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Myhill's work in recursion theory

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Abstract

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In this paper we discuss the following contributions to recursion theory made by John Myhill: (1) two sets are recursively isomorphic iff they are one-one equivalent; (2) two sets are recursively isomorphic iff they are recursively equivalent and their complements are also recursively equivalent; (3) every two creative sets are recursively isomorphic; (4) the recursive analogue of the Cantor–Bernstein theorem; (5) the notion of a combinatorial function and its use in the theory of recursive equivalence types.

1. Introduction

Though he received his graduate degrees in philosophy, John Myhill was also a gifted mathematician who contributed significantly to recursion theory and other parts of mathematics. From 1953 to 1962 he published about fifteen papers which deal with recursive functions. We have selected four of his achievements for a detailed discussion in this paper. These we consider as his most important and original contributions to the theory of recursive functions, though this may of course be a matter of taste. They are:

(A) his isomorphism theorems;

(B) his theorem on creative sets;

(C) his recursive analogue of the Cantor–Bernstein theorem;

(D) his notion of a combinatorial function and its use in the arithmetic of recursive equivalence types.

† Deceased 15 June 1988.

2. Notations and terminology

Nonnegative integers will be called *numbers,* collections of numbers *sets,* and collections of sets *classes*. The symbol $\emptyset$ stands for the empty set. Inclusion is denoted by $\subset$, proper inclusion by $\subset_+$. If S is any collection, S^n stands for the collection of all ordered n-tuples of elements of S. We write ε for the set of all numbers. A *function* is a mapping f from a subcollection of ε^n into ε; its domain and its range are denoted by δf and ρf, respectively. A function is called *finite,* if its domain is finite. We abbreviate 'partial recursive' to 'p.r.' and 'recursively enumerable' to 'r.e.' We need a recursive function which maps ε^2 one-one onto ε. For this purpose we use the function

$$j(x, y) = \tfrac{1}{2}(x + y)(x + y + 1) + x.$$

The functions $k(n)$ and $l(n)$ are defined by the identity $j[k(n), l(n)] = n$, for $n \in \varepsilon$. If α and β are sets we write $\alpha - \beta = \{x \in \varepsilon \mid x \in \alpha \;\&\; x \notin \beta\}$ and $\alpha' = \varepsilon - \alpha$. The class of all r.e. sets is denoted by F and the class of all finite sets by Q.

3. The isomorphism theorems

A *recursive permutation* is a recursive function which maps ε one-one onto itself. The sets σ and τ are *recursively isomorphic* ($\sigma \cong \tau$), if there is a recursive permutation which maps σ onto τ, hence σ' onto τ'. The sets σ and τ are *recursively equivalent* ($\sigma \simeq \tau$), if there is a one-one p.r. function q with $\sigma \subset \delta q$ and $q(\sigma) = \tau$. The relations $\cong$ and $\simeq$ are clearly reflexive, symmetric and transitive. The equivalence classes into which they partition the class of all sets are called *recursive isomorphism types* (RITs) and *recursive equivalence types* (RETs), respectively. Obviously, $\sigma \cong \tau \Rightarrow \sigma \simeq \tau$. Every two infinite r.e. sets are recursively equivalent, but they need not be recursively isomorphic. In fact, even two infinite, recursive sets need not be recursively isomorphic. For let $\alpha_n = (n, n+1, \ldots)$, for $n \in \varepsilon$; then the finite, recursive sets $\alpha_0, \alpha_1, \ldots$ are mutually not recursively isomorphic. We do, however, have

$$\alpha \cong \beta \;\Leftrightarrow\; \alpha \simeq \beta \;\&\; \alpha' \simeq \beta', \quad \text{for } \alpha, \beta \in F,$$

which implies $\alpha \cong \beta \Leftrightarrow \alpha' \simeq \beta'$, for α, $\beta \in F - Q$. The RITs of two infinite r.e. sets are therefore characterized by the RETs of their complements.

Post [27] introduced the notions of one-one reducibility and one-one equivalence. A set α is *one-one reducible* to β, '$\alpha \leqslant_1 \beta$', if there is a one-one recursive function f such that $x \in \alpha$ iff $f(x) \in \beta$, i.e., such that $f(\alpha) \subset \beta$ and $f(\alpha') \subset \beta'$. A set α is *one-one equivalent* to β, '$\alpha \equiv_1 \beta$', if $\alpha \leqslant_1 \beta$ and $\beta \leqslant_1 \alpha$. We shall refer to the following two theorems as the *isomorphism* theorems, though Myhill's theorem about creative sets (see Section 5) also involves recursive isomorphisms.

Theorem A1. *Two sets are recursively isomorphic iff they are one-one equivalent.*

Theorem A2. *Two sets σ and τ are recursively isomorphic iff $\sigma \simeq \tau$ and $\sigma' \simeq \tau'$.*

A surprising feature of these two theorems is that the two sets involved need not be r.e. Theorem A2 was obtained simultaneously and independently by Carol Karp (unpublished).

4. Proofs of the isomorphism theorems

We now present proofs of Theorems A1 and A2 by a uniform method. Its generality promises other applications to the theory of RITs. Our proof of Theorem A1 is somewhat longer than Myhill's, but that of Theorem A2 is shorter. A function has only one variable unless otherwise stated.

Let f, g be p.r. functions. For numbers x and y,

$$L(x, y) =_{\mathrm{df}} \langle x, y\rangle \in f \vee \langle x, y\rangle \in g \vee \langle y, x\rangle \in f \vee \langle y, x\rangle \in g,$$

$x \sim y =_{\mathrm{df}}$ there is a finite sequence $\langle z_0, \ldots, z_n\rangle$ of numbers such that $x = z_0$, $y = z_n$ and $\forall i < n\, L(z_i, z_{i+1})$.

The relation $\sim$ is an equivalence relation in ε. Write $\lambda(x)$ for $\{y \in \varepsilon \mid y \sim x\}$. For every number x the (finite or infinite) set $\lambda(x)$ is r.e. and given x we can effectively generate $\lambda(x)$. Let $|\alpha|$ denote the cardinality of α.

Lemma L1. *Let f, g be p.r. functions and suppose $L(x, y)$, $\sim$, $\lambda(x)$ are defined in terms of f and g. If α and β are r.e. sets such that*

$$|\lambda(x) \cap \alpha| = |\lambda(x) \cap \beta|, \quad \text{for } x \in \alpha \cup \beta, \tag{4.1}$$

there is a one-one p.r. function q with $\delta q = \alpha$, $\rho q = \beta$, and $x \sim q(x)$, for $x \in \alpha$.

Proof. The function q will be defined as the limit (union) of a monotone increasing sequence $\langle q^s\rangle$ of finite functions. Let *gen-first* mean: first in the order of effective generation. Define $q^0 = \emptyset$. Suppose the one-one finite function q^s has been defined, where $\delta q^s \subset \alpha$, $\rho q^s \subset \beta$, and $x \sim q^s(x)$, for $x \in \delta q^s$. We distinguish two cases.

Case I: s is even. Search for the gen-first $x \in \alpha - \delta q^s$ and the gen-first $y \in [\lambda(x) \cap \beta] - \rho q^s$. If there is such an ordered pair $\langle x, y\rangle$, let $q^{s+1} = q^s \cup (\langle x, y\rangle)$, otherwise let $q^{s+1} = q^s$.

Case II: s is odd. The same as above, but reverse the roles of α and β, δq^s and ρq^s, and x and y.

Define $q = \bigcup \{q^s \mid s \in \varepsilon\}$. It is clear that q is one-one, $\delta q \subset \alpha$, $\rho q \subset \beta$ and $x \sim q(x)$, for $x \in \delta q$. We now prove that $\delta q = \alpha$. Assume not and let x be the gen-first element of $\alpha - \delta q$. Then there is an even number s such that x is the gen-first element of $\alpha - \delta q^s$. Since $x \in \lambda(x)$ this implies $\lambda(x) \cap \delta q^s \subset_+ \lambda(x) \cap \alpha$ and hence $|q^s[\lambda(x) \cap \delta q^s]| < |\lambda(x) \cap \beta|$, because q^s is one-one. Moreover, $q^s[\lambda(x) \cap \delta q^s] = \lambda(x) \cap \rho q^s$ so that $|\lambda(x) \cap \rho q^s| < |\lambda(x) \cap \beta|$. However, $\lambda(x) \cap \rho q^s \subset \lambda(x) \cap \beta$, hence $\lambda(x) \cap \rho q^s \subset_+ \lambda(x) \cap \beta$ and $[\lambda(x) \cap \beta] - \rho q^s \neq \emptyset$. Let y be the gen-first element of $[\lambda(x) \cap \beta] - \rho q^s$, then $\langle x, y \rangle \in q^{s+1}$ and $q^s \subset_+ q^{s+1}$ by Case I. However, $\langle x, y \rangle \in q$ implies $x \in \delta q$, contradicting the hypothesis that $x \in \alpha - \delta q$. We have therefore proved that $\delta q = \alpha$. We know that $\rho q \subset \beta$. By taking s to be odd and using Case II we can show that $\rho q = \beta$. We now show that the function q is p.r. Note that q maps α one-one onto β. Thus either α and β are both finite or both are infinite. In the former case q has a finite domain and q is p.r. In the latter case we have $q^s \subset_+ q^{s+1}$, for $s \in \varepsilon$ and the constructions described in Cases I and II are effective. Again, q is p.r. □

Agreement. In the remainder of this section $\alpha = (0, 2, 4, \ldots)$ and $\beta = (1, 3, 5 \ldots)$.

Lemma L2. *Let f, g be one-one p.r. functions such that $\delta f = \alpha$, $\rho f \subset \beta$, $\delta g = \beta$, and $\rho g \subset \alpha$. If σ and τ are sets with $\sigma \subset \alpha$, $\tau \subset \beta$ and*

$$x \in \sigma \Leftrightarrow f(x) \in \tau, \quad \text{for } x \in \alpha, \qquad y \in \tau \Leftrightarrow g(y) \in \sigma, \quad \text{for } y \in \beta,$$

then there is a one-one p.r. function q such that $\delta q = \alpha$, $\rho q = \beta$ and $q(\sigma) = \tau$.

Proof. Define $L(x, y)$, $x \sim y$, $\lambda(x)$ in terms of f and g. We first note that for x, $y \in \varepsilon$,

$$\text{if } x \sim y, \text{ then } x \in \sigma \cup \tau \Leftrightarrow y \in \sigma \cup \tau. \tag{4.2}$$

This is proved by induction from its special case

$$\text{if } L(x, y), \text{ then } x \in \sigma \cup \tau \Leftrightarrow y \in \sigma \cup \tau.$$

For $x \in \alpha \cup \beta$ we have $f(\lambda(x) \cap \alpha) \subset \lambda(x) \cap \beta$ and $g(\lambda(x) \cap \beta) \subset \lambda(x) \cap \alpha$. Since f and g are one-one,

$$|\lambda(x) \cap \alpha| = |\lambda(x) \cap \beta|, \quad \text{for } x \in \alpha \cup \beta.$$

According to Lemma L1 there is a one-one p.r. function q with $\delta q = \alpha$, $\rho q = \beta$, and $x \sim q(x)$, for $x \in \alpha$. Using (4.2) one can now show that $q(\sigma) \subset \tau$ and $\tau \subset q(\sigma)$. Hence $q(\sigma) = \tau$, which completes the proof. □

Theorem A1. *Two sets are recursively isomorphic iff they are one-one equivalent.*

Proof. Let σ_0, τ_0 be two sets. Trivially, $\sigma_0 \cong \tau_0$ implies $\sigma_0 \equiv_1 \tau_0$, hence we only have to prove its converse. Assume $\sigma_0 \equiv_1 \tau_0$. Then there are one-one *recursive* functions f_0 and g_0 such that for all x, $y \in \varepsilon$,

$$x \in \sigma_0 \Leftrightarrow f_0(x) \in \tau_0, \qquad y \in \tau_0 \Leftrightarrow g_0(y) \in \sigma_0.$$

Define

$$\sigma = \{2x \in \varepsilon \mid x \in \sigma_0\}, \qquad \tau = \{2y + 1 \in \varepsilon \mid y \in \tau_0\},$$

$$\begin{cases} f(x) = 2f_0([\tfrac{1}{2}x]) + 1, & \text{for } x \in \alpha, \text{ i.e., } x \text{ even}, \\ g(y) = 2g_0([\tfrac{1}{2}y]), & \text{for } y \in \beta, \text{ i.e., y odd.} \end{cases}$$

Then f, g, σ, τ satisfy the hypothesis of Lemma L2. Thus there is a one-one p.r. function q with $\delta q = \alpha$, $\rho q = \beta$, and $q(\sigma) = \tau$. Define

$$q_0(x) = [\tfrac{1}{2}q(2x)], \quad \text{for } x \in \varepsilon.$$

It can now be shown that q_0 is a recursive permutation of ε which maps σ_0 onto τ_0. Thus $\sigma_0 \cong \tau_0$. □

Lemma L3. *Let f, g be one-one p.r. functions such that $\delta f \subset \alpha$, $\rho f \subset \beta$, $\delta g \subset \beta$, $\rho g \subset \alpha$. If σ and τ are sets such that $\sigma \subset \delta f$, $f(\sigma) = \tau$, $\beta - \tau \subset \delta g$ and $g(\beta - \tau) = \alpha - \sigma$, then there is a one-one p.r. function q such that $\delta q = \alpha$, $\rho q = \beta$ and $q(\sigma) = \tau$.*

Proof. Assume the hypothesis. Define $L(x, y)$, $x \sim y$ and $\lambda(x)$ in terms of f and g. We have (4.2) as before and we now prove (4.1). We distinguish two cases.

Case I. $\lambda(x) \cap \alpha$ is infinite. Then the relation

$$\lambda(x) \cap \alpha = [\lambda(x) \cap \sigma] \cup [\lambda(x) \cap (\alpha - \sigma)]$$

implies that either $\lambda(x) \cap \sigma$ or $\lambda(x) \cap (\alpha - \sigma)$ is infinite. In the former case $f[\lambda(x) \cap \sigma]$ is an infinite subset of $\lambda(x) \cap \beta$ and in the latter case $g^{-1}[\lambda(x) \cap (\alpha - \sigma)]$ is an infinite subset of $\lambda(x) \cap \beta$. Thus $\lambda(x) \cap \beta$ is infinite and (4.1) holds, since $\lambda(x) \cap \alpha$ and $\lambda(x) \cap \beta$ are both denumerable.

Case II. $\lambda(x) \cap \alpha$ is finite, say $\lambda(x) \cap \alpha = (x_0, \ldots, x_n)$. Then the elements of $\lambda(x) \cap \alpha$ can be denoted in such a way that the diagram of Fig. 1 holds, i.e., such that $f(x_i) = y_i$, $g(y_i) = x_{i+1}$, for $0 \leq i < n$. Now consider the element $x_0 \in \alpha$. Then either (i) $x_0 \in \sigma$, or (ii) $x_0 \in \alpha - \sigma$. If (i) holds, we have by (4.2),

$$x_n \in \sigma, \qquad \sigma \subset \delta f, \qquad f(x_n) \in [\lambda(x) \cap \beta] - \{y_i \mid i < n\},$$

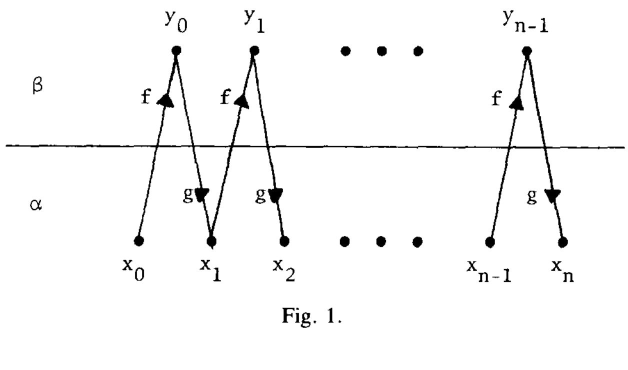

Fig. 1.

while if (ii) holds,

$$\alpha - \sigma \subset \rho g, \qquad g^{-1}(x_0) \in [\lambda(x) \cap \beta] - \{y_i \mid i < n\}.$$

In each case $|\lambda(x) \cap \alpha| \leq |\lambda(x) \cap \beta|$. By symmetry considerations we also have $|\lambda(x) \cap \beta| \leq |\lambda(x) \cap \alpha|$. This implies (4.1). The proof of Lemma L3 can now be completed as the proof of Lemma L2. □

Theorem A2. *Two sets are recursively isomorphic iff they are recursively equivalent and their complements are recursively equivalent.*

Proof. Trivially, $\sigma_0 \cong \tau_0$ implies $\sigma_0 \simeq \tau_0$ & $\sigma_0' \simeq \tau_0'$, hence we only have to prove its converse. Assume $\sigma_0 \simeq \tau_0$ and $\varepsilon - \sigma_0 \simeq \varepsilon - \tau_0$. Then there are one-one p.r. functions f_0, g_0 with $\sigma_0 \subset \delta f_0$, $f_0(\sigma_0) = \tau_0$, $\varepsilon - \tau_0 \subset \delta g_0$ and $g_0(\varepsilon - \tau_0) = \varepsilon - \sigma_0$. Define σ, τ, f, g by:

$$\sigma = \{2x \in \varepsilon \mid x \in \sigma_0\}, \qquad \tau = \{2y + 1 \in \varepsilon \mid y \in \tau_0\},$$

$$\delta f = \{2x \in \varepsilon \mid x \in \delta f_0\}, \qquad f(x) = 2f_0([\tfrac{1}{2}x]) + 1,$$

$$\delta g = \{2y + 1 \in \varepsilon \mid y \in \delta g_0\}, \qquad g(y) = 2g_0([\tfrac{1}{2}y]).$$

Then f, g, σ, τ satisfy the hypothesis of Lemma L3. Thus there is a one-one p.r. function q such that $\delta q = \alpha$, $\rho q = \beta$, and $q(\sigma) = \tau$. Define

$$q_0(x) = [\tfrac{1}{2}q(2x)], \quad \text{for } x \in \varepsilon.$$

We can now show as in the proof of Theorem A1 that q_0 is a recursive permutation of ε which maps σ_0 onto τ_0. Thus $\sigma_0 \cong \tau_0$. □

5. Creative sets

Henceforth we denote the value of a function f at a number $n \in \delta f$ by $f(n)$ or f_n. Recall that F stands for the class of all r.e. sets and Q for the class of all finite sets. Let E denote the class of all recursive sets. Thus

$$Q \subset_+ E \subset_+ F \quad \text{and} \quad \alpha \in E \iff \alpha, \alpha' \in F. \tag{5.1}$$

Let $\omega_0, \omega_1, \ldots$ be a *standard* enumeration of the class F, i.e., an enumeration such that gives the Gödel number of (the list of instructions for the computation of) a p.r. function f, we can find a number t such that $\rho f = \omega_t$; we also assume that the set ω_0 is empty.

A set π is *productive*, if there is a p.r. function p such that

$$\omega_n \subset \pi \Rightarrow n \in \delta p \;\&\; p(n) \in \pi - \omega_n. \tag{5.2}$$

The following three properties of productive sets follow from their definition:

$$\pi \text{ productive } \Rightarrow \pi \text{ not r.e.}, \tag{5.3}$$

$$\pi \text{ productive } \Rightarrow \pi \text{ has an infinite r.e. subset}, \tag{5.4}$$

$$\pi \text{ productive } \& \ \gamma \subset \pi \ \& \ \gamma \text{ r.e. } \Rightarrow \pi - \gamma \text{ productive}. \tag{5.5}$$

For let π be productive. Then $\omega_n \subset \pi$ implies $\omega_n \subset_+ \pi$; this proves (5.3). As far as (5.4) is concerned, $\omega_0 \subset \pi$, hence $(p_0) \subset \pi$. From p_0 we can find a number q_0 such that $\omega_{q(0)} = (p_0)$, so that $p(q_0) \in \pi - (p_0)$; we have now found two elements of π, namely p_0 and $p(q_0)$. Then we can find a number q_1 such that $(p_0, p(q_0)) = \omega_{q(1)}$ and a third element of π, namely $p(q_1)$. Continuing this procedure we obtain an infinite r.e. subset of π. Now assume the hypothesis of (5.5). Given an r.e. subset ω_n of $\pi - \gamma$, we can find a number q_n such that $\omega_{q(n)} = \omega_n \cup \gamma$, hence an element $p(q_n) \in \pi - (\omega_n \cup \gamma)$. However, $\pi - (\omega_n \cup \gamma) = (\pi - \gamma) - \omega_n$ and this implies the conclusion of (5.5).

A set α is *creative,* if α is r.e. and α' productive. In view of (5.3) and the second part of (5.1) we see that every creative set belongs to $F - E$. The classical example of a creative set is the set $\delta = \{n \in \varepsilon \mid n \in \omega_n\}$. Note that $n \in (\delta - \omega_n) \cup (\omega_n - \delta)$, for $n \in \varepsilon$.

Theorem B. *Every two creative sets are recursively isomorphic.*

Since every set recursively isomorphic to a creative set is itself creative, Theorem B implies that the creative sets form a single RIT. Myhill obtained this result by proving that every two creative sets are one-one equivalent and then using Theorem A1. A surprising consequence of Theorem B is the following. Let α be a creative set. By (5.4) there is an infinite r.e. subset of α', say γ. Put $\beta = \alpha \cup \gamma$, then β is r.e. and β' is productive by (5.5). Hence β is creative. Thus in spite of the fact that α and β differ in an infinite r.e. set, namely γ, there is a recursive permutation of ε which maps α onto β.

Theorem B is important for the study of formal systems. Let $\mathscr{F}$ be a formal system which for each number n contains a name $\bar{n}$ of n. Let

$W_{\mathscr{F}}$ = the collection of all wffs of $\mathscr{F}$,

G = a Gödel-numbering of $W_{\mathscr{F}}$,

$\mathrm{Th}_{\mathscr{F}}$ = the collection of all theorems of $\mathscr{F}$.

We call $\mathscr{F}$ *axiomatizable* if $G(\mathrm{Th}_{\mathscr{F}})$ is r.e., *adequate for arithmetic,* if for every r.e. set α there is a formula $\phi_\alpha(n) \in W_{\mathscr{F}}$ with exactly one free variable, namely n such that

$$n \in \alpha \Leftrightarrow \phi_\alpha(\bar{n}) \in \mathrm{Th}_{\mathscr{F}}, \quad \text{for } n \in \varepsilon.$$

It follows that if $\mathscr{F}$ is adequate for arithmetic, then $\mathscr{F}$ is consistent. Let F be called *creative,* if the set $G(\mathrm{Th}_{\mathscr{F}})$ is creative. Myhill proved that every formal system which is axiomatizable and adequate for arithmetic is creative. Theorem B therefore implies that for any two such formal systems $\mathscr{F}$ and $\mathscr{F}'$,

$$G(\mathrm{Th}_{\mathscr{F}}) \cong G(\mathrm{Th}_{\mathscr{F}'}). \tag{5.6}$$

Let us call the formal systems $\mathscr{F}$ and $\mathscr{F}'$ *isomorphic,* if (5.6) holds. It is known that every consistent axiomatizable extension of Peano arithmetic ($\mathscr{P}$) is adequate for arithmetic. One such system is Zermelo–Fraenkel set theory ($\mathscr{Z}\mathscr{F}$). Thus $\mathscr{P}$ and $\mathscr{Z}\mathscr{F}$ are isomorphic, though $\mathrm{Th}_{\mathscr{Z}\mathscr{F}} \cap W_{\mathscr{P}}$ is much larger than $\mathrm{Th}_{\mathscr{P}}$. This result is extended in [1], where Boykan Pour-El showed that the isomorphism between $\mathscr{P}$ and $\mathscr{Z}\mathscr{F}$ can be chosen so as to preserve negation. Let a *degree* mean: a Turing degree of unsolvability. Creative sets are of highest r.e. degree. In [9] Feferman showed that for every r.e. set α, there is a formal system $\mathscr{F}$ such that α and $G(\mathrm{Th}_{\mathscr{F}})$ have the same degree. In general, these systems require infinitely many axioms. Myhill conjectured that every finitely axiomatizable, undecidable theory was creative. This conjecture was dashed by Hanf [11] who showed that for each r.e. degree there was a finitely axiomatizable theory of that degree.

6. A recursive analogue of the Cantor–Bernstein theorem

We need the following notations:

$$\mathrm{Req}(\alpha) = \{\sigma \subset \varepsilon \mid \sigma \simeq \alpha\} = \text{the RET of } \alpha, \quad \text{for } \alpha \subset \varepsilon,$$
$$\Omega = \{\mathrm{Req}(\alpha) \mid \alpha \subset \varepsilon\} = \text{the collection of all RETs.}$$

Every function with a finite domain is p.r. Thus *two finite sets are recursively equivalent iff they have the same cardinality.* We therefore identify the RET $\mathrm{Req}\{x \in \varepsilon \mid x < n\}$ with the number n, so that $\varepsilon \subset_+ \Omega$. Let c denote the cardinality of the continuum. It can be shown that $|\Omega| = c$, hence $|\Omega - \varepsilon| = c$. The operations of addition and multiplication can be extended from ε to Ω in a natural manner. For α, $\beta \subset \varepsilon$,

$$\mathrm{Req}\ \alpha + \mathrm{Req}\ \beta =_{\mathrm{df}} \mathrm{Req}(\{2x \in \varepsilon \mid x \in \alpha\} \cup \{2x+1 \in \varepsilon \mid x \in \beta\}),$$
$$\mathrm{Req}\ \alpha \cdot \mathrm{Req}\ \beta =_{\mathrm{df}} \mathrm{Req}\ j(\alpha \times \beta).$$

These operations are well-defined and the system $\langle \Omega, +, \cdot \rangle$ has the following properties:

(a) $+$ and $\cdot$ are associative and commutative, while $\cdot$ is distributive over $+$,

(b) $X + 0 = X$, $X \cdot 1 = X$, $X \cdot Y = 0 \Leftrightarrow (X = 0 \vee Y = 0)$.

The role played by disjointness in cardinal arithmetic is played by the relation of separability in the arithmetic of RETs. The sets α and β are *separable* ($\alpha \mid \beta$), if $\alpha \subset \bar{\alpha}$ and $\beta \subset \bar{\beta}$, for two disjoint r.e. sets $\bar{\alpha}$ and $\bar{\beta}$. Intuitively, $\alpha \mid \beta$ means that there is an effective procedure which when presented with an element $x \in \alpha \cup \beta$ decides whether $x \in \alpha$ or $x \in \beta$. Since every finite set is r.e., *two finite sets are separable iff they are disjoint.* Note that separability was already involved in the definition of addition for RETs; every two RETs Req α and Req β have separable representatives, e.g., $\{2x \in \varepsilon \mid x \in \alpha\}$ and $\{2x+1 \in \varepsilon \mid x \in \beta\}$.

The classical Cantor–Bernstein theorem states that for arbitrary collections A and B: if A is equivalent to a subcollection of B and vice versa, then A and B are equivalent. Algebraically,

$$|A| \leqslant |B| \;\&\; |B| \leqslant |A| \;\Rightarrow\; |A| = |B|.$$

Theorem C. *If the set α is recursively equivalent to some subset γ of β with $\gamma \mid \beta - \gamma$ and vice versa, then α and β are recursively equivalent.*

Let us define $A \leqslant B =_{\mathrm{df}} \exists X\,[A + X = B]$ for RETs A and B. The $\leqslant$-relation in Ω is trivially reflexive and transitive. According to Myhill's Theorem C it is also antisymmetric, hence a partial ordering relation. We mention in passing that $\leqslant$ is not a total ordering relation. In fact, there exist collections of c RETs, any two of which are incomparable under $\leqslant$. Theorem C has been subsequently obtained by Nerode as a consequence of one of his metatheorems.

7. Proof of Theorem C

We present a proof of Theorem C which is essentially different from Myhill's proof [6, pp. 75–78]. The methods used are similar to those employed in Section 4.

The sets $\alpha_1, \ldots, \alpha_n$ are *separable,* if there are mutually disjoint r.e. sets $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ such that $\alpha_i \subset \bar{\alpha}_i$, for $1 \leqslant i \leqslant n$. We now define α^s, for a r.e. set α and a number s. In case α is nonempty we assume that a recursive function a_x ranging over α has been selected. If $\alpha = \emptyset$, $\alpha^s = \emptyset$, for each s. If $\alpha \neq \emptyset$, $\alpha^s = \{a_x \in \varepsilon \mid x < s\}$. In this way we associate with an r.e. set α a monotone increasing sequence $\langle \alpha^s \rangle$ of finite sets with α as union. Now suppose that f is a p.r. function and s a number. Then

$$\alpha_f = \{j(x,y) \in \varepsilon \mid f(x) = y\}, \qquad f^s = \{\langle x, y\rangle \mid j(x,y) \in \alpha_f^s\}.$$

In this way we associate with a p.r. function f a monotone increasing sequence $\langle f^s \rangle$ of finite functions with f as union. We no longer assume as in the latter part of Section 4 that $\alpha = (0, 2, 4, \ldots)$ and $\beta = (1, 3, 5, \ldots)$.

Lemma L4. *Let α, β, σ, τ be separable sets and f, g be one-one p.r. functions such that*

$$\alpha \cup \sigma \subset \delta f, \quad f(\alpha \cup \sigma) = \beta, \qquad \beta \cup \tau \subset \delta g, \quad g(\beta \cup \tau) = \alpha.$$

Then there is a one-one p.r. function q such that $\alpha \subset \delta q$ and $q(\alpha) = \beta$.

Proof. Let $\bar{\alpha}$, $\bar{\beta}$, $\bar{\sigma}$, $\bar{\tau}$ be mutually disjoint r.e. sets with $\alpha \subset \bar{\alpha}$, $\beta \subset \bar{\beta}$, $\sigma \subset \bar{\sigma}$, $\tau \subset \bar{\tau}$. Define for $s, x, y \in \varepsilon$,

$$L^s(x, y) =_{\mathrm{df}} \langle x, y \rangle \in f^s \vee \langle x, y \rangle \in g^s \vee \langle y, x \rangle \in f^s \vee \langle y, x \rangle \in g^s,$$

$$x \sim^s y =_{\mathrm{df}} \text{there is a finite sequence } \langle z_0, \ldots, z_n \rangle \text{ of numbers with } x = z_0, \; y = z_n \text{ and } \forall i < n \; L^s(z_i, z_{i+1}).$$

For each number s the $\sim^s$-relation is an equivalence relation in ε. We write $\lambda^s(x)$ for $\{y \in \varepsilon \mid y \sim^s x\}$. The ternary relation $x \sim^s y$ is recursive. Moreover, for each ordered pair $\langle s, x \rangle$ the set $\lambda^s(x)$ is finite and both its elements and cardinality can be computed from $\langle s, x \rangle$. We claim that for $x, y \in \varepsilon$,

$$\text{if } x, y \in \bar{\alpha} \cup \bar{\beta} \text{ and } x \sim^s y, \text{ we have } x \in \alpha \cup \beta \Leftrightarrow y \in \alpha \cup \beta. \tag{7.1}$$

Relation (7.1) can be proved by induction from its special case

$$\text{if } x, y \in \bar{\alpha} \cup \bar{\beta} \text{ and } L^s(x, y), \text{ we have } x \in \alpha \cup \beta \Leftrightarrow y \in \alpha \cup \beta.$$

We write $x \sim y$ for $\exists s\, [x \sim^s y]$. The desired function q will be defined as the union of a monotone increasing sequence $\langle q^s \rangle$ of finite functions. Let $q^0 = \emptyset$. Assuming that the function q^s has been defined and

$$\delta q^s \subset \bar{\alpha}, \qquad \rho q^s \subset \bar{\beta}, \qquad x \sim^s q(x), \quad \text{for } x \in \delta q^s,$$

we search for elements x and y such that

$$x \in (\bar{\alpha}^s \cap \delta f^s) - \delta q^s, \qquad y \in (\bar{\beta}^s \cap \delta g^s) - \rho q^s, \qquad x \sim^s y$$

and $j(x, y)$ is minimal. Given $\bar{\alpha}^s$, $\bar{\beta}^s$, f^s, g^s, q^s we can decide whether such an ordered pair $\langle x, y \rangle$ exists and if so, compute it. If it exists q^{s+1} is defined as $q^s \cup (\langle x, y \rangle)$, if not, $q^{s+1} = q^s$. Using induction on s, we can prove that for every s, q^s is a one-one finite function whose definition can be computed from s. This implies that q is a one-one p.r. function such that

$$\delta q \subset \bar{\alpha}, \qquad \rho q \subset \bar{\beta}, \qquad x \sim q(x), \quad \text{for } x \in \delta q.$$

It follows from (7.1) that $x \in \alpha \Leftrightarrow q(x) \in \beta$, for $x \in \delta q$. This implies

$$q(\alpha \cap \delta q) = \beta \cap \rho q. \tag{7.2}$$

Suppose we could prove (a) $\alpha \subset \delta q$, and (b) $\beta \subset \rho_q$. Then we would have $\alpha \cap \delta q = \alpha$ and $\beta \cap \rho q = \beta$. Substitution in (7.2) would then yield the desired relation $q(\alpha) = \beta$. We shall write $\nu(x)$ for $\{t \in \varepsilon \mid t < x\}$.

Re (a). Suppose (a) were false, i.e., $\alpha - \delta q \neq \emptyset$. Let $x =_{\mathrm{df}} \min(\alpha - \delta q)$. Then

$$x \in [\bar{\alpha}^s \cap \delta f^s] - \delta q^s \;\&\; \nu(x) \cap \delta q^s = \nu(x) \cap \delta q, \quad \text{for some } s. \tag{7.3}$$

Let $\gamma =_{\mathrm{df}} \lambda^s(x) \cap \bar{\alpha}^s \cap \delta f^s$. Using (7.1) we obtain $\gamma \subset \lambda^s(x) \cap \bar{\alpha} \subset \alpha$ and hence $f^s(\gamma) \subset \lambda^s(x) \cap \beta$. Now $\lambda^s(x) \cap \delta q^s \subset_+ \gamma$, because $x \in \gamma - \delta q^s$. Thus

$$|q^s[\lambda^s(x) \cap \delta q^s]| < |f^s(\gamma)|,$$

since q^s and f^s are one-to-one. However,

$$q^s[\lambda^s(x) \cap \delta q^s] = \lambda^s(x) \cap \rho q^s,$$

so that

$$|\lambda^s(x) \cap \rho q^s| < |\lambda^s(x) \cap \beta| \quad \text{and} \quad [\lambda^s(x) \cap \beta] - \rho q^s \neq \emptyset.$$

Let $y =_{\text{df}} \min([\lambda^s(x) \cap \beta] - \rho q^s)$. We claim that $y \in \rho q$. For suppose not, then there is a number $t \geq s$ such that

$$y \in [\lambda^t(x) \cap \bar{\beta}^t \cap \delta g^t] - \rho q^t \;\;\&\;\; \nu(y) \cap \rho q^t = \nu(y) \cap \rho q. \tag{7.4}$$

Then $\langle x, y \rangle \in q^{t+1}$, where $q^{t+1} \subset q$, because (7.3) also holds when we replace s by t, while $j(x, y)$ is iminimal. Hence $y \in \rho q$. We proved that $y \notin \rho q$ implies its own negation, hence that $y \notin \rho q$ is false. Thus $y \in \rho q$. If $\langle x', y \rangle \in q^{u+1} - q^u$, then $u \geq s$ and hence $x' = x$ by the minimality of $j(x, y)$. This implies $x \in \delta q$ contrary to $x = \min(\alpha - \delta q)$. The assumption that $\alpha - \delta q$ is nonempty has therefore led to a contradiction. Thus $\alpha - \delta q$ is empty and $\alpha \subset \delta q$.

Re (b). In view of symmetry considerations this can be proved by an argument similar to that used in the proof of (a). This completes the proof of Lemma L4. □

Theorem C readily follows from Lemma L4. For let A, $B \in \Omega$, $A \leq B$ and $B \leq A$, say $A + S = B$ and $B + T = A$. Suppose $\alpha_0 \in A$, $\sigma_0 \in S$, $\beta_0 \in B$, $\tau_0 \in T$. Let

$$\alpha =_{\text{df}} \{4x \in \varepsilon \mid x \in \alpha_0\}, \qquad \sigma =_{\text{df}} \{4x + 1 \in \varepsilon \mid x \in \sigma_0\},$$
$$\beta =_{\text{df}} \{4x + 2 \mid x \in \beta_0\}, \qquad \tau =_{\text{df}} \{4x + 3 \in \varepsilon \mid x \in \tau_0\}.$$

Then the sets α, β, σ, τ are separable and there are one-one p.r. functions f and g such that $\alpha \cup \sigma \subset \delta f$, $f(\alpha \cup \sigma) = \beta$, $\beta \cup \tau \subset \delta g$, $g(\beta \cup \tau) = \alpha$. Then $\alpha \simeq \beta$, i.e., $A = B$ by Lemma L4. □ (Theorem C)

8. Isols

A set is *isolated*, if it is not recursively equivalent to a proper subset or equivalently, if it has no infinite r.e. subset. Since every infinite r.e. set has an infinite recursive subset, a set is isolated iff it has no *infinite recursive* subset. A set is called *immune*, if it is infinite and isolated. We define

$$\Lambda =_{\text{df}} \{X \in \Omega \mid X \neq X + 1\}, \quad \text{i.e.,} \ \Lambda =_{\text{df}} \{\text{Req}\, \alpha \mid \alpha \text{ is isolated}\}.$$

The members of Λ are called *isols*. An isol is *finite*, if it consists of finite sets (hence is identified with a nonnegative integer), infinite, if it consists of immune sets. There is no analogue of infinite isols in cardinal arithmetic (i.e., *with* the axiom of choice) so that we are here dealing with objects of a different type. If A,

B, C, $D \in \Omega$, we write $A < B$ for $A \leq B$ & $A \neq B$, $C \geq D$ for $D \leq C$ and $C > D$ for $D < C$. Some basic properties of isols are the following. For A, B, $C \in \Lambda$,

(a) Λ is closed under $+$ and $\cdot$;
(b) $\varepsilon \subset_+ \Lambda \subset_+ \Omega$, where $\Lambda - \varepsilon$, Λ, $\Omega - \Lambda$ have cardinality c;
(c) $A + C = B + C \Rightarrow A = B$;
(d) $AC = BC$ & $C \neq 0 \Rightarrow A = B$.

If $B \leq A$, we define $A - B$ as the unique solution of the equation $B + X = A$. Two other basic properties of isols are:

(e) $A \in \Lambda - \varepsilon \Leftrightarrow A > A - 1 > A - 2 > \cdots$;
(f) $A \in \Lambda - \varepsilon \Leftrightarrow \exists B$ [neither $A \leq B$ nor $B \leq A$].

A set α is *simple*, if α is r.e. and α' immune. The existence of simple sets was first proved by Post [27]. As far as RITs are concerned, simple sets are poles apart from creative sets. For while the creative sets form a single RIT by Theorem B, we can show using (e) that the simple sets represent denumerably many RITs. For let α be simple, $\xi = \alpha'$ and p_n a one-one function from ε onto ξ. Put $X = \text{Req}\, \xi$ and $X_{n+1} = \text{Req}[\xi - (p_0, \ldots, p_n)]$, for $n \geq 0$. Since X is an isol, we have $X_n = X - n$, for $n \geq 1$ and by (e),

$$X > X - 1 > X - 2 > \cdots.$$

The complements of the simple sets α, $\alpha \cup (p_0)$, $\alpha \cup (p_0, p_1)$, ... belong therefore to different RETs. This implies by Theorem A2 that the simple sets α, $\alpha \cup (p_0)$, $\alpha \cup (p_0, p_1)$, ... belong to different RITs. The Turing degree of an r.e., but not recursive set can be represented by a simple set. Since the structure of these Turing degrees is quite complicated, so are the RITs of the simple sets which represent them.

In view of the cancellation laws (c) and (d) for isols, we see that isolic arithmetic is much closer to ordinary arithmetic than to cardinal arithmetic. On the other hand, properties (e) and (f) show that there are important differences. Myhill raised the following two questions.

(I) *Which recursive functions can be extended in a natural way from ε to Λ?*
(II) *Which algebraic properties of ε carry over to Λ?*

He answered the first question by introducing the family of combinatorial functions and by defining them in a set-theoretic way, which enabled him to extend them to Λ. He also obtained many basic theorems involving the second question, thereby changing a heterogeneous collection of results into a part of mathematics.

9. Combinatorial functions

We need an effective enumeration without repetitions of the class Q of all finite sets. There are denumerably many such enumerations, but we shall use ρ_0,

$\rho_1, \ldots,$ where

$$\rho_0 = \emptyset,$$

$$\rho_n = (a_1, \ldots, a_t), \text{ where } a_1, \ldots, a_t \text{ are the distinct numbers such that } n = 2^{a(1)} + \cdots + 2^{a(t)}, \quad \text{for } n \geq 1.$$

We define r_n as $|\rho_n|$. Note that given n we can compute the elements and cardinality of ρ_n. Thus r_n is a recursive function. The collection of all integers is denoted by ε^*, the set $\{x \in \varepsilon \mid x < n\}$ by v_n and the class of all sets (i.e., subsets of ε) by V. The binomial coefficient 'n choose i' is denoted by $C(n; i)$; the usual vertical notation is only used in displayed formulas. For $\alpha, \beta, \gamma, \delta \in Q$ and $(\alpha, \beta), (\gamma, \delta) \in Q^2$,

$$\alpha \sim \beta \text{ means: } |\alpha| = |\beta|;$$

$$(\alpha, \beta) \sim (\gamma, \delta) \text{ means: } \alpha \sim \gamma \,\&\, \beta \sim \delta;$$

$$(\alpha, \beta) \neq (\gamma, \delta) \text{ means: } \alpha \neq \gamma \vee \beta \neq \delta;$$

$$(\alpha, \beta) \subset (\gamma, \delta) \text{ means: } \alpha \subset \gamma \,\&\, \beta \subset \delta;$$

$$(\alpha, \beta) \subset_+ (\gamma, \delta) \text{ means: } (\alpha, \beta) \subset (\gamma, \delta) \,\&\, (\alpha, \beta) \neq (\gamma, \delta).$$

We recall that j, k, l are recursive functions such that $j(x, y)$ maps ε^2 one-to-one onto ε and $j[k(n), l(n)] = n$, for $n \in \varepsilon$. A mapping from V into V is an *operator of one variable*, a mapping from V^2 into V is an *operator of two variables.*

Definition 1. An operator Φ of one variable is *numerical*, if
(a) $\alpha \in Q \Rightarrow \Phi(\alpha) \in Q$;
(b) $\alpha \sim \beta \Rightarrow \Phi(\alpha) \sim \Phi(\beta)$, for $\alpha, \beta \in Q$.
An operator Φ of two variables is *numerical*, if
(a) $(\alpha, \beta) \in Q^2 \Rightarrow \Phi(\alpha, \beta) \in Q$;
(b) $(\alpha, \beta) \sim (\gamma, \delta) \Rightarrow \Phi(\alpha, \beta) \sim \Phi(\gamma, \delta)$, for $(\alpha, \beta), (\gamma, \delta) \in Q^2$.

Definition 2. For a numerical operator Φ of one variable, $f_\Phi(n) = |\Phi(v_n)| =$ the function *induced by Φ*. For a numerical operator Φ of two variables, $f_\Phi(m, n) = |\Phi(v_m, v_n)| =$ the function *induced by Φ.*

Definition 3. A *combinatorial operator of one variable* is a mapping Φ from V into V such that
(a) Φ is numerical;
(b) for $\Phi^\varepsilon = \bigcup \{\Phi(\alpha) \mid \alpha \in V\}$ there is a mapping Φ^{-1} from Φ^ε into Q such that

$$x \in \Phi(\alpha) \Leftrightarrow \Phi^{-1}(x) \subset \alpha, \quad \text{for } x \in \Phi^\varepsilon,\ \alpha \in V.$$

Definition 4. A *combinatorial operator of two variables* is a mapping Φ from V^2 into V such that

(a) Φ is numerical;

(b) for $\Phi^\varepsilon = \bigcup\{\Phi(\alpha, \beta) \mid (\alpha, \beta) \in V^2\}$ there is a mapping Φ^{-1} from Φ^ε into Q^2 such that

$$x \in \Phi(\alpha, \beta) \iff \Phi^{-1}(x) \subset (\alpha, \beta), \quad \text{for } x \in \Phi^\varepsilon,\ (\alpha, \beta) \in V^2.$$

Both in the one-variable case and in the two-variable case Φ^{-1} is called a *quasi-inverse of* Φ. It is not an ordinary inverse, since (i) Φ need not be one-to-one, and (ii) while Φ maps Q or Q^2 into Q, the domain of Φ^{-1} is a set of numbers (i.e., nonnegative integers).

Definition 5. A function f from ε into ε (from ε^2 into ε) is *combinatorial,* if it is induced by a combinatorial operator of one variable (of two variables).

We give five examples of combinatorial operators and the combinatorial functions they induce. We assume that $\alpha, \beta \in V$.

(A) $\Phi_1(\alpha) = \{x \in \varepsilon \mid \rho_x \subset \alpha\}, \quad f_1(n) = 2^n,$

(B) $\Phi_2(\alpha) = \{x \in \varepsilon \mid \rho_x \subset \alpha \,\&\, r_x = t\}, \quad f_2(n) = \binom{n}{t},$

(C) $\Phi_3(\alpha, \beta) = \{2x \in \varepsilon \mid x \in \alpha\} \cup \{2x+1 \mid x \in \beta\}, \quad f_3(m, n) = m + n,$

(D) $\Phi_4(\alpha, \beta) = j(\alpha \times \beta), \quad f_4(m, n) = m \cdot n,$

(E) $\Phi_5(\alpha, \beta) = \{j(x, y) \in \varepsilon \mid \rho_x \subset \alpha \,\&\, \rho_y \subset \beta \,\&\, r_x = r_y\},$

$$f_5(m, n) = \binom{m}{0}\binom{n}{0} + \binom{m}{1}\binom{n}{1} + \cdots = \frac{(m+n)!}{m!n!}.$$

Three elementary properties of a combinatorial operator Φ of one variable are: for $\alpha, \beta, \gamma \in V$,

$$\alpha \subset \beta \Rightarrow \Phi(\alpha) \subset \Phi(\beta), \tag{9.1}$$

$$\Phi(\alpha \cap \beta) = \Phi(\alpha) \cap \Phi(\beta), \tag{9.2}$$

$$\Phi(\alpha) = \bigcup\{\Phi(\gamma) \mid \gamma \subset \alpha\} = \bigcup\{\Phi(\rho_n) \mid \rho_n \subset \alpha\}. \tag{9.3}$$

We claim that

every combinatorial function of one variable is monotone increasing, (9.4)

$$\Phi^\varepsilon = \Phi(\varepsilon), \quad \text{for a combinatorial operator } \Phi \text{ of one variable.} \tag{9.5}$$

Re (9.4). $m \leqslant n \Rightarrow \nu_m \subset \nu_n \Rightarrow \Phi(\nu_m) \subset \Phi(\nu_n) \Rightarrow |\Phi(\nu_m)| \leqslant |\Phi(\nu_n)|$
$\Rightarrow f_\Phi(m) \leqslant f_\Phi(n).$

Re (9.5). $\Phi(\varepsilon) = \bigcup\{\Phi(\gamma) \mid \gamma \subset \varepsilon\} = \bigcup\{\Phi(\gamma) \mid \gamma \in V\} = \Phi^\varepsilon.$

Some of the simplest functions of combinatorics are not monotone increasing, e.g., $C(5; n)$. Thus the function $C(5; n)$ is not combinatorial (i.e., in the sense of Myhill). Relations (9.1)–(9.5) are readily generalized to the two-variable case. Then (9.4) and (9.5) become:

every combinatorial function of two variables is monotone increasing in each variable, (9.4*)

$$\Phi^{\varepsilon} = \Phi(\varepsilon, \varepsilon). \tag{9.5*}$$

According to (9.4*) the function $C(m; n)$ of two variables is not combinatorial.

The following two theorems are special cases of Newton's approximation theorem.

Theorem N. *For every function f from ε into ε there is exactly one function c from ε into ε^* such that*

$$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i}, \tag{9.6}$$

namely the function

$$c_i = \Delta^i f(0) = [\Delta^i f(n)]_{n=0}, \tag{9.7}$$

where $\Delta f(n) = f(n+1) - f(n)$.

The function c_i related to $f(n)$ by (9.6) or equivalently by (9.7) is called the function *associated* with $f(n)$.

Theorem N*. *For every function f from ε^2 into ε there is exactly one function c from ε^2 into ε^* such that*

$$f(m, n) = \sum_{i=0}^{m} \sum_{k=0}^{n} c_{ik} \binom{m}{i}\binom{n}{k}, \tag{9.6*}$$

namely the function

$$c_{ik} = \Delta_m^i \Delta_n^k f(0, 0) = [\Delta_m^i \Delta_n^k f(m, n)]_{m,n=0}, \tag{9.7*}$$

where

$$\Delta_m f(m, n) = f(m+1, n) - f(m, n), \qquad \Delta_n f(m, n) = f(m, n+1) - f(m, n).$$

The function c_{ik} related to $f(m, n)$ by (9.6*) or equivalently by (9.7*) is called the function *associated* with $f(m, n)$.

Theorem D1. *A function*

$$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i}$$

from ε into ε is combinatorial iff $\forall i\ [c_i \geqslant 0]$. *If f is combinatorial, f is induced by the combinatorial operator Φ_f, where*

$$\Phi_f(\alpha) = \{j(x, y) \in \varepsilon \mid \rho_x \subset \alpha \;\&\; y < c_{r(x)}\}, \quad \textit{for } \alpha \in V. \tag{9.8}$$

In the next theorem we need a recursive function which maps ε^3 one-one onto ε. Let $j_3(x, y, z) = j[j(x, y), z]$; then j_3 is such a function.

Theorem D1*. *A function*

$$f(m, n) = \sum_{i=0}^{m} \sum_{k=0}^{n} c_{ik} \binom{m}{i} \binom{n}{k}$$

from ε^2 into ε is combinatorial iff $\forall i\ \forall k\ [c_{ik} \geqslant 0]$. *If f is combinatorial, f is induced by the combinatorial operator Φ_f, where*

$$\Phi_f(\alpha, \beta) = \{j_3(x, y, z) \in \varepsilon \mid \rho_x \subset \alpha \;\&\; \rho_y \subset \beta \;\&\; z < c_{r(x), r(y)}\}, \quad \textit{for } \alpha, \beta \in V. \tag{9.8*}$$

Note that if c_i and c_{ik} are the functions associated with the combinatorial functions $f(n)$ and $f(m, n)$, respectively,

$$f(n) \text{ recursive} \iff c_i \text{ recursive}, \tag{9.9}$$

$$f(m, n) \text{ recursive} \iff c_{ik} \text{ recursive}. \tag{9.9*}$$

Let us compute the functions c_i and c_{ik} associated with the five functions $f_1, \ldots, f_5$ we gave as examples of combinatorial functions:

(A) $f_1(n) = 2^n$; then $c_i = 1$, for $i \geqslant 0$;

(B) $f_2(n) = \binom{n}{t}$; then $c_t = 1$, $c_i = 0$, for $i \neq t$;

(C) $f_3(m, n) = \binom{m}{1} + \binom{n}{1}$; then $c_{10} = c_{01} = 1$, $c_{ik} = 0$, otherwise;

(D) $f_4(m, n) = \binom{m}{1}\binom{n}{1}$; then $c_{11} = 1$, $c_{ik} = 0$, otherwise;

(E) $f_5(m, n) = \dfrac{(m+n)!}{m!n!}$; then $c_{ii} = 1$, $c_{ik} = 0$, for $i \neq k$.

We also claim that the function $f(n) = n!$ is combinatorial. For $f(n)$ is the cardinality of the family of all permutations of ν_n. Every permutation p of ν_n is characterized by $p \mid \mu_p$, where μ_p is the set of all nonfixed points of p. It follows that

$$n! = \sum_{i=0}^{n} d_i \binom{n}{i}, \tag{9.10}$$

where d_i is the number of *derangements* (i.e., permutations without fixed points) of a finite set with cardinality i. Since $d_i \geqslant 0$, for $i \in \varepsilon$, (9.10) implies by Theorem D1 that the function $n!$ is combinatorial.

Combinatorial operators and functions can be defined for any finite number n of variables. To simplify the exposition we have only considered the most important cases, namely $n = 1$ and $n = 2$. If c_i is the function associated with a function f from ε into ε, the numbers c_i are called the *combinatorial coefficients* of f. Similarly for the numbers c_{ik} associated with a function f from ε^2 into ε. According to Theorems D1 and D1* the (for our purposes) crucial property of functions from ε (or ε^2) into ε is that they are combinatorial iff all their combinatorial coefficients are nonnegative.

We now discuss Myhill's method for extending combinatorial functions from ε^n into ε to functions from Ω^n into Ω. We again restrict our attention to the cases $n = 1$ and $n = 2$. Let f be a combinatorial function of one or two variables. The operator Φ_f is defined as in (9.8) or (9.8*). It is called the *normal* operator which induces the function f or the *normal* operator of the function f.

Theorem D2. *For a combinatorial function f from ε into ε,*

(a) $\quad |\alpha| = n \;\Rightarrow\; |\Phi_f(\alpha)| = f(n), \quad$ *for* $\alpha \in Q$,

(b) $\quad \alpha \simeq \beta \;\Rightarrow\; \Phi_f(\alpha) \simeq \Phi_f(\beta), \quad$ *for* $\alpha, \beta \in V$,

(c) $\quad \alpha$ *isolated* $\;\Rightarrow\; \Phi_f(\alpha)$ *isolated.* $\quad$ *for* $\alpha \in V$.

Theorem D2*. *For a combinatorial function f from ε^2 into ε,*

(a) $\quad |\alpha| = m \;\&\; |\beta| = n \;\Rightarrow\; |\Phi_f(\alpha, \beta)| = f(m, n), \quad$ *for* $\alpha, \beta \in Q$,

(b) $\quad \alpha \simeq \gamma \;\&\; \beta \simeq \delta \;\Rightarrow\; \Phi_f(\alpha, \beta) \simeq \Phi_f(\gamma, \delta), \quad$ *for* $(\alpha, \beta), (\gamma, \delta) \in V^2$,

(c) $\quad \alpha, \beta$ *isolated* $\;\Rightarrow\; \Phi_f(\alpha, \beta)$ *isolated,* $\quad$ *for* $(\alpha, \beta) \in V^2$.

Definition 6. For a combinatorial function f from ε into ε,

$$f_\Omega(A) = \text{Req}\ \Phi_f(\alpha), \quad \text{for } \alpha \in A,\ A \in \Omega.$$

Definition 7. For a combinatorial function f from ε^2 into ε,

$$f_\Omega(A, B) = \text{Req}\ \Phi_f(\alpha, \beta), \quad \text{for } \alpha \in A,\ \beta \in B,\ (A, B) \in \Omega^2.$$

Parts (b) of Theorems D2 and D2* tell us that f_Ω is well-defined, parts (c) guarantee that f_Ω maps Λ (or Λ^2) into Λ and parts (a) that f_Ω is an extension of f. We call f_Ω the *canonical extension* of f to Ω. Its restriction to Λ (or Λ^2) is called the *canonical extension* of f to Λ and is denoted by f_Λ.

10. Some metatheorems involving isols

Theorem D3. *Identities between recursive combinatorial functions carry over from ε to Λ, i.e.,*

$$f(x)=g(x) \text{ in } \varepsilon \Rightarrow f_\Lambda(X)=g_\Lambda(X) \text{ in } \Lambda.$$

Similarly for recursive combinatorial functions of two or more variables.

Examples. (a) $+$ and $\cdot$ are associative and commutative, $\cdot$ is distributive over $+$.

(b) Let $f(x, y)=(x+y)!/x!y!$ and $g(x)=x!$ We have seen in Section 9 that the recursive functions f and g are combinatorial. Write $X!$ for $g_\Lambda(X)$. Then

$$x!y!f(x, y)=(x+y)! \Rightarrow X!Y!f_\Lambda(X, Y)=(X+Y)!$$

In view of the cancellation law for multiplication this enables us to define

$$\binom{X+Y}{X}=\frac{(X+Y)!}{X!Y!}, \quad \text{for } X, Y \in \Lambda,$$

$$\binom{N}{K}=\frac{N!}{K!(N-K)!}, \quad \text{for } K \leqslant N,\ K, N \in \Lambda.$$

The basic properties of $C(n; k)$ carry over from ε to Λ, i.e., for $N, K \in \Lambda$,

$$\binom{N}{0}=\binom{N}{N}=1, \qquad \binom{N}{K}=\binom{N}{N-K}, \quad \text{for } K \leqslant N,$$

$$\binom{N+1}{K+1}=\binom{N}{K+1}+\binom{N}{K}, \quad \text{for } K+1 \leqslant N.$$

Theorem D4. *For recursive combinatorial functions $f(x)$ and $g(x)$,*

(a) *if $f(x)=g(x)$ has only finitely many solutions in ε, then $f_\Lambda(X)=g_\Lambda(x)$ has no new solutions in Λ;*

(b) *if $f(x)=g(x)$ has infinitely many solutions in ε, then $f_\Lambda(X)=g_\Lambda(X)$ has infinitely many new solutions in Λ.*

Similarly for recursive combinatorial functions of two or more variables.

Examples. The equation $2X=2Y+1$ has no solutions in ε, hence none in Λ. This can be rephrased as: *no isol is both even and odd.* Note the difference with cardinal arithmetic. There the notion of parity is not introduced, since $2\alpha = 2\alpha+1=\alpha$, for every infinite cardinal α. We shall prove in Section 11 that *there are isols which are neither even nor odd.* The equation $X^2=2Y^2$ has only the trivial solution $X=Y=0$ in ε, hence this is also its only solution in Λ. On the other hand, the Pell equation $X^2=2Y^2+1$ has infinitely many solutions in ε, hence infinitely many solutions in $\Lambda-\varepsilon$.

Theorem D5. *Let $f(x, y)$ be a recursive combinatorial function and $k \in \varepsilon$. If the cancellation law*

$$f(x, z) = f(y, z) \mathbin{\&} z \geqslant k \Rightarrow x = y$$

holds in ε, it also holds in Λ, i.e.,

$$f_\Lambda(X, Z) = f_\Lambda(Y, Z) \mathbin{\&} Z \geqslant k \Rightarrow X = Y.$$

Examples. (i) $X + Z = Y + Z \Rightarrow X = Y$;
(ii) $XZ = YZ \mathbin{\&} Z \geqslant 1 \Rightarrow X = Y$.

11. Myhill's category method

Myhill had always been interested in applications of the Baire category theorem to recursion theory; see [19]. Such methods proved invaluable in the theory of isols. In this section we shall show how the Baire category theorem can be used to obtain two of Myhill's results which he had previously obtained by other methods, namely:

(i) there are isols which are neither even nor odd;
(ii) the ring Λ^* has zero divisors.

The relevant definitions are as follows. An isol X is *even*, if $X = 2Y$, for some isol Y, while X is *odd*, if $X = 2Y + 1$, for some isol Y. We mentioned in Section 10 as an illustration of Theorem D4 that an isol cannot be both even and odd. The system $\langle \Lambda, +, \cdot \rangle$ can be extended to a ring in the same way as $\langle \varepsilon, +, \cdot \rangle$ can be extended to the ring $\langle \varepsilon^*, +, \cdot \rangle$ of integers. We denote this extension by Λ^*. It is called the ring of *isolic integers*.

Let again $\emptyset$ denote the empty set, V the class of all sets (i.e., of subsets of ε) and Q the class of all finite sets. We define

$$N(\alpha, \beta) = \{\xi \in V \mid \alpha \subset \xi \mathbin{\&} \xi \cap \beta = \emptyset\}, \quad \text{for } \alpha, \beta \in Q, \ \alpha \cap \beta = \emptyset,$$
$$\mathcal{B} = \{N(\alpha, \beta) \subset V \mid \alpha, \beta \in Q \mathbin{\&} \alpha \cap \beta = \emptyset\}.$$

A member of $\mathcal{B}$ is called a *neighborhood* (nbh), though a superclass of an nbh need not be an nbh. The family $\mathcal{B}$ has the property

$$U_1, U_2 \in \mathcal{B} \mathbin{\&} \eta \in U_1 \cap U_2 \Rightarrow \exists W[W \in \mathcal{B} \mathbin{\&} \eta \in W \mathbin{\&} W \subset U_1 \cap U_2]. \quad (*)$$

For assume the hypothesis, say $U_1 = N(\alpha_1, \beta_1)$ and $U_2 = N(\alpha_2, \beta_2)$. Then the finite sets $\alpha_1 \cup \alpha_2$ and $\beta_1 \cup \beta_2$ are disjoint because $\eta \in U_1 \cap U_2$. Then the nbh $W = N(\alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2)$ satisfies the requirements. Since $\mathcal{B}$ satisfies $(*)$, it is the basis of some topology in V. It is the only topology in V we shall use. V is a Hausdorff space. For let $\eta_1, \eta_2 \in V$ and $\eta_1 \neq \eta_2$; then $\eta_1 - \eta_2$ or $\eta_2 - \eta_1$ is nonempty and we may assume without limitation of generality that $\eta_1 - \eta_2$ is nonempty. Let $p \in \eta_1 - \eta_2$; then $N_1 = N((p), \emptyset)$ and $N_2 = N(\emptyset, (p))$ are disjoint nbhs with $\eta_1 \in N_1$ and $\eta_2 \in N_2$. A sequence $\langle \sigma_n \rangle$ *converges* to σ, if every nbh of σ contains almost all the sets $\sigma_0, \sigma_1, \ldots$ or equivalently, if for $p, q \in \varepsilon$,

$$p \in \sigma \Rightarrow \exists N \, \forall n [n > N \Rightarrow p \in \sigma_n]$$

and

$$q \notin \sigma \Rightarrow \exists N \, \forall n [n > N \Rightarrow q \notin \sigma_n].$$

Every combinatorial operator Φ from V into V is continuous, i.e.,

$$\langle \sigma_n \rangle \text{ converges to } \sigma \Rightarrow \langle \Phi(\sigma_n) \rangle \text{ converges to } \Phi(\sigma).$$

Let $S \subset V$. Then S is *nowhere dense*, if every nbh has a subnbh which is disjoint from S. The class S is *meager*, if it is the union of countably many nowhere dense sets. According to the Baire category theorem,

$$S \subset V \,\&\, S \text{ meager} \Rightarrow |V - S| = c.$$

Lemma M1. *The class of all supersets of some infinite set is nowhere dense.*

Proof. Let $\sigma \in V - Q$ and let S be the class of all supersets of σ. We wish to prove that

$$N \text{ nbh} \Rightarrow N \text{ has a subnbh disjoint from } S.$$

Assume the hypothesis, say $N = N(\alpha, \beta)$, where $\alpha, \beta \in Q$ and $\alpha \cap \beta = \emptyset$. Since α is finite and σ infinite, there is a number $p \in \sigma - \alpha$. We claim

(i) $N(\alpha, \beta \cup (p))$ is a nbh;
(ii) $N(\alpha, \beta \cup (p)) \subset N(\alpha, \beta)$;
(iii) $N(\alpha, \beta \cup (p))$ is disjoint from S.

Re (i). $\alpha \cap \beta$ is empty and $p \notin \alpha$.
Re (ii). Immediate.
Re (iii). Suppose there were a set $\xi_0 \in N(\alpha, \beta \cup (p)) \cap S$. Then

$$\xi_0 \in N(\alpha, \beta \cup (p)) \Rightarrow p \notin \xi_0,$$

$$\xi_0 \in S \,\&\, p \in \sigma \,\&\, \sigma \subset \xi_0 \Rightarrow p \in \xi_0.$$

Hence $N(\alpha, \beta \cup (p)) \cap S$ must be empty. □

Lemma M2. *Let f, g be combinatorial functions such that $f(x) \neq g(x)$, for infinitely many numbers x and let p be a one-one p.r. function. Put*

$$S = \{\xi \in V \mid \Phi_f(\xi) \subset \delta p \,\&\, p\Phi_f(\xi) = \Phi_g(\xi)\},$$

i.e.,

$$S = \{\xi \in V \mid \Phi_f(\xi) \simeq \Phi_g(\xi) \textit{ under } p\}.$$

Then S is nowhere dense.

Proof. Assume the hypothesis. Then $f(x)<g(x)$, for infinitely many x or $f(x)>g(x)$, for infinitely many x. We may assume without limitation of generality that $f(x)>g(x)$, for infinitely many x. Let $N(\alpha, \beta)$ be any nbh, hence α and β be disjoint finite sets. We wish to prove that $N(\alpha, \beta)$ has a subnbh disjoint from S.

Case 1. There is a finite set $\gamma \in N(\alpha, \beta)$ such that

$$\text{either not}[\Phi_f(\gamma) \subset \delta p] \quad \text{or} \quad (\Phi_f(\gamma) \subset \delta p \;\&\; \text{not}[p\Phi_f(\gamma) \subset \Phi_g(\varepsilon)]). \tag{11.1}$$

We claim that in this case (a) $N(\gamma, \beta) \subset N(\alpha, \beta)$, (b) $N(\gamma, \beta)$ disjoint from S.

Re (a). Let $\delta \in N(\gamma, \beta)$, i.e., $\gamma \subset \delta \;\&\; \delta \cap \beta = \emptyset$. Note that $\gamma \in N(\alpha, \beta)$ implies $\alpha \subset \gamma$. Thus $\alpha \subset \delta \;\&\; \delta \cap \beta = \emptyset$, i.e., $\delta \in N(\alpha, \beta)$.

Re (b). Let $\delta \in N(\gamma, \beta)$; then $\gamma \subset \delta$. If not$[\Phi_f(\delta) \subset \delta p]$, we have $\delta \notin S$ by the definition of S. Now assume $\Phi_f(\delta) \subset \delta p$. Then $\Phi_f(\gamma) \subset \delta p$, since $\gamma \subset \delta$ implies $\Phi_f(\gamma) \subset \Phi_f(\delta)$. By (11.1) the relation $\Phi_f(\gamma) \subset \delta p$ implies not$[p\Phi_f(\gamma) \subset \Phi_g(\varepsilon)]$. Taking into account that $\Phi_f(\gamma) \subset \Phi_f(\delta)$, we see that $p\Phi_f(\gamma) \subset p\Phi_f(\delta)$. Hence not$[p\Phi_f(\delta) \subset \Phi_g(\varepsilon)]$. This implies not$[p\Phi_f(\delta) = \Phi_g(\delta)]$, hence again $\delta \notin S$.

Case 2. There is no finite set $\gamma \in N(\alpha, \beta)$ such that (11.1) holds, hence

$$\forall \gamma[\gamma \in N(\alpha, \beta) \cap Q \Rightarrow \Phi_f(\gamma) \subset \delta p \;\&\; p\Phi_f(\gamma) \subset \Phi_g(\varepsilon)]. \tag{11.2}$$

Since $f(x)>g(x)$, for infinitely many x and the class $N(\alpha, \beta)$ contains finite sets of arbitrarily high cardinality, $N(\alpha, \beta)$ must contain a finite set γ with $f(|\gamma|)>g(|\gamma|)$, say $\gamma=\gamma^*$. Then

$$\Phi_f(\gamma^*) \subset \delta p \;\&\; p\Phi_f(\gamma^*) \subset \Phi_g(\varepsilon) \;\&\; f(|\gamma^*|)>g(|\gamma^*|). \tag{11.3}$$

Let $|\gamma^*|=s$; then $f(s)>g(s)$. Suppose $\langle c_i \rangle$ and $\langle d_i \rangle$ are associated with Φ_f and Φ_g, respectively, then (11.3) implies

$$p\{j(x, y) \in \varepsilon \mid \rho_x \subset \gamma \;\&\; y < c_{r(x)}\} \subset \{j(x, y) \in \varepsilon \mid \rho_x \subset \varepsilon \;\&\; y < d_{r(x)}\}. \tag{11.4}$$

If $\rho_{k(t)} \subset \gamma^*$, for all $t \in p\Phi_f(\gamma)$, (11.4) would imply the false statement $f(s) \leq g(s)$. Thus not$[\rho_{k(t)} \subset \gamma^*]$, for some t, say $t=t^*$. Let $z \in \rho_{k(t^*)} - \gamma^*$. Since $\gamma^* \in N(\alpha, \beta)$, it is disjoint from β, hence also from $\beta \cup (z)$. Thus $N(\gamma^*, \beta \cup (z))$ is an interval, in fact a subinterval of $N(\alpha, \beta)$. Let $\delta \in N(\gamma^*, \beta \cup (z))$ and suppose $\delta \in S$, i.e.,

$$\Phi_f(\delta) \subset \delta p \;\&\; p\Phi_f(\delta) = \Phi_g(\delta). \tag{11.5}$$

Since $\gamma^* \subset \delta$, hence $\Phi_f(\gamma^*) \subset \Phi_f(\delta)$, relation (11.5) implies

$$\Phi_f(\gamma^*) \subset \delta p \;\&\; p\Phi_f(\gamma^*) \subset \Phi_g(\varepsilon),$$

contrary to (11.3). Thus (11.5) is false, hence $\delta \notin S$. We proved that the subinterval $N(\gamma^*, \beta \cup (z))$ of $N(\alpha, \beta)$ is disjoint from S. □

Theorem S1 (Myhill). *Let $f_1, \ldots, f_n, g_1, \ldots, g_n$ be combinatorial functions such that $\{x \in \varepsilon \mid f_i(x) \neq g_i(x)\}$ is infinite for $1 \leq i \leq n$. Denote the canonical extensions* *of f_i and g_i to Λ by F_i and G_i, respectively, for $1 \leq i \leq n$. Then there is an infinite isol X such that $F_i(X) \neq G_i(X)$, for $1 \leq i \leq n$.*

Proof. We apply Lemma M1 letting the infinite set σ range over the class of all infinite r.e. sets and Lemma M2 letting p range over the family of all one-one p.r. functions. Define

$$S = \{\xi \in V \mid \xi \text{ is not isolated or there is an } i \leq n \text{ and a one-one p.r. function } p \text{ such that } \Phi_{f(i)}(\xi) \subset \delta p \ \& \ p\Phi_{f(i)}(\xi) = \Phi_{g(i)}(\xi)\}.$$

Then S is the union of denumerably many nowhere dense sets, i.e., S is meager. By the Baire category theorem the class $V - S$ is nonempty, in fact, has cardinality c. Let $\xi_0 \notin Q$, $\xi_0 \in V - S$ and $X_0 = \text{Req}\ \xi_0$. We claim that X_0 satisfies the requirements. For first of all the fact that $\xi_0 \notin S$ implies that ξ_0 has no infinite r.e. subset so that ξ_0 is isolated. Secondly, $\xi_0 \notin S$ implies that each of the n relations $\Phi_{f(i)}(\xi) \simeq \Phi_{g(i)}(\xi)$, for $1 \leq i \leq n$, is false. Thus $F_i(X_0) \neq G_i(X_0)$, for $1 \leq i \leq n$. □

Theorem S2 (Myhill–Nerode). *Let $f_1, \ldots, f_n, g_1, \ldots, g_n$, $\bar{f}$ and $\bar{g}$ be recursive combinatorial functions of k variables, say of $X_k = \langle x_1, \ldots, x_k \rangle$. If the sentence*

$$\forall i \leq n[f_i(X_k) = g_i(X_k)] \Rightarrow \bar{f}(X_k) = \bar{g}(X_k)$$

holds in ε, i.e., for all $X_k \in \varepsilon^k$, then it also holds in Λ, i.e., for all $X_k \in \Lambda^k$, when we replace each of the $2n+2$ functions $f_1, \ldots, f_n, g_1, \ldots, g_n, \bar{f}, \bar{g}$ by its canonical extension to Λ.

We do not prove this here, but note that a short elegant proof can be given along the lines of Nerode's proof of a similar theorem for arithmetic isols [24, (2.6)]. In Theorem S1 the functions $f_1, \ldots, f_n, g_1, \ldots, g_n$ need not be recursive, while in Theorem S2 we need the hypothesis that $f_1, \ldots, f_n, g_1, \ldots, g_n, \bar{f}, \bar{g}$ are recursive. Using Theorems S1 and S2 it is possible to obtain Nerode's conditions for the truth in Λ of a first-order universally quantified sentence whose atomic formulas are equations between recursive functions [22, Theorem 11.1]. It is too intricate to do this here. We shall content ourselves with using Theorems S1 and S2 to prove the statements (i) and (ii) mentioned in the beginning of this section.

Theorem S3. *There are isols which are neither even nor odd.*

Proof. Construct two recursive combinatorial functions f and g such that

$$\begin{cases} \forall x\ \forall y[x = 2y \Rightarrow f(x) = g(x)], \\ \forall x\ \forall y\ [x = 2y + 1 \Rightarrow f(x) + 1 = g(x)]. \end{cases} \tag{11.6}$$

This can be done using the technique used in the proof of [5, Theorem T3]. Then $\{x \in \varepsilon \mid f(x) \neq g(x)\}$ is an infinite set. By Theorem S1 there is an infinite isol X such that

$$f_\Lambda(X) \neq g_\Lambda(X) \quad \text{and} \quad f_\Lambda(X) + 1 \neq g_\Lambda(X). \tag{11.7}$$

Since (11.7) holds in ε, it also holds in Λ by Theorem S2. If X were even, we would have $f_\Lambda(X) = g_\Lambda(X)$. If X were odd, we would have $f_\Lambda(X) + 1 = g_\Lambda(X)$. We conclude by (11.7) that X is neither even nor odd. □

Theorem S4. *The ring Λ^* of isolic integers has zero-divisors.*

Proof. We use the recursive combinatorial functions f and g and the infinite isol X mentioned in the proof of Theorem S3. Put $U = f_\Lambda(X) - g_\Lambda(X)$; then $U \in \Lambda^*$. Moreover $U \notin (0, 1)$ by (11.7). Consider the two sentences

$$[g(x) - f(x)]^2 = g(x) - f(x), \tag{11.8}$$

$$g(x)^2 + f(x)^2 + f(x) = 2f(x)g(x) + g(x). \tag{11.9}$$

Sentence (11.8) is true in ε, since $g(x) - f(x) \in (0, 1)$, hence so is sentence (11.9). Using Theorem D3 of Section 10 we see that (11.9) is true in Λ, if we replace the recursive combinatorial functions f and g by their respective canonical extensions f_Λ and g_Λ to Λ. Thus $U^2 = U$, where $U \notin (0, 1)$, hence $U(U-1) = 0$, $U \neq 0$ and $U - 1 \neq 0$. We proved that U is a zero-divisor (and a nontrivial idempotent) of the ring Λ^*. □

12. The composition of combinatorial functions

Let f be a combinatorial function, c_i its associated function and Φ_f its normal operator, i.e.,

$$\Phi_f(\alpha) = \{j(x, y) \in \varepsilon \mid \rho_x \subset \alpha \;\&\; y < c_{r(x)}\}, \quad \text{for } \alpha \in V.$$

According to Theorem D2 of Section 9 we have

$$\alpha \simeq \beta \;\Rightarrow\; \Phi_f(\alpha) \simeq \Phi_f(\beta), \quad \text{for } \alpha, \beta \in V.$$

The canonical extension f_Ω of f to Ω is therefore well defined by

$$f_\Omega(A) = \text{Req } \Phi_f(\alpha), \quad \text{for } \alpha \in A,\; A \in \Omega,$$

even when the combinatorial function f is not recursive. We also have

$$A \leqslant B \;\Rightarrow\; f_\Omega(a) \leqslant f_\Omega(B), \quad \text{for } A, B \in \Omega.$$

Myhill [18, p. 375] stated that for $h = fg$,

$$\text{if } f \text{ and } g \text{ are recursive and combinatorial, so is } h, \tag{12.1}$$

and he knew that $h_\Omega = f_\Omega g_\Omega$ under the hypothesis of (12.1). However, the composition of two combinatorial functions f and g is combinatorial even when f and g are not recursive. This raises the question whether for $h = fg$,

$$\text{if } f \text{ and } g \text{ are combinatorial, then } h_\Omega = f_\Omega g_\Omega. \tag{12.2}$$

We shall show below that the answer is negative. It clearly suffices to prove the existence of combinatorial functions f and g such that $h_\Lambda \neq f_\Lambda g_\Lambda$, for $h = fg$.

Lemma M3. *Let f, g, h be combinatorial functions, $fg = h$, h a recursive, but not constant function, ξ an immune set and $\Phi_f \Phi_g(\xi) \simeq \Phi_h(\xi)$. Then f and g are recursive in ξ.*

Proof. This is [8, Lemma 2]. □

Corollary. *Under the same hypotheses for f, g and h we have: if $f_\Lambda g_\Lambda(X) = h_\Lambda(X)$, for every $X \in \Lambda$, then f and g are recursive.*

Proof. This is [8, Lemma 3]. □

Lemma M4. *There is a recursive function $t(n)$ such that $\langle \rho_{t(n)} \rangle$ is a sequence of mutually disjoint doubletons and if f is any choice function for $\langle \rho_{t(n)} \rangle$, i.e., $f(n) \in \rho_{t(n)}$, for all n, then f is combinatorial.*

Proof. This is proved in the proof of [13, Lemma 2.1]. □

We now give our counterexample. Let $t(x)$ be a recursive function with the properties listed in Lemma M4 and let δ be any nonrecursive set. Define the function f as follows: for $x \in \varepsilon$,

$$f(2x) = \max \rho_{t(2x)}, \qquad f(2x+1) = \begin{cases} \max \rho_{t(2x+1)}, & \text{if } x \in \delta, \\ \min \rho_{t(2x+1)}, & \text{if } x \notin \delta. \end{cases}$$

The function f is combinatorial by Lemma M4. Moreover, the function f and the set δ have the same degree, hence f is not recursive. Let $g(x) = 2x$, for $x \in \varepsilon$ and $h = fg$. Since $h(x) = \max \rho_{t(2x)}$, for $x \in \varepsilon$, the function h is recursive. Also, h is not a constant function and since both f and g are combinatorial, so is h. If $f_\Lambda g_\Lambda(X)$ were equal to $h_\Lambda(X)$, for every $X \in \Lambda$, the function f would be recursive by the Corollary of Lemma M3. However, f is not recursive, hence $f_\Lambda g_\Lambda(X) \neq h_\Lambda(X)$, for at least one isol X. We conclude that $f_\Lambda g_\Lambda \neq h_\Lambda$.

13. Combinatorial functions and $\Omega - \Lambda$

Let

$$f(n) = \sum_{i=0}^{n} c_i \binom{n}{i}$$

be a recursive combinatorial function Φ_f its normal operator and f_Ω its canonical extension. Although these extensions are a crucial tool for the study of Λ, they degenerate very badly on $\Omega - \Lambda$. Let us write R for Req ε, i.e., for the class of all infinite r.e. sets. Then

$$A \in \Omega - \Lambda \iff R \leqslant A, \quad \text{for } A \in \Omega, \tag{13.1}$$

since a set is not isolated iff it has an infinite recursive subset. For the definition of A^B see [6, Definition 64]. Some, but not all properties of R are similar to those of $\aleph_0$ in cardinal arithmetic:

$$R + n = n \cdot R = R^n = B, \quad \text{for } n \in \varepsilon,\ n \geqslant 1,\ 2^R = R^R = R.$$

Two other properties of R are according to [6, Corollary of Theorem 12 and Theorem 86],

$$A + B = A \iff BR \leqslant A, \quad \text{for } A, B \in \Omega, \tag{13.2}$$

$$AB = A \iff AB^R = A, \quad \text{for } A, B \in \Omega. \tag{13.3}$$

The cancellation laws for addition and multiplication fail in Ω, since $2 + R = 1 + R = R$ and $2 \cdot R = 1 \cdot R = R$. However, we do have

$$nA = nB \iff A = B, \quad \text{for } n \in \varepsilon,\ n > 0,\ A, B \in \Omega. \tag{13.4}$$

The nontrivial part of (13.4), i.e., the conditional from the left to the right is Friedberg's cancellation law [10]. According to [6, Theorem 113] we have

$$A(A-1) \cdot \ldots \cdot (A - K + 1) = k!\binom{A}{k}, \quad \text{for } 1 \leqslant k \leqslant A,\ A \in \Omega. \tag{13.5}$$

Since $A \in \Omega - \Lambda$ implies $A - i = A$, for $i \in \varepsilon$, relation (13.5) yields

$$A^k = k!\binom{A}{k}, \quad \text{for } k \in \varepsilon,\ A \in \Omega - \Lambda. \tag{13.6}$$

We write $A \mid B$ for $\exists X\,[A \cdot X = B]$. According to (13.6) $k!$ divides A^k, for $A \in \Omega - \Lambda$. In view of (13.4) we can therefore define $A^k/k!$ as the unique $X \in \Omega$ such that $A^k = k!X$. Hence by (13.6),

$$\binom{A}{k} = \frac{A^k}{k!}, \quad \text{for } k \in \varepsilon,\ A \in \Omega - \Lambda. \tag{13.7}$$

Theorem S5 (Myhill). *Let $f(n)$ be a recursive combinatorial function. Then f_Ω reduces on $\Omega - \Lambda$ to a function of one of the following three types*:

(i) *$f_\Omega(A) = c$, for some finite constant c,*

(ii) *$f_\Omega(A) = mA^n/n!$, for some finite constants m and n,*

(iii) $f_\Omega(A) = 2^A$.

Proof. Myhill did not publish a proof of this theorem, but he might have done it as follows. Let $f(n)$ be a recursive combinatorial function, c_i its associated function and $A \in \Omega - \Lambda$.

Case 1. $c_i = 0$, for $i > 0$. Then $f_\Omega(A) = c_0$ and f_Ω is of type (i).

Case 2. There is a number $n > 0$ such that $c_n > 0$ and $c_i = 0$, for $i > n$. Then we have

$$\begin{aligned} A^n = (A+R)^n &= A^n + \binom{n}{1} A^{n-1}R + \cdots + \binom{n}{n-1} AR^{n-1} + R^n \\ &= A^n + A^{n-1}R + \cdots + AR^{n-1} + R^n, \quad \text{since } pR = R, \ \text{for } p > 0, \\ &= A^n + A^{n-1}R + \cdots + AR^{n-1}, \end{aligned}$$

since $R^n = R$ and $A^n + R = A^n$, because $A^n \in \Omega - \Lambda$. Thus

$$A^n = A^n + R(A + A^2 + \cdots + A^{n-1}).$$

Hence $A^n \geqslant RA^i$, for $1 \leqslant i \leqslant n-1$, so that

$$A^n + A^i = A^n, \quad \text{for } 1 \leqslant i \leqslant n-1, \tag{13.8}$$

by (13.2). Let $p(x)$ be a polynomial in x of degree n, say

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad \text{for } a_0, \ldots, a_n \in \varepsilon, \ a_n > 0. \tag{13.9}$$

Relation (13.8) implies $A^n + a_i A^i = A^n$, for $1 \leqslant i \leqslant n$, hence (13.8) yields

$$\begin{aligned} p_\Omega(A) &= (a_n - 1)A^n + A^n + a_0 + a_1 A + \cdots + a_{n-1}A^{n-1} \\ &= (a_n - 1)A^n + A^n, \end{aligned} \tag{13.10}$$

$$p_\Omega(A) = a_n A^n.$$

Since

$$f(x) = c_0 + c_1 \binom{x}{1} + \cdots + c_n \binom{x}{n},$$

we can write $f(x)$ in the form $p(x)/n!$, where $p(x)$ is a polynomial in x of degree n. Thus (13.10) implies

$$f_\Omega(a) = \frac{c_n A^n}{n!}$$

and $f_\Omega(a)$ is of type (ii).

Case 3. $c_i > 0$, for infinitely many values of i. We write $\sigma \subset^* \tau$, for $\sigma \subset \tau \mathbin{\&} \sigma \mid \tau - \sigma$. Let $\alpha \in A$. Since

$$\begin{aligned} \Phi_f(\alpha) &= \{ j(x, y) \in \varepsilon \mid \rho_x \subset \alpha \mathbin{\&} y < c_{r(x)} \}, \\ j(2^\alpha \times \varepsilon) &= \{ j(x, y) \in \varepsilon \mid \rho_x \subset \alpha \mathbin{\&} y \in \varepsilon \}, \end{aligned}$$

we have

$$\Phi_f(\alpha) \subset^* j(2^\alpha \times \varepsilon),$$

so that

$$f_\Omega(A) \leqslant R \cdot 2^A. \tag{13.11}$$

Since $R = 2^R$ we have $R \cdot 2^A = 2^R \cdot 2^A = 2^{A+R} = 2^A$. Thus by (13.11)

$$f_\Omega(A) \leqslant 2^A. \tag{13.12}$$

Let b_x be the strictly increasing recursive function which ranges over the infinite recursive set $\{i \in \varepsilon \mid c_i > 0\}$, then $c_{b(0)}, c_{b(1)}, \ldots$ are all positive. If $c_0, c_1, \ldots$ are all positive, we have $b_x = x$, for all x, while if $c_0, c_1, \ldots$ are not all positive and $k = (\mu i)[c_i = 0]$, then

$$x \leqslant b_x, \quad \text{for all } x, \qquad x = b_x, \quad \text{for } x < k, \qquad \text{while } x < b_x, \quad \text{for } x \geqslant k. \tag{13.13}$$

Suppose $\sigma = (0, 2, \ldots)$ and $\tau = (1, 3, \ldots)$. We may assume without limitation of generality that $\alpha \subset \sigma$, for if not, we can replace α by $\{2x \in \varepsilon \mid x \in \alpha\}$. Recall that $r_x = |\rho_x|$. Define the function $g(x)$ as follows: $\delta g = 2^\sigma$ and for $x \in 2^\sigma$, i.e., $\rho_x \subset \sigma$, $g(x) = j(y, 0)$, where

$$\rho_y = \begin{cases} \rho_x, & \text{if } c_0, \ldots, c_{r(x)} \text{ are all positive,} \\ \rho_x \cup (1, 3, \ldots, 2(b_{r(x)} - r_x) - 1), & \text{if } 0 \in (c_0, \ldots, c_{r(x)}). \end{cases}$$

If the first clause applies, $b_{r(x)} = r_x$ and $r_y = r_x$, hence $r_y = b_{r(x)}$. If the second clause applies, $b_{r(x)} > r_x$ and ρ_y is obtained by adjoining the smallest $b_{r(x)} - r_x$ odd numbers to ρ_x. In that case $r_y = r_x + (b_{r(x)} - r_x) = b_{r(x)}$. Thus $r_y = b_{r(x)}$, no matter which clause applies. Since c_i is positive for every $i \in \rho b$ according to the definition of the function b, we conclude

$$x \in 2^\sigma \,\&\, g(x) = j(y, 0) \;\Rightarrow\; 0 < c_{r(y)},$$

hence

$$x \in 2^\alpha \,\&\, g(x) = j(y, 0) \;\Rightarrow\; 0 < c_{r(y)}. \tag{13.14}$$

We make two more claims concerning the function g, namely:

$$g \text{ is one-one and p.r.,} \tag{13.15}$$

$$g(2^\alpha) \subset^* \Phi_f(\alpha \cup \tau). \tag{13.16}$$

Re (13.15). The set $\delta g = 2^\sigma$ is recursive and given an element $x \in \delta g$ we can compute the number y such that $g(x) = j(y, 0)$, since the functions c_i, b_x and r_x are recursive. Thus g is a p.r. function. Now assume p, $q \in \delta g$ and $g(p) = j(y_p, 0)$, $g(q) = j(y_q, 0)$. Then

$$\begin{aligned} g(p) = g(q) \;&\Rightarrow\; j(y_p, 0) = j(y_q, 0) \;\Rightarrow\; y_p = y_q \;\Rightarrow\; \rho_{y(p)} = \rho_{y(q)} \\ &\Rightarrow\; \rho_{y(p)} \cap \sigma = \rho_{y(q)} \cap \sigma \;\Rightarrow\; \rho_p = \rho_q \;\Rightarrow\; p = q. \end{aligned}$$

The function g is therefore also one-one.

Re (13.16). Assume $x \in 2^\alpha$, i.e., $\rho_x \subset \alpha$. Then we have by (13.14)

$$g(x) = j(y, 0) \in \{j(y, z) \in \varepsilon \mid \rho_y \subset \alpha \cup \tau \,\&\, z < c_{r(y)}\},$$

hence $g(x) \in \Phi_f(\alpha \cup \tau)$. We proved that $g(2^\alpha) \subset \Phi_f(\alpha \cup \tau)$. Given any element $j(y, z) \in \Phi_f(\alpha \cup \tau)$, we can decide whether $z = 0$, hence $g(2^\alpha) \subset^* \Phi_f(\alpha \cup \tau)$.

From $2^\alpha \subset \delta g$ we conclude by (13.15) that $\mathrm{Req}\, g(2^\alpha) = \mathrm{Req}\, 2^\alpha = 2^A$. Moreover, since $\alpha \subset \sigma$ and $\sigma \mid \tau$, we have $\alpha \mid \tau$, hence $\mathrm{Req}(\alpha \cup \tau) = A + R$. Thus $\Phi_f(\alpha \cup \tau) = f_\Omega(A + R) = f_\Omega(A)$ and relation (13.16) implies

$$2^A \leqslant f_\Omega(a). \tag{13.17}$$

The $\leqslant$-relation in Ω is antisymmetric by Theorem C of Section 6. Hence relations (13.16) and (13.17) imply that $f_\Omega(A) = 2^A$, i.e., that f_Ω is of type (iii). □

14. Guide to the references

For Myhill's theorems A1 see [17, Theorem 18], B see [17, Theorem 19], A2 see [6, Corollary 2, p. 207], C see [6, Theorem 13(b)], D1, D1*, D2, D2*, D3, D4, D5, see [18, 20].

For Section 11 see [19], Section 12 see [8, 13], Section 13 see [20, 21]. The definitions and basic theorems about recursive functions are discussed in the books [3, 12, 26, 28, 29]. Creative sets and simple sets were introduced in Post's seminal paper [27]. This paper also led to the books by Davis [3], Rogers [28], Soare [29] and Odifreddi [26]. A unified treatment of isols is given in Nerode's paper [22] and McLaughlin's book [14]. If one replaces the family of all one-one partial recursive functions by the family of all one-one partial arithmetic functions and the class of all isolated sets by the class of all arithmetically isolated sets (i.e., sets with no infinite arithmetic subset), one can develop a theory of arithmetic isols [24]. For a probably complete list of publications on RETs up to 1985, see [16, section D 50, pp. 182–186].

References

[1] M. Boykan Pour-El, "Recursive isomorphism" and effectively extensible theories, Bull. Amer. Math. Soc. 71 (1965) 551–555.
[2] J.N. Crossely and A. Nerode, Combinatorial Functors (Springer, New York, 1974).
[3] M. Davis, Computability and Unsolvability (McGraw-Hill, New York, 1958).
[4] J.C.E. Dekker, A non-constructive extension of the number system, two abstracts, J. Symbolic Logic 20 (1955) 204–205.
[5] J.C.E. Dekker and E. Ellentuck, Isols and the pigeonhole principle, J. Symbolic Logic 54 (1989) 833–846.
[6] J.C.E. Dekker and J. Myhill, Recursive equivalence types, Univ. California Publ. in Math. (N.S.) 3 (1960) 67–214.
[7] E. Ellentuck, The universal properties of Dedekind finite cardinals, Ann. of Math. 82 (1965) 225–248.
[8] E. Ellentuck, Incompatible extensions of combinatorial functions, J. Symbolic Logic 48 (1983) 752–755.

[9] S. Feferman, Degrees of unsolvability associated with classes of formalized theories, J. Symbolic Logic 22 (1957) 161–175.
[10] R.M. Friedberg, The uniqueness of finite division for recursive equivalence types, Math. Z. 75 (1961) 3–7.
[11] W. Hanf, Model-theoretic methods in the study of elementary logic, in: J. Addison, L. Henkin and A. Tarski, eds., Symp. on the Theory of Models (North-Holland, Amsterdam, 1965) 132–145.
[12] S. C. Kleene, Introduction to Metamathematics (Van Nostrand, New York, 1952).
[13] T.G. McLaughlin, On the divergence of extension procedures in isol theory, Proc. Amer. Math. Soc. 83 (1981) 769–773.
[14] T.G. McLaughlin, Regressive Sets and the Theory of Isols (Marcel Dekker, New York, 1982).
[15] T.G. McLaughlin, Some properties of $\forall\exists$ models in the isols, Proc. Amer. Math. Soc. 97 (1986) 495–502.
[16] G.H. Müller and W. Lenski, Ω-Bibliography of Mathematical Logic 4 (Springer, Berlin, 1987).
[17] J. Myhill, Creative sets, Z. Math. Logik Grundlag. Math. 1 (1955) 97–108.
[18] J. Myhill, Recursive equivalence types and combinatorial functions, Part 1, Bull. Amer. Math. Soc. 64 (1958) 373–376.
[19] J. Myhill, Category methods in recursion theory, Pacific J. Math. 11 (1961) 1479–1486.
[20] J. Myhill, Recursive equivalence types and combinatorial functions, Part 2, Logic, methodology and philosophy of science, in: E. Nagel, P. Suppes and A. Tarski eds., Proc. 1960 Internat. Congress (Stanford Univ. Press, Stanford, CA, 1962) 46–55.
[21] J. Myhill, $\Omega - \Lambda$, in: Proc. Symp. on recursive function theory (Amer. Mathematical, Soc, Providence, RI, 1962) 97–104.
[22] A. Nerode, Extensions to isols, Ann. of Math. 73 (1961) 362–403.
[23] A. Nerode, Extensions to isolic integers, Ann. of Math. 75 (1962) 419–448.
[24] A. Nerode, Arithmetically isolated sets and non-standard models, in: Proc. Symp. on recursive function theory (Amer. Mathematical Soc., Providence, RI, 1962) 105–116.
[25] A. Nerode, Diophantine correct non-standard models in the isols, Ann. of Math. 84 (1966) 421–432.
[26] P.G. Odifreddi, Classical Recursion Theory, Stud. Logic Found. Math. 125 (North-Holland, Amsterdam, 1989).
[27] E.L. Post, Recursively enumerable sets of positive integers and their decision problems, Bull. Amer. Math. Soc. 50 (1944) 284–316.
[28] H. Rogers, Theory of Recursive Functions and Effective Computability (McGraw-Hill, New York, 1967).
[29] R.I. Soare, Recursively Enumerable Sets and Degrees (Springer, New York, 1987).