = N^- \mathcal{H} \oplus N^+ \mathcal{H}$ as*

$$B = \begin{bmatrix} B_{\perp\perp} & B_{\perp\|} \\ B_{\|\perp} & B_{\|\|} \end{bmatrix}.$$

If $f_\| = N^+ f$ and $f_\perp = N^- f$, then

$$(B_{\perp\perp} f_\perp, f_\perp) = (B_{\|\|} f_\|, f_\|).$$

In particular $\|f\| \approx \|N^+ f\| \approx \|N^- f\|$.

Proof. We obtain from Proposition 3.8 that $\mathrm{Re}(B_{\perp\perp} f_\perp + B_{\perp\|} f_\|, f_\perp) = \mathrm{Re}(B_{\|\perp} f_\perp + B_{\|\|} f_\|, f_\|)$. The corollary now follows since $B_{\perp\perp}, B_{\|\|} \geqslant \kappa_B$ and $B_{\perp\|}^* = B_{\|\perp}$. □

Next we turn to quadratic estimates for the operator T_B. As before we assume that B is as in Definition 2.8 and that $B = B^*$. For the rest of this section we shall also assume that

$$B_{\|\perp}^j = B_{\perp\|}^j = 0, \quad \text{for } j \geqslant 2. \tag{3.1}$$

Note that in particular this is true if $B = I \oplus A \oplus I \oplus \cdots \oplus I$, where $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^1))$.

To prove quadratic estimates, we shall use Proposition 2.36 where we verify the hypothesis separately on the subspaces $\hat{\mathcal{H}}_B^k$ and $\check{\mathcal{H}}_B$, which is possible by Lemma 2.46.

Lemma 3.10. *Assume that $B^* = B$ and* (3.1) *holds. Then*

$$\|f\| \lesssim |||t\partial_t F_t|||_\pm, \quad \textit{for all } f \in E_B^\pm \check{\mathcal{V}}_B,$$

where $F_t = e^{\mp t|T_B|} f$.

Proof. As the two estimates are similar, we only show the estimate for $f \in E_B^+ \check{\mathcal{V}}_B$. Note that $F(t,x)$ satisfies (2.12), i.e.

$$\begin{cases} dF = e_0 \lrcorner\, \partial_t F, \\ d^*(BF) = -e_0 \wedge (B\partial_t F). \end{cases}$$

Splitting both sides of both equations into normal and tangential parts, with notation as in Corollary 3.9, we obtain

$$dF_\| = m\partial_t F_\perp, \tag{3.2}$$

$$dF_\perp = 0, \tag{3.3}$$

$$d^*(B_{\perp\perp} F_\perp + B_{\perp\|} F_\|) = -m(B_{\|\perp}\partial_t F_\perp + B_{\|\|}\partial_t F_\|), \tag{3.4}$$

$$d^*(B_{\|\perp} F_\perp + B_{\|\|} F_\|) = 0. \tag{3.5}$$

We have here used that $e_0 \lrcorner\, f = mf$ if f is normal, and $e_0 \wedge f = mf$ if f is tangential. A key observation is that the first term on the right-hand side in (3.4) vanishes since $B_{\|\perp} F_\perp = 0$. To see this, note that

$$B_{\|\perp} F_\perp = B_{\|\perp}^1 F_\perp^1 + B_{\|\perp}^2 F_\perp^2 + \cdots + B_{\|\perp}^{n+1} F_\perp^{n+1}.$$

By hypothesis (3.1), $B_{\|\perp}^2 = \cdots = B_{\|\perp}^{n+1} = 0$. Furthermore, writing $F_\perp^1 = F_0 e_0$ with the function F_0 being scalar, we get from (3.3) that $0 = d(F_0 e_0) = (\nabla F_0) \wedge e_0$, so F_0 is constant and therefore vanishes, and thus so does $F_\perp^1$. Eq. (3.4) reduces to

$$B_{\|\|}\partial_t F_\| = -md^*(B_{\perp\perp} F_\perp + B_{\perp\|} F_\|). \tag{3.6}$$

We calculate

$$\|f\|^2 \lesssim (B_{\|\|} f_\|, f_\|) = -2\operatorname{Re}\int_0^\infty (B_{\|\|}\partial_t F_\|, F_\|)\,dt = 2\operatorname{Re}\int_0^\infty \big(d^*(B_{\perp\perp} F_\perp + B_{\perp\|} F_\|), mF_\|\big)\,dt$$

$$= -2\operatorname{Re}\int_0^\infty \Big(\big(d^*(B_{\perp\perp}\partial_t F_\perp + B_{\perp\|}\partial_t F_\|), mF_\|\big) + \big(d^*(B_{\perp\perp} F_\perp + B_{\perp\|} F_\|), m\partial_t F_\|\big)\Big)t\,dt$$

$$= 2\operatorname{Re}\int_0^\infty \big((B_{\perp\perp}\partial_t F_\perp + B_{\perp\|}\partial_t F_\|, \partial_t F_\perp) + (mB_{\|\|}\partial_t F_\|, m\partial_t F_\|)\big)t\,dt$$

$$= 2\,\mathrm{Re}\int_0^\infty (B\partial_t F, \partial_t F)t\,dt \approx \|\!|t\partial_t F|\!\|^2.$$

Here in the third step we use (3.6), in the fourth step we integrate by parts, in the fifth step we use duality and that $dmF_\| = -mdF_\| = -\partial_t F_\perp$ by (3.2) for the first term and again (3.6) for the second term. Finally in step 6 we use that m is an isometry and that $B_{\|\perp}F_\perp = 0$. □

Next we turn to the subspace $\hat{\mathcal{H}}^1$.

Lemma 3.11. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^1))$ be real symmetric and $A \geqslant \kappa > 0$, and let $B = I \oplus A \oplus I \oplus \cdots \oplus I$. Then*

$$\|f\| \lesssim \|\!|t\partial_t F|\!\|_\pm, \quad \textit{for all } f \in E_B^\pm \hat{\mathcal{V}}_B^1,$$

where $F = e^{\mp t|T_B|}f$.

Proof. Recall that if $f \in E_B^\pm \hat{\mathcal{V}}_B^1$ and $F = F_0 e_0 + F_\| = e^{\mp t|T_B|}f$, then Lemma 2.12 shows that F_0 satisfies the equation

$$\mathrm{div}_{t,x}\, A(x)\nabla_{t,x} F_0(t,x) = 0.$$

Therefore, by the square function estimate of Dahlberg, Jerison and Kenig [13] and the estimates of harmonic measure of Jerison and Kenig [17], we have estimates $\|f_0\| \lesssim \|\!|t\nabla_{t,x}F_0|\!\|_\pm$. Hence applying the Rellich estimates in Proposition 3.8, we obtain

$$\|f\| \approx \|f_0\| \lesssim \|\!|t\nabla_{t,x}F_0|\!\|_\pm \lesssim \|\!|t\partial_t F|\!\|_\pm,$$

since $\partial_i F_0 = \partial_0 F_i$, as $d_{t,x}F = 0$. □

We are now in position to prove the main result of this section.

Theorem 3.12. *Let $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^1))$ be real symmetric and $A \geqslant \kappa > 0$, and let $B = I \oplus A \oplus I \oplus \cdots \oplus I$. Then T_B has quadratic estimates, so that in particular $E_B = \mathrm{sgn}(T_B) : \mathcal{H} \to \mathcal{H}$ is bounded. Furthermore we have isomorphisms*

$$I \pm E_B N_B : \mathcal{H} \to \mathcal{H}.$$

Proof. To prove that T_B has quadratic estimates it suffices by Proposition 2.36 to prove

$$\|f\| \lesssim \|\!|t\partial_t F|\!\|_\pm, \quad \text{for all } f \in E_B^\pm \mathcal{V}_B.$$

To this end, we split

$$f = f_0 + f_1 + \cdots + f_{n+1} + \check{f}$$

where $f_j \in \hat{\mathcal{V}}_B^j$ and $\check{f} \in \check{\mathcal{V}}_B$. Lemma 2.46 shows that $e^{\mp t|T_B|}$ preserves these subspaces so that similarly

$$F = F_0 + F_1 + \cdots + F_{n+1} + \check{F},$$

where $(F_j)_t \in \hat{\mathcal{V}}_B^j$ and $\check{F}_t \in \check{\mathcal{V}}_B$. It thus suffices to prove that $\|f_j\| \lesssim |||t\partial_t F_j|||_\pm$, $j = 0, 1, \ldots, n+1$, and $\|\check{f}\| \lesssim |||t\partial_t \check{F}|||_\pm$. Lemma 3.10 shows that $\|\check{f}\| \lesssim |||t\partial_t \check{F}|||_\pm$ and Lemma 3.11 shows that $\|f_1\| \lesssim |||t\partial_t F_1|||_\pm$. Furthermore, for $j \neq 1$ we observe that $T_B = \hat{T}_B = T_I$ on the subspace $\hat{\mathcal{V}}_B^j$, where I denotes the identity matrix. Thus $F_j = e^{\mp t|T_I|} f_j$ and it follows from Proposition 3.5 that $\|f_j\| \lesssim |||t\partial_t F_j|||_\pm$.

Applying Propositions 2.36 and 2.32 now shows that $E_B = \operatorname{sgn}(T_B) : \mathcal{H} \to \mathcal{H}$ is a bounded operator. To show that $I \pm E_B \hat{N}_B : \mathcal{H} \to \mathcal{H}$ is invertible, note that the adjoint with respect to the duality $\langle\cdot,\cdot\rangle_B$ from Definition 2.16 is

$$I \mp \check{N}_B E_B = \check{N}_B (I \mp E_B \check{N}_B) \check{N}_B,$$

according to Proposition 2.17. Having established the boundedness of E_B, the Rellich estimates clearly extends to a priori estimates for all eight restricted projections

$$\hat{N}_B^\pm : E_B^\pm \mathcal{H} \to \hat{N}_B^\pm \mathcal{H}, \qquad \check{N}_B^\pm : E_B^\pm \mathcal{H} \to \check{N}_B^\pm \mathcal{H}.$$

As in Remark 2.54 this translates to a priori estimates for $I \pm E_B \hat{N}_B$ and their adjoints, which proves that they are isomorphisms as in Remark 2.18. $\square$

Remark 3.13. Using instead Corollary 3.9, we also prove as in Theorem 3.12 that $I \pm E_B N : \mathcal{H} \to \mathcal{H}$, where N is the unperturbed operator, are isomorphisms.

4. Quadratic estimates for perturbed operators

In Section 3 we proved that T_{B_0} satisfies quadratic estimates

$$\left|\left|\left| Q_t^{B_0} f \right|\right|\right| \approx \|f\|, \quad f \in \mathcal{H}, \tag{4.1}$$

for certain unperturbed coefficients B_0:

(b) Block coefficients

$$B_0 = \begin{bmatrix} (B_0)_{\perp\perp} & 0 \\ 0 & (B_0)_{\parallel\parallel} \end{bmatrix}.$$

(c) Constant coefficients $B_0(x) = B_0$, $x \in \mathbf{R}^n$, of the form $B_0 = I \oplus A_0 \oplus I \oplus \cdots \oplus I$, i.e. B_0 only acts non-trivially on the vector part.
(s) Real symmetric coefficients of the form $B_0 = I \oplus A_0 \oplus I \oplus \cdots \oplus I$.

Note that for (c), we did prove quadratic estimates for general constant coefficients B_0 in Proposition 3.5. However, since we only prove invertibility $I \pm E_{B_0} N_{B_0}$ on the subspace $\hat{\mathcal{H}}^1$

in Theorem 3.6, we shall only prove perturbation results for constant coefficients of the form $B_0 = I \oplus A_0 \oplus I \oplus \cdots \oplus I$.

In this section we let $B_0 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ be a fixed accretive coefficient matrix with properties (b), (c) or (s). Constants C in estimates or implicit in notation $\lesssim$ and $\approx$ in this section will be allowed to depend only on $\|B_0\|_\infty$, κ_{B_0} and dimension n. Note that this is indeed the case for the constants implicit in (4.1).

We now consider a small perturbation $B \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ of B_0:

$$\|B - B_0\|_{L_\infty(\mathbf{R}^n;\mathcal{L}(\bigwedge))} \leqslant \varepsilon_0.$$

Throughout this section, we assume in particular that ε_0 is chosen small enough so that B has properties as in Definition 2.8 with $\|B\|_\infty \leqslant 2\|B_0\|_\infty$ and $\kappa_B \geqslant \frac{1}{2}\kappa_{B_0}$. The goal is to show that

$$\left|\!\left|\!\left| Q_t^B f \right|\!\right|\!\right| \leqslant C\|f\|, \quad \text{whenever } \|B - B_0\|_\infty \leqslant \varepsilon,\ f \in \mathcal{H}, \tag{4.2}$$

where $C = C(\|B_0\|_\infty, \kappa_{B_0}, n)$ and $\varepsilon = \varepsilon(\|B_0\|_\infty, \kappa_{B_0}, n) \leqslant \varepsilon_0$. Since the properties of B_0 are stable under taking adjoints, we see that the quadratic estimate (2.7) follows from (4.2) and Lemma 2.35. In order to prove (4.2) we make use of the following identity, where we note that each of the six terms consists of three factors, the first being one of the operators from Corollary 2.23, the second being a multiplication operator $\mathcal{E}$ with norm $\|\mathcal{E}\|_\infty \lesssim \varepsilon_0$ and the last factor being one of the operators from Corollary 2.23 but with B replaced by B_0:

$$\begin{aligned} Q_t^B - Q_t^{B_0} &= P_t^B\big(\big(M_B^{-1} - M_{B_0}^{-1}\big)M_{B_0}\big)Q_t^{B_0} && (4.3)\\ &\quad - iP_t^B\big(M_B^{-1}\big(B^{-1} - B_0^{-1}\big)B_0\big)\big(t\underline{d}_{B_0}^* P_t^{B_0}\big) && (4.4)\\ &\quad - i\big(P_t^B M_B^{-1} t\underline{d}_B^*\big)\big(B^{-1}(B - B_0)\big)P_t^{B_0} && (4.5)\\ &\quad - Q_t^B\big(\big(M_B^{-1} - M_{B_0}^{-1}\big)M_{B_0}\big)\big(tT_{B_0}Q_t^{B_0}\big) && (4.6)\\ &\quad + iQ_t^B\big(M_B^{-1}\big(B^{-1} - B_0^{-1}\big)B_0\big)\big(t\underline{d}_{B_0}^* Q_t^{B_0}\big) && (4.7)\\ &\quad + i\big(Q_t^B M_B^{-1} t\underline{d}_B^*\big)\big(B^{-1}(B - B_0)\big)Q_t^{B_0}. && (4.8) \end{aligned}$$

Recall that

$$T_B = -iM_B^{-1}\big(\underline{d} + B^{-1}\underline{d}^* B\big), \quad \text{where } \underline{d} = imd,\ M_B = N^+ - B^{-1}N^- B,$$

$P_t^B = (1 + t^2T_B^2)^{-1}$ and $Q_t^B = tT_B(1 + t^2T_B^2)^{-1}$ and similarly for B_0. The identity (4.3)–(4.8) is established by using

$$Y\big(1 + Y^2\big)^{-1} - X\big(1 + X^2\big)^{-1} = \big(1 + Y^2\big)^{-1}\big((Y - X) - Y(Y - X)X\big)\big(1 + X^2\big)^{-1}$$

with $Y = tT_B$ and $X = tT_{B_0}$, and then inserting

$$\begin{aligned} Y - X &= T_B - T_{B_0}\\ &= \big(M_B^{-1} - M_{B_0}^{-1}\big)M_{B_0} tT_{B_0} - i\big(M_B^{-1}\big(B^{-1} - B_0^{-1}\big)B_0\big)\big(t\underline{d}_{B_0}^*\big)\\ &\quad - i\big(M_B^{-1}t\underline{d}_B^*\big)\big(B^{-1}(B - B_0)\big). \end{aligned}$$

Lemma 4.1. *Assume that for all $\mathcal{E} \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ and all operator families $(\widetilde{Q}_t)_{t>0}$ with L_2 off-diagonal bounds $\|\widetilde{Q}_t\|_{\text{off}} \leqslant C$ as in Definition 2.24, we have the two estimates*

$$\big|\big|\big|\widetilde{Q}_t \mathcal{E}\big(t\underline{d}\, Q_t^{B_0}\big)\big|\big|\big|_{\text{op}} \leqslant C\big(|||\widetilde{Q}_t|||_{\text{op}} + 1\big)\|\mathcal{E}\|_\infty, \tag{4.9}$$

$$\big|\big|\big|\widetilde{Q}_t \mathcal{E}\big(t\underline{d}^*_{B_0} Q_t^{B_0}\big)\big|\big|\big|_{\text{op}} \leqslant C\big(|||\widetilde{Q}_t|||_{\text{op}} + 1\big)\|\mathcal{E}\|_\infty. \tag{4.10}$$

Then there exists $\varepsilon > 0$ such that (4.2) *holds.*

Proof. First consider the terms (4.3), (4.4) and (4.8). Using the uniform boundedness of P_t^B and $Q_t^B M_B^{-1} t\underline{d}^*_B$ from Corollary 2.23, we deduce that

$$\big|\big|\big|(4.3)\big|\big|\big|_{\text{op}} \lesssim \|B - B_0\|_\infty \big|\big|\big|Q_t^{B_0}\big|\big|\big|_{\text{op}},$$
$$\big|\big|\big|(4.4)\big|\big|\big|_{\text{op}} \lesssim \|B - B_0\|_\infty \big|\big|\big|t\underline{d}^*_{B_0} P_t^{B_0}\big|\big|\big|_{\text{op}},$$
$$\big|\big|\big|(4.8)\big|\big|\big|_{\text{op}} \lesssim \|B - B_0\|_\infty \big|\big|\big|Q_t^{B_0}\big|\big|\big|_{\text{op}}.$$

Observe that $t\underline{d}^*_{B_0} P_t^{B_0} = i\mathbf{P}^2_{B_0} M_{B_0} Q_t^{B_0}$ as in Corollary 2.23, where the Hodge projection $\mathbf{P}^2_{B_0}$ and M_{B_0} are bounded. Thus we see from (4.1) that

$$\big|\big|\big|(4.3)\big|\big|\big|_{\text{op}} + \big|\big|\big|(4.4)\big|\big|\big|_{\text{op}} + \big|\big|\big|(4.8)\big|\big|\big|_{\text{op}} \lesssim \|B - B_0\|_\infty.$$

To handle the terms (4.5)–(4.7) we introduce the truncated operator families $\widetilde{Q}_t^1 := \chi(t) Q_t^B$ and $\widetilde{Q}_t^2 := \chi(t) P_t^B M_B^{-1} t\underline{d}^*_B$, where $\chi(t)$ denotes the characteristic function of the interval $[\tau^{-1}, \tau]$ for some large τ. Note that $P_t^B M_B^{-1} t\underline{d}^*_B = i Q_t^B \mathbf{P}^1_B$ as in Corollary 2.23, and therefore $|||\widetilde{Q}_t^2|||_{\text{op}} = |||\widetilde{Q}_t^1 \mathbf{P}^1_B|||_{\text{op}} \lesssim |||\widetilde{Q}_t^1|||_{\text{op}}$. To use the hypothesis on the terms (4.5)–(4.7) we note that the last factors are

$$P_t^{B_0} = I + i M_{B_0}^{-1}\big(t\underline{d}\, Q_t^{B_0}\big) + i M_{B_0}^{-1}\big(t\underline{d}^*_{B_0} Q_t^{B_0}\big),$$
$$t T_{B_0} Q_t^{B_0} = -i M_{B_0}^{-1}\big(t\underline{d}\, Q_t^{B_0}\big) - i M_{B_0}^{-1}\big(t\underline{d}^*_{B_0} Q_t^{B_0}\big),$$
$$-it\underline{d}^*_{B_0} Q_t^{B_0} = -i\big(t\underline{d}^*_{B_0} Q_t^{B_0}\big),$$

respectively. Thus we get from (4.3)–(4.8), after multiplication with $\chi(t)$, that

$$\big|\big|\big|\chi(t)\big(Q_t^B - Q_t^{B_0}\big)\big|\big|\big|_{\text{op}} \leqslant C\big(\big|\big|\big|\chi(t) Q_t^B\big|\big|\big|_{\text{op}} + 1\big)\|B - B_0\|_\infty,$$

and thus if $\|B - B_0\|_\infty \leqslant \varepsilon := 1/(2C)$ that

$$\big|\big|\big|\chi Q_t^B\big|\big|\big|_{\text{op}} \leqslant \frac{C\|B - B_0\|_\infty + |||Q_t^{B_0}|||_{\text{op}}}{1 - C\|B - B_0\|_\infty} \leqslant C',$$

since $|||\chi(t) Q_t^B|||_{\text{op}} < \infty$. Since this estimate is independent of τ, it follows that $|||Q_t^B f||| \lesssim \|f\|$. $\square$

Before turning to the proofs of (4.9) and (4.10), we summarise fundamental techniques from harmonic analysis that we shall need. We use the following dyadic decomposition of $\mathbf{R}^n$. Let $\triangle = \bigcup_{j=-\infty}^{\infty} \triangle_{2^j}$ where $\triangle_t := \{2^j(k+(0,1]^n)\colon k \in \mathbf{Z}^n\}$ if $2^{j-1} < t \leqslant 2^j$. For a dyadic cube $Q \in \triangle_{2^j}$, denote by $l(Q) = 2^j$ its *sidelength*, by $|Q| = 2^{nj}$ its *volume* and by $R_Q := Q \times (0, 2^j] \subset \mathbf{R}^{n+1}_+$ the associated *Carleson box*. Let the *dyadic averaging operator* $A_t : \mathcal{H} \to \mathcal{H}$ be given by

$$A_t u(x) := u_Q := \fint_Q u(y)\,dy = \frac{1}{|Q|}\int_Q u(y)\,dy$$

for every $x \in \mathbf{R}^n$ and $t > 0$, where $Q \in \triangle_t$ is the unique dyadic cube that contains x.

We now survey known results for a family of operators $\Theta_t : \mathcal{H} \to \mathcal{H}$, $t > 0$. For the proofs we refer to [4] and [10].

Definition 4.2. By the *principal part* of $(\Theta_t)_{t>0}$ we mean the multiplication operators γ_t defined by

$$\gamma_t(x)w := (\Theta_t w)(x)$$

for every $w \in \bigwedge$. We view w on the right-hand side of the above equation as the constant function defined on $\mathbf{R}^n$ by $w(x) := w$. We identify $\gamma_t(x)$ with the (possibly unbounded) multiplication operator $\gamma_t : f(x) \mapsto \gamma_t(x)f(x)$.

Lemma 4.3. *Assume that Θ_t has L_2 off-diagonal bounds with exponent $M > n$. Then Θ_t extends to a bounded operator $L_\infty \to L_2^{\mathrm{loc}}$. In particular we have well-defined functions $\gamma_t \in L_2^{\mathrm{loc}}(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ with bounds*

$$\fint_Q \left|\gamma_t(y)\right|^2 dy \lesssim \|\Theta_t\|_{\mathrm{off}}^2$$

for all $Q \in \Delta_t$. Moreover $\|\gamma_t A_t\| \lesssim \|\Theta_t\|_{\mathrm{off}}$ uniformly for all $t > 0$.

We have the following principal part approximation $\Theta_t \approx \gamma_t$.

Lemma 4.4. *Assume that Θ_t has L_2 off-diagonal bounds with exponent $M > 3n$ and let $F_t : \mathbf{R}^n \to \bigwedge$ be a family of functions. Then*

$$\left|\!\left|\!\left|(\Theta_t - \gamma_t A_t)F_t\right|\!\right|\!\right| \lesssim \|\Theta_t\|_{\mathrm{off}} |||t\nabla F_t|||,$$

where $\nabla f = \nabla \otimes f = \sum_{j=1}^n e_i \otimes (\partial_j f)$ denotes the full differential of f. Moreover, if P_t is a standard Fourier mollifier (*we shall use $P_t = (1+t^2\Pi^2)^{-1}$ where $\Pi = \Gamma + \Gamma^*$ and Γ is an exact nilpotent, homogeneous first order partial differential operator with constant coefficients as in* [10]) *and $F_t = P_t f$ for some $f \in \mathcal{H}$, then $|||\gamma_t A_t(P_t - I)f||| \lesssim \|\Theta_t\|_{\mathrm{off}}\|f\|$ and $|||t\nabla P_t|||_{\mathrm{op}} \leqslant C$. Thus*

$$\left|\!\left|\!\left|(\Theta_t P_t - \gamma_t A_t)f\right|\!\right|\!\right| \lesssim \|\Theta_t\|_{\mathrm{off}}\|f\|.$$

Definition 4.5. A function $\gamma(t,x) : \mathbf{R}^{n+1}_+ \to \bigwedge$ is called a *Carleson function* if there exists $C < \infty$ such that

$$\iint_{R_Q} \left|\gamma(t,x)\right|^2 \frac{dx\,dt}{t} \leqslant C^2|Q|$$

for all cubes $Q \subset \mathbf{R}^n$. Here $R_Q := Q \times (0, l(Q)]$ is the Carleson box over Q. We define the Carleson norm $\|\gamma_t\|_C$ to be the smallest constant C.

We use Carleson's lemma in the following form.

Lemma 4.6. *Let $\gamma_t(x) = \gamma(t,x) : \mathbf{R}^{n+1}_+ \to \bigwedge$ be a Carleson function and let $F_t(x) = F(t,x) : \mathbf{R}^{n+1}_+ \to \bigwedge$ be a family of functions. Then*

$$|||\gamma_t F_t||| \lesssim \|\gamma_t\|_C \left\|N_*(F_t)\right\|,$$

where $N_*(F_t)(x) := \sup_{|y-x|<t} |F_t(y)|$ denotes the non-tangential maximal function of F_t. In particular, if $F_t = A_t f$ for some $f \in \mathcal{H}$, then $\|N_*(A_t f)\| \lesssim \|M(f)\| \lesssim \|f\|$, where $M(f)(x) := \sup_{r>0} ⨍_{B(x,r)} |f(y)|\,dy$ denotes the Hardy–Littlewood maximal function, and thus

$$|||\gamma_t A_t f||| \lesssim \|\gamma_t\|_C \|f\|.$$

Lemma 4.7. *Assume that Θ_t has L_2 off-diagonal bounds with exponent $M > n$. Then*

$$\|\Theta_t f\|_C \lesssim \left(|||\Theta_t|||_{\mathrm{op}} + \|\Theta_t\|_{\mathrm{off}}\right)\|f\|_\infty,$$

for every $f \in L_\infty(\mathbf{R}^n; \bigwedge)$. In particular, choosing $f = w =$ constant we obtain

$$\|\gamma_t\|_C \lesssim |||\Theta_t|||_{\mathrm{op}} + \|\Theta_t\|_{\mathrm{off}}.$$

4.1. Perturbation of block coefficients

In this section we assume that B_0 is a block matrix, i.e. we assume that

$$B_0 = \begin{bmatrix} (B_0)_{\perp\perp} & 0 \\ 0 & (B_0)_{\|\|} \end{bmatrix},$$

in the splitting $\mathcal{H} = N^-\mathcal{H} \oplus N^+\mathcal{H}$. Our goal is to prove Theorem 1.4 by verifying the hypothesis of Lemma 4.1. We recall from Lemma 3.1 that

$$T_{B_0} = \Pi_{B_0} = \Gamma + B_0^{-1}\Gamma^* B_0, \quad \text{where } \Gamma := -iN\underline{d},$$

is an operator of the form treated in [10]. Thus with a slight change of notation for $\mathcal{E}$, we need to prove the following.

Theorem 4.8. *If B_0 is a block matrix and if $(\widetilde{Q}_t)_{t>0}$ is an operator family with L_2 off-diagonal bounds $\|\widetilde{Q}_t\|_{\mathrm{off}} \leqslant C$ as in Definition* 2.24*, then with notation as above*

$$\big|\!\big|\!\big|\widetilde{Q}_t \mathcal{E}\big(t^2 \Gamma \Gamma_{B_0}^* P_t^{B_0}\big)\big|\!\big|\!\big|_{\mathrm{op}} \leqslant C\big(|\!|\!|\widetilde{Q}_t|\!|\!|_{\mathrm{op}} + 1\big)\|\mathcal{E}\|_\infty, \tag{4.11}$$

$$\big|\!\big|\!\big|\widetilde{Q}_t \mathcal{E}\big(t^2 \Gamma_{B_0}^* \Gamma P_t^{B_0}\big)\big|\!\big|\!\big|_{\mathrm{op}} \leqslant C\big(|\!|\!|\widetilde{Q}_t|\!|\!|_{\mathrm{op}} + 1\big)\|\mathcal{E}\|_\infty, \tag{4.12}$$

where $P_t^{B_0} = (1 + t^2 \Pi_{B_0}^2)^{-1}$.

We note that if (4.11) holds with Γ replaced by Γ^* and with B_0 replaced by B_0^{-1}, then also (4.12) holds. This follows from the conjugation formula

$$\begin{aligned} &\widetilde{Q}_t \mathcal{E}\big(t^2 \Gamma_{B_0}^* \Gamma P_t^{B_0}\big) \\ &\quad = B_0^{-1}\big(B_0 \widetilde{Q}_t B_0^{-1}\big)\big(B_0 \mathcal{E} B_0^{-1}\big)\big(t^2 \Gamma^* \Gamma_{B_0^{-1}}\big(1 + t^2\big(\Gamma^* + B_0 \Gamma B_0^{-1}\big)^2\big)^{-1}\big)B_0. \end{aligned}$$

Thus it suffices to prove (4.11), as long as we only use properties of (Γ, B_0) shared with (Γ^*, B_0^{-1}). To this end, we let Θ_t be the operator

$$\Theta_t := \widetilde{Q}_t \mathcal{E}\big(t^2 \Gamma \Gamma_{B_0}^* P_t^{B_0}\big),$$

and denote by $\gamma_t(x)$ its principal part as in Definition 4.2. We note that we have a Hodge type splitting $\mathcal{H} = \mathsf{N}(\Gamma) \oplus \mathsf{N}(\Gamma_{B_0}^*)$ by Lemma 2.21(i), and since $\Theta_t|_{\mathsf{N}(\Gamma_{B_0}^*)} = 0$ it suffices to bound $|\!|\!|\Theta_t f|\!|\!|$ for $f \in \mathsf{N}(\Gamma)$. We do this by writing

$$\Theta_t f = \Theta_t(I - P_t)f + (\Theta_t P_t - \gamma_t A_t)f + \gamma_t A_t f, \tag{4.13}$$

where $\Pi := \Gamma + \Gamma^*$ is the corresponding unperturbed operator and $P_t := (1 + t^2\Pi^2)^{-1}$ and $Q_t := t\Pi(1 + t^2\Pi^2)^{-1}$.

Lemma 4.9. *We have, for all $f \in \mathsf{N}(\Gamma)$, the estimate*

$$\big|\!\big|\!\big|\Theta_t(I - P_t)f\big|\!\big|\!\big| \leqslant C\big(|\!|\!|\widetilde{Q}_t|\!|\!|_{\mathrm{op}} + 1\big)\|\mathcal{E}\|_\infty \|f\|.$$

Proof. If $f \in \mathsf{N}(\Gamma)$, then $(I - P_t)f = t\Gamma Q_t f \in \mathsf{N}(\Gamma)$, which shows that $t^2 \Gamma \Gamma_{B_0}^* P_t^{B_0}(I - P_t)f = (I - P_t^{B_0})(I - P_t)f$. To prove the estimate, we write

$$\Theta_t(I - P_t)f = \widetilde{Q}_t \mathcal{E}\big(I - P_t^{B_0}\big)t\Gamma Q_t f = \widetilde{Q}_t \mathcal{E} f - \widetilde{Q}_t \mathcal{E} P_t f - \widetilde{Q}_t \mathcal{E} Q_t^{B_0}\big(\Pi_{B_0}^{-1}\Gamma\big)Q_t f,$$

where we recall that $Q_t^{B_0} = t\Pi_{B_0} P_t^{B_0}$. Clearly $|\!|\!|\widetilde{Q}_t \mathcal{E} f|\!|\!| \lesssim |\!|\!|\widetilde{Q}_t|\!|\!| \|\mathcal{E}\|_\infty \|f\|$. For the second term, we write $\widetilde{\Theta}_t := \widetilde{Q}_t \mathcal{E}$ with principal part $\widetilde{\gamma}_t$ as in Definition 4.2 and estimate

$$\begin{aligned} |\!|\!|\widetilde{\Theta}_t P_t f|\!|\!| &\leqslant \big|\!\big|\!\big|(\widetilde{\Theta}_t P_t - \widetilde{\gamma}_t A_t)f\big|\!\big|\!\big| + |\!|\!|\widetilde{\gamma}_t A_t f|\!|\!| \\ &\lesssim \|\mathcal{E}\|_\infty \|f\| + \|\widetilde{\gamma}_t\|_C \|f\| \lesssim \|\mathcal{E}\|_\infty \|f\| + \big(|\!|\!|\widetilde{Q}_t|\!|\!|_{\mathrm{op}} + 1\big)\|\mathcal{E}\|_\infty \|f\|. \end{aligned}$$

In the second step, we used for the first term that $\|\widetilde{\Theta}_t\|_{\text{off}} \lesssim \|\mathcal{E}\|_\infty$ in Lemma 4.4 and for the second term we used Lemma 4.6. In the last step, we used Lemma 4.7 on the last term and that $\|\!|\widetilde{\Theta}_t|\!\|_{\text{op}} \lesssim \|\!|\widetilde{Q}_t|\!\|_{\text{op}}\|\mathcal{E}\|_\infty$. Finally we note that

$$\big|\!\big|\!\big| \widetilde{Q}_t \mathcal{E} Q_t^{B_0}\big(\Pi_{B_0}^{-1}\Gamma\big) Q_t f \big|\!\big|\!\big| \lesssim \|\mathcal{E}\|_\infty |\!|\!| Q_t f |\!|\!| \lesssim \|\mathcal{E}\|_\infty \|f\|,$$

since the Hodge projection $\Pi_{B_0}^{-1}\Gamma$ is bounded by Lemma 2.21(i). □

To estimate the last term in (4.13) we shall apply a local $T(b)$ theorem as in [4,10]. We here give an alternative construction of test functions to those used in [10], more in the spirit of the original proof of the Kato square root problem [4].

Lemma 4.10. *Let Γ be a nilpotent operator in $\mathcal{H}$, which is a homogeneous, first order partial differential operator with constant coefficients. Denote by $\bigwedge_\Gamma \subset \bigwedge$ the image of the linear functions $u : \mathbf{R}^n \to \bigwedge$ under Γ, where we identify $\bigwedge$ with the constant functions $\mathbf{R}^n \to \bigwedge$.*

Then for each $w \in \bigwedge_\Gamma$ with $|w| = 1$, each cube $Q \subset \mathbf{R}^n$ and each $\varepsilon > 0$, there exists a test function $f_{Q,\varepsilon}^w \in \mathcal{H}$ such that $f_{Q,\varepsilon}^w \in \mathsf{R}(\Gamma)$, $\|f_{Q,\varepsilon}^w\| \lesssim |Q|^{1/2}$,

$$\big\|\Gamma_{B_0}^* f_{Q,\varepsilon}^w\big\| \lesssim \frac{1}{\varepsilon l(Q)}|Q|^{1/2} \quad \textit{and} \quad \left|\fint_Q f_{Q,\varepsilon}^w - w\right| \lesssim \varepsilon^{1/2}.$$

Proof. Let $u(x)$ be a linear function such that $w = \Gamma u$ and $\sup_{3Q}|u(x)| \lesssim l(Q)$, and define $w_Q := \Gamma(\eta_Q u)$, where η_Q is a smooth cutoff such that $\eta_Q|_{2Q} = 1$, $\operatorname{supp}(\eta_Q) \subset 3Q$ and $\|\nabla \eta_Q\|_\infty \lesssim 1/l(Q)$. It follows that

$$w_Q \in \mathsf{R}(\Gamma), \qquad w_Q|_{2Q} = w, \qquad \operatorname{supp} w_Q \subset 3Q \quad \text{and} \quad \|w_Q\|_\infty \leqslant C.$$

Next we define the test function $f_{Q,\varepsilon}^w := P_{\varepsilon l}^{B_0} w_Q$, where we write $l = l(Q)$. Using Corollary 2.23, it follows that $\|f_{Q,\varepsilon}^w\| \lesssim |Q|^{1/2}$ and $\|\Gamma_{B_0}^* f_{Q,\varepsilon}^w\| \lesssim \frac{1}{\varepsilon l(Q)}|Q|^{1/2}$ and since Γ commutes with $P_{\varepsilon l}^{B_0}$, it follows that $f_{Q,\varepsilon}^w \in \mathsf{R}(\Gamma)$. To verify the accretivity property, we make use of [10, Lemma 5.6] which shows that

$$\left|\fint_Q f_{Q,\varepsilon}^w - w\right| = \left|\fint_Q \big(I - P_{\varepsilon l}^{B_0}\big) w_Q\right| = \left|\fint_Q \varepsilon l \Gamma\big(Q_{\varepsilon l}^{B_0} w_Q\big)\right|$$
$$\lesssim \varepsilon^{1/2}\left(\fint_Q \big|Q_{\varepsilon l}^{B_0} w_Q\big|^2\right)^{1/4}\left(\fint_Q \big|\varepsilon l \Gamma Q_{\varepsilon l}^{B_0} w_Q\big|^2\right)^{1/4} \lesssim \varepsilon^{1/2},$$

where we used that $w_Q \in \mathsf{R}(\Gamma)$ in the second step. □

Proof of Theorem 4.8. We have seen that it suffices to prove (4.11), and to bound each term in (4.13) for $f \in \mathsf{N}(\Gamma)$. The first term is estimated by Lemma 4.9 and the second by Lemma 4.4.

To prove that the last term has estimate $|||\gamma_t A_t f||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty \|f\|$, it suffices by Lemma 4.6 to prove that

$$\|\gamma_t\|_C \lesssim \big(|||\widetilde{Q}_t|||_{\mathrm{op}} + 1\big)\|\mathcal{E}\|_\infty.$$

To this end, we apply the local $T(b)$ argument and stopping time argument in [10, Section 5.3]. Note that $\mathsf{R}(\Gamma) \subset L_2(\mathbf{R}^n; \bigwedge_\Gamma)$ and thus, since Γ is an exact nilpotent operator, it follows that $\mathsf{N}(\Gamma) \subset L_2(\mathbf{R}^n; \bigwedge_\Gamma)$. Furthermore $A_t f \in L_2(\mathbf{R}^n; \bigwedge_\Gamma)$ if $f \in L_2(\mathbf{R}^n; \bigwedge_\Gamma)$. Thus it suffices to bound the Carleson norm of $\gamma_t(x)$ seen as a linear operator $\gamma_t(x) : \bigwedge_\Gamma \to \bigwedge$. The conical decomposition $\bigcup_{\nu \in \mathcal{V}} K_\nu$ of the space of matrices performed in [10, Section 5.3], here decomposes the space $\mathcal{L}(\bigwedge_\Gamma; \bigwedge)$ and for the fixed unit matrix $\nu \in \mathcal{L}(\bigwedge_\Gamma; \bigwedge)$ we choose $w \in \bigwedge_\Gamma$ and $\hat{w} \in \bigwedge$ such that $|\hat{w}| = |w| = 1$ and $\nu^*(\hat{w}) = w$. With the stopping time argument in [10, Section 5.3], using the new test functions f_Q^w from Lemma 4.10, we obtain

$$\|\gamma_t\|_C^2 \lesssim \sup_{Q, w \in \bigwedge_\Gamma, |w|=1} \frac{1}{|Q|} \iint_{R_Q} \big|\gamma_t(x) A_t f_Q^w(x)\big|^2 \frac{dx\,dt}{t},$$

where $f_Q^w = f_{Q,\varepsilon}^w$ for a small enough but fixed ε. To estimate the right-hand side we use (4.13) with f replaced with the test function f_Q^w, estimate the first two terms with Lemma 4.9 (which works since $f_Q^w \in \mathsf{R}(\Gamma)$) and Lemma 4.4, and obtain

$$\begin{aligned} \iint_{R_Q} \big|\gamma_t A_t f_Q^w\big|^2 \frac{dx\,dt}{t} &\lesssim \iint_{R_Q} \big|\gamma_t A_t f_Q^w - \Theta_t f_Q^w\big|^2 \frac{dx\,dt}{t} + \iint_{R_Q} \big|\Theta_t f_Q^w\big|^2 \frac{dx\,dt}{t} \\ &\lesssim \big(|||\widetilde{Q}_t|||_{\mathrm{op}} + 1\big)^2 \|\mathcal{E}\|_\infty^2 |Q| + \iint_{R_Q} \big|\Theta_t f_Q^w\big|^2 \frac{dx\,dt}{t}. \end{aligned}$$

Using that $\|\Gamma_{B_0}^* f_Q^w\| \lesssim \frac{1}{l(Q)} |Q|^{1/2}$ we then get

$$\big\|\Theta_t f_Q^w\big\| = \big\|\widetilde{Q}_t \mathcal{E}\big(t\Gamma P_t^{B_0}\big) t\big(\Gamma_{B_0}^* f_Q^w\big)\big\| \lesssim \|\mathcal{E}\|_\infty \frac{t}{l(Q)} |Q|^{1/2}.$$

This yields

$$\iint_{R_Q} \big|\Theta_t f_Q^w\big|^2 \frac{dx\,dt}{t} \lesssim |Q| \|\mathcal{E}\|_\infty^2 \int_0^{l(Q)} \left(\frac{t}{l(Q)}\right)^2 \frac{dt}{t} \lesssim |Q| \|\mathcal{E}\|_\infty^2,$$

which proves that

$$\iint_{R_Q} \big|\gamma_t A_t f_Q^w\big|^2 \frac{dx\,dt}{t} \lesssim \big(|||\widetilde{Q}_t|||_{\mathrm{op}} + 1\big)^2 \|\mathcal{E}\|_\infty^2 |Q|. \qquad \square$$

Remark 4.11. (i) Note that using the new test function from Lemma 4.10 simplifies the estimate of the term $(\gamma_t A_t - \Theta_t) f_Q^w$ in the above proof as compared with the proof of [10, Proposition 5.9]. The useful new property of the test functions from Lemma 4.10 is that they belong to $\mathsf{R}(\Gamma)$.

(ii) In the proof of Theorem 4.8, we only estimate the Carleson norm of the restriction of the matrix γ_t to the subspace $\bigwedge_\Gamma$ as this suffices since we want to bound the quadratic norm of $\gamma_t A_t f$, and $A_t f$ is always $\bigwedge_\Gamma$-valued. However, to prove (4.11) and (4.12), we use Γ being either Nmd or Nmd^*. In these two cases, the space $\bigwedge_\Gamma$ is either the orthogonal complement of $\text{span}\{1, e_0\}$ or $\text{span}\{e_{0,1,\dots,n}, e_{1,\dots,n}\}$, respectively. Note also that block matrices preserve these spaces $\bigwedge_\Gamma^\perp$. It is seen that in these two cases $\gamma_t = 0$ on $\bigwedge_\Gamma^\perp$, so actually we do get an estimate of the Carleson norm of the whole matrix γ_t.

4.2. Perturbation of vector coefficients

In this section we assume that the unperturbed coefficients B_0 are of the form

$$B_0 = I \oplus A_0 \oplus I \oplus \cdots \oplus I,$$

i.e. B_0 only acts non-trivially on the vector part, and that A_0 is a matrix such that T_{B_0} has quadratic estimates. Note that this hypothesis is true if A_0 is either real symmetric, constant or of block form, by Theorem 3.12, Proposition 3.5 and Theorem 3.3, respectively. Our goal is to prove Theorem 1.1, by verifying the hypothesis of Lemma 4.1, as well as proving Theorem 1.3. We start by reformulating Lemma 4.1 in terms of $e^{-t|T_{B_0}|}$, acting only on functions f in one of the two Hardy spaces $E^\pm_{B_0}\mathcal{H}$, instead of $P_t^{B_0}$.

Theorem 4.12. *If B_0 is as above and if $\|\widetilde{Q}_t\|_{\text{off}} \leqslant C$, then for all $f \in E^+_{B_0}\mathcal{H}$ we have estimates*

$$|||\widetilde{Q}_t\mathcal{E}(F_t - f)||| \leqslant C\big(|||\widetilde{Q}_t|||_{\text{op}} + 1\big)\|\mathcal{E}\|_\infty \|f\|, \tag{4.14}$$

$$\left|\left|\left|\widetilde{Q}_t\mathcal{E}\underline{d}\int_0^t F_s\,ds\right|\right|\right| \leqslant C\big(|||\widetilde{Q}_t|||_{\text{op}} + 1\big)\|\mathcal{E}\|_\infty \|f\|, \tag{4.15}$$

where $F_t := e^{-t|T_{B_0}|} f$ is the extension of f as in Lemma 2.49. *The corresponding estimates for $f \in E^-_{B_0}\mathcal{H}$ also hold.*

Proof that Theorem 4.12 implies (4.9) and (4.10). We first note that it suffices to prove (4.9) and (4.10) for all $f \in E^+_{B_0}\mathcal{H}$ and all $f \in E^-_{B_0}\mathcal{H}$ since we have a Hardy space splitting $\mathcal{H} = E^+_{B_0}\mathcal{H} \oplus E^-_{B_0}\mathcal{H}$. We only consider $f \in E^+_{B_0}\mathcal{H}$ since the proof for $f \in E^-_{B_0}\mathcal{H}$ is similar.

Now let $f \in E^+_{B_0}\mathcal{H}$ and use Proposition 2.31, which shows that if $\psi \in \Psi(S^o_\nu)$, then $|||\psi_t(T_{B_0})|||_{\text{op}} \lesssim |||Q_t^{B_0}|||_{\text{op}} \leqslant C$. For the estimate (4.9), we write

$$\widetilde{Q}_t\mathcal{E}t\underline{d}Q_t^{B_0} = \widetilde{Q}_t\mathcal{E}\big(\underline{d}T_{B_0}^{-1}\big)\big(I - e^{-t|T_{B_0}|}\big) + \widetilde{Q}_t\mathcal{E}\big(\underline{d}T_{B_0}^{-1}\big)\psi_t(T_{B_0}),$$

where $\psi(z) = e^{-|z|} - (1+z^2)^{-1}$. Note for the first term that $T_{B_0}^{-1}(I - e^{-t|T_{B_0}|})f = T_{B_0}^{-1}(f - F_t) = -T_{B_0}^{-1}\int_0^t \partial_s F_s\,ds = \int_0^t F_s\,ds$. Therefore Theorem 4.12, the boundedness of $\underline{d}T_{B_0}^{-1}$ and Proposition 2.31 give the estimate (4.9).

For the estimate (4.10), note that it suffices to estimate

$$\widetilde{Q}_t \mathcal{E} t T_{B_0} Q_t^{B_0} = \widetilde{Q}_t \mathcal{E}\big(I - P_t^{B_0}\big)$$

since $i M_{B_0} T_{B_0} = \underline{d} + \underline{d}^*_{B_0}$. But this follows immediately from (4.14) since

$$\widetilde{Q}_t \mathcal{E}\big(I - P_t^{B_0}\big) f = \widetilde{Q}_t \mathcal{E}(f - F_t) + \widetilde{Q}_t \mathcal{E} \psi_t(T_{B_0}) f,$$

with the same ψ as above. □

Remark 4.13. In the case when B_0 is a constant matrix, we can estimate $\widetilde{Q}_t \mathcal{E}(I - P_t^{B_0}) f$ directly. We prove that

$$\big|\!\big|\!\big| \widetilde{Q}_t \mathcal{E}\big(I - P_t^{B_0}\big) f \big|\!\big|\!\big| \lesssim \big(|\!|\!| \widetilde{Q}_t |\!|\!|_{\mathrm{op}} + 1\big) \|\mathcal{E}\|_\infty \|f\|$$

as follows. Clearly $|\!|\!|\widetilde{Q}_t \mathcal{E}|\!|\!|_{\mathrm{op}} \lesssim |\!|\!|\widetilde{Q}_t|\!|\!|_{\mathrm{op}} \|\mathcal{E}\|_\infty$. For the second term we write $\Theta_t := \widetilde{Q}_t \mathcal{E} P_t^{B_0}$. Inserting a standard Fourier mollifier P_t, we write $\Theta_t = \Theta_t(I - P_t) + \Theta_t P_t$. Here $|\!|\!|\Theta_t(I - P_t) f|\!|\!| \lesssim |\!|\!|P_t^{B_0}(I - P_t) f|\!|\!| \lesssim \|f\|$ is easily verified using the Fourier transform. On the other hand,

$$|\!|\!|\Theta_t P_t f|\!|\!| \leqslant \big|\!\big|\!\big|(\Theta_t P_t - \gamma_t A_t) f\big|\!\big|\!\big| + |\!|\!|\gamma_t A_t f|\!|\!| \lesssim \|f\| + \|\gamma_t\|_C \|f\|,$$

using Lemmas 4.4 and 4.6. However, since B_0 is constant, we have that $T_{B_0} w = 0$ if w is a constant function, and therefore

$$\gamma_t w = \widetilde{Q}_t \mathcal{E}\big(P_t^{B_0} w\big) = \widetilde{Q}_t \mathcal{E} w.$$

Therefore Lemma 4.7 shows that $\|\gamma_t\|_C \lesssim (|\!|\!|\widetilde{Q}_t|\!|\!|_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty$.

We now set up some notation for the proof of Theorem 4.12. We decompose the function $F_t := e^{-t|T_{B_0}|} f$, where $f \in E^+_{B_0}\mathcal{H}$, as

$$F_t = F_t^0 + \big(F_t^{1,\perp} + F_t^{1,\parallel}\big) + F_t^2 + \cdots + F_t^{n+1}, \tag{4.16}$$

and similarly for $f = \lim_{t\to 0^+} F_t$. It is important to note the special property that the normal component of the vector part $F_t^{1,\perp} = F_t^{1,0} e_0$ has by Lemma 2.12: it satisfies the divergence form second order equation $\mathrm{div}_{t,x} A_0(x) \nabla_{t,x} F^{1,0} = 0$. Furthermore, we decompose the matrix A_0 as

$$A_0 = \begin{bmatrix} a_{\perp\perp} & a_{\perp\parallel} \\ a_{\parallel\perp} & a_{\parallel\parallel} \end{bmatrix},$$

in the splitting $\mathcal{H} = N^-\mathcal{H} \oplus N^+\mathcal{H}$. We view the components $a_{\perp\perp}$, $a_{\perp\parallel}$, $a_{\parallel\perp}$ and $a_{\parallel\parallel}$ as operators, and write $a_{\perp\perp}(f^{1,0} e_0) = (a_{00} f^{1,0}) e_0$, $a_{\perp\parallel} f^{1,\parallel} = (a_{0\parallel} \cdot f^{1,\parallel}) e_0$ and $a_{\parallel\perp}(f^{1,0} e_0) = f^{1,0} a_{\parallel 0}$, where a_{00} is a scalar and $a_{0\parallel}$ and $a_{\parallel 0}$ are vectors.

We introduce an auxiliary block matrix

$$\hat{B}_0 = I \oplus \hat{A}_0 \oplus I \oplus \cdots \oplus I, \qquad \hat{A}_0 = \begin{bmatrix} a_{\perp\perp} & 0 \\ 0 & a_{\|\|} \end{bmatrix}.$$

Lemma 4.14. *Let $F_t := e^{-t|T_{B_0}|} f$, where $f \in E^+_{B_0}\mathcal{H}$, so that $(\partial_t + T_{B_0})F_t = 0$. Then*

$$(\partial_t + T_{\hat{B}_0})F_t = -i\big(a_{00}^{-1} a_{\perp\|}\big)\underline{d} F_t^{1,\perp} - i\underline{d}^*_{\hat{B}_0}\big(a_{\|\|}^{-1} a_{\|\perp}\big)F_t^{1,\perp} = \partial_t F_t^{1,\perp} + i\underline{d}^*_{\hat{B}_0} F_t^{1,\|}.$$

Proof. To prove the first identity, note that

$$\begin{aligned}
(\partial_t + T_{\hat{B}_0})F_t &= (T_{\hat{B}_0} - T_{B_0})F_t \\
&= -i\big(\hat{B}_0 N^+ - N^- \hat{B}_0\big)^{-1}(\hat{B}_0 \underline{d} + \underline{d}^* \hat{B}_0)F_t \\
&\quad + i\big(B_0 N^+ - N^- B_0\big)^{-1}(B_0 \underline{d} + \underline{d}^* B_0)F_t \\
&= i\big(\hat{B}_0 N^+ - N^- \hat{B}_0\big)^{-1}\big((B_0 - \hat{B}_0)\underline{d} + \underline{d}^*(B_0 - \hat{B}_0)\big)F_t
\end{aligned}$$

since $\hat{B}_0 N^+ - N^- \hat{B}_0 = B_0 N^+ - N^- B_0$. The vector part of this matrix is $\left[\begin{smallmatrix} -a_{\perp\perp} & 0 \\ 0 & a_{\|\|} \end{smallmatrix}\right]$. Furthermore

$$B_0 - \hat{B}_0 = 0 \oplus \begin{bmatrix} 0 & a_{\perp\|} \\ a_{\|\perp} & 0 \end{bmatrix} \oplus 0 \oplus \cdots \oplus 0,$$

which shows that

$$(B_0 - \hat{B}_0)\underline{d} F_t = a_{\perp\|}\underline{d} F_t^{1,\perp} \quad \text{and} \quad \underline{d}^*(B_0 - \hat{B}_0)F_t = \underline{d}^* a_{\|\perp} F_t^{1,\perp},$$

using the mapping properties of $\underline{d}$ and $\underline{d}^*$ from Remark 2.6. Thus

$$\begin{aligned}
(\partial_t + T_{\hat{B}_0})F_t &= -i a_{\perp\perp}^{-1}\big(a_{\perp\|}\underline{d} F_t^{1,\perp} + \underline{d}^* a_{\|\perp} F_t^{1,\perp}\big) \\
&= -i\big(a_{00}^{-1} a_{\perp\|}\big)\underline{d} F_t^{1,\perp} - i\underline{d}^*_{\hat{B}_0}\big(a_{\|\|}^{-1} a_{\|\perp}\big)F_t^{1,\perp},
\end{aligned}$$

since $\hat{B}_0 N^+ - N^- \hat{B}_0 = -a_{\perp\perp}$ on normal vector fields.

To prove the second identity, we multiply the Dirac equation $(\partial_t + T_{B_0})F_t = 0$ by $B_0 N^+ - N^- B_0$ to obtain

$$\big(B_0 N^+ - N^- B_0\big)\partial_t F_t - i(B_0 \underline{d} + \underline{d}^* B_0)F_t = 0.$$

The normal component of the vector part on the left-hand side is

$$-a_{\perp\perp}\partial_t F_t^{1,\perp} - i\big(a_{\perp\|}\underline{d} F_t^{1,\perp} + \underline{d}^*\big(a_{\|\perp} F_t^{1,\perp} + a_{\|\|} F_t^{1,\|}\big)\big) = 0.$$

Here we have used the expression for the vector part of $B_0 N^+ - N^- B_0$ above for the first term. For the other terms we recall from Remark 2.6 that the vector part of $\underline{d} F_t$ is $\underline{d} F_t^{1,\perp}$ which is tangential, and that the normal vector part of $\underline{d}^* B_0 F_t$ is $\underline{d}^*(B_0 F_t)^{1,\parallel}$. Therefore, after multiplying the equation with a_{00}^{-1}, we obtain

$$-i\big(a_{00}^{-1} a_{\perp\parallel}\big)\underline{d} F_t^{1,\perp} - i\underline{d}^*_{\hat{B}_0}\big(a_{\parallel\parallel}^{-1} a_{\parallel\perp}\big) F_t^{1,\perp} = \partial_t F_t^{1,\perp} + i\underline{d}^*_{\hat{B}_0} F_t^{1,\parallel},$$

which proves the lemma. □

As in Lemma 3.1 we note that since $\hat{B}_0$ is a block matrix

$$T_{\hat{B}_0} = \Pi_{\hat{B}_0} = \Gamma + \hat{B}_0^{-1}\Gamma^* \hat{B}_0, \quad \text{where } \Gamma := -iN\underline{d}.$$

To prove Theorem 4.12 we shall need the following corollary to Theorem 4.8.

Corollary 4.15. *Let $\hat{B}_0$ be the block matrix defined above, assume that $\|\widetilde{Q}_t\|_{\mathrm{off}} \leqslant C$ and let $v \in L_\infty(\mathbf{R}^n; \bigwedge)$ be a function with norm $\|v\|_\infty \leqslant C$. Then for all $f \in E^+_{B_0}\mathcal{H}$ we have the estimates*

$$\big|\!\big|\!\big| \widetilde{Q}_t \mathcal{E} t^2 \Gamma \Gamma^*_{\hat{B}_0} P_t^{\hat{B}_0}\big(F_t^{1,0} v\big) \big|\!\big|\!\big| \lesssim \big(|\!|\!| \widetilde{Q}_t |\!|\!|_{\mathrm{op}} + 1\big) \|\mathcal{E}\|_\infty \|f\|, \tag{4.17}$$

$$\big|\!\big|\!\big| \widetilde{Q}_t \mathcal{E} P_t^{\hat{B}_0}\big(F_t^{1,0} v\big) \big|\!\big|\!\big| \lesssim \big(|\!|\!| \widetilde{Q}_t |\!|\!|_{\mathrm{op}} + 1\big) \|\mathcal{E}\|_\infty \|f\|, \tag{4.18}$$

where $F_t := e^{-t|T_{B_0}|} f$ and $F_t^{1,0} = (F_t, e_0)$. The corresponding estimates for $f \in E^-_{B_0}\mathcal{H}$ also hold.

We defer the proof until the end of this section, and turn to a lemma in preparation for the proof of Theorem 4.12.

Lemma 4.16. *If $f \in E^+_{B_0}\mathcal{H}$ and $F_t := e^{-t|T_{B_0}|} f$, then*

$$\left|\!\left|\!\left| \int_0^t \frac{s}{t}(s\partial_s F_s)\frac{ds}{s} \right|\!\right|\!\right| \leqslant |\!|\!| t\partial_t F_t |\!|\!| \lesssim \|f\|, \tag{4.19}$$

$$|\!|\!| t\underline{d} F_t |\!|\!| \lesssim |\!|\!| t\partial_t F_t |\!|\!| \lesssim \|f\|. \tag{4.20}$$

The corresponding estimates for $f \in E^-_{B_0}\mathcal{H}$ also hold.

Proof. The proof of (4.19) uses Schur estimates. Applying Cauchy–Schwarz inequality, we estimate the square of the left-hand side by

$$\int_0^\infty \left\| \int_0^t \frac{s}{t}(s\partial_s F_s)\frac{ds}{s} \right\|^2 \frac{dt}{t} \leqslant \int_0^\infty \left(\int_0^t \frac{s}{t}\frac{ds}{s} \right)\left(\int_0^t \frac{s}{t}\|s\partial_s F_s\|^2 \frac{ds}{s} \right)^2 \frac{dt}{t}$$
$$= \int_0^\infty \left(\int_s^\infty \frac{s}{t}\frac{dt}{t} \right)\|s\partial_s F_s\|^2 \frac{ds}{s} = \big|\!\big|\!\big| \psi_t(T_{B_0}) f \big|\!\big|\!\big|^2 \lesssim \|f\|^2,$$

where $\psi(z) := z e^{-|z|}$.

To prove (4.20), we note that $iM_{B_0}T_{B_0} = \underline{d} + \underline{d}^*_{B_0}$ and write $\mathbf{P}^1_{B_0}$, $\mathbf{P}^2_{B_0}$ for the two Hodge projections corresponding to the splitting $\mathcal{H} = \mathsf{N}(\underline{d}) \oplus \mathsf{N}(\underline{d}^*_{B_0})$. Note that these projections are bounded by Lemma 2.21(i). We get

$$t\underline{d}F_t = i\mathbf{P}^1_{B_0}M_{B_0}(tT_{B_0}F_t) = i\mathbf{P}^1_{B_0}M_{B_0}\psi_t(T_{B_0})f,$$

from which (4.20) follows. □

Proof that Corollary 4.15 implies the estimate (4.14). By inserting $I = P_t^{\hat{B}_0} + (I - P_t^{\hat{B}_0})$ we write the left-hand side in (4.14) as

$$\widetilde{Q}_t\mathcal{E}P_t^{\hat{B}_0}(F_t - f) + \widetilde{Q}_t\mathcal{E}\big(I - P_t^{\hat{B}_0}\big)F_t - \widetilde{Q}_t\mathcal{E}\big(I - P_t^{\hat{B}_0}\big)f =: X_1 + X_2 - X_3.$$

For X_3 we get from Theorem 4.8 the estimate $|||X_3||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty\|f\|$ since $I - P_t^{\hat{B}_0} = t^2\Gamma\Gamma^*_{\hat{B}_0}P_t^{\hat{B}_0} + t^2\Gamma^*_{\hat{B}_0}\Gamma P_t^{\hat{B}_0}$. For the term X_2 we use the first identity in Lemma 4.14 to obtain

$$\begin{aligned}
X_2 &= \widetilde{Q}_t\mathcal{E}Q_t^{\hat{B}_0}tT_{\hat{B}_0}F_t \\
&= -\widetilde{Q}_t\mathcal{E}Q_t^{\hat{B}_0}(t\partial_t F_t) - i\widetilde{Q}_t\mathcal{E}Q_t^{\hat{B}_0}a_{00}^{-1}a_{\perp\parallel}\big(t\underline{d}F_t^{1,\perp}\big) \\
&\quad - i\widetilde{Q}_t\mathcal{E}\big(Q_t^{\hat{B}_0}t\underline{d}^*_{\hat{B}_0}\big)\big(F_t^{1,0}\big(a_{\parallel\parallel}^{-1}a_{\parallel\perp}e_0\big)\big) =: -X_4 - iX_5 - iX_6.
\end{aligned}$$

We have the estimate $|||X_4||| \lesssim \|\mathcal{E}\|_\infty|||t\partial_t F_t||| \lesssim \|\mathcal{E}\|_\infty\|f\|$. For X_5, we see from Remark 2.6 that $\underline{d}F_t^{1,\perp}$ is the vector part of $\underline{d}F_t$. Thus $|||X_5||| \lesssim \|\mathcal{E}\|_\infty|||t\underline{d}F_t||| \lesssim \|\mathcal{E}\|_\infty\|f\|$ by (4.20). To handle the term X_6 we note that $Q_t^{\hat{B}_0}t\underline{d}^*_{\hat{B}_0} = -Q_t^{\hat{B}_0}t\Gamma^*_{\hat{B}_0}N = -t^2\Gamma\Gamma^*_{\hat{B}_0}P_t^{\hat{B}_0}N$ using Remark 3.2. Thus we obtain from Corollary 4.15, with $v = a_{\parallel\parallel}^{-1}a_{\parallel\perp}e_0$, the estimate $|||X_6||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty\|f\|$.

It remains to estimate the term X_1. To handle this, we separate the normal vector part as

$$X_1 = \widetilde{Q}_t\mathcal{E}P_t^{\hat{B}_0}\big(F_t^{1,0}e_0\big) - \widetilde{Q}_t\mathcal{E}P_t^{\hat{B}_0}\big(f^{1,0}e_0\big) + \widetilde{Q}_t\mathcal{E}P_t^{\hat{B}_0}(G_t - g) =: X_7 - X_8 + X_9,$$

where $G_t := F_t - F_t^{1,\perp}$ and $g = f - f^{1,\perp}$. From Corollary 4.15, with $v = e_0$, we get the estimate $|||X_7||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty\|f\|$. For the term X_8, we write $P_t^{\hat{B}_0} = I - t^2\Gamma\Gamma^*_{\hat{B}_0}P_t^{\hat{B}_0} - t^2\Gamma^*_{\hat{B}_0}\Gamma P_t^{\hat{B}_0}$. From Theorem 4.8 we obtain the estimate $|||X_8||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty\|f\|$. For the term X_9, we integrate by parts to obtain

$$X_9 = \widetilde{Q}_t\mathcal{E}P_t^{\hat{B}_0}(t\partial_t G_t) - \widetilde{Q}_t\mathcal{E}P_t^{\hat{B}_0}\left(\int_0^t s\partial_s^2 G_s\,ds\right) =: X_{10} - X_{11}.$$

We have the estimate $|||X_{10}||| \lesssim \|\mathcal{E}\|_\infty |||t\partial_t F_t||| \lesssim \|\mathcal{E}\|_\infty \|f\|$. For the term X_{11} we apply ∂_t to the last expression for $(\partial_t + T_{\hat{B}_0})F_t$ in Lemma 4.14 and get

$$\partial_s^2 G_s + \partial_s T_{\hat{B}_0} F_s = i\underline{d}^*_{\hat{B}_0} \partial_s F_s^{1,\parallel}. \tag{4.21}$$

Thus

$$\begin{aligned} X_{11} &= -\widetilde{Q}_t \mathcal{E}\big(P_t^{\hat{B}_0} t T_{\hat{B}_0}\big)\left(\int_0^t \frac{s}{t}(s\partial_s F_s)\frac{ds}{s}\right) + i\,\widetilde{Q}_t\mathcal{E}\big(P_t^{\hat{B}_0} t\underline{d}^*_{\hat{B}_0}\big)\left(\int_0^t \frac{s}{t}\big(s\partial_s F_s^{1,\parallel}\big)\frac{ds}{s}\right) \\ &=: -X_{12} + iX_{13}. \end{aligned}$$

Both $|||X_{12}|||$ and $|||X_{13}|||$ can now be estimated with $\|\mathcal{E}\|_\infty\|f\|$ by (4.19) since $P_t^{\hat{B}_0} t T_{\hat{B}_0} = Q_t^{\hat{B}_0}$ and $P_t^{\hat{B}_0} t\underline{d}^*_{\hat{B}_0}$ are uniformly bounded by Corollary 2.23. This proves the estimate (4.14). □

Proof that Corollary 4.15 implies the estimate (4.15). We write the left-hand side in (4.15), using integration by parts, as

$$\widetilde{Q}_t\mathcal{E}\underline{d}\int_0^t F_s\,ds = \widetilde{Q}_t\mathcal{E}t\underline{d}F_t - \widetilde{Q}_t\mathcal{E}\underline{d}\int_0^t s\partial_s F_s\,ds =: X_1 - X_2.$$

For X_1 we have $|||\widetilde{Q}_t\mathcal{E}(t\underline{d}F_t)||| \lesssim \|\mathcal{E}\|_\infty\|f\|$ by (4.20). For X_2, we write $I = P_t^{\hat{B}_0} + (I - P_t^{\hat{B}_0})$ and get

$$X_2 = \widetilde{Q}_t\mathcal{E}\big(t\underline{d}P_t^{\hat{B}_0}\big)\int_0^t \frac{s}{t}(s\partial_s F_s)\frac{ds}{s} + \widetilde{Q}_t\mathcal{E}t\underline{d}Q_t^{\hat{B}_0}\int_0^t s\partial_s T_{\hat{B}_0}F_s\,ds =: X_3 + X_4.$$

Using that $\|t\underline{d}P_t^{\hat{B}_0}\| \leqslant C$ by Corollary 2.23, and (4.19) shows that $|||X_3||| \lesssim \|\mathcal{E}\|_\infty\|f\|$. To handle X_4, we use the identity (4.21), which gives

$$X_4 = \widetilde{Q}_t\mathcal{E}t\underline{d}Q_t^{\hat{B}_0}\left(-\int_0^t s\partial_s^2 G_s\,ds + i\int_0^t s\underline{d}^*_{\hat{B}_0}\partial_s F_s^{1,\parallel}\,ds\right) =: -X_5 + iX_6.$$

For X_6 we note that $\underline{d}Q_t^{\hat{B}_0}\underline{d}^*_{\hat{B}_0} = N\underline{d}Q_t^{\hat{B}_0}N\underline{d}^*_{\hat{B}_0} = -\Gamma Q_t^{\hat{B}_0}\Gamma^*_{\hat{B}_0} = -\Gamma^2 Q_t^{\hat{B}_0} = 0$ using Remark 3.2, and thus $X_6 = 0$. To handle X_5, we rewrite this with an integration by parts as

$$X_5 = \widetilde{Q}_t\mathcal{E}\big(t\underline{d}Q_t^{\hat{B}_0}\big)(t\partial_t G_t) - \widetilde{Q}_t\mathcal{E}t\underline{d}Q_t^{\hat{B}_0}G_t + \widetilde{Q}_t\mathcal{E}\big(t\underline{d}Q_t^{\hat{B}_0}g\big) =: X_6 - X_7 + X_8,$$

where $g = f - f^{1,\perp}$. Using Lemma 4.16 and that $\|t\underline{d}Q_t^{\hat{B}_0}\| \leqslant C$ by Corollary 2.23, we get $|||X_6||| \lesssim \|\mathcal{E}\|_\infty\|f\|$. For X_8, we note that $t\underline{d}Q_t^{\hat{B}_0} = iNt^2\Gamma\Gamma^*_{\hat{B}_0}P_t^{\hat{B}_0}$. Thus we can apply Theorem 4.8 to obtain $|||X_8||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty\|f\|$.

We now write $G_t = F_t - F_t^{1,0} e_0$ and get

$$X_7 = \widetilde{Q}_t \mathcal{E} t \underline{d} P_t^{\hat{B}_0}(t T_{\hat{B}_0} F_t) - \widetilde{Q}_t \mathcal{E} t \underline{d} Q_t^{\hat{B}_0}\big(F_t^{1,0} e_0\big) =: X_9 - X_{10}.$$

Again noting that $t\underline{d} Q_t^{\hat{B}_0} = iNt^2 \Gamma \Gamma^*_{\hat{B}_0} P_t^{\hat{B}_0}$ we obtain from Corollary 4.15, with $v = e_0$, the estimate $|||X_{10}||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty \|f\|$. The term X_9 remains, on which we use the first identity in Lemma 4.14 to obtain

$$\begin{aligned} X_9 = {} & -\widetilde{Q}_t \mathcal{E} t \underline{d} P_t^{\hat{B}_0}(t\partial_t F_t) - i\widetilde{Q}_t \mathcal{E}\big(t\underline{d} P_t^{\hat{B}_0}\big) a_{00}^{-1} a_{\perp\|}\big(t\underline{d} F_t^{1,\perp}\big) \\ & - i\widetilde{Q}_t \mathcal{E}\big(t\underline{d} P_t^{\hat{B}_0} t\underline{d}^*_{\hat{B}_0}\big)\big(F_t^{1,0}\big(a_{\|\|}^{-1} a_{\|\perp} e_0\big)\big) =: -X_{11} - iX_{12} - iX_{13}. \end{aligned}$$

Using that $\|t\underline{d} P_t^{\hat{B}_0}\| \leqslant C$ by Corollary 2.23, shows that $|||X_{11}||| \lesssim \|\mathcal{E}\|_\infty \|f\|$. For X_{12} we see from Remark 2.6 that $\underline{d} F_t^{1,\perp}$ is the vector part of $\underline{d} F_t$. Thus $|||X_{12}||| \lesssim \|\mathcal{E}\|_\infty |||t\underline{d} F_t||| \lesssim \|\mathcal{E}\|_\infty \|f\|$ by (4.20). To handle the final term X_{13} we note that $\underline{d} P_t^{\hat{B}_0} \underline{d}^*_{\hat{B}_0} = \underline{d} N P_t^{\hat{B}_0} N \underline{d}^*_{\hat{B}_0} = -\Gamma P_t^{\hat{B}_0} \Gamma^*_{\hat{B}_0} = -\Gamma \Gamma^*_{\hat{B}_0} P_t^{\hat{B}_0}$ using Remark 3.2. Thus we obtain from Corollary 4.15, with $v = a_{\|\|}^{-1} a_{\|\perp} e_0$, the estimate $|||X_{13}||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty \|f\|$. This proves the estimate (4.15). □

Proof of Corollary 4.15. Let $f \in E^+_{B_0}\mathcal{H}$ and consider the functions $F_t = e^{-t|T_{B_0}|} f$ and $G_t := P_t^{B_0} f$. We note that $F_t - G_t = \psi_t(T_{B_0}) f$, where $\psi(z) = e^{-|z|} - (1+z^2)^{-1} \in \Psi(S^o_\nu)$. Thus it suffices to prove the estimate $|||\Theta_t^i(G_t^{1,0})||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty \|f\|$, $i = 1, 2$, for the two families of operators

$$\Theta_t^1 := \widetilde{Q}_t \mathcal{E} t^2 \Gamma \Gamma^*_{\hat{B}_0} P_t^{\hat{B}_0} M_v, \tag{4.22}$$

$$\Theta_t^2 := \widetilde{Q}_t \mathcal{E} P_t^{\hat{B}_0} M_v, \tag{4.23}$$

where M_v denotes the multiplication operator $M_v(f) := vf$. To this end, we write

$$\Theta_t^i\big(G_t^{1,0}\big) = \big(\Theta_t^i - \gamma_t^i A_t\big) G_t^{1,0} + \gamma_t^i A_t G_t^{1,0},$$

where $\gamma_t^i(x)$ is the principal part of the operator family Θ_t^i as in Definition 4.2. Using the principal part approximation Lemma 4.4 and Carleson's lemma 4.6 we obtain the estimate

$$\big|\big|\big|\Theta_t^i\big(G_t^{1,0}\big)\big|\big|\big| = \big\|\Theta_t^i\big\|_{\mathrm{off}} \big|\big|\big|t\nabla\big(G_t^{1,0}\big)\big|\big|\big| + \big\|\gamma_t^i\big\|_C \big\|N_*\big(A_t G_t^{1,0}\big)\big\|.$$

By Proposition 2.25 and Lemma 2.26, we have $\|\Theta_t^i\|_{\mathrm{off}} \lesssim \|\mathcal{E}\|_\infty$. Furthermore, by Lemma 4.7 and Theorem 4.8 we have

$$\big\|\gamma_t^i\big\|_C \lesssim \big|\big|\big|\Theta_t^i\big|\big|\big|_{\mathrm{op}} + \big\|\Theta_t^i\big\|_{\mathrm{off}} \lesssim \big(|||\widetilde{Q}_t|||_{\mathrm{op}} + 1\big)\|\mathcal{E}\|_\infty.$$

For Θ_t^2 we have used that $\Theta_t^2 = \widetilde{Q}_t \mathcal{E}(I - t^2 \Gamma \Gamma^*_{\hat{B}_0} P_t^{\hat{B}_0} - t^2 \Gamma^*_{\hat{B}_0} \Gamma P_t^{\hat{B}_0}) M_v$.

To bound $|||t\nabla(G_t^{1,0})|||$, we note that the vector part of $\underline{d}G_t$ is

$$\underline{d}G_t^{1,\perp} = imd\big(e_0 G_t^{1,0}\big) = -id\big(G_t^{1,0}\big) = -i\nabla\big(G_t^{1,0}\big).$$

Thus, similar to the proof of (4.20), it follows that

$$\big|\big|\big|t\nabla\big(G_t^{1,0}\big)\big|\big|\big| \lesssim |||t\underline{d}G_t||| \lesssim \big|\big|\big|tT_{B_0}P_t^{B_0}f\big|\big|\big| \lesssim \|f\|.$$

Finally, to bound $\|N_*(A_t G_t^{1,0})\|$ we write $G_t = \frac{1}{2}(H_t + H_{-t})$, where $H_t := (1 + itT_{B_0})^{-1}f$. We now observe that the divergence form equation and estimate for $H_t^{1,0}$ in Lemma 2.57 in fact holds for all $f \in \mathcal{H}$, by inspection of the proof. We get

$$\begin{aligned} N_*\big(A_t H_t^{1,0}\big)(x) &= \sup_{|y-x|<t} \left| \fint_{Q(y,t)} H_t^{1,0}(z)\,dz \right| \\ &\lesssim \sup_{t>0} \bigg(\fint_{B(x,r_0t)} \big|H_t^{1,0}(z)\big|^q\,dz \bigg)^{1/q} \lesssim \big(M\big(|f|^p\big)(x)\big)^{1/p}, \end{aligned}$$

where $Q(y,t)$ denotes the dyadic cube $Q \in \Delta_t$ which contains y. Using the boundedness of the Hardy–Littlewood maximal function on $L_{2/p}(\mathbf{R}^n)$, we obtain

$$\big\|N_*\big(A_t H_t^{1,0}\big)\big\|_2 \lesssim \big\|\big(M\big(|f|^p\big)\big)^{1/p}\big\|_2 = \big\|M\big(|f|^p\big)\big\|_{2/p}^{1/p} \lesssim \big\||f|^p\big\|_{2/p}^{1/p} = \|f\|_2.$$

A similar argument shows that $\|N_*(A_t H_{-t}^{1,0})\| \lesssim \|f\|$, and thus $\|N_*(A_t G_t^{1,0})\| \lesssim \|f\|$. We have proved that $|||\Theta_t^i(G_t^{1,0})||| \lesssim (|||\widetilde{Q}_t|||_{\mathrm{op}} + 1)\|\mathcal{E}\|_\infty\|f\|$, and therefore Corollary 4.15. □

4.3. *Proof of main theorems*

We are now in position to prove Theorems 1.1, 1.3 and 1.4 stated in the introduction.

Proof of Theorem 1.4. Given a perturbation $B^k \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ of the unperturbed coefficients $B_0^k \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$, we introduce $B := I \oplus \cdots \oplus B^k \oplus \cdots \oplus I \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ and $B_0 := I \oplus \cdots \oplus B_0^k \oplus \cdots \oplus I \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ acting in all $\mathcal{H}$. By Theorem 3.3(i), T_{B_0} satisfies quadratic estimates and by Lemma 4.1 and Theorem 4.8 there exists $\varepsilon > 0$ such that we have quadratic estimates

$$\big|\big|\big|Q_t^B f\big|\big|\big| \approx \|f\|, \quad \text{whenever } \|B - B_0\|_\infty \leqslant \varepsilon,\ f \in \mathcal{H}.$$

Therefore, by Propositions 2.32 and 2.42 we have when $\|B - B_0\|_\infty \leqslant \varepsilon/2$ well-defined and bounded operators $E_B = \operatorname{sgn}(T_B)$ which depend Lipschitz continuously on B, i.e.

$$\|E_{B_2} - E_{B_1}\| \leqslant C\|B_2 - B_1\|_\infty, \quad \text{when } \|B_i - B_0\| < \varepsilon/2,\ i = 1, 2.$$

To prove that (Tr-$B^k\alpha^\pm$) is well posed, note that by Lemma 2.51 it suffices to show that $\lambda - E_B N_B$ is invertible since then in particular $\lambda - E_{B^k}N_{B^k} = (\lambda - E_B N_B)|_{\hat{\mathcal{H}}_B^k}$ is invertible. Here the spectral parameter is $\lambda = (\alpha+1)/(\alpha-1)$ and $\alpha := \alpha^+/\alpha^-$. By Theorem 3.3(ii), the unperturbed operator $\lambda - E_{B_0}N_{B_0}$ is invertible when $\lambda^2+1 \neq 0$. For the perturbed operator we write

$$\lambda - E_B N_B = (\lambda - E_{B_0}N_{B_0})\left(I + \frac{1}{\lambda^2+1}(\lambda - N_{B_0}E_{B_0})(E_{B_0}N_{B_0} - E_B N_B)\right).$$

Here $\|E_{B_0}N_{B_0} - E_B N_B\| \lesssim \|B - B_0\|_\infty$ since we clearly have Lipschitz continuity $\|N_{B_2} - N_{B_1}\| \lesssim \|B_2 - B_1\|_\infty$. It follows that $\lambda - E_B N_B$ is invertible when $\|B - B_0\|_\infty \leqslant C|\lambda^2+1|$. Under the assumption $\|B - B_0\|_\infty < \varepsilon/2$ with ε small we can replace $\lambda^2 + 1 = 2(\alpha^2+1)/(\alpha-1)^2$ with α^2+1.

This proves that for each boundary function g there exists a unique solution $f = f^+ + f^-$ satisfying the jump conditions in (Tr-$B^k\alpha^\pm$) and $\|f^+\| + \|f^-\| \approx \|f\| \approx \|g\|$. The norm estimates for $F^\pm$ in Theorem 1.4 now follows from Lemma 2.49. A formula for the solution is

$$F^\pm(t,x) = 2e^{\mp t|T_B|}E_B^\pm\big((\alpha^+ + \alpha^-)E_B - (\alpha^+ - \alpha^-)N_B\big)^{-1}g(x).$$

This completes the proof of Theorem 1.4. □

Before turning to the proofs of Theorems 1.1 and 1.3, we note some corollaries of Theorem 1.4. First, if we let $k=1$ in Theorem 1.4, then it proves that the following Neumann-regularity transmission problem is well posed for small L_∞ perturbations A of a block matrix A_0.

Transmission problem (Tr-$A\alpha^\pm$). Let $\alpha^\pm \in \mathbf{C}$ be given jump parameters. Given scalar functions $\psi, \phi : \mathbf{R}^n \to \mathbf{C}$ with $\nabla_x\psi \in L_2(\mathbf{R}^n;\mathbf{C}^n)$ and $\phi \in L_2(\mathbf{R}^n;\mathbf{C})$, find gradient vector fields $F^\pm(t,x) = \nabla_{t,x}U^\pm(t,x)$ in $\mathbf{R}^{n+1}_\pm$ such that $F_t^\pm \in C^1(\mathbf{R}_\pm; L_2(\mathbf{R}^n;\mathbf{C}^{n+1}))$ and $F^\pm$ satisfies (1.2) for $\pm t > 0$, and furthermore $\lim_{t\to\pm\infty} F_t^\pm = 0$ and $\lim_{t\to 0^\pm} F_t^\pm = f^\pm$ in L_2 norm, where the traces $f^\pm$ satisfy the jump conditions

$$\begin{cases} \alpha^-\nabla_x U^+(0,x) - \alpha^+\nabla_x U^-(0,x) = \nabla_x\psi(x), \\ \alpha^+\dfrac{\partial U^+}{\partial\nu_A}(0,x) - \alpha^-\dfrac{\partial U^-}{\partial\nu_A}(0,x) = \phi(x), \end{cases}$$

where $\nabla_x U^\pm(0,x) = f_\parallel^\pm(x)$ and $\frac{\partial U^\pm}{\partial\nu_A} = (Af^\pm, e_0)$ denotes the conormal derivative.

Secondly, Theorem 1.4 give perturbation results for the following boundary value problems for k-vector fields.

Normal BVP (Nor-B^k). Given a k-vector field $g \in \hat{\mathcal{H}}^k_B$, find a k-vector field $F(t,x)$ in $\mathbf{R}^{n+1}_+$ such that $F_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \bigwedge^k))$ and F satisfies (1.8) for $t > 0$, and furthermore $\lim_{t\to\infty} F_t = 0$ and $\lim_{t\to 0} F_t = f$ in L_2 norm, where f satisfies

$$e_0 \lrcorner \left(B^k f\right) = e_0 \lrcorner \left(B^k g\right) \quad \text{on } \mathbf{R}^n = \partial\mathbf{R}^{n+1}_+.$$

Tangential BVP (Tan-B^k). Given a k-vector field $g \in \hat{\mathcal{H}}^k_B$, find a k-vector field $F(t,x)$ in $\mathbf{R}^{n+1}_+$ such that $F_t \in C^1(\mathbf{R}_+; L_2(\mathbf{R}^n; \bigwedge^k))$ and F satisfies (1.8) for $t > 0$, and furthermore $\lim_{t\to\infty} F_t = 0$ and $\lim_{t\to 0} F_t = f$ in L_2 norm, where f satisfies

$$e_0 \wedge f = e_0 \wedge g \quad \text{on } \mathbf{R}^n = \partial\mathbf{R}^{n+1}_+.$$

Corollary 4.17. *Let $B_0^k = B_0^k(x) \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ be accretive and assume that B_0^k is a block matrix. Then there exists $\varepsilon > 0$ depending only on the constants $\|B_0^k\|_\infty$ and $\kappa_{B_0^k}$ and dimension n, such that if $B^k \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^k))$ satisfies $\|B^k - B_0^k\|_\infty < \varepsilon$, then Normal and Tangential boundary value problems* (Nor-B^k) *and* (Tan-B^k) *above are well posed.*

Proof. (i) For (Nor-B^k) in $\mathbf{R}_+^{n+1}$, we let $\alpha^- = 0$ and $\alpha^+ = 1$ in (Tr-$B^k\alpha^\pm$). Then we obtain two decoupled jump conditions

$$\begin{cases} -e_0 \wedge f^- = e_0 \wedge g, \\ e_0 \lrcorner \big(B^k f^+\big) = e_0 \lrcorner \big(B^k g\big). \end{cases}$$

Discarding the solution F^-, we obtain a unique solution $F = F^+$ to (Nor-B^k).

(ii) For (Tan-B^k) in $\mathbf{R}_+^{n+1}$, we let $\alpha^- = 1$ and $\alpha^+ = 0$ in (Tr-$B^k\alpha^\pm$). Then we obtain two decoupled jump conditions

$$\begin{cases} e_0 \wedge f^+ = e_0 \wedge g, \\ -e_0 \lrcorner \big(B^k f^-\big) = e_0 \lrcorner \big(B^k g\big). \end{cases}$$

Discarding the solution F^-, we obtain a unique solution $F = F^+$ to (Tan-B^k). □

Note that well-posedness of (Neu-A) and (Reg-A) for Theorem 1.1(b) is the special case $k = 1$ of Corollary 4.17, as well as the special cases $(\alpha^+, \alpha^-) = (1, 0)$ and $(0, 1)$, respectively, of (Tr-$A\alpha^\pm$).

Proof of Theorem 1.3. Let A_0 be such that T_{B_0} has quadratic estimates in $\mathcal{H}$, where $B_0 = I \oplus A_0 \oplus I \oplus \cdots \oplus I$. Thus by Lemma 4.1 and Theorem 4.12 there exists $\varepsilon > 0$ such that we have quadratic estimates

$$|||Q_t^B f||| \approx \|f\|, \quad \text{whenever } \|B - B_0\|_\infty < \varepsilon,\ f \in \mathcal{H}.$$

Therefore, by Proposition 2.32, we have when $\|B - B_0\|_\infty < \varepsilon$ well defined and bounded operators $E_B = \operatorname{sgn}(T_B)$. With Lemma 2.46, these restricts to bounded operators E_A in $\hat{\mathcal{H}}^1$. In particular $f \in \hat{\mathcal{H}}^1$ can be decomposed as $f = f^+ + f^-$, where $f^\pm := E_A^\pm f$ and $\|f\| \approx \|f^+\| + \|f^-\|$. Moreover, Lemma 2.49 and Proposition 2.56 proves the stated norm equivalences for $F^\pm$. □

Proof of Theorem 1.1. Given a perturbation $A \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^1))$ of the unperturbed coefficients $A_0 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge^1))$, which we assume are either of block form, real symmetric or constant, we introduce $B := I \oplus A \oplus I \oplus \cdots \oplus I \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ and $B_0 := I \oplus A_0 \oplus I \oplus \cdots \oplus I \in L_\infty(\mathbf{R}^n; \mathcal{L}(\bigwedge))$ acting in all $\mathcal{H}$. That T_{B_0} satisfies quadratic estimates follows from Theorem 3.3(i), Theorem 3.12 and Proposition 3.5, respectively. Theorem 1.3 now shows that we have quadratic estimates for T_B when $\|B - B_0\|_\infty < \varepsilon$. In case A_0 is a block matrix, we note that this result follows already from Theorem 4.8. By Propositions 2.32 and 2.42 we have when $\|B - B_0\|_\infty < \varepsilon/2$ well-defined and bounded operators $E_B = \operatorname{sgn}(T_B)$ which depend Lipschitz continuously on B, so that

$$\|E_{B_2} - E_{B_1}\| \leqslant C\|B_2 - B_1\|_\infty, \quad \text{when } \|B_i - B_0\| < \varepsilon/2,\ i = 1, 2.$$

To prove that (Neu-A) and (Reg-A) are well posed, note that by Lemma 2.53 it suffices to show that $I \pm E_A N_A : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$ are invertible, where $E_A = E_B|_{\hat{\mathcal{H}}^1}$ and $N_A = N_B|_{\hat{\mathcal{H}}^1}$. By Theorems 3.3(ii), 3.12 and 3.6, respectively, the unperturbed operators $I \pm E_{A_0} N_{A_0} : \hat{\mathcal{H}}^1 \to \hat{\mathcal{H}}^1$ are invertible. For the perturbed operator we write

$$I \pm E_A N_A = (I \pm E_{A_0} N_{A_0})\big(I \pm (I \pm E_{A_0} N_{A_0})^{-1}(E_A N_A - E_{A_0} N_{A_0})\big).$$

Here $\|E_A N_A - E_{A_0} N_{A_0}\| \lesssim \|A - A_0\|_\infty$ since we clearly have Lipschitz continuity $\|N_{A_2} - N_{A_1}\| \lesssim \|A_2 - A_1\|_\infty$. It follows that $I \pm E_A N_A$ are invertible when $\|A - A_0\|_\infty < \varepsilon'$.

The well-posedness of (Neu$^\perp$-A) is a consequence of Proposition 2.52, since our hypothesis is stable when taking adjoints $A \mapsto A^*$. Alternatively, we can replace N_{A_0} with N above, proving that $I \pm E_A N$ is an isomorphism, using Theorem 3.3(ii), Remarks 3.13 and 3.7. This proves that for (Neu-A), (Reg-A) and (Neu$^\perp$-A) and each boundary function g (being ϕ and $\nabla_x \psi$, respectively), there exists a unique solution $f = E_A^+ \hat{\mathcal{H}}^1$ satisfying the boundary condition and $\|f\| \approx \|g\|$. Lemma 2.49 and Proposition 2.56 proves that the stated norms of f and $F(t,x) := (e^{-t|T_A|} f)(x)$ are equivalent.

The well-posedness of (Dir-A), as well as the first three norm estimates, follows from Lemma 2.55 and (Neu$^\perp$-A). To show that $|||t\nabla_x U_t||| \approx |||t\partial_t U_t|||$, we consider the gradient vector field $G_t = \nabla_{t,x} U_t$ as in the proof of the lemma. From (Reg-A) and (Neu$^\perp$-A), it follows that for all $t > 0$, we have $\|\partial_t U_t\| = \|N^- G_t\| \approx \|N^+ G_t\| = \|\nabla_x U_t\|$, from which the square function estimate for $\nabla_x U_t$ follows. To show that $\|u\| \approx \|\widetilde{N}_*(U)\|$, we consider the vector field F_t of conjugate functions from the proof of the lemma. Proposition 2.56 shows that $\|u\| \approx \|f\| \approx \|\widetilde{N}_*(F)\| \gtrsim \|\widetilde{N}_*(U)\|$. Moreover, the proof of the reverse estimate $\|u\| \lesssim \|\widetilde{N}_*(U)\|$ is similar to the proof of $\|f\| \lesssim \|\widetilde{N}_*(F)\|$ in Proposition 2.56, using the uniform boundedness of $\mathcal{P}_t$.

Finally we note that the solution operators for (Neu-A), (Neu$^\perp$-A), (Reg-A) and (Dir-A) are

$$F_t = 2e^{-t|T_A|}(E_A - N_A)^{-1}\big(a_{00}^{-1}\phi e_0\big), \qquad F_t = 2e^{-t|T_A|}(E_A - N)^{-1}(\phi e_0),$$
$$F_t = 2e^{-t|T_A|}(E_A + N)^{-1}(\nabla_x \psi), \qquad U_t = 2\big(e^{-t|T_A|}(E_A - N)^{-1}(u e_0), e_0\big).$$

The Lipschitz continuity of $u \mapsto U$ is a consequence of the corresponding result for (Neu$^\perp$-A). For the norms $\sup_{t>0} \|F_t\|$ and $|||t\partial_t F_t|||$, Lipschitz continuity for the solution operators follows from Proposition 2.42. It remains to show Lipschitz continuity for the norm $\|F\|_{\mathcal{X}} = \|\widetilde{N}_*(F)\|_2$. To this end, we consider

$$f(x) \mapsto F_{A_z} = \big(e^{-t|T_{A_z}|} E_{A_z}^+ f\big)(x) : \hat{\mathcal{H}}^1 \to \mathcal{X}$$

and the truncations $f(x) \mapsto F_{A_z}^k(t,x) = \chi_k(t) F_{A_z}(t,x)$, where χ_k denotes the characteristic function for $(1/k, k)$ as in the proof of Lemma 2.41(iii). We claim that it suffices to show that, for each fixed k, the operator $f(x) \mapsto F_{A_z}^k(t,x) : \hat{\mathcal{H}}^1 \to \mathcal{X}$ depends holomorphically on z. Indeed, using Schwarz' lemma as in Proposition 2.42, we obtain the Lipschitz estimate

$$\big\|F_{A_2}^k - F_{A_1}^k\big\|_{\mathcal{X}} \leqslant C\|A_2 - A_1\|_\infty \|f\|_2,$$

uniformly for all k, since $\|F^k\|_{\mathcal{X}} \leqslant \|F\|_{\mathcal{X}} \leqslant C\|f\|$ by Proposition 2.56. Furthermore, by the monotone convergence theorem we have

$$\left\|F^k_{A_2} - F^k_{A_1}\right\|_{\mathcal{X}} \nearrow \|F_{A_2} - F_{A_1}\|_{\mathcal{X}}, \quad k \to \infty,$$

so the desired Lipschitz continuity follows after taking limits.

To prove that $f(x) \mapsto F^k_{A_z}(t,x)$ is holomorphic, we note that

$$\|F^k\|^2_{\mathcal{X}} \lesssim \int_{\mathbf{R}^n} \left(\sup_{t>0} \int_{|s-t|<c_0 t} \int_{|y-x|<c_1 s/(1-c_0)} \left|F^k(s,y)\right|^2 \frac{ds\,dy}{s^{n+1}} \right) dx$$
$$\lesssim \int_{\mathbf{R}^n} \left(\int_{1/k}^{k} \int_{|y-x|<c_1 s/(1-c_0)} \left|F^k(s,y)\right|^2 \frac{ds\,dy}{s^{n+1}} \right) dx \approx \int_{1/k}^{k} \|F^k_s\|_2^2\, ds,$$

for fixed k, where in the last step we use that $1/k \leqslant s^{-1} \leqslant k$. Since $f(x) \mapsto F^k(t,x) : \hat{\mathcal{H}}^1 \to L_2(\mathbf{R}^n \times (1/k,k))$ is holomorphic by Lemma 2.41(ii) and the embedding $L_2(\mathbf{R}^n \times (1/k,k)) \hookrightarrow \mathcal{X}$ is continuous and independent of z, it follows that $f(x) \mapsto F^k(t,x) : \hat{\mathcal{H}}^1 \to \mathcal{X}$ is holomorphic for each fixed k. This completes the proof of Theorem 1.1. □

Acknowledgments

This work was mainly conducted at the University of Orsay while Axelsson was post-doctoral fellow supported by the French Ministry of Teaching and Research (Ministère de l'Enseignement et de la Recherche) and then by the European IHP network "Harmonic Analysis and Related Problems" (contract HPRN-CT-2001-00273-HARP). Axelsson gratefully acknowledges the University of Orsay for providing facilities to work. Hofmann also thanks the University of Orsay for partial support of a one week visit that jump started this project.

References

[1] D. Albrecht, X. Duong, A. M^{c}Intosh, Operator theory and harmonic analysis, in: Instructional Workshop on Analysis and Geometry, Part III, Canberra, 1995, in: Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 34, Austral. Nat. Univ., Canberra, 1996, pp. 77–136.

[2] M. Alfonseca, P. Auscher, A. Axelsson, S. Hofmann, S. Kim, Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients, preprint.

[3] P. Auscher, A. M^{c}Intosh, A. Nahmod, The square root problem of Kato in one dimension, and first order elliptic systems, Indiana Univ. Math. J. 46 (3) (1997) 659–695.

[4] P. Auscher, S. Hofmann, M. Lacey, A. M^{c}Intosh, P. Tchamitchian, The solution of the Kato square root problem for second order elliptic operators on $\mathbf{R}^n$, Ann. of Math. (2) 156 (2) (2002) 633–654.

[5] A. Axelsson, Oblique and normal transmission problems for Dirac operators with strongly Lipschitz interfaces, Comm. Partial Differential Equations 28 (11–12) (2003) 1911–1941.

[6] A. Axelsson, Transmission problems for Dirac's and Maxwell's equations with Lipschitz interfaces, PhD thesis, The Australian National University, 2003.

[7] A. Axelsson, Transmission problems and boundary operator algebras, Integral Equations Operator Theory 50 (2) (2004) 147–164.

[8] A. Axelsson, Transmission problems for Maxwell's equations with weakly Lipschitz interfaces, Math. Methods Appl. Sci. 29 (6) (2006) 665–714.

[9] A. Axelsson, A. McIntosh, Hodge decompositions on weakly Lipschitz domains, in: Advances in Analysis and Geometry, in: Trends Math., Birkhäuser Boston, Boston, MA, 2004, pp. 3–29.
[10] A. Axelsson, S. Keith, A. McIntosh, Quadratic estimates and functional calculi of perturbed Dirac operators, Invent. Math. 163 (3) (2006) 455–497.
[11] R. Coifman, P. Jones, S. Semmes, Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves, J. Amer. Math. Soc. 2 (3) (1989) 553–564.
[12] B. Dahlberg, Estimates of harmonic measure, Arch. Ration. Mech. Anal. 65 (3) (1977) 275–288.
[13] B. Dahlberg, D. Jerison, C. Kenig, Area integral estimates for elliptic differential operators with nonsmooth coefficients, Ark. Mat. 22 (1) (1984) 97–108.
[14] E. Fabes, D. Jerison, C. Kenig, Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure, Ann. of Math. (2) 119 (1) (1984) 121–141.
[15] S. Hofmann, J. Martell, L^p bounds for Riesz transforms and square roots associated to second order elliptic operators, Publ. Mat. 47 (2) (2003) 497–515.
[16] D. Jerison, C. Kenig, The Neumann problem on Lipschitz domains, Bull. Amer. Math. Soc. (N.S.) 4 (2) (1981) 203–207.
[17] D.S. Jerison, C.E. Kenig, The Dirichlet problem in nonsmooth domains, Ann. of Math. (2) 113 (2) (1981) 367–382.
[18] T. Kato, Perturbation Theory for Linear Operators, second ed., Grundlehren Math. Wiss., vol. 132, Springer-Verlag, Berlin, 1976.
[19] C. Kenig, Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Reg. Conf. Ser. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1994.
[20] C. Kenig, J. Pipher, The Neumann problem for elliptic equations with nonsmooth coefficients, Invent. Math. 113 (3) (1993) 447–509.
[21] C. Kenig, D. Rule, The regularity and Neumann problem for non-symmetric elliptic operators, preprint.
[22] C. Kenig, H. Koch, J. Pipher, T. Toro, A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations, Adv. Math. 153 (2) (2000) 231–298.
[23] A. McIntosh, T. Qian, Convolution singular integral operators on Lipschitz curves, in: Harmonic Analysis, Tianjin, 1988, in: Lecture Notes in Math., vol. 1494, Springer, Berlin, 1991, pp. 142–162.
[24] I.J. Šneĭberg, Spectral properties of linear operators in interpolation families of Banach spaces, Mat. Issled. 9 2 (32) (1974) 214–229, 254–255.
[25] E. Stein, G. Weiss, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, Acta Math. 103 (1960) 25–62.
[26] G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains, J. Funct. Anal. 59 (3) (1984) 572–611.