Computational Geometry 44 (2011) 95–99

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

On graph thickness, geometric thickness, and separator theorems

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ARTICLE INFO

Article history:
Received 11 November 2009
Accepted 23 September 2010
Available online 25 September 2010
Communicated by W. Evans

Keywords:
Graph thickness
Geometric thickness
Arboricity
Planar separators

ABSTRACT

We investigate the relationship between geometric thickness, thickness, outerthickness, and arboricity of graphs. In particular, we prove that all graphs with arboricity two or outerthickness two have geometric thickness $O(\log n)$. The technique used can be extended to other classes of graphs so long as a separator theorem exists. For example, we can apply it to show the known bound that thickness two graphs have geometric thickness $O(\sqrt{n})$, yielding a simple construction in the process.

1. Introduction

In many applications of graph visualization and graph theory, it is often useful to draw the edges of a graph with multiple colors or in multiple layers. A *drawing* of a graph $G = (V, E)$ is a one-to-one mapping of the vertices in $V(G)$ to points in the plane and a one-to-one mapping of the edges in $E(G)$ to simple curves in the plane such that the curve for each edge uv intersects no vertices except u and v at the endpoints. Two edges *cross* if their corresponding curves intersect at a point other than a common endpoint. A *plane drawing* is a drawing where no edges cross. A graph for which there exists a plane drawing is *planar*. An *(edge) coloring* of a graph is a many-to-one mapping of the edges in $E(G)$ to a set of colors. A *partitioning* of a graph is a partitioning of the edges in $E(G)$ into disjoint sets.

The general class of thickness problems deals with determining the minimum number of colors needed to draw a graph under various conditions. The *(graph) thickness* of a graph G is defined to be the minimum number of planar subgraphs whose union forms G. A graph G has thickness k if and only if G admits a drawing such that the edges of G are k-colored and no two edges with the same color cross [1,2]. To see this, place the vertices of G in unique positions on the plane. Since any planar graph can be drawn without crossings and with its vertices at prespecified locations, albeit with the edges having many bends [3], each of the k planar subgraphs can be drawn on the same vertex set using a unique edge color for each subgraph. Initially, the notion of thickness derived from early work on *biplanar* graphs, graphs with thickness two [4–6], which was subsequently extended to graphs of thickness k by Tutte [7]. The research in graph thickness problems is too rich to summarize here, and the interested reader is referred to the survey by Mutzel et al. [8].

By adding the constraint that all edges in the drawing must be represented by straight-line segments, we arrive at the *geometric thickness* problem [1,9]. Further constraining the problem such that the vertices must lie in convex position yields the *book thickness* [10] problem. Because of the added constraints for each variant, we see that

$$\text{thickness}(G) \leqslant \text{geometric thickness}(G) \leqslant \text{book thickness}(G).$$

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0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2010.09.005

In analyzing the tightness of these inequalities, Eppstein [11,12] showed that there are graphs with thickness three that have unbounded geometric thickness and that there are graphs with geometric thickness two that have unbounded book thickness.

We can constrain the graph thickness in other ways. For example, the *arboricity* of a graph G is defined to be the smallest number of forests whose union is G [13–15]. In other words, the minimum number of edge colors such that the subgraph induced by each color is a forest of trees. In *linear arboricity* the resulting colored subgraphs must be pairwise disjoint paths [16,17]. In *outerthickness*, the resulting colored subgraphs must be outerplanar graphs [18,19]. Outerplanar graphs are precisely the graphs with book thickness one; however, outerthickness and book thickness are not identical.

Another way to look at the thickness problem is to divide the edges of the graph into different layers and draw the layers independently as planar graphs such that the vertex positions are identical in each layer. In this case, the objective is to minimize the number of layers. A common related application of this problem is in VLSI design where wires are placed in various layers to avoid crossings, see for example [20]. Minimizing the number of layers reduces cost and improves performance of the created circuits.

Another related problem is the area of simultaneous embeddings. In simultaneous embedding problems, the edges are already assigned to various layers, and one must determine a placement of vertices to realize each drawing without crossings, if possible [21].

1.1. Related work

Our work is motivated by and related to recent results about geometric thickness. In particular, Eppstein [12] presents a class of graphs having arboricity three and thickness three, for which the geometric thickness grows as a function of n, the number of vertices. The proof relies on Ramsey theory and so the lower bound on the geometric thickness is a very slow growing function of n. Eppstein [11] also describes a class of graphs with arboricity two and geometric thickness two having a book thickness that grows as another slow function of n.

Using results on the simultaneous embedding of two paths [21], Duncan et al. [22] proved that graphs with linear arboricity two have geometric thickness two. Arboricity-two graphs are not as simple. Geyer et al. [23] show two trees which cannot be embedded simultaneously. In the context of geometric thickness, this would imply that one cannot simply take a graph of arboricity two, partition it into two forests arbitrarily, and then embed the two graphs simultaneously. However, because the *union* of the two trees described does have geometric thickness two, whether arboricity two implies geometric thickness two or any constant t is still an open problem.

Malitz [24] proved that the book thickness of graphs with m edges is $O(\sqrt{m})$. This result immediately implies that thickness-t graphs have geometric thickness $O(\sqrt{tn})$. There has also been work on bounding the geometric thickness of a graph in terms of its degree. In particular, graphs with maximum degree at most two trivially have geometric thickness one, graphs with maximum degree three and four have geometric thickness two [22], and for any $\epsilon > 0$ and $\Delta \geqslant 9$, there exist graphs with maximum degree Δ having geometric thickness at least $c\sqrt{\Delta}n^{1/2-4/\Delta-\epsilon}$ for sufficiently large n and some constant c [25].

Dujmović and Wood [26] discuss the relationship between geometric thickness and graph treewidth. In particular, they show that graphs with treewidth k have geometric thickness at most $\lceil k/2 \rceil$. This is complementary to our work as even planar graphs can have arbitrarily large treewidth. For example, the $n \times n$ grid graph has treewidth n.

Blankenship and Oporowski [27,28] showed that every K_h-minor free graph has book thickness (and hence geometric thickness) that is bounded by $f(h)$, for some large function f. Since f does not depend on n but only on h, if we consider h to be a constant, this result implies that the book thickness (and geometric thickness) is also a constant.

1.2. Our results

In this paper, we further analyze the relationship between thickness and geometric thickness. In particular, Corollary 4 states that graphs with arboricity two have geometric thickness $O(\log n)$. We show this by providing a more generalized solution (Theorem 2) for graphs that can be partitioned into two subgraphs having some separation property. Using this generalized solution, we show in Corollary 3 that graphs with thickness two have geometric thickness $O(\sqrt{n})$, which is also immediately implied by the results from Malitz [24]. Additionally, if the graph can be partitioned into *two* K_h-minor free graphs, then the graph has geometric thickness $O(\sqrt{h^3 n})$ (Corollary 6), again implied by Malitz [24] if we assume h is constant. Note however that this result is *not* superseded by Blankenship and Oporowski's results [27,28] as a graph that can be separated into two K_h-minor free graphs may have an arbitrarily large complete minor [29, Lemma 1]. We can however improve for the case when $h = 4$ and prove such graphs, including those with outerthickness two, have geometric thickness $O(\log n)$ (Corollary 7).

2. Using a separator theorem

A *cut set* for a graph $G = (V, E)$ is a set of vertices $C \subset V(G)$ such that $G - C$ is disconnected. Thus $G - C$ consists of two non-empty subgraphs G_1 and G_2 with no edges of G between them. Let E_C be the set of edges having at least one endpoint in C. For convenience, we also let $V_1 = V(G_1)$ and $V_2 = V(G_2)$.

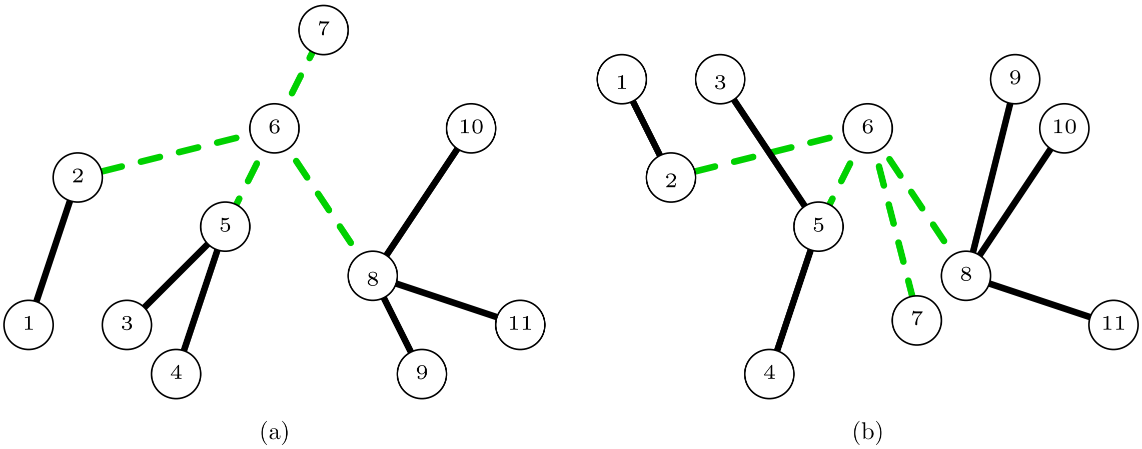

Fig. 1. (a) A very simple assignment of the x-values to the vertices and colors (solid and dashed lines) to the edges of a tree graph. (b) Notice that even if the y-values change, there are no same-color crossings.

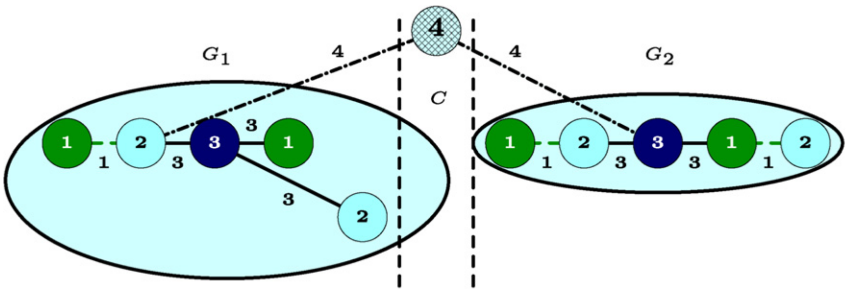

Fig. 2. A tree that has been partitioned into three parts G_1, G_2, and C. The vertices and edges of G_1 and G_2 have been separately colored. The x-coordinates have been assigned so that the two subgraphs do not overlap. The cut vertex and the two edges in E_C have been assigned a distinct color. Notice that there can be no edge between G_1 and G_2 as C was a cut set.

For two functions f and g and constant c, let $\mathcal{G}_{f,g}$ be a (hereditary) class of graphs having a separator property on f and g that states for any graph $G \in \mathcal{G}$ with $n = |V(G)| > c$, there exists a cut set C such that $G_1, G_2 \in \mathcal{G}_{f,g}$ and $|V_i| \leqslant f(n)$ for $i = 1, 2$ and $|C| \leqslant g(n)$. In other words, any sufficiently large graph $G \in \mathcal{G}_{f,g}$ has a *relatively small* cut set that divides G into *relatively small* components. Our primary result uses the following key lemma, which we illustrate in Fig. 1:

Lemma 1. *Let G be a graph in $\mathcal{G}_{f,g}$ with n vertices. There exists an assignment of colors to $e \in E(G)$ in the range 1 to $\mathcal{F}(n)$, and of unique x-coordinate values to $v \in V(G)$, in the range 1 to n, such that for any assignment of y-coordinates to v, with all vertex points being in general position, no two edges with the same color assignment intersect, except at common endpoints, when drawn as straight-line segments from their respective endpoints. Here $\mathcal{F}(n)$ is defined by the following recurrence relation*:

$$\mathcal{F}(n) = \begin{cases} n & \text{if } n \leqslant c, \\ \mathcal{F}(f(n)) + g(n) & \text{otherwise.} \end{cases}$$

Proof. In our arguments, we shall also color each vertex with color $c(v)$ so that an edge $uv \in E(G)$ has color either $c(u)$ or $c(v)$.

We prove this lemma inductively. If $n \leqslant c$, number the vertices $v_1 \dots v_n$ and color each edge $v_i v_j$ by i or j. This process requires the assignment of at most c colors. Since edges assigned the same color also share a common endpoint, the only color crossings possible are between adjacent edges, but if the points are in general position, the only intersection is at the endpoints. Therefore, our lemma holds for the base case.

Assume now that the lemma holds for all graphs with order less than n. Let G have n vertices. Since G belongs to the class $\mathcal{G}_{f,g}$ we can use the separator property to partition $V(G)$ into the three sets V_1, V_2, C. Let $n_1 = |V_1|$, $n_2 = |V_2|$, $n_c = |C|$.

Next, we compute color and x-coordinate assignments separately for G_1 and G_2. From our inductive assumption, both G_1 and G_2 can be assigned values independently without any invalid crossings in their respective graphs.

To combine the two assignments and provide assignments for the remaining vertices and edges, we proceed as follows, see Fig. 2. Let V_1 have x-coordinate assignments ranging from 1 to n_1 and V_2 have x-coordinate assignments ranging from 1 to n_2. We assign the x-coordinates of C in (arbitrary) order from $n_1 + 1$ to $n_1 + n_c$. We shift the x-coordinates of V_2 over by $n_1 + n_c$. Notice that shifting the values of V_2 does not affect the intersection properties of G_2, as the entire graph is moved. Let c_1 and c_2 be the number of colors assigned to G_1 and G_2. Let $c' = \max(c_1, c_2)$. We color the vertices of C with n_c distinct colors ranging from $c' + 1$ to $c' + n_c$. We then color the edges in E as follows. If $e \in E(G_1 \cup G_2)$, we use the color assigned during the construction of G_1 and G_2. Otherwise, $e \in E_C$ and let $v \in C$ be an endpoint of e, because the separation property guarantees that there are no edges between V_1 and V_2. We then color the edge e with the value $c(v)$. If both endpoints are in C, the choice of v is arbitrary.

This assignment process guarantees that the vertices have x-coordinates in the range of 1 to n. To see that there are no crossing violations, observe that from our inductive assumption there are no crossing violations between edges in G_1 or between edges in G_2. In addition, because of the separation in the placement of the vertices for V_1 and V_2, there can be no edge crossings between an edge in G_1 and an edge in G_2. Therefore, any crossing violations must involve at least one edge in E_C. Since edges in E_C are colored differently than edges in G_1 or G_2, the other edge must also be in E_C. However, two edges in E_C with the same color must also have a common endpoint in C and so cannot intersect if the vertices are in general position. Therefore, there can be no crossing violations.

To complete the proof, recall that $n_c \leqslant g(n)$ and $n_1, n_2 \leqslant f(n)$. The number of colors used is consequently bounded by $c' + n_c \leqslant \mathcal{F}(f(n)) + g(n)$. □

We now use this lemma to prove our main theorem.

Theorem 2. *Assume we have a graph H whose edges can be colored into two layers H_1 and H_2 such that $H_1, H_2 \in \mathcal{G}_{f,g}$. Then H has geometric thickness at most $2\mathcal{F}(|V(H)|)$ where $\mathcal{F}$ is defined as in the preceding lemma.*

Proof. From Lemma 1, we know that there exists an assignment of colors and x-coordinates for both H_1 and H_2 separately. For H_2 we simply transpose the x and y coordinates. Therefore, each vertex $v \in V(H)$ has its x-coordinate defined by H_1's assignment and its y-coordinate defined by H_2's assignment. From Lemma 1, we know that the choice of y-coordinates does not introduce crossings into the drawing of H_1 and symmetrically for H_2. The only caveat is that the vertices may not be in general position, which could cause overlap. A simple solution is to perturb the positions slightly resulting in no new crossings and eliminating any overlapping edges. The key observation is that in the proof for Lemma 1 the exact position of the vertices is not important in avoiding crossings merely that the x ordering of the vertices be preserved. Therefore, each vertex can be randomly displaced within a small area around its original position so that the likelihood of any three points being collinear is almost zero while maintaining the relative ordering.

The colors used in the two assignments are kept distinct. That is, we color the edges in H_1 with a different color set than that used for the edges in H_2, thereby entirely avoiding any crossing violations between the two subgraphs. □

3. Specific examples

In this section, we show specific examples of graphs with varying thickness values. From [30,31], we know that every planar graph has a separator property with $f(n) = 2n/3$ and $g(n) = 3\sqrt{2n}/2$. Solving for $\mathcal{F}(n)$ yields the following (known) corollary:

Corollary 3. *Any graph with (graph) thickness two has geometric thickness at most $3\sqrt{2n}/(1 - \sqrt{2/3}) < 24\sqrt{n}$.*

It is also well known that trees have centroid vertices yielding a separator property with $f(n) = 2n/3$ and $g(n) = 1$. Solving for $\mathcal{F}(n)$ yields the following corollary:

Corollary 4. *Any graph with arboricity two has geometric thickness at most $2\lceil \log_{\frac{3}{2}} n \rceil$.*

Graphs with treewidth k have separators with $f(n) = 2n/3$ and $g(n) = k + 1$ [32]. This class includes regular trees ($k = 1$), outerplanar graphs ($k = 2$), and h-outerplanar graphs ($k = 3h - 1$). For further information on the treewidth, see the survey by Bodlaender [32]. This leads to the following result:

Corollary 5. *Any graph that can be partitioned into two subgraphs each with treewidth at most k has geometric thickness at most $2(k + 1)\lceil \log_{\frac{3}{2}} n \rceil$.*

DeVos et al. [33] prove that every graph with no K_h-minor can be partitioned into two graphs with tree-width $f(h)$, for some large function f dependent on h but not n. The previous corollary implies then that any K_h-minor-free graph has geometric thickness at most $2 * (f(h) + 1)\lceil \log_{\frac{3}{2}} n \rceil$. Although this result is not better than Blankenship and Oporowski's result [27,28], which shows *book* thickness $O(f(h))$, it is a simple consequence of our main theorem.

Alon et al. [34] show that any graph with n vertices and no K_h-minor has a separator with $f(n) = 2n/3$ and $g(n) = h^{3/2}n^{1/2}$. This yields the following:

Corollary 6. *For any $h > 0$, any graph that can be partitioned into two K_h-minor-free graphs has geometric thickness at most* $2h^{3/2}n^{1/2}/(1 - \sqrt{2/3}) < 11h^{3/2}n^{1/2}$.

K_4-minor-free graphs including series-parallel and outerplanar graphs have separators with $f(n) = 2n/3$ and $g(n) = 2$ [35].

Corollary 7. *Any graph that can be partitioned into two K_4-minor-free graphs has geometric thickness at most* $4\lceil \log_{\frac{3}{2}} n \rceil$.

4. Closing remarks

We have shown that for certain classes of graphs the geometric thickness can be bounded from above by a non-trivial function of the number of vertices in n. In particular, we have proven that graphs with arboricity two or outerthickness two have geometric thickness $O(\log n)$. Given that some arboricity three, and hence thickness three, graphs have been shown to have $\omega(1)$ geometric thickness, albeit using a very slow-growing function, it would be interesting to show a non-trivial upper bound for arboricity three graphs. Another remaining problem is to tighten the bounds on the geometric thickness for arboricity two graphs. Currently, the only known (and trivial) lower bound is two, and our upper bound of $O(\log n)$ still leaves a wide gap in our understanding of this subject.

Acknowledgements

We would like to thank David Eppstein and David Wood for several helpful discussions on this topic leading to the results in this paper and to the various referees for their helpful comments.

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