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Inverse nodal problem for differential pencils

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ABSTRACT

An inverse nodal problem is studied for the diffusion operator with real-valued coefficients on a finite interval with Dirichlet boundary conditions. The oscillation of the eigenfunctions corresponding to large modulus eigenvalues is established and an asymptotic of the nodal points is obtained. The uniqueness theorem is proved and a constructive procedure for solving the inverse problem is given.

1. Introduction

Inverse nodal problems consist in recovering operators from given nodes (zeros) of their eigenfunctions. From the physical point of view this corresponds to finding, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. McLaughlin seems to be the first to consider this sort of inverse problem (see [1]). Later on, some remarkable results were obtained. For example, X. F. Yang got the uniqueness for general boundary conditions using the same method as McLaughlin (see [2]); C. K. Law and Ching-Fu Yang (see [3]) have reconstructed the potential function and its derivatives from nodal data. Besides, the readers can refer to [4–9] (see also the references therein). In the references cited above, the inverse nodal problems were studied for second-order differential equations with a linear dependence on the spectral parameter. In the present paper we investigate the inverse nodal problem for the differential pencil $L = L(q_0(x), q_1(x))$ of the form

$$y'' + (\rho^2 - 2\rho q_1(x) - q_0(x))y = 0, \quad 0 < x < 1, \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where ρ is a spectral parameter and $q_m(x) \in W_1^m[0, 1]$, $m = 1, 2$, are real-valued functions. Differential equations with a nonlinear dependence on the spectral parameter frequently appear in mathematics as well as in applications, for example, the diffusion operator (see [10–21] for details). First we prove that, being numbered in a natural way, the nth eigenfunction of L (here $n \in \mathbb{Z} \setminus \{0\}$) has exactly $|n| - 1$ nodes in the interval $(0, 1)$ for sufficiently large $|n|$, which is an analog of the classical Sturm's oscillation theorem for the Sturm–Liouville operator. Further, we study the inverse problem of recovering L from the nodal points. Note that for any $C \equiv$ const the modified pencil

$$L_C = L(q_0(x) + 2Cq_1(x) - C^2, q_1(x) - C)$$

possesses the same eigenfunctions as L does. Indeed, the pencil L_C is obtained from L by the shift of the spectral parameter $\rho \to \rho + C$. Below (see Theorem 4) we prove that it is a unique modification of L leaving the nodal points unchanged,

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provided that $q_1(x)$ is not a constant. Otherwise one deals with two-parametric set of pencils $L(q_0(x) + C_0, C_1)$ possessing the same nodal points. In what follows without loss of generality we assume that

$$\omega_0 := \int_0^1 q_1(x)\,dx = 0 \tag{3}$$

and also we exclude the Sturm–Liouville operator ($q_1(x) \equiv 0$) from the consideration, i.e.

$$q_1(x) \neq \text{const.} \tag{4}$$

Under these assumptions we prove the uniqueness of recovering the functions $q_0(x),\ q_1(x)$ from a dense set of nodal points and obtain a constructive procedure for solving the inverse nodal problem. The analogous results can be obtained also for other types of boundary conditions (2). We note that in [15] the uniqueness of recovering the function $q_0(x) - \int_0^1 q_0(t)\,dt$ from the nodal points was studied, provided that $q_1(x)$ was known a priori.

There are some connections between inverse nodal problems and the classical inverse spectral theory (see [22–28]). We note that for differential pencils some aspects of inverse spectral problems were studied in [10,11,13,17,19–21] and other works.

In the next section we investigate the oscillation of the eigenfunctions and derive a detailed asymptotic formula for the nodal points. In Section 3 we prove the uniqueness theorem and point out an algorithm for solving the inverse nodal problem. The main results of the paper are contained in Theorems 3 and 4.

2. Oscillation theorem. Asymptotics of the nodal points

Let $S(x, \rho)$ be a solution of Eq. (1) satisfying the initial conditions $S(0, \rho) = 0$, $S'(0, \rho) = 1$. The eigenvalues of L coincide with the zeros of its characteristic function $\Delta(\rho) := S(1, \rho)$. Denote

$$Q(x) := \int_0^x q_1(t)\,dt. \tag{5}$$

Using the standard approach (see, e.g., [23]) one can establish the asymptotics

$$S(x, \rho) = \frac{\sin(\rho x - Q(x))}{\rho} + \xi(x, \rho), \tag{6}$$

where

$$\xi^{(\nu)}(x, \rho) = O\left(\frac{1}{\rho^{2-\nu}} \exp(|\mathrm{Im}\rho| x)\right), \quad \nu = 0, 1,\ |\rho| \to \infty, \tag{7}$$

uniformly with respect to $x \in [0, 1]$. According to (6) and (7) we have

$$\Delta(\rho) = \frac{\sin \rho}{\rho} + O\left(\frac{1}{\rho^2} \exp(|\mathrm{Im}\rho|)\right), \quad |\rho| \to \infty. \tag{8}$$

By the well-known method (see, e.g., [23]) using (8) and Rouché's theorem one can prove that L has infinitely many eigenvalues $\rho_n,\ n \in \mathbb{Z} \setminus \{0\}$, of the form

$$\rho_n = \pi n + O\left(\frac{1}{n}\right), \quad |n| \to \infty. \tag{9}$$

Theorem 1. *For sufficiently large $|n|$ the eigenfunction $y_n(x) := S(x, \rho_n)$ has exactly $|n| - 1$ zeros x_n^j in the interval $(0, 1)$:*

$$0 < x_n^1 < x_n^2 < \cdots < x_n^{n-1} < 1 \quad \text{for } n > 0$$

and

$$0 < x_n^{-1} < x_n^{-2} < \cdots < x_n^{n+1} < 1 \quad \text{for } n < 0.$$

Moreover,

$$x_n^j = \frac{j}{n} + \frac{Q(\frac{j}{n})}{\pi n} + O\left(\frac{1}{n^2}\right), \quad |n| \to \infty, \tag{10}$$

uniformly with respect to j.

Proof. First we note that ρ_n are real for sufficiently large $|n|$. Indeed, according to (9) for large $|n|$ in the domain $D_n := \{\rho : |\rho - \pi n| \le 1\}$ there is exactly one eigenvalue ρ_n. Taking into account the real-valuedness of $q_0(x),\ \ q_1(x)$ we conclude that there is also an eigenvalue $\overline{\rho_n} \in D_n$, and hence $\rho_n = \overline{\rho_n}$. Therefore, the functions $y_n(x)$ are real-valued for large $|n|$. Substituting (9) in (6) we arrive at

$$\rho_n y_n(x) = \sin(\pi n x - Q(x)) + \varepsilon_n(x), \tag{11}$$

where in view of (7)

$$\varepsilon_n^{(\nu)}(x) = O\left(\frac{1}{n^{1-\nu}}\right), \quad \nu = 0, 1, \ |n| \to \infty, \tag{12}$$

uniformly on [0, 1]. Consider on (0, 1) the equation $y_n(x) = 0$, which according to (11) and (12) is equivalent for large $|n|$ to the aggregate of equations

$$x = \chi_n^j(x) := \frac{j}{n} + \frac{Q(x)}{\pi n} + \varepsilon_n^j(x), \quad j \in \mathbb{Z}, \tag{13}$$

where $\varepsilon_n^j(x) = (-1)^{j+1}\frac{\arcsin \varepsilon_n(x)}{\pi n}$. The estimates (12) give

$$(\varepsilon_n^j)^{(\nu)}(x) = O\left(\frac{1}{n^{2-\nu}}\right), \quad \nu = 0, 1, \ |n| \to \infty, \tag{14}$$

uniformly for $j \in \mathbb{Z}$ and $x \in [0, 1]$. Put $q_1(x) = 0$ on $(-\infty, 0) \cup (1, +\infty)$ and continue $\varepsilon_n^j(x)$ on $(-\infty, 0) \cup (1, +\infty)$ by differentiability in any way to satisfy (14) uniformly in $x \in \mathbb{R}$ and $j \in \mathbb{Z}$. For example one can take $\varepsilon_n^j(x) = 0$ on $(-\infty, -\frac{1}{|n|}] \cup [1 + \frac{1}{|n|}, +\infty)$ and

$$\varepsilon_n^j(x) = \begin{cases} |n|^3 \left(x + \dfrac{1}{|n|}\right)^2 \left(\varepsilon_n^j(0)\left(\dfrac{1}{|n|} - 2x\right) + (\varepsilon_n^j)'(0)\dfrac{x}{|n|}\right), & x \in \left(-\dfrac{1}{|n|}, 0\right), \\ n^2 \left(x - 1 - \dfrac{1}{|n|}\right)^2 \left(\varepsilon_n^j(1)\left(1 + 2|n|(x-1)\right) + (\varepsilon_n^j)'(1)(x-1)\right), & x \in \left(1, 1 + \dfrac{1}{|n|}\right). \end{cases}$$

Consider Eq. (13) in $\mathbb{R}$. According to (14) and the formula

$$\chi_n^j(x_1) - \chi_n^j(x_2) = \frac{1}{\pi n}\int_{x_2}^{x_1} q_1(t)\,dt + (\varepsilon_n^j)'(\theta)(x_1 - x_2), \quad \theta \in (x_1, x_2),$$

there exists n_0 such that for $|n| \geq n_0$ the function $\chi_n^j(x)$ is a contracting mapping in $\mathbb{R}$ for all $j \in \mathbb{Z}$. Let $|n| \geq n_0$. Thus, for each $j \in \mathbb{Z}$ Eq. (13) has a unique solution in $\mathbb{R}$, which we denote by x_n^j. Substituting x_n^j in (13) we arrive by (14) at the formula

$$x_n^j = \frac{j}{n} + \frac{Q(x_n^j)}{\pi n} + O\left(\frac{1}{n^2}\right), \quad |n| \to \infty, \tag{15}$$

uniformly with respect to $j \in \mathbb{Z}$. In particular, we have

$$x_n^j = \frac{j}{n} + O\left(\frac{1}{n}\right), \quad |n| \to \infty,$$

uniformly in j. Substituting this on the right-hand side of (15) we arrive at (10), which, in turn, gives

$$x_n^{j+1} - x_n^j = \frac{1}{n} + O\left(\frac{1}{n^2}\right), \quad |n| \to \infty,$$

uniformly in j. Consequently, for large $|n|$ we have $x_n^j < x_n^{j+1}$ for positive n and $x_n^j > x_n^{j+1}$ for negative n. For $j = 0, \pm 1, n, n \pm 1$ formula (10) gives

$$x_n^{-1} = -\frac{1}{n} + O\left(\frac{1}{n^2}\right), \qquad x_n^0 = O\left(\frac{1}{n^2}\right), \qquad x_n^1 = \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$
$$x_n^{n-1} = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right), \qquad x_n^n = 1 + O\left(\frac{1}{n^2}\right), \qquad x_n^{n+1} = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Thus, according to the boundary conditions (2) and the order of x_n^j we conclude that $x_n^0 = 0$, $x_n^n = 1$ for large $|n|$. Hence, exactly $|n| - 1$ zeros lie in (0, 1), namely: x_n^j, $j = \overline{1, n-1}$, for positive n and x_n^j, $j = \overline{n+1, -1}$, for negative n. □

Corollary 1. *From (10) it follows that the set X of all nodal points is dense in* [0, 1].

For convenience we agree that in what follows $x_n^0 = 0,\ x_n^n = 1$. For solving the inverse nodal problem we need a more detailed asymptotics of the nodal points, which is established in the following theorem.

Theorem 2. *In the representation*

$$x_n^j = \frac{j}{n} + \frac{Q(x_n^j)}{\pi n} + \delta_n^j \tag{16}$$

the term δ_n^j has the form

$$\delta_n^j = \frac{1}{2(\pi n)^2}\left(\int_0^{x_n^j}(q_0(t)+q_1^2(t))\,dt - \omega_1 x_n^j - (A_n^j - A_n^n x_n^j)\right) + \frac{1}{2(\pi n)^3}\left(\int_0^{x_n^j}(q_0(t)+q_1^2(t))q_1(t)\,dt - \omega_2 x_n^j\right) + o\left(\frac{1}{n^3}\right), \quad |n|\to\infty, \tag{17}$$

uniformly with respect to j, where

$$\omega_1 = \int_0^1 (q_0(t)+q_1^2(t))\,dt, \qquad \omega_2 = \int_0^1 (q_0(t)+q_1^2(t))q_1(t)\,dt,$$
$$A_n^j = \int_0^{x_n^j}(q_0(t)+q_1^2(t))\cos(2\pi nt - 2Q(t))\,dt - \int_0^{x_n^j} q_1'(t)\sin(2\pi nt - 2Q(t))\,dt. \tag{18}$$

Proof. Let us first calculate a more detailed asymptotics of the spectrum. In (6) we have more precisely

$$\begin{aligned}\xi(x,\rho) = {}& \frac{1}{2\rho^2}\Bigg\{(q_1(0)+q_1(x))\sin(\rho x - Q(x)) - \cos(\rho x - Q(x))\int_0^x (q_0(t)+q_1^2(t))\,dt \\ &+ \int_0^x (q_0(t)+q_1^2(t))\cos(\rho(x-2t) - Q(x) + 2Q(t))\,dt + \int_0^x q_1'(t)\sin(\rho(x-2t) - Q(x)+2Q(t))\,dt\Bigg\} \\ &+ \frac{1}{4\rho^3}\Bigg\{\left[q_1^2(0) + q_1^2(x) + \frac{(q_1(0)+q_1(x))^2}{2} - \frac{1}{2}\left(\int_0^x (q_0(t)+q_1^2(t))\,dt\right)^2\right]\sin(\rho x - Q(x)) \\ &- \cos(\rho x - Q(x))\int_0^x (q_0(t)+q_1^2(t))(q_1(0)+q_1(x)+2q_1(t))\,dt\Bigg\} + o\left(\frac{1}{\rho^3}\exp(|\mathrm{Im}\rho x|)\right), \quad |\rho|\to\infty,\end{aligned} \tag{19}$$

uniformly in $x\in[0,1]$. This gives

$$\begin{aligned}\Delta(\rho) = {}& \frac{\sin\rho}{\rho} + \frac{1}{2\rho^2}\Bigg\{(q_1(0)+q_1(1))\sin\rho - \omega_1\cos\rho + \int_0^1 (q_0(t)+q_1^2(t))\cos(\rho(1-2t)+2Q(t))\,dt \\ &+ \int_0^1 q_1'(t)\sin(\rho(1-2t)+2Q(t))\,dt\Bigg\} + \frac{1}{4\rho^3}\Bigg\{\left[q_1^2(0)+q_1^2(1) + \frac{(q_1(0)+q_1(1))^2 - \omega_1^2}{2}\right]\sin\rho \\ &- [(q_1(0)+q_1(1))\omega_1 + 2\omega_2]\cos\rho\Bigg\} + o\left(\frac{1}{\rho^3}\exp(|\mathrm{Im}\rho|)\right), \quad |\rho|\to\infty.\end{aligned} \tag{20}$$

Substituting (9) in (20) and taking $\Delta(\rho_n) = 0$ into account we arrive at

$$\rho_n = \pi n + \frac{\omega_1 - A_n^n}{2\pi n} + \frac{\omega_2}{2(\pi n)^2} + o\left(\frac{1}{n^2}\right), \quad |n|\to\infty. \tag{21}$$

Substituting (21) in (6), (19) we get

$$\begin{aligned}\rho_n y_n(x) = {}& \sin(\pi nx - Q(x)) \\ &- \frac{1}{2\pi n}\Bigg\{\left[\int_0^x (q_0(t)+q_1^2(t))\,dt - \omega_1 x\right]\cos(\pi nx - Q(x)) - [q_1(0)+q_1(x)]\sin(\pi nx - Q(x)) \\ &- \Bigg[\int_0^x (q_0(t)+q_1^2(t))\cos(\pi n(x-2t) - Q(x) + 2Q(t))\,dt \\ &+ \int_0^x q_1'(t)\sin(\pi n(x-2t) - Q(x) + 2Q(t))\,dt - A_n^n x\cos(\pi nx - Q(x))\Bigg]\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{(2\pi n)^2}\Bigg\{\Bigg[(q_1(0)+q_1(x))\left(\int_0^x (q_0(t)+q_1^2(t))\,\mathrm{d}t-\omega_1 x\right)\\
&+2\int_0^x (q_0(t)+q_1^2(t))q_1(t)\,\mathrm{d}t-2\omega_2 x\Bigg]\cos(\pi n x-Q(x))\\
&-\Bigg[q_1^2(0)+q_1^2(x)+\omega_1 x\int_0^x (q_0(t)+q_1^2(t))\,\mathrm{d}t+\frac{(q_1(0)+q_1(x))^2-\omega_1^2x^2}{2}\\
&-\frac{1}{2}\left(\int_0^x (q_0(t)+q_1^2(t))\,\mathrm{d}t\right)^2\Bigg]\sin(\pi n x-Q(x))\Bigg\}+o\left(\frac{1}{n^2}\right),\quad |n|\to\infty,
\end{aligned}\tag{22}
$$

uniformly in $x\in[0,1]$. Substituting (16) in (22) and taking (15), $S(x_n^j,\rho_n)=0$ into account we get (17). □

3. Uniqueness theorem. Solution of the inverse nodal problem

Consider the following inverse problem.

Problem 1. Given the set of nodal points X, find the functions $q_0(x),\ q_1(x)$.

Here and below the notion "set of nodal points" is understood with account of their indices. In other words, $\{x_n^j\}_{(n,j)\in I}=\{\tilde{x}_n^j\}_{(n,j)\in\tilde{I}}$ if and only if $I=\tilde{I}$ and $x_n^j=\tilde{x}_n^j$ for all $(n,j)\in I$.

Denote

$$
\tilde{A}_n^j=\int_0^{x_n^j}(\tilde{p}(t)+q_1^2(t))\cos(2\pi nt-2Q(t))\,\mathrm{d}t-\int_0^{x_n^j}q_1'(t)\sin(2\pi nt-2Q(t))\,\mathrm{d}t,
$$

where

$$
\tilde{p}(x)=q_0(x)-\int_0^1 q_0(t)\,\mathrm{d}t.
$$

According to (18) we have

$$
A_n^j=\tilde{A}_n^j+o\left(\frac{1}{n}\right),\quad |n|\to\infty,
$$

uniformly in j. Thus, the following assertion is an immediate corollary of Theorem 2.

Lemma 1. *Fix $x\in[0,1]$. Choose $\{j_n\}$ such that $x_n^{j_n}\to x$ as $|n|\to\infty$. Then there exist finite limits and the corresponding equalities hold:*

$$
Q(x)=\pi\lim_{|n|\to\infty}(nx_n^{j_n}-j_n),\tag{23}
$$

$$
f(x):=2\pi\lim_{|n|\to\infty}n(\pi(nx_n^{j_n}-j_n)-Q(x_n^{j_n})),\tag{24}
$$

$$
g(x):=\pi\lim_{|n|\to\infty}n(2\pi n(\pi nx_n^{j_n}-\pi j_n-Q(x_n^{j_n}))-f(x_n^{j_n})+\tilde{A}_n^{j_n}-\tilde{A}_n^n x_n^{j_n})\tag{25}
$$

and

$$
f(x)=\int_0^x(q_0(t)+q_1^2(t))\,\mathrm{d}t-x\int_0^1(q_0(t)+q_1^2(t))\,\mathrm{d}t,\tag{26}
$$

$$
g(x)=\int_0^x(q_0(t)+q_1^2(t))q_1(t)\,\mathrm{d}t-x\int_0^1(q_0(t)+q_1^2(t))q_1(t)\,\mathrm{d}t.\tag{27}
$$

Let us prove the uniqueness theorem for the solution of the inverse nodal problem.

Theorem 3. *The specification of any dense subset $X_0\subset X$ uniquely determines the functions $q_0(x),\ q_1(x)$, which can be found by the following algorithm.*

Algorithm 1. Let a dense subset X_0 of the nodal points be given. Then

(i) for each $x\in[0,1]$ choose a sequence $\{x_n^{j_n}\}\subset X_0$ such that $x_n^{j_n}\to x$ as $|n|\to\infty$;
(ii) find the function $Q(x)$ via (23) and calculate

$$
q_1(x)=Q'(x);\tag{28}
$$

(iii) calculate $f(x)$ by formula (24) and determine

$$p(x) = f'(x) - q_1^2(x) + \int_0^1 q_1^2(t)\,dt; \tag{29}$$

(iv) fix an arbitrary $x \in [0, 1]$ such that $Q(x) \neq 0$, find $g(x)$ via (25) and put

$$A = \frac{1}{Q(x)}\left(g(x) - \int_0^x (p(t) + q_1^2(t))q_1(t)\,dt + x\int_0^1 (p(t) + q_1^2(t))q_1(t)\,dt\right); \tag{30}$$

(v) finally, calculate the function $q_0(x)$ by the formula

$$q_0(x) = p(x) + A. \tag{31}$$

Proof. Formula (28) follows from (5). Further, differentiating (26) and comparing with (29) we obtain

$$q_0(x) = p(x) + \int_0^1 q_0(t)\,dt.$$

Substituting this in (27) and taking (3) into account we get

$$g(x) = \int_0^x (p(t) + q_1^2(t))q_1(t)\,dt - x\int_0^1 (p(t) + q_1^2(t))q_1(t)\,dt + Q(x)\int_0^1 q_0(t)\,dt.$$

According to (4) there exists $x \in [0, 1]$ such that $Q(x) \neq 0$. Comparing the latter relation with (30) we arrive at

$$A = \int_0^1 q_0(t)\,dt$$

and hence formula (31) holds. □

Let us point out an alternative algorithm using the notion of nodal length $l_n^j := x_n^{j+1} - x_n^j$ analogously to the case $q_1(x) \equiv 0$ (see [7]), which allows one to approximate $q_0(x)$, $q_1(x)$ directly, i.e. not via their primitive functions. Consider the step-function $j_n(x)$:

$$j_n(x) = \begin{cases} 0, & x = 0, \\ \max\limits_{x_n^j < x} j, & x \in (0, 1], \end{cases} \quad \text{for } n > 0, \qquad j_n(x) = \begin{cases} -1, & x = 0, \\ \max\limits_{x_n^j \geq x} j, & x \in (0, 1], \end{cases} \quad \text{for } n < 0.$$

Clearly, $x_n^{j_n(x)} \to x$ as $|n| \to \infty$. First we prove the following assertion.

Lemma 2. *There exists a finite limit and the corresponding equality holds:*

$$q_1(x) = \pi \lim_{|n|\to\infty} n(nl_n^{j_n(x)} - 1). \tag{32}$$

Moreover, for almost all $x \in (0, 1)$ there exists a finite limit

$$v(x) := 2\pi \lim_{|n|\to\infty} n^2 \left(\pi n l_n^{j_n(x)} - \pi - \int_{x_n^{j_n(x)}}^{x_n^{j_n(x)+1}} q_1(t)\,dt\right) \tag{33}$$

and

$$v(x) = q_0(x) - \int_0^1 q_0(t)\,dt + q_1^2(x) - \int_0^1 q_1^2(t)\,dt. \tag{34}$$

Proof. According to (16) and (17) we have

$$l_n^j = \frac{1}{n} + \frac{1}{\pi n}\int_{x_n^j}^{x_n^{j+1}} q_1(t)\,dt + o\left(\frac{1}{n^2}\right), \quad n = \frac{1}{l_n^j}\left(1 + o\left(\frac{1}{n}\right)\right).$$

Hence,

$$\pi n(nl_n^j - 1) = n\int_{x_n^j}^{x_n^{j+1}} q_1(t)\,dt + o(1) = q_1(\xi_n^j) + o(1), \quad \xi_n^j \in (x_n^j, x_n^{j+1}).$$

Thus, if $x_n^j \to x$, then $q_1(\xi_n^j) \to q_1(x)$, and (32) is proved. Further, as in [7] we obtain that for any function $f(x) \in L(0, 1)$

$$\int_{x_n^{j_n(x)}}^{x_n^{j_n(x)+1}} f(t)\cos(2\pi nt)\,dt, \quad \int_{x_n^{j_n(x)}}^{x_n^{j_n(x)+1}} f(t)\sin(2\pi nt)\,dt = o\left(\frac{1}{n}\right),$$

and hence $n(A_n^{j_n(x)+1} - A_n^{j_n(x)}) = o(1)$. Therefore, (16) and (17) yield

$$l_n^{j_n(x)} = \frac{1}{n} + \frac{1}{\pi n}\int_{x_n^{j_n(x)}}^{x_n^{j_n(x)+1}} q_1(t)\,dt + \frac{1}{2(\pi n)^2}\int_{x_n^{j_n(x)}}^{x_n^{j_n(x)+1}} (q_0(t) + q_1^2(t) - \omega_1)\,dt + o\left(\frac{1}{n^3}\right).$$

Since

$$\frac{1}{l_n^j} = n + o\left(\frac{1}{n}\right)$$

and, for all Lebesgue points of $f(x) \in L(0, 1)$,

$$\frac{1}{l_n^{j_n(x)}}\int_{x_n^{j_n(x)}}^{x_n^{j_n(x)+1}} f(t)\,dt = f(x) + o(1),$$

we have

$$n\int_{x_n^{j_n(x)}}^{x_n^{j_n(x)+1}} f(t)\,dt = f(x) + o(1).$$

Hence, we get (33) and (34). □

Thus, we arrive at the following alternative algorithm for solving Problem 1, which unlike the first one contains no differentiation.

Algorithm 2. Let the set of nodal points X be given. Then

(i) find the function $q_1(x)$ via (32);
(ii) calculate $v(x)$ by (33) and find the function

$$p(x) = v(x) - q_1^2(x) + \int_0^1 q_1^2(t)\,dt;$$

(iii) find the mean value $A = \int_0^1 q_0(t)\,dt$ as is done in Algorithm 1 and calculate the function $q_0(x)$ by (31).

Finally, let us give a generalization of the uniqueness theorem for the case when (3) is not assumed any more while (4) being kept. For this purpose together with L we consider the pencil $\tilde{L} = L(\tilde{q}_0(x), \tilde{q}_1(x))$ with, generally speaking, other functions $\tilde{q}_0(x)$, $\tilde{q}_1(x)$. We agree that if a certain symbol α denotes an object related to L then the same symbol with tilde $\tilde{\alpha}$ denotes the analogous object corresponding to $\tilde{L}$, and $\hat{\alpha} := \alpha - \tilde{\alpha}$.

Theorem 4. *If* $X_0 = \tilde{X}_0$ *then*

$$q_1(x) = \tilde{q}_1(x) + \hat{\omega}_0, \qquad q_0(x) = \tilde{q}_0(x) - 2\hat{\omega}_0\tilde{q}_1(x) - \hat{\omega}_0^2. \tag{35}$$

Proof. Note that the pencils

$$L_1 = L(q_0(x) + 2\omega_0 q_1(x) - \omega_0^2, q_1(x) - \omega_0), \qquad \tilde{L}_1 = L(\tilde{q}_0(x) + 2\tilde{\omega}_0\tilde{q}_1(x) - \tilde{\omega}_0^2, \tilde{q}_1(x) - \tilde{\omega}_0)$$

possess the same nodal points as L, $\tilde{L}$ do respectively. Since

$$\int_0^1 (q_1(t) - \omega_0)\,dt = \int_0^1 (\tilde{q}_1(t) - \tilde{\omega}_0)\,dt = 0,$$

according to Theorem 3 we have

$$q_1(x) - \omega_0 = \tilde{q}_1(x) - \tilde{\omega}_0, \qquad q_0(x) + 2\omega_0 q_1(x) - \omega_0^2 = \tilde{q}_0(x) + 2\tilde{\omega}_0\tilde{q}_1(x) - \tilde{\omega}_0^2,$$

which is equivalent to (35). □

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