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PI degree parity in q-skew polynomial rings ☆

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Abstract

For k a field of arbitrary characteristic, and R a k-algebra, we show that the PI degree of an iterated skew polynomial ring $R[x_1;\tau_1,\delta_1]\cdots[x_n;\tau_n,\delta_n]$ agrees with the PI degree of $R[x_1;\tau_1]\cdots[x_n;\tau_n]$ when each (τ_i,δ_i) satisfies a q_i-skew relation for $q_i \in k^\times$ and extends to a higher q_i-skew τ_i-derivation. We confirm the quantum Gel'fand–Kirillov conjecture for various quantized coordinate rings, and calculate their PI degrees. We extend these results to completely prime factor algebras.

Keywords: Noncommutative rings; Skew polynomial rings; Quantum algebras

1. Introduction

Presented here is a new technique for analyzing skew polynomial rings satisfying a polynomial identity with an eye toward discovering their PI degrees. It combines and extends the methods of Jøndrup [16] and Cauchon [5], who introduced techniques of deleting derivations in skew polynomial rings, by means of which they showed that some properties of certain types of iterated skew polynomial ring $A = k[x_1][x_2;\tau_2,\delta_2]\cdots[x_n;\tau_n,\delta_n]$ are determined by the corresponding ring $A' = k[x_1][x_2;\tau_2]\cdots[x_n;\tau_n]$. Jøndrup's results imply that A and A' have the same PI degree under certain hypotheses, including characteristic zero for the base field. Cauchon developed an algorithm that gives an isomorphism between certain localizations of A and A', but this requires a q_i-skew condition on each (τ_i,δ_i) with q_i not a root of unity, which usually pre-

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cludes A from satisfying a polynomial identity. We relax the restrictions placed on the base field and its chosen scalars by Jøndrup and Cauchon, respectively, by introducing the notion of a *higher q-skew τ-derivation*.

Our main theorem addresses both PI degree parity and the structure of division rings of fractions, thus confirming some cases of the *quantum Gel'fand–Kirillov conjecture*. For more information on that conjecture and proofs of conditions under which the result holds, see [1,6,17,21,25,26].

The first section sets up the conventions under which we work, including definitions and an established result concerning the PI degree of quantum affine space. A comprehensive discussion of any unfamiliar terms can be found in [4,11] and [20]. In the second section we define *higher τ-derivations* and give necessary and sufficient conditions for their existence. In the third section we present a structure theorem for a localization of q-skew polynomial rings. This extends the work of Cauchon [5], and the calculations are simplified by the presence of higher q-skew τ-derivations. In the fourth section we deal with the structure of iterated skew polynomial rings. The main theorem there asserts that if A is an iterated q-skew polynomial ring with certain higher τ-derivations, then there is a finitely generated Ore set $T \subseteq A$ such that AT^{-1} is isomorphic to a localization of a much "nicer" iterated skew polynomial ring. In the fifth section, we use the tools developed in the previous sections to confirm certain cases of the quantum Gel'fand–Kirillov conjecture and to find the PI degree of some quantized coordinate rings and quantized Weyl algebras. In the last section, we follow up with a structure theorem for completely prime factors of iterated skew polynomial rings. Many routine arguments have been omitted; details can be found in [12].

Throughout, k will denote a field of arbitrary characteristic, $q \in k$ a nonzero element. The following assumptions apply to all skew polynomial rings that we will consider:

- all coefficient rings are k-algebras,
- all automorphisms are k-algebra automorphisms,
- all skew derivations are k-linear,
- in all skew polynomial rings $R[x; \tau, \delta]$, τ is an automorphism, not just an endomorphism.

To say that $R[x; \tau, \delta]$ is a *q-skew polynomial ring* means that the automorphism and skew derivation satisfy the relation $\delta\tau = q\tau\delta$. The reader will note that this is opposite to Cauchon's conventions, but it matches the presentation in [8] and others. To say that δ is *locally nilpotent* means that for every $r \in R$ there is an integer $n_r \geqslant 0$ such that $\delta^{n_r}(r) = 0$, and $\delta^p(r) \neq 0$ for $p < n_r$. Such n_r is called the *δ-nilpotence index of r*. The symbol $\mathbb{N}$ refers to the set of positive integers. For a real number m we use the notation $\lfloor m \rfloor$ in section five to indicate the integer part of m.

Definition 1.1. We say that two rings R and S exhibit *PI degree parity* when these two conditions are satisfied:

(1) R is a PI ring if and only if S is a PI ring,
(2) $\operatorname{PIdeg} R = \operatorname{PIdeg} S$.

For a field k and multiplicatively antisymmetric $\boldsymbol{\lambda} \in M_n(k)$, the corresponding *multiparameter quantum affine space* is the k-algebra $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ with generators $x_1, \ldots, x_n$ and relations $x_i x_j = \lambda_{ij} x_j x_i$ for all i, j. The corresponding *multiparameter quantum torus* is the k-algebra $\mathcal{O}_{\boldsymbol{\lambda}}((k^\times)^n)$

given by generators $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ and the same relations. The multiplicative set generated by $x_1, \dots, x_n$ in $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ is a denominator set, and $\mathcal{O}_{\boldsymbol{\lambda}}((k^\times)^n)$ is a localization of $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ with respect to this set.

In this paper we will show that iterated skew polynomial algebras covering a large class of standard examples have PI degree parity with $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ for an appropriately chosen $\boldsymbol{\lambda}$.

Theorem 1.2. *Let $A = k[x_1; \tau_1, \delta_1] \cdots [x_n; \tau_n, \delta_n]$, where each τ_i is a k-linear automorphism of $k\langle x_i, \dots, x_{i-1}\rangle$ such that $\tau_i(x_j) = \lambda_{ij} x_j$ for all i, j with $1 \leqslant j < i \leqslant n$ and some $\lambda_{ij} \in k^\times$, and where each δ_i is a k-linear τ_i-derivation. Assume that there exist elements $q_i \in k^\times$ with $q_i \neq 1$ such that $\delta_i \tau_i = q_i \tau_i \delta_i$, and that δ_i extends to a locally nilpotent, iterative higher q_i-skew τ_i-derivation on $k\langle x_i, \dots, x_{i-1}\rangle$ for $i = 1, \dots, n$. Set $\lambda_{ji} = \lambda_{ij}^{-1}$, $\boldsymbol{\lambda} = (\lambda_{ij}) \in M_n(k^\times)$. Then*

(1) *A and $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ have isomorphic division rings of fractions.*
(2) *A is a PI-algebra if and only if all the λ_{ij} are roots of unity, in which case A and $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ have the same PI degree.*

To find out what that PI degree may be, we utilize a result of De Concini and Procesi. In [7, Proposition 7.1], they establish the following formula for calculating the PI degree of a quantum affine space $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$. Their assumption of characteristic zero from [7, Section 4] is not used in this result.

Theorem 1.3 *(De Concini–Procesi). Let $\boldsymbol{\lambda} = (\lambda_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix over k.*

(1) *The quantum affine space $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ is a PI ring if and only if all the λ_{ij} are roots of unity. In this case, there exist a primitive root of unity $q \in k^\times$ and integers a_{ij} such that $\lambda_{ij} = q^{a_{ij}}$ for all i, j.*
(2) *Suppose $\lambda_{ij} = q^{a_{ij}}$ for all i, j, where $q \in k$ is a primitive ℓth root of unity and the $a_{ij} \in \mathbb{Z}$. Let h be the cardinality of the image of the homomorphism*

$$\mathbb{Z}^n \xrightarrow{(a_{ij})} \mathbb{Z}^n \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^n$$

where π denotes the canonical epimorphism. Then $\text{PI-deg}(\mathcal{O}_{\boldsymbol{\lambda}}(k^n)) = \sqrt{h}$.

2. Higher q-skew τ-derivations

Before the featured definition, a brief discussion of a tool used to study q-skew polynomial rings is needed. Having the q-skew relation $\delta\tau = q\tau\delta$ in place allows us to group terms of the same degree when we do skew polynomial arithmetic. The means to do this are provided by the q-Liebnitz rules.

Definition 2.1. For an indeterminate t, and integers $n \geqslant m \geqslant 0$, we define the following polynomial functions:

$$(m)_t = t^{m-1} + t^{m-2} + \cdots + t + 1, \tag{1}$$

$$(m)!_t = (m)_t (m-1)_t \cdots (1)_t, \quad \text{and} \quad (0)!_t = 1, \tag{2}$$

$$\binom{n}{m}_t = \frac{(n)!_t}{(m)!_t(n-m)!_t}. \tag{3}$$

The expressions $\binom{n}{m}_t$ are called the *t-binomial coefficients*, or *Gaussian polynomials*. The t-binomial coefficients satisfy identities similar to those of the regular binomial coefficients. More information can be found in combinatorics texts such as [30]. When we evaluate the t-binomial coefficients at $t = q$, we obtain the q-binomial coefficients that we need for studying q-skew polynomial rings.

As shown in [8, Section 6], the following q-Liebnitz rules hold for any q-skew polynomial ring $R[x; \tau, \delta]$:

$$\delta^n(rs) = \sum_{i=0}^{n} \binom{n}{i}_q \tau^{n-i}\delta^i(r)\delta^{n-i}(s) \quad \text{for all } r,\ s \in R \text{ and } n = 0, 1, 2, \ldots,$$

$$x^n r = \sum_{i=0}^{n} \binom{n}{i}_q \tau^{n-i}\delta^i(r)x^{n-i} \quad \text{for all } r \in R \text{ and } n = 0, 1, 2, \ldots.$$

Now, taking a cue from the study of Schmidt differential operator rings, for instance [18], we define a sequence of k-linear maps that allows us to broaden the class of rings for which we may derive results like those of Jøndrup and Cauchon.

Definition 2.2. A *higher q-skew τ-derivation* (h.q-s.τ-d.) on a k-algebra R is a sequence $d_0, d_1, d_2, \ldots$ of k-linear operators on R such that

$$d_0 \text{ is the identity,}$$

$$d_n(rs) = \sum_{i=0}^{n} \tau^{n-i} d_i(r) d_{n-i}(s) \quad \text{for all } r, s \in R \text{ and all } n,$$

$$d_i \tau = q^i \tau d_i \quad \text{for all } i.$$

If a sequence of k-linear maps satisfies the first two conditions, we refer to it as a *higher τ-derivation*. We abbreviate the sequence $\{d_i\}_{i=0}^{\infty}$ usually as just $\{d_i\}$. A h.q-s.τ-d. is *locally nilpotent* if for all $r \in R$, there exists an integer $n \geqslant 0$ such that $d_i(r) = 0$ for all $i \geqslant n$, and $d_p(r) \neq 0$ for $p < n$. In this case, n is called the *d-nilpotence index of r*. A h.q-s.τ-d. is *iterative* if $d_i d_j = \binom{i+j}{j}_q d_{i+j}$ for all i, j. This implies that the d_i commute with each other. A q-skew τ-derivation δ on R *extends to a h.q-s.τ-d.* if there is a h.q-s.τ-d. $\{d_i\}$ on R with $d_1 = \delta$.

For example, consider the k-algebra with two generators x and y, and one relation $xy - qyx = 1$, where $q \in k^\times$. We will assume that $q \neq 1$ and recognize this algebra as a q-skew polynomial ring $k[y][x; \tau, \delta]$ with $\tau(y) = qy$ and $\delta(y) = 1$, commonly known as a *quantized Weyl algebra* and denoted $A_1^q(k)$. If q is not a root of unity, then the maps

$$d_i = \frac{\delta^i}{(i)!_q} \tag{4}$$

comprise an iterative higher q-skew τ-derivation that extends δ on $k[y]$. The properties of a higher q-skew τ-derivation follow directly from the fact that δ is a q-skew τ-derivation and the first q-Liebnitz rule. This particular h.q-s.τ-d. is also locally nilpotent because

$$d_i(y^n) = \begin{cases} \binom{n}{i}_q y^{n-i} & \text{when } i \leqslant n, \\ 0 & \text{when } i > n. \end{cases} \tag{5}$$

Proposition 2.3. *Let $\{d_i\}$ be a sequence of k-linear maps on a k-algebra R with $d_0 = \mathrm{id}_R$, and let $R[\![x;\tau^{-1}]\!]$ be the skew power series ring where τ is a k-linear automorphism of R, the coefficients are written on the right of the variable x, and $rx = x\tau(r)$ for all $r \in R$.*

(a) *Then $\{d_i\}$ is a higher τ-derivation on R if and only if the map $\Psi : R \to R[\![x;\tau^{-1}]\!]$ given by $r \mapsto \sum_{i=0}^{\infty} x^i d_i(r)$ is a ring homomorphism.*
(b) *Extend τ to an automorphism of $R[\![x;\tau^{-1}]\!]$ such that $\tau(x) = xq$. Assume that $\{d_i\}$ is a higher τ-derivation. Then the sequence $\{d_i\}$ is a h.q-s.τ-d. if and only if this diagram is commutative*:

$$\begin{tikzcd} R[\![x;\tau^{-1}]\!] \arrow[r, "\tau"] & R[\![x;\tau^{-1}]\!] \\ R \arrow[u, "\Psi"] \arrow[r, "\tau"] & R. \arrow[u, "\Psi"'] \end{tikzcd}$$

Proof. The proof is elementary. Hence, it is left to the reader. □

Remark 2.4. If $\{d_i\}$ is locally nilpotent on R, we observe that claims analogous to the proposition can be made for the map $\Psi : R \to R[x;\tau^{-1}]$.

Proposition 2.5. *Let $\{d_i\}$ be a h.q-s.τ-d. on a k-algebra R, where τ is an automorphism, and let S be a right denominator set in R with $\tau(S) = S$. Then $\{d_i\}$ can be uniquely extended to a h.q-s.τ-d. on RS^{-1}.*

Proof. It has been established that τ and d_1 extend uniquely to RS^{-1} by $\tau(rs^{-1}) = \tau(r)\tau(s)^{-1}$ and $d_1(rs^{-1}) = d_1(r)s^{-1} - \tau(rs^{-1})d_1(s)s^{-1}$ in [8, Lemma 1.3]. Suppose that $\{d_i\}$ extends to a h.q-s.τ-d. on RS^{-1}. For $r \in R$ and $s \in S$, we apply d_n to the equation $r1^{-1} = (rs^{-1})(s1^{-1})$ to get

$$\begin{aligned} d_n(r)1^{-1} = d_n\big((rs^{-1})(s1^{-1})\big) &= \sum_{j=0}^{n} \tau^{n-j} d_j\big(rs^{-1}\big) d_{n-j}\big(s1^{-1}\big) \\ &= \tau^n\big(rs^{-1}\big) d_n(s)1^{-1} + \cdots + d_n\big(rs^{-1}\big)s1^{-1}. \end{aligned}$$

This implies that

$$d_n\big(rs^{-1}\big) = \left[d_n(r) - \sum_{j=0}^{n-1} \tau^{n-j} d_j\big(rs^{-1}\big) d_{n-j}(s) \right] s^{-1}.$$

So we have uniqueness in case of existence.

To show existence, let $\Psi : R \to R[\![x;\tau^{-1}]\!]$ be the map defined in Proposition 2.3, and let $\phi : R[\![x;\tau^{-1}]\!] \to RS^{-1}[\![x;\tau^{-1}]\!]$ be the natural map. Consider the composite map $\Phi = \phi\Psi : R \to RS^{-1}[\![x;\tau^{-1}]\!]$. For any $s \in S$, the constant term of $\Phi(s)$ is a unit. So we may inductively solve for the coefficients of an inverse for $\Phi(s)$ in $RS^{-1}[\![x;\tau^{-1}]\!]$. Details, as in [29, 1.2], are left to the reader. Hence, Φ extends to a ring homomorphism $\Phi' : RS^{-1} \to RS^{-1}[\![x;\tau^{-1}]\!]$ such that $\Phi'(rs^{-1}) = \Phi(r)\Phi(s)^{-1}$, and we consider the diagram

$$\begin{tikzcd} RS^{-1}[\![x;\tau^{-1}]\!] \arrow[r, "\tau"] & RS^{-1}[\![x;\tau^{-1}]\!] \\ RS^{-1} \arrow[u, "\Phi'"] \arrow[r, "\tau"] & RS^{-1} \arrow[u, "\Phi'"'] \end{tikzcd}$$

where τ has been extended to an automorphism of $RS^{-1}[\![x;\tau^{-1}]\!]$ as in Proposition 2.3.

Since $\Phi(r) = \sum_{i=0}^{\infty} x^i d_i(r)1^{-1}$, and $\{d_i\}$ is a h.q-s.τ-d. on R, we have

$$\tau\Phi(r) = \sum_{i=0}^{\infty} x^i q^i \tau d_i(r)1^{-1} = \sum_{i=0}^{\infty} x^i d_i\big(\tau(r)\big)1^{-1} = \Phi\tau(r)$$

for all $r \in R$. It follows directly that $\tau\Phi'(rs^{-1}) = \Phi'\tau(rs^{-1})$. So, indeed, the diagram is commutative.

Define a sequence $\{d_i\}$ on RS^{-1} such that $d_i(t)$ equals the coefficient of x^i in $\Phi'(t)$ for all $t \in RS^{-1}$. Then by Proposition 2.3 we conclude that this sequence is a h.q-s.τ-d. on RS^{-1} extending $\{d_i\}$ on R. $\square$

The following lemmas are proved by induction on the length of monomials in the given generators.

Lemma 2.6. *Let A be a k-algebra, $B \subseteq A$ a k-subalgebra generated by $\{b_1, b_2, \ldots\}$, τ a k-linear automorphism of A, and $\{d_i\}$ a higher τ-derivation on A. If $d_i(b_j) \in B$ and $\tau(b_j) \in B$, for all $i, j \in \mathbb{N}$, then $d_i(B) \subseteq B$ for all i.*

Lemma 2.7. *Let A be a k-algebra with a set $\{x_j\}$ of generators, τ an automorphism of A, and $\{d_i\}$ a h.q-s.τ-d. on A. If $\{d_i\}$ is locally nilpotent for all x_j, then $\{d_i\}$ is locally nilpotent on A.*

Consider again the quantized Weyl algebra $A_1^q(k)$. In case q is an ℓth root of unity, the d_ℓ given in (5) would be undefined due to the occurrence of a zero denominator. However, realizing $A_1^q(k)$ as a factor of a quantized Weyl algebra over $k[t^{\pm 1}]$ allows us to define a h.q-s.τ-d. on $A_1^q(k)$ nonetheless. The $k[t^{\pm 1}]$-algebra $A_1^t(k[t^{\pm 1}])$ has generators x and y and one relation $xy - tyx = 1$. This is a t-skew polynomial ring $k[t^{\pm 1}][y][x; \bar\tau, \bar\delta]$ where $\bar\tau(y) = ty$, $\bar\tau(t) = t$, $\bar\delta(y) = 1$, and $\bar\delta(t) = 0$. Note that

$$\bar\delta^i(y^n) = \begin{cases} \frac{(n)!_t}{(n-i)!_t} y^{n-i} & \text{when } i \leqslant n, \\ 0 & \text{when } i > n \end{cases}$$

implying that $\bar{\delta}^i\big(k[t^{\pm 1}][y]\big) \subseteq (i)!_t k[t^{\pm 1}][y]$. So the assignment

$$\bar{d}_i = \frac{\bar{\delta}^i}{(i)!_t}$$

defines an iterative, locally nilpotent h.t-s.$\bar{\tau}$-d. $\{\bar{d}_i\}$ on $k[t^{\pm 1}][y]$. Now, the relation $xy - tyx = 1$ is equivalent to the relation $xy - qyx = 1$ modulo $\langle t - q\rangle$. Hence we have

$$A_1^t\big(k\big[t^{\pm 1}\big]\big)/\langle t - q\rangle \cong A_1^q(k).$$

When q is an ℓth root of unity, we have $\bar{\delta}^\ell\big(k[t^{\pm 1}][y]\big) \subseteq \langle t - q\rangle k[t^{\pm 1}][y]$. Nonetheless, the h.$t$-s.$\bar{\tau}$-d. $\{\bar{d}_i\}$ on $k[t^{\pm 1}][y]$ induces a h.q-s.τ-d. $\{d_i\}$ on $k[y]$, also iterative and locally nilpotent, with $d_1 = \delta$. Note that even though $\delta^\ell = 0$ in this algebra, we have $d_i(y^i) = 1$ for all i.

This phenomenon is not unique to the quantized Weyl algebras. The conditions that drive it are codified in the following theorem.

Theorem 2.8. *Let R be a k-algebra and $R[x;\tau,\delta]$ a q-skew polynomial ring where $q \in k$, $q \neq 1$. Suppose there exists a torsion-free $k[t^{\pm 1}]$-algebra $\overline{R}$ and $\overline{R}[x;\bar{\tau},\bar{\delta}]$ a t-skew polynomial ring such that $\overline{R}/\langle t - q\rangle\overline{R} \cong R$, with $\bar{\tau}$ and $\bar{\delta}$ reducing to τ and δ. Suppose further that $\bar{\delta}^i(\overline{R}) \subseteq (i)!_t\overline{R}$ for all i. Then δ extends to an iterative h.q-s.τ-d. $\{d_i\}$ on R. If $\bar{\delta}$ is locally nilpotent, then so is $\{d_i\}$. If q is not a root of unity, then $d_i = \frac{\delta^i}{(i)!_q}$ for all i. If q is a primitive ℓ^{th} root of unity, then $d_i = \frac{\delta^i}{(i)!_q}$ for $i < \ell$.*

Proof. The assumption $\bar{\delta}^i(\overline{R}) \subseteq (i)!_t\overline{R}$ for all i implies that the sequence of maps $\bar{d}_i = \frac{\bar{\delta}^i}{(i)!_t}$ make up a well-defined iterative h.t-s.$\bar{\tau}$-d. on $\overline{R}$, and also implies that $\bar{\delta}^\ell(\overline{R}) \subseteq \langle t - q\rangle\overline{R}$ because $(\ell)_t \equiv (\ell)_q = 0$ modulo $\langle t - q\rangle$. Since $\bar{\tau}$ and $\bar{\delta}$ reduce to τ and δ modulo $\langle t - q\rangle$, we have an isomorphism $\overline{R}/\langle t - q\rangle[x;\bar{\tau},\bar{\delta}] \cong R[x;\tau,\delta]$ whereby $\{\bar{d}_i\}$ induces an iterative h.q-s.τ-d. $\{d_i\}$ on R. The reduction of the maps from $\overline{R}$ to R also implies the remaining results. $\square$

We will find that all of the conditions assumed above are satisfied by the common quantized coordinate rings and related examples, which will be discussed in a subsequent section.

3. The τ-derivation removing homomorphism

The propositions in this section were shown to hold when q is not a root of unity in [5]. The proofs, which are computational in nature and related to those presented in detail by Cauchon, have been omitted. Let $A = R[x;\tau,\delta]$, and suppose that δ is locally nilpotent. Set $S = \{x^n \mid n \in \mathbb{N} \cup \{0\}\} \subset A$. The following result is well known.

Lemma 3.1. *The set S is a denominator set in A.*

Suppose also that the derivation δ extends to an iterative, locally nilpotent higher q-skew τ-derivation $\{d_i\}$ on R and that $q \neq 1$. Denote $\widehat{A} = AS^{-1} = S^{-1}A$, the localization of A with respect to S, and define a map $f : R \to \widehat{A}$ by

$$f(r) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (q-1)^{-n} d_n \tau^{-n}(r) x^{-n},$$

noting that $\{d_i\}$ is locally nilpotent and that $q-1$ is invertible. If q is not a root of unity and $\{d_i\}$ is obtained from a q-skew τ-derivation δ as in (5), the formula for f can be rewritten as

$$f(r) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \frac{(q-1)^{-n}}{(n)!_q} \delta^n \tau^{-n}(r) x^{-n}.$$

The rewritten formula matches the one presented in [5, Section 2] when q is replaced by q^{-1} to account for the difference between $\delta\tau = q\tau\delta$ (used here) and $\tau\delta = q\delta\tau$ (used in [5]). We will show that f is a homomorphism and that the multiplication in $\operatorname{im} f$ is made simpler than that in A by removing the derivation, as seen in the following.

Proposition 3.2. *If $r \in R$, then $xf(r) = f(\tau(r))x$ in $\widehat{A}$.*

From Proposition 3.2, it follows by routine induction that

$$x^m f(r) = f\big(\tau^m(r)\big)x^m \quad \forall m \in \mathbb{Z}. \tag{6}$$

This is what we need in order to show that our map is indeed a k-algebra homomorphism.

Proposition 3.3. *The map $f : R \to \widehat{A}$ is a k-algebra homomorphism.*

Proposition 3.4.

(1) *The map f extends uniquely to an algebra homomorphism, also denoted f, of $R[y;\tau]$ to $\widehat{A}$ satisfying $f(y) = x$.*
(2) *The extended homomorphism is injective.*

Definition 3.5. The algebra homomorphism $f : R[y;\tau] \to \widehat{A} = AS^{-1}$ is called the *derivation removing homomorphism*. The image of f, call it A', is the subalgebra of $\widehat{A} = AS^{-1}$ generated by x and $f(R)$, and is isomorphic (as an algebra) to $R[y;\tau]$ by the derivation removing homomorphism f.

Observe that A' contains the multiplicative system $S = \{x^n \mid n \in \mathbb{N} \cup \{0\}\}$. Since Eq. (6) holds and $f(y) = x$, the elements of this set are normal in A'. Hence, S satisfies the (two-sided) Ore condition in A'. The elements of S are regular in A' because they are regular in $\widehat{A}$, and thus:

Proposition 3.6. $A'S^{-1} = AS^{-1}$.

This equality of quotient rings reveals that if A is a PI ring, then

$$\operatorname{PIdeg} A = \operatorname{PIdeg} A' = \operatorname{PIdeg} R[y;\tau],$$

with the second equality arising from the derivation removing homomorphism f. This recovers the result of Jøndrup [16] without the assumption that k has characteristic zero. We summarize the results of this section in the following theorem.

Theorem 3.7. *Let k be a field, R a k-algebra and $A = R[x;\tau,\delta]$ a q-skew polynomial ring in which δ extends to a locally nilpotent, iterative h.q-s.τ-d. $\{d_i\}$ on R for some $q \in k^\times$, $q \neq 1$. Let S be the Ore set in A generated by x, and define a map $f : R \to AS^{-1}$ by $f(r) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (q-1)^{-n} d_n \tau^{-n}(r) x^{-n}$. Then f is a k-algebra homomorphism, and it extends to an injective homomorphism $f : R[y;\tau] \to AS^{-1}$ sending y to x. Furthermore, the extension $f : R[y^{\pm 1};\tau] \to AS^{-1}$ is an isomorphism. So there is PI degree parity between A and $R[y;\tau]$. Moreover, if R is a noetherian domain, then* $\operatorname{Fract} A \cong \operatorname{Fract} R[y;\tau]$.

4. Main theorem

In the case where A is an iterated skew polynomial ring, we would like to apply repeatedly the method presented above to remove all of the derivations and compare the resulting Ore localizations. We must first establish some facts about the behavior of h.q-s.τ-d. when the variables adjoined to the coefficient ring are rearranged, and about iterated localization. The results of these lemmas will ensure that after the induction step in the proof of the main theorem we are left with a ring to which the method of the preceding section applies.

The first parts of the following lemmas hold in a broader class of skew polynomial rings and also when the q-skew condition is imposed. The final parts assert that h.q-s.τ-d. are preserved when rearranging of the variables is permissible.

Lemma 4.1. *Let $S = R[x;\tau,\delta]$, $A = R[x;\tau,\delta][y;\sigma]$, and $\widehat{A} = R[x;\tau,\delta][y^{\pm 1};\sigma]$, where $\sigma(R) = R$ and $\sigma(x) = \lambda x$ for some $\lambda \in k^\times$.*

1. *Then $A = R[y;\sigma'][x;\tau';\delta']$, and $\widehat{A} = R[y^{\pm 1};\sigma'][x;\tau';\delta']$, where $\sigma' = \sigma|_R$, $\tau'|_R = \tau$, $\delta'|_R = \delta$, $\tau'(y) = \lambda^{-1} y$, and $\delta'(y) = 0$.*
2. *If (τ,δ) is q-skew, then so is (τ',δ').*
3. *Suppose further that δ extends to a h.q-s.τ-d. $\{d_i\}$ on R, and that $\sigma d_i = \lambda^i d_i \sigma$ for all i. Then the τ'-derivation δ' extends to a h.q-s.τ'-d. $\{d_i'\}$ on $R[y^{\pm 1};\sigma']$ such that the restrictions of the d_i' to R coincide with d_i, and $d_i'(y) = 0$ for all $i \geqslant 1$. Moreover, $\{d_i'\}$ restricts to a h.q-s.τ'-d. on $R[y;\sigma']$.*
   - (a) *If $\{d_i\}$ is iterative, then $\{d_i'\}$ is iterative.*
   - (b) *If $\{d_i\}$ is locally nilpotent, then $\{d_i'\}$ is locally nilpotent.*

Proof. (1) Routine details omitted so as not to try the patience of the reader.

(2) Suppose that (τ,δ) is q-skew on R. It is easy to check that the two τ'-derivations $\tau'^{-1}\delta'\tau'$ and $q\delta'$ agree on a set of generators for $R[y^{\pm 1};\sigma']$.

(3) Define a sequence of maps $d_i' : R[y^{\pm 1}; \sigma'] \to R[y^{\pm 1}; \sigma']$ by

$$d_i'\left(\sum_{j=-m}^{m} r_j y^j\right) = \sum_{j=-m}^{m} d_i(r_j) y^j.$$

It is not difficult to verify that this sequence is a h.q-s.τ'-d. $\square$

Lemma 4.2. *Let*

$$A = R[x_1; \tau_1, \delta_1][x_2; \tau_2, \delta_2] \cdots [x_n; \tau_n, \delta_n][y; \sigma],$$
$$\widehat{A} = R[x_1; \tau_1, \delta_1][x_2; \tau_2, \delta_2] \cdots [x_n; \tau_n, \delta_n]\big[y^{\pm 1}; \sigma\big],$$

where $\sigma(R) = R$, and for all $i \in \{1, \ldots, n\}$, $\sigma(x_i) = \lambda_i x_i$ for some nonzero $\lambda_i \in k$. Let $A_j = R[x_1; \tau_1; \delta_1][x_2; \tau_2, \delta_2] \cdots [x_j; \tau_j, \delta_j]$ for $j = 1, 2, \ldots, n$, and $A_0 = R$.

(1) *Then*

$$A = R[y; \sigma^*]\big[x_1; \tau_1', \delta_1'\big]\big[x_2; \tau_2', \delta_2'\big] \cdots \big[x_n; \tau_n', \delta_n'\big],$$
$$\widehat{A} = R\big[y^{\pm 1}; \sigma^*\big]\big[x_1; \tau_1', \delta_1'\big]\big[x_2; \tau_2', \delta_2'\big] \cdots \big[x_n; \tau_n', \delta_n'\big],$$

where $\sigma^* = \sigma|_R$, $\tau_i'|_{A_j} = \tau_i$, $\delta_i'|_{A_j} = \delta_i$, $\tau_i'(y) = \lambda_i^{-1} y$, and $\delta_i'(y) = 0$ for all $1 \leqslant i \leqslant n$ and $j \leqslant i - 1$.

(2) *If (τ_i, δ_i) is q_i-skew for any $1 \leqslant i \leqslant n$, then (τ_i', δ_i') is also q_i-skew.*

(3) *Suppose that each δ_i extends to a h.q_i-s.τ_i-d. $\{d_{i,p}\}_{p=0}^{\infty}$, and that $\sigma d_{i,p} = \lambda_i^p d_{i,p} \sigma$ on A_{i-1} for all i and p. Then each δ_i' extends to a h.q_i-s.τ_i'-d. $\{d_{i,p}'\}_{p=0}^{\infty}$ on the algebra $R\langle y, y^{-1}, x_1, \ldots, x_{i-1}\rangle$, where $d_{i,p}'$ coincides with $d_{i,p}$ on A_j, for $j < i$, and $d_{i,p}'(y) = 0$ for $p \geqslant 1$. Moreover, $\{d_{i,p}'\}$ restricts to a h.q_i-s.τ_i'-d. on $R\langle y, x_1, \ldots, x_{i-1}\rangle$.*
   (a) *If $\{d_{i,p}\}$ is iterative for any $1 \leqslant i \leqslant n$, then $\{d_{i,p}'\}$ is iterative.*
   (b) *If $\{d_{i,p}\}$ is locally nilpotent for any $1 \leqslant i \leqslant n$, then $\{d_{i,p}'\}$ is locally nilpotent.*

Proof. (1) The condition $\sigma(x_i) = \lambda_i x_i$ for all i implies that $\sigma(A_i) = A_i$. The result can then be proved using induction on n and Lemma 4.1.

(2) Consider the two τ_i'-derivations $\tau_i'^{-1} \delta_i' \tau_i'$ and $q_i \delta_i'$ on the ring

$$R\big[y^{\pm 1}; \sigma^*\big]\big[x_1; \tau_1', \delta_1'\big] \cdots \big[x_{i-1}; \tau_{i-1}', \delta_{i-1}'\big]$$

for $1 \leqslant i \leqslant n$. They agree on a full set of generators, giving the result.

(3) Suppose the result holds for the algebra $R[x_1; \tau_1, \delta_1] \cdots [x_{n-1}; \tau_{n-1}, \delta_{n-1}][y^{\pm 1}; \sigma]$. Then Lemma 4.1 may be applied, with A_{n-1} providing the coefficients, to get

$$A_{n-1}[x_n; \tau_n, \delta_n]\big[y^{\pm 1}; \sigma\big] = A_{n-1}\big[y^{\pm 1}; \sigma'\big]\big[x_n; \tau_n', \delta_n'\big],$$

where δ_n' extends to a h.q_n-s.τ_n'-d. $\{d_{n,p}'\}$ on $A_{n-1}[y^{\pm 1}]$. The induction hypothesis gives the result. $\square$

Definition 4.3. For a k-algebra A and $a, b \in A$, we say that a and b *scalar commute* if there is an element $\alpha \in k^\times$ such that $ab = \alpha ba$. We may also say that a and b α-commute.

In the following two lemmas, we let D denote the division ring of fractions for the noetherian domain A. When comparing localizations of A, we identify them as subrings of D.

Lemma 4.4. *Let A be a noetherian domain, $S \subseteq A \setminus \{0\}$ an Ore set. Let T be an Ore set in $AS^{-1} \setminus \{0\}$ with $S \subseteq T$.*

(1) *Then there exists an Ore set $\widetilde{T} \subseteq A \setminus \{0\}$ with $S \subseteq \widetilde{T}$ such that $A\widetilde{T}^{-1} = (AS^{-1})T^{-1}$.*
(2) *Suppose A is a k-algebra and S is generated by $s_1, \dots, s_n$ satisfying $s_i s_j = \gamma_{ij} s_j s_i$ for all i, j and some $\gamma_{ij} \in k^\times$. Further suppose that T is generated by $S \cup t$ for some $t \in AS^{-1}$ that satisfies $s_i t = \lambda_i t s_i$ for all i and some $\lambda_i \in k^\times$. Then there exist a cyclic Ore set $\widehat{T} \subseteq A \setminus \{0\}$ and an $(n+1)$-generator Ore set $\widehat{S} \subseteq A \setminus \{0\}$ such that $S \subseteq \widehat{S}$, and $(AS^{-1})T^{-1} = A\widehat{T}^{-1} = A\widehat{S}^{-1}$.*

Proof. (1) Consider $T \cap A$, the subset in T of elements with a denominator of 1. Clearly, this is a multiplicative set in A which contains S. Set $\widetilde{T} = T \cap A$. Let $a \in \widetilde{T}$ and $\alpha \in A$. Then $a \in T$, and since $\alpha \in AS^{-1}$, there exist $b' \in T$ and $\beta' \in AS^{-1}$ such that $a\beta' = \alpha b'$. By [11, 10.2], there exist $y \in S$, and $b, \beta \in A$ such that $\beta' = \beta y^{-1}$ and $b' = by^{-1}$; hence, $a\beta y^{-1} = \alpha b y^{-1}$ in AS^{-1}. It follows that $a\beta = \alpha b$ in A. So $\widetilde{T}$ satisfies the right Ore condition in A, and the left Ore condition by symmetry. By the universal property, $A\widetilde{T}^{-1} \cong (AS^{-1})T^{-1}$. As subrings of D, we have $A\widetilde{T}^{-1} = (AS^{-1})T^{-1}$.

(2) The generating element t has the form $t = \bar{a}(s_1^{m_1} s_2^{m_2} \cdots s_n^{m_n})^{-1}$ for some $m_i \in \mathbb{N}$, and $\bar{a} \in A$. For any $s_i \in S$, we have

$$s_i \bar{a}\bigl(s_1^{m_1} s_2^{m_2} \cdots s_n^{m_n}\bigr)^{-1} = \lambda_i \bar{a}\bigl(s_1^{m_1} s_2^{m_2} \cdots s_n^{m_n}\bigr)^{-1} s_i = \mu\lambda_i \bar{a} s_i \bigl(s_1^{m_1} s_2^{m_2} \cdots s_n^{m_n}\bigr)^{-1},$$

where μ is a product of powers of the γ_{ij}. So $\bar{a}$ scalar commutes with the generators of S via the relations $s_i \bar{a} = \mu\lambda_i \bar{a} s_i$. Let $\widehat{S}$ be the multiplicative set generated by $\bar{a}, s_1, \dots, s_n$ in A, and $\widehat{T}$ the multiplicative set generated by $\bar{a} s_1 s_2 \cdots s_n$ in A. We have $(AS^{-1})T^{-1} = A\widetilde{T}^{-1}$, where $\widetilde{T} = T \cap A$ as in part (1). From the scalar commuting relations it follows that any element $a\tilde{t}^{-1} \in A\widetilde{T}^{-1}$ may be written in the form $b(\bar{a} s_1 \cdots s_n)^{-m}$ for some $m \in \mathbb{N} \cup \{0\}$, $b \in A$, or the form $c\bar{a}^{-\ell_{n+1}} s_1^{-\ell_1} \cdots s_n^{-\ell_n}$, for $\ell_j \in \mathbb{N} \cup \{0\}$, $c \in A$. So we conclude that $\widehat{S}$ and $\widehat{T}$ are Ore sets in A and that $(AS^{-1})T^{-1} = A\widehat{T}^{-1} = A\widehat{S}^{-1}$. $\square$

Lemma 4.5. *Let A be a noetherian domain, $S_1 \subseteq A \setminus \{0\}$ an Ore set, and for integers $j = 2, \dots, n$ let S_j be an Ore set in $((AS_1^{-1}) \cdots)S_{j-1}^{-1} \setminus \{0\}$ with $S_{j-1} \subseteq S_j$.*

(1) *Then there exists an Ore set $T \subseteq A \setminus \{0\}$ such that*

$$AT^{-1} = \bigl(\bigl(\bigl(AS_1^{-1}\bigr)S_2^{-1}\bigr) \cdots\bigr)S_n^{-1}.$$

(2) *Suppose A is a k-algebra, S_1 is generated by s_1, and for $j = 2, \dots, n$, S_j is generated by $S_{j-1} \cup \{s_j\}$, where $s_i s_j = \gamma_{ij} s_j s_i$ for some multiplicatively antisymmetric $(\gamma_{ij}) \in M_n(k^\times)$. Then there are a cyclic Ore set $\widehat{T} \subseteq A$ and an n-generator Ore set $\widehat{S} \subseteq A$ such that $S_1 \subseteq \widehat{S}$, and $((AS_1^{-1})S_2^{-1}) \cdots S_n^{-1} = A\widehat{T}^{-1} = A\widehat{S}^{-1}$.*

Proof. These results are proved by induction on n. □

In the proof of the main theorem, we will use without mention the facts gathered here. For greater details on these statements, see [11, 10X, 10Y] and [8, 1.4].

(1) Given a noetherian ring A and a normal element $x \in A$, the multiplicative set generated by x is an Ore set.
(2) The multiplicative set generated by a nonempty family of right Ore sets is right Ore.
(3) Let $A = R[x; \tau, \delta]$, and S a right denominator set in R such that $\tau(S) = S$. Then S is a right denominator set in A and the identity map on AS^{-1} extends to an isomorphism of AS^{-1} onto $(RS^{-1})[x; \tau, \delta]$ sending $x1^{-1}$ to x. Note that if A is a k-algebra, τ, δ are k-linear, and $\tau(k^\times S) = k^\times S$, then the result holds because S is a denominator set if and only if $k^\times S$ is a denominator set.

Theorem 4.6. *Let R be a k-algebra and noetherian domain,*

$$A = R[x_1; \tau_1, \delta_1] \cdots [x_n; \tau_n, \delta_n],$$

where each τ_i is a k-linear automorphism of $R\langle x_i, \ldots, x_{i-1}\rangle$ such that $\tau_i(x_j) = \lambda_{ij} x_j$ for all i, j with $1 \leqslant j < i \leqslant n$ and some $\lambda_{ij} \in k^\times$, and where each δ_i is a k-linear τ_i-derivation. Assume that there exist elements $q_i \in k^\times$ with $q_i \neq 1$ such that $\delta_i \tau_i = q_i \tau_i \delta_i$, and that δ_i extends to a locally nilpotent, iterative h.q_i-s.τ_i-d. on $R\langle x_i, \ldots, x_{i-1}\rangle$ for $i = 1, \ldots, n$.

(1) *Then there exists an Ore set $T \subseteq A$ generated by n elements of A such that*

$$AT^{-1} \cong R\big[y_1^{\pm 1}; \tau_1\big]\big[y_2^{\pm 1}; \tau_2'\big] \cdots \big[y_n^{\pm 1}; \tau_n'\big]$$

where $\tau_i'|_R = \tau_i$ and $\tau_i'(y_j) = \lambda_{ij} y_j$ for all i, j with $1 \leqslant j < i \leqslant n$.
(2) *There is PI degree parity between A and $R[y_1; \tau_1][y_2; \tau_2'] \cdots [y_n; \tau_n']$. Moreover, these algebras have isomorphic division rings of fractions.*

Proof. (a) Suppose, inductively, that we have

$$R[x_1; \tau_1, \delta_1]\big[y_2^{\pm 1}; \tau_2\big] \cdots \big[y_n^{\pm 1}; \tau_n'\big] \cong AS_2^{-1}$$

where the restriction of τ_i' to $R\langle x_1\rangle$ coincides with τ_i, $\tau_i'(y_m) = \lambda_{im} y_m$ for $2 \leqslant i \leqslant n$ and $1 < m < i$, and S_2 is an Ore set in A generated by $n-1$ elements from A. Then by Lemma 4.2

$$AS_2^{-1} \cong R\big[y_2^{\pm 1}; \tau_2''\big] \cdots \big[y_n^{\pm 1}; \tau_n''\big]\big[x_1; \tau_1', \delta_1'\big] \tag{7}$$

where the restrictions of τ_1' and δ_1' to R coincide with τ_1 and δ_1, $\tau_1'(y_j) = \lambda_{j1}^{-1} y_j$, $\delta_1'(y_j) = 0$, and τ_i'' coincides with the restriction of τ_i to $R\langle y_2, \ldots, y_{i-1}\rangle$ for $2 \leqslant i \leqslant n$. Observe that by Lemmas 4.2 and 2.7 we also have $\delta_1' \tau_1' = q_1 \tau_1' \delta_1'$, and that δ_1' extends to a locally nilpotent iterative h.q_1-s.τ-d. on $R\langle y_2^{\pm 1}, \ldots, y_n^{\pm 1}\rangle$. Then applying the derivation removing homomorphism to the right-hand side of (7) gives an isomorphism

$$\big(AS_2^{-1}\big)T_1^{-1} \cong R\big[y_2^{\pm 1}; \tau_2'\big] \cdots \big[y_n^{\pm 1}; \tau_n'\big]\big[y_1^{\pm 1}; \tau_1'\big]$$

where $T_1 \subseteq AS_2^{-1}$ is an Ore set generated by one element of AS_2^{-1}. Then Lemma 4.5 and a reordering of variables show the existence of an Ore set $T \subseteq A$, generated by n elements of A, such that $AT^{-1} \cong R[y_1^{\pm 1}; \tau_1][y_2^{\pm 1}; \tau_2'] \cdots [y_n^{\pm 1}; \tau_n']$.

(2) This follows from part (1). □

Corollary 4.7. *Let $A = k[x_1; \tau_1, \delta_1] \cdots [x_n; \tau_n, \delta_n]$ with the hypotheses as in Theorem 4.6. Set $\boldsymbol{\lambda} = (\lambda_{ij})$. Then*

(1) *A and $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ have isomorphic division rings of fractions.*
(2) *A is a PI-algebra if and only if all the λ_{ij} are roots of unity, in which case A and $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ have the same PI degree.*

5. Examples

We will demonstrate how each of the following k-algebras satisfies all the conditions of Theorem 2.8. Then Corollary 4.7 is applied to obtain an isomorphism of quotient division rings (thereby confirming the quantum Gel'fand–Kirillov conjecture) and PI degree parity with a multiparameter quantum affine space. For concreteness, we will mainly discuss the single parameter cases of these examples; multiparameter cases are addressed in [12].

When calculating the PI degree of a quantum affine space, we encounter an antisymmetric, or *skew-symmetric*, integral matrix. As proved in [23, Theorem IV.1], such a matrix is congruent to a matrix in *skew normal form*.

Theorem 5.1 *(Newman). Let A be a skew-symmetric matrix of rank r which belongs to $M_n(R)$, where the commutative principal ideal domain R is not of characteristic 2. Then $r = 2s$, and A is congruent to a block diagonal matrix $S = \operatorname{diag}(B_1, \dots, B_s, 0, \dots, 0)$, where $B_i = \left(\begin{smallmatrix} 0 & h_i \\ -h_i & 0 \end{smallmatrix}\right)$ and $h_i \mid h_{i+1}$, $1 \leqslant i \leqslant s-1$.*

The same result, in the language of alternating bilinear forms, can be found in [3, Section 5.1].

The matrix S in Theorem 5.1 is clearly equivalent to the more familiar Smith normal form, $\operatorname{diag}(h_1, h_1, h_2, h_2, \dots, h_s, h_s, 0, 0, \dots, 0)$, where the diagonal entries are the invariant factors of the matrix A. In the examples that follow, we outline the operations necessary to obtain the Smith normal form.

Definition 5.2. Let $A = k[x_1; \tau_1, \delta_1] \cdots [x_n; \tau_n, \delta_n]$ and $A' = k[x_1; \tau_1] \cdots [x_n; \tau_n]$ be iterated skew polynomial rings. (1) If there exists $Q = (q_1, \dots, q_n) \in (k^\times)^n$ such that $\delta_i \tau_i = q_i \tau_i \delta_i$ for $i = 1, \dots, n$, then A is called an *iterated Q-skew polynomial ring*. (2) If there exist $\lambda_{ji} \in k^\times$ such that $\tau_j(x_i) = \lambda_{ji} x_i$ for all $i < j$, then set $\lambda_{ij} = \lambda_{ji}^{-1}$ and $\lambda_{ii} = 1$ for all i. We call $\Lambda = (\lambda_{ij}) \in M_n(k^\times)$ *the matrix of relations for A'*.

The following lemma is proved by induction on the length of monomials in the given generators.

Lemma 5.3. *Let C be a commutative k-algebra, A a C-algebra, $B \subseteq A$ a C-subalgebra generated by $\{b_1, b_2, \dots\}$. Let τ be a C-algebra automorphism of A, and δ a u-skew τ-derivation on A for some unit $u \in C$. If $\tau(b_j) \in B$ and $\delta^n(b_j) \in (n)!_u B$ for all j, n, then $\delta^n(B) \subseteq (n)!_u B$ for all n.*

For a first family of examples, we take odd-dimensional quantum Euclidean spaces. The even-dimensional ones will be covered in Example 5.4. With appropriate modifications to the parameters of this first example, the omitted details of the successors become clear.

5.1. *The coordinate ring of odd-dimensional quantum Euclidean space;* $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$

For $q \in k^\times$, assuming q has a (fixed) square root $q^{1/2} \in k$, the k-algebra $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$ may be presented as an iterated skew polynomial ring

$$k[w][y_1;\sigma_1][x_1;\tau_1,\delta_1]\cdots[y_n;\sigma_n][x_n;\tau_n,\delta_n]$$

with automorphisms σ_i, τ_i and derivations δ_i defined by

$$\begin{aligned}
&\sigma_i(w) = q^{-1}w, && \text{all } i,\\
&\tau_i(w) = qw, && \text{all } i,\\
&\sigma_i(y_j) = q^{-1}y_j, && j<i,\\
&\sigma_i(x_j) = q^{-1}x_j, && j<i,\\
&\tau_i(y_j) = qy_j, && i \neq j,\\
&\tau_i(x_j) = qx_j, && j<i,\\
&\tau_i(y_i) = y_i, && \text{all } i,\\
&\delta_i(w) = \delta_i(x_j) = \delta_i(y_j) = 0, && j<i,\\
&\delta_i(y_i) = \bigl(q^{1/2}-q^{3/2}\bigr)w^2 + \bigl(1-q^2\bigr)\sum_{\ell<i} y_\ell x_\ell, && \text{all } i.
\end{aligned}$$

Quantum Euclidean spaces have been studied since 1990 when they were introduced by Reshetikhin et al. in [28]. The three-dimensional case has applications to the structure of space–time at small distances. Musson simplified the original set of relations in [22], and Oh further simplified them, renaming the generators ω, x_i, y_i in [24]. Here, we have made a change to Oh's variables, $y_i \mapsto q^i y_i$, to obtain the relations in our presentation of $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$.

Routine computations show that $\tau_i^{-1}\delta_i\tau_i(y_i) = q^{-2}\delta_i(y_i)$ for all i, and so we conclude that each (τ_i,δ_i) is a q^{-2}-skew derivation. We may present the analogous $k[t^{\pm 1}]$-algebra $\mathcal{O}_t(\mathfrak{o}k[t^{\pm 1}]^{2n+1})$ as an iterated skew polynomial ring with coefficient ring $k[t^{\pm 1}]$ and generators w, y_i, x_i for $i = 1,\dots,n$,

$$k\bigl[t^{\pm 1}\bigr][w][y_1;\bar\sigma_1][x_1;\bar\tau_1,\bar\delta_1]\cdots[y_n;\bar\sigma_n][x_n;\bar\tau_n,\bar\delta_n]$$

where the automorphisms and derivations are defined analogously to those of the algebra $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$ with $t \in k[t^{\pm 1}]$ replacing $q \in k^\times$. So each $(\bar\tau_i,\bar\delta_i)$ is a t^{-2}-skew derivation. It is immediate that

$$\mathcal{O}_t\bigl(\mathfrak{o}k\bigl[t^{\pm 1}\bigr]^{2n+1}\bigr)/\langle t-q\rangle \cong \mathcal{O}_q\bigl(\mathfrak{o}k^{2n+1}\bigr)$$

with each $\bar\tau_i$ and $\bar\delta_i$ reducing to τ_i and δ_i respectively.

Let A_j denote the $k[t^{\pm 1}]$-subalgebra generated by w, y_m, x_m for $m < j$, and y_j. To show that $\bar{\delta}^i_j(A_j) \subseteq (i)!_{t^{-2}} A_j$, we apply Lemma 5.3 noting that $\bar{\delta}^i_j(y_j)$ has been given for $i = 1$ and is zero for $i > 1$. So, by Theorem 2.8, each δ_i in our presentation of $\mathcal{O}_q(\mathfrak{o}k^{2n+1})$ extends to an iterative, locally nilpotent h.q^{-2}-s.τ_i-d. on an appropriate subalgebra. Then Corollary 4.7 gives

$$\operatorname{Fract} \mathcal{O}_q\big(\mathfrak{o}k^{2n+1}\big) \cong \operatorname{Fract} \mathcal{O}_B\big(k^{2n+1}\big),$$

where the matrix of relations is

$$B = \begin{pmatrix} 1 & q & q^{-1} & q & q^{-1} & \cdots & q & q^{-1} \\ q^{-1} & 1 & 1 & q & q^{-1} & \cdots & q & q^{-1} \\ q & 1 & 1 & q & q^{-1} & \cdots & q & q^{-1} \\ q^{-1} & q^{-1} & q^{-1} & 1 & 1 & \cdots & q & q^{-1} \\ q & q & q & 1 & 1 & \cdots & q & q^{-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q^{-1} & q^{-1} & q^{-1} & q^{-1} & q^{-1} & \cdots & 1 & 1 \\ q & q & q & q & q & \cdots & 1 & 1 \end{pmatrix}.$$

If $q \in k^{\times}$ is a root of unity, we may assume without loss of generality that it is a primitive rth root of unity. Then the powers of q from the matrix B become the entries of a $(2n+1) \times (2n+1)$ integer matrix

$$B' = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 & \cdots & 1 & -1 \\ -1 & 0 & 0 & 1 & -1 & \cdots & 1 & -1 \\ 1 & 0 & 0 & 1 & -1 & \cdots & 1 & -1 \\ -1 & -1 & -1 & 0 & 0 & \cdots & 1 & -1 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & -1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Now, $\operatorname{PIdeg} \mathcal{O}_q(\mathfrak{o}k^{2n+1})$ can be computed from Theorem 1.3(2) using the matrix B'. The cardinality of the image will not be changed if we first perform some row reductions on B'. Letting $N = 2n + 1$, $n > 2$, we manipulate the rows as follows.

- For $i = 2, 4, 6, \ldots, N - 1$, replace row i with row i + row $(i + 1)$.
- For $i = N, N - 2, N - 4, \ldots, 5$, replace row i with row i − row $(i - 2)$.
- Replace row 5 with row 5 − row 1.
- For $i = 2, 4, 6, \ldots, N - 5$, replace row i with row i − 2row $(i + 5)$.
- Multiply the even numbered rows, except row $2n - 2$, by -1.

The resulting matrix has $2n$ pivots and one zero row. We put the rows in this order

$$3, 1, 5, 7, 2, 9, 4, 11, 6, 13, \ldots, 2i, 2i + 7, \ldots, N, N - 5, N - 3, N - 1$$

to place the pivots on the main diagonal and the zero row in the last position. Then we have a matrix of this form

$$\left(\begin{array}{cccccccccccc}
1 & & & & & & & & & & & \\
 & 1 & -1 & & & & & & & & & \\
 & & 2 & & & & & & & & & \\
 & & & 1 & 1 & & & & & * & & \\
 & & & & 4 & & & & & & & \\
 & & & & & 1 & 1 & & & & & \\
 & & & & & & 4 & & & & & \\
 & & & \mathbf{0} & & & & \ddots & & & & \\
 & & & & & & & & 1 & 1 & & \\
 & & & & & & & & & 4 & & \\
 & & & & & & & & & & 2 & -2 \\
 & & & & & & & & & & 0 & 0
\end{array}\right).$$

The diagonal entries of this echelon matrix do not yet reveal the size of its image because the pivot in row three does not divide all of the (suppressed) entries in its row when $n \geqslant 3$. So more row reduction is needed.

First replace row 3 with row $3 + \sum_{i=1}^{\lfloor \frac{n-2}{2} \rfloor} \mathrm{row}(4i+2)$.

For n even and $j = 5, 7, 9, \ldots, 2n-3$, replace row j as follows:

$$\text{for } j = 4p+1,\ p \geqslant 1,\ \text{ use row } j + \sum_{i=p+1}^{\frac{n-2}{2}} 2 \cdot \mathrm{row}(4i) + \mathrm{row}(2n);$$

$$\text{for } j = 4p+3,\ p \geqslant 1,\ \text{ use row } j + \sum_{i=p+1}^{\frac{n-2}{2}} 2 \cdot \mathrm{row}(4i+2).$$

For n odd and $j = 5, 7, 9, \ldots, 2n-5$, replace row j as follows:

$$\text{for } j = 4p+1,\ p \geqslant 1,\ \text{ use row } j + \sum_{i=p+1}^{\lfloor \frac{n}{2} \rfloor} 2 \cdot \mathrm{row}(4i) + 2 \cdot \mathrm{row}(2n);$$

$$\text{for } j = 4p+3,\ p \geqslant 1,\ \text{ use row } j + \sum_{i=p+1}^{\lfloor \frac{n-2}{2} \rfloor} 2 \cdot \mathrm{row}(4i+2) + \mathrm{row}(2n).$$

Then add $\mathrm{row}(2n)$ to $\mathrm{row}(2n-3)$, and add $2 \cdot \mathrm{row}(2n)$ to $\mathrm{row}(2n-1)$. For integers $4 \leqslant j \leqslant 2n-1$, with $j \not\equiv 2 \pmod 4$, add $(-1)^j$ col 3 to col j. Subtract $\mathrm{col}(2n+1)$ from col 3; add row 3 to $\mathrm{row}(2n-2)$; and subtract $2 \cdot \mathrm{row}\,3$ from $\mathrm{row}(2n)$. The result is an upper echelon matrix in which each pivot divides all the nonzero entries in its row. So it is trivial to diagonalize by column operations. The Smith normal form for n odd is $\mathrm{diag}(1, 1, \ldots, 1, 4, 4, \ldots, 4, 0)$ with $n+1$ ones and $n-1$ fours. The Smith normal form for n even is $\mathrm{diag}(1, 1, \ldots, 1, 2, 2, 4, 4, \ldots, 4, 0)$ with n ones, two twos, and $n-2$ fours.

For the cases $n = 1, 2$, the row-reduced matrices are, respectively,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence we have, for all $n > 0$,

$$\operatorname{PIdeg} \mathcal{O}_q\big(\mathfrak{o}k^{2n+1}\big) = \begin{cases} r^n, & r \text{ odd}, \\ r^n/2^{\lfloor \frac{n}{2} \rfloor}, & r \text{ even}, r \notin 4\mathbb{Z}, \\ r^n/2^{n-1}, & r \in 4\mathbb{Z}. \end{cases}$$

5.2. The multiparameter quantized Weyl algebras; $A_n^{Q,\Gamma}(k)$

For $Q = (q_1, \ldots, q_n) \in (k^\times)^n$ and $\Gamma = (\gamma_{ij})$ a multiplicatively antisymmetric $n \times n$ matrix over k, the algebra $A_n^{Q,\Gamma}(k)$, studied in [17] and [19], may be presented as an iterated skew polynomial ring

$$k[y_1][x_1; \tau_1, \delta_1][y_2; \sigma_2][x_2; \tau_2, \delta_2] \cdots [y_n; \sigma_n][x_n; \tau_n, \delta_n]$$

where the automorphisms and derivations are defined by

$$\begin{aligned} \sigma_i(y_j) &= \gamma_{ji} y_j, && j < i, \\ \sigma_i(x_j) &= \gamma_{ij} x_j, && j < i, \\ \tau_i(y_j) &= q_j \gamma_{ji} y_j, && j < i, \\ \tau_i(x_j) &= q_j^{-1} \gamma_{ij} x_j, && j < i, \\ \tau_i(y_i) &= q_i y_i, && \text{all } i, \\ \delta_i(x_j) &= \delta_i(y_j) = 0, && j < i, \\ \delta_i(y_i) &= 1 + \sum_{\ell < i} (q_\ell - 1) y_\ell x_\ell, && \text{all } i. \end{aligned}$$

Corollary 4.7 gives $\operatorname{Fract} A_n^{Q,\Gamma}(k) \cong \operatorname{Fract} \mathcal{O}_\Lambda(k^{2n})$, where the $2n \times 2n$ matrix of relations Λ is comprised of 2×2 blocks

$$B_{ii} = \begin{pmatrix} 1 & q_i^{-1} \\ q_i & 1 \end{pmatrix}, \quad \text{for all } i;$$

$$B_{ij} = \begin{pmatrix} \gamma_{ji} & q_i^{-1}\gamma_{ji} \\ \gamma_{ij} & q_i \gamma_{ij} \end{pmatrix}, \quad \text{for } i < j;$$

$$B_{ij} = \begin{pmatrix} \gamma_{ji} & \gamma_{ij} \\ q_j \gamma_{ji} & q_j^{-1} \gamma_{ij} \end{pmatrix}, \quad \text{for } i > j.$$

Consider the single parameter case, denoted $A_n^q(k)$, where $q_i = q$ for all i, and $\gamma_{ij} = 1$ for $i < j$, relegating the σ_i to identity maps. Assuming that q is a primitive rth root of unity, then $\delta_i(y_i^r) = 0$ and $\tau_i(y_i^r) = y_i^r$ for all i, implying that y_i^r is central. The definition of the τ_i, along with the q-Liebnitz rule, implies that x_i^r is central for all i. So the algebra $A_n^q(k)$ is a finitely generated module over the central subring $k[y_i^r, x_1^r, \dots, y_n^r, x_n^r]$. Using Corollary 4.7 and the suitable matrix of relations verifies that $\operatorname{PIdeg} A_n^q(k) = r^n$.

5.3. The multiparameter coordinate ring of quantum $n \times n$ matrices; $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$

The multiparameter coordinate ring of quantum $n \times n$ matrices was introduced by Artin, Schelter, and Tate in [2]. The k-algebra $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ is defined by generators x_{ij} for $i, j = 1, \dots, n$ and relations

$$x_{\ell m} x_{ij} = \begin{cases} p_{\ell i} p_{jm} x_{ij} x_{\ell m} + (\lambda - 1) p_{\ell i} x_{im} x_{lj} & (\ell > i,\ m > j), \\ \lambda p_{\ell i} p_{jm} x_{ij} x_{\ell m} & (\ell > i, m \leqslant j), \\ p_{jm} x_{ij} x_{\ell m} & (\ell = i, m > j), \end{cases}$$

where $\lambda \in k^\times$ and $\boldsymbol{p} = (p_{ij}) \in M_{n^2}(k^\times)$ is multiplicatively antisymmetric. It can also be presented as an iterated skew polynomial ring

$$k[x_{11}][x_{12}; \tau_{12}] \cdots [x_{ij}; \tau_{ij}, \delta_{ij}] \cdots [x_{nn}; \tau_{nn}, \delta_{nn}]$$

where each $\tau_{\ell m}$ and $\delta_{\ell m}$ is k-linear and satisfies

$$\tau_{\ell m}(x_{ij}) = \begin{cases} p_{\ell i} p_{jm} x_{ij} & \text{when } \ell > i \text{ and } m \neq j, \\ \lambda p_{\ell i} p_{jm} x_{ij} & \text{when } \ell > i \text{ and } m = j, \\ p_{jm} x_{ij} & \text{when } \ell = i \text{ and } m > j, \end{cases}$$

$$\delta_{\ell m}(x_{ij}) = \begin{cases} (\lambda - 1) p_{\ell i} x_{im} x_{\ell j} & \text{when } \ell > i \text{ and } m > j, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 4.7 gives $\operatorname{Fract} \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k)) \cong \operatorname{Fract} \mathcal{O}_\Lambda(k^{n^2})$, where the matrix of relations $\Lambda = (b_{ij}) \in M_{n^2}(k)$ is comprised of $n \times n$ blocks

$$B_{ii} = \begin{pmatrix} 1 & p_{21} & p_{31} & \cdots & p_{n1} \\ p_{12} & 1 & p_{32} & \cdots & p_{n2} \\ p_{13} & p_{23} & 1 & \cdots & p_{n3} \\ \vdots & \vdots & & \ddots & \vdots \\ p_{1n} & p_{2n} & p_{3n} & \cdots & 1 \end{pmatrix} \quad \text{for all } i,$$

$$B_{ij} = \begin{pmatrix} \lambda^{-1} p_{ij} & p_{ij} p_{21} & p_{ij} p_{31} & \cdots & p_{ij} p_{n1} \\ \lambda^{-1} p_{ij} p_{12} & \lambda^{-1} p_{ij} & p_{ij} p_{32} & \cdots & p_{ij} p_{n2} \\ \lambda^{-1} p_{ij} p_{13} & \lambda^{-1} p_{ij} p_{23} & \lambda^{-1} p_{ij} & \cdots & p_{ij} p_{n3} \\ \vdots & \vdots & & \ddots & \vdots \\ \lambda^{-1} p_{ij} p_{1n} & \lambda^{-1} p_{ij} p_{2n} & \lambda^{-1} p_{ij} p_{3n} & \cdots & \lambda^{-1} p_{ij} \end{pmatrix}, \quad \text{for } i < j,$$

$$B_{ij} = \begin{pmatrix} \lambda p_{ij} & \lambda p_{ij} p_{21} & \lambda p_{ij} p_{31} & \cdots & \lambda p_{ij} p_{n1} \\ p_{ij} p_{12} & \lambda p_{ij} & \lambda p_{ij} p_{32} & \cdots & \lambda p_{ij} p_{n2} \\ p_{ij} p_{13} & p_{ij} p_{23} & \lambda p_{ij} & \cdots & \lambda p_{ij} p_{n3} \\ \vdots & \vdots & & \ddots & \vdots \\ p_{ij} p_{1n} & p_{ij} p_{2n} & p_{ij} p_{3n} & \cdots & \lambda p_{ij} \end{pmatrix}, \quad \text{for } i > j.$$

The single parameter quantized coordinate ring of $n \times n$ matrices, $\mathcal{O}_q(M_n(k))$, is defined over k analogously to $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$, but with relations that are recovered by setting $\lambda = q^{-2}$ and $p_{ij} = q$ for all $i > j$. When k has characteristic zero and q is a primitive mth root of unity *for m odd*, Jakobsen and Zhang found in [15] that $\operatorname{PIdeg} \mathcal{O}_q(M_n(k)) = m^{\frac{n(n-1)}{2}}$ by using De Concini and Procesi's tool given in Theorem 1.3. This result is reproved in [14] using results of De Concini and Procesi and also Jøndrup's work from [16]. Now we can recover $\operatorname{PIdeg} \mathcal{O}_q(M_n(k))$ without the assumption that k has characteristic zero.

The matrix of relations can be reduced through row operations to an upper triangular $n^2 \times n^2$ matrix with $2n-2$ ones, $(n-1)(n-2)$ twos, and n zeroes on the diagonal. Assuming that $q \in k$ is a primitive mth root of unity, and recalling Theorem 1.3, the cardinality of the image in $(\mathbb{Z}/m\mathbb{Z})^{n^2}$ is m^{n^2-n} if m is odd. Thus we conclude that $\operatorname{PIdeg} \mathcal{O}_q M_n(k) = m^{\frac{n(n-1)}{2}}$, recovering the result of Jakobsen and Zhang [15] in characteristic zero. By similar methods, one can show that $\operatorname{PIdeg} \mathcal{O}_q M_n(k) = m^{\frac{n(n-1)}{2}} / 2^{\frac{(n-1)(n-2)}{2}}$ when m is even. For details on this result see [15] or [12].

5.4. *The algebra $K_{n,\Gamma}^{P,Q}(k)$, which generalizes the coordinate rings of even-dimensional quantum Euclidean space and quantum symplectic space*

For $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ in $(k^\times)^n$ with $p_i \neq q_i$ for all $i = 1, \ldots, n$, and $\Gamma = (\gamma_{ij}) \in M_n(k^\times)$ multiplicatively antisymmetric, the k-algebra $K_{n,\Gamma}^{P,Q}(k)$ introduced in [13] is defined by generators x_i, y_i for $i = 1, \ldots, n$ and relations

$$\begin{aligned}
y_i y_j &= \gamma_{ij} y_j y_i, && \text{all } i, j, \\
x_i x_j &= q_i p_j^{-1} \gamma_{ij} x_j x_i, && i < j, \\
x_i y_j &= p_j \gamma_{ji} y_j x_i, && i < j, \\
x_i y_j &= q_j \gamma_{ji} y_j x_i, && i > j, \\
x_i y_i &= q_i y_i x_i + \sum_{\ell < i} (q_\ell - p_\ell) y_\ell x_\ell, && \text{all } i.
\end{aligned}$$

(An odd-dimensional analogue of $K_{n,\Gamma}^{P,Q}(k)$ is developed in [9].)

This algebra may be presented in the form of an iterated skew polynomial ring

$$k[y_1][x_1; \tau_1][y_2; \sigma_2][x_2; \tau_2, \delta_2] \cdots [y_n; \sigma_n][x_n; \tau_n, \delta_n]$$

where the automorphisms τ_i, σ_i and derivations δ_i are defined by

$$\begin{aligned}
\sigma_i(y_j) &= \gamma_{ij} y_j, && j < i, \\
\sigma_i(x_j) &= p_i^{-1} \gamma_{ji} x_j, && j < i, \\
\tau_i(y_j) &= q_j \gamma_{ji} y_j, && j < i,
\end{aligned}$$

$$\begin{aligned}
\tau_i(x_j) &= q_j^{-1} p_i \gamma_{ij} x_j, && j < i,\\
\tau_i(y_i) &= q_i y_i, && \text{all } i,\\
\delta_i(x_j) &= \delta_i(y_j) = 0, && j < i,\\
\delta_i(y_i) &= \sum_{\ell < i} (q_\ell - p_\ell) y_\ell x_\ell, && \text{all } i.
\end{aligned}$$

Corollary 4.7 gives $\operatorname{Fract} K_{n,\Gamma}^{P,Q}(k) \cong \operatorname{Fract} \mathcal{O}_\Lambda(k^{2n})$, where the $2n \times 2n$ matrix of relations $\Lambda = (B_{ij})$ is comprised of 2×2 blocks

$$B_{ii} = \begin{pmatrix} 1 & q_i^{-1} \\ q_i & 1 \end{pmatrix}, \quad \text{for all } i;$$

$$B_{ij} = \begin{pmatrix} \gamma_{ij} & q_i^{-1}\gamma_{ji} \\ p_j\gamma_{ji} & q_i p_j^{-1}\gamma_{ij} \end{pmatrix}, \quad \text{for } i < j;$$

$$B_{ij} = \begin{pmatrix} \gamma_{ij} & p_i^{-1}\gamma_{ij} \\ q_j\gamma_{ji} & q_j^{-1} p_i\gamma_{ij} \end{pmatrix}, \quad \text{for } i > j.$$

Suppose q_i is an r_ith root of unity, p_i is an s_ith root of unity, and γ_{ij} is an r_{ij}th root of unity for all i, j. Let $r = \operatorname{lcm}\{r_i, s_i, \gamma_{ij} \mid i, j = 1, \dots, n\}$.

The coordinate ring of quantum Euclidean $2n$-space over k, $\mathcal{O}_q(\mathfrak{o}k^{2n})$, is formed by setting $q_i = 1$, $p_i = q^{-2}$ for all i, and $\gamma_{ij} = q^{-1}$ for $i < j$ in the parameters Q, P, and Γ (see [13, Example 2.6]). By a method similar to that used in Example 5.1, suppressed here in the interest of saving space but listed explicitly in [12], we obtain

$$\operatorname{PIdeg} \mathcal{O}_q\bigl(\mathfrak{o}k^{2n}\bigr) = \begin{cases} r^{n-1}, & r \text{ odd},\\ r^{n-1}/2^{\lfloor \frac{n-1}{2} \rfloor}, & r \text{ even}, r \notin 4\mathbb{Z},\\ r^{n-1}/2^{n-2}, & r \in 4\mathbb{Z}. \end{cases}$$

As a specific case of $K_{n,\Gamma}^{P,Q}(k)$, quantum symplectic space $\mathcal{O}_q(\mathfrak{sp}(k^{2n}))$ is formed by setting $q_i = q^{-2}$ and $p_i = 1$ for all i, and $\gamma_{ij} = q$ for $i < j$ (see [13, Example 2.4]). With these parameters we have, for all n,

$$\operatorname{PIdeg} \mathcal{O}_q\bigl(\mathfrak{sp}\bigl(k^{2n}\bigr)\bigr) = \begin{cases} r^{n}, & r \text{ odd},\\ r^{n}/2^{\lfloor \frac{n+1}{2} \rfloor}, & r \text{ even}, r \notin 4\mathbb{Z},\\ r^{n}/2^{n}, & r \in 4\mathbb{Z}. \end{cases}$$

6. Prime factor localizations

In this section we present a structure theorem for completely prime factors of iterated skew polynomial rings analogous to the main theorem of section four. Applying this result to the algebras studied in section five, we would like to strengthen it to the form of the quantum Gel'fand–Kirillov conjecture. Recall that the assumptions about skew polynomial rings from section one are still in effect.

Theorem 6.1. *Let $A = R[x; \tau, \delta]$, where R is noetherian and $\delta\tau = q\tau\delta$ for some $q \in k^\times$. Assume that δ extends to a locally nilpotent, iterative h.q-s.τ-d., $\{d_i\}$, on R. Let $P \in \operatorname{spec} A$ be completely prime. Then*

(1) *there exists a cyclic Ore set S in A/P such that $(A/P)S^{-1}$ is isomorphic to $(R[y; \tau]/Q)Y^{-1}$ for some completely prime $Q \in \operatorname{spec} R[y; \tau]$ and cyclic Ore set Y,*
(2) $\operatorname{Fract} A/P \cong \operatorname{Fract} R[y; \tau]/Q$.

Proof. The completely prime ideal P naturally satisfies one of two cases: $x \in P$ or $x \notin P$. If $x \in P$, then $xA \subseteq P$ and $Ax \subseteq P$. So the relation $xr = \tau(r)x + \delta(r)$ implies that $\delta(r) \in P$ for all $r \in R$. Hence, there is a completely prime ideal $I \in R$ such that $A/P \cong R/I \cong R[y; \tau]/(I + \langle y \rangle)$. In this case, we can take $S = Y = \{1\}$ and localize. If $x \notin P$, then $x^i \notin P$ for all $i \in \mathbb{N} \cup \{0\}$ because A/P is a domain. Letting $S = \{1, x, x^2, \ldots\}$, which is a known denominator set in A, we have $P \cap S = \varnothing$. Since extension and contraction provide inverse bijections between the sets $\operatorname{spec} AS^{-1}$ and $\{I \in \operatorname{spec} A \mid I \cap S = \varnothing\}$, we know that $P^e \in \operatorname{spec} AS^{-1}$. From Theorem 3.7, we have $AS^{-1} \cong R[y^{\pm 1}; \tau]$, a localization of $R[y; \tau]$. So there is a completely prime ideal $\bar{Q} \lhd R[y^{\pm 1}; \tau]$ such that $AS^{-1}/P^e \cong R[y^{\pm 1}; \tau]/\bar{Q}$. Setting $Y = \{1, y, y^2, \ldots, \}$, contraction to $R[y; \tau]$ gives a completely prime ideal Q, where $Q \cap Y = \varnothing$, such that $R[y^{\pm 1}; \tau]/\bar{Q} \cong (R[y; \tau]/Q)Y^{-1}$. The canonical projection $\pi : AS^{-1} \to (A/P)S^{-1}$ gives $AS^{-1}/P^e \cong (A/P)S^{-1}$. Thus $(A/P)S^{-1} \cong (R[y; \tau]/Q)Y^{-1}$. □

Theorem 6.2. *Let R be a noetherian k-algebra, and let*

$$A = R[x_1, \tau_1, \delta_1] \cdots [x_n; \tau_n, \delta_n]$$

be an iterated skew polynomial ring where, for $j < i$ and $\lambda_{ij} \in k^\times$, $\tau_i(x_j) = \lambda_{ij}x_j$, and δ_i is a q_i-skew τ_i-derivation, $q_i \neq 1$, which extends to a locally nilpotent, iterative h.q_i-s.τ_i-d. $\{d_{i,p}\}_{p=0}^{\infty}$ on $R[x_1; \tau_1, \delta_1] \cdots [x_{i-1}; \tau_{i-1}, \delta_{i-1}]$ for all i. Let $A' = R[y_1; \tau_1'][y_2; \tau_2'] \cdots [y_n; \tau_n']$ where $\tau_i'(y_j) = \lambda_{ij}y_j$ for all i with $j < i$ and the same units λ_{ij} as above. Let P be a completely prime ideal in A. Then

(1) *there exists a finitely generated Ore set S_n in A/P such that $(A/P)S_n^{-1}$ is isomorphic to $(A'/Q)Y_n^{-1}$ for some completely prime ideal $Q \subseteq A'$ and finitely generated Ore set Y_n,*
(2) $\operatorname{Fract} A/P \cong \operatorname{Fract} A'/Q$.

Proof. The result is obtained using Theorem 6.1, Lemma 4.2, and an induction argument. □

When R is replaced by k, we have the following result.

Corollary 6.3. *Let $A = k[x_1, \tau_1, \delta_1] \cdots [x_n; \tau_n, \delta_n]$, where $\tau_i(x_j) = \lambda_{ij}x_j$ and $\delta_i\tau_i = q_i\tau_i\delta_i$, $q_i \neq 1$, for $\lambda_{ij}, q_i \in k^\times$ and all i with $j < i$. Assume that each δ_i extends to a locally nilpotent, iterative h.q_i-s.τ_i-d. $\{d_{i,m}\}_{m=0}^{\infty}$ on the subalgebra $k[x_1; \tau_1, \delta_1] \cdots [x_{i-1}; \tau_{i-1}, \delta_{i-1}]$. Let P be a completely prime ideal in A and set $\lambda_{ii} = 1$ and $\lambda_{ji} = \lambda_{ij}^{-1}$. Then for $\boldsymbol{\lambda} = (\lambda_{ij}) \in M_n(k)$, and an appropriate completely prime ideal $Q \subseteq \mathcal{O}_{\boldsymbol{\lambda}}(k^n)$, we have*

$$\operatorname{Fract} A/P \cong \operatorname{Fract} \mathcal{O}_{\boldsymbol{\lambda}}(k^n)/Q.$$

We summarize how this applies to the k-algebras of quantized coordinate type.

Corollary 6.4. *Let A be any of the examples discussed in Sections 5.1–5.4, and let P be a completely prime ideal of A. Then there exist a positive integer N, a multiplicatively antisymmetric $N \times N$ matrix $\boldsymbol{\lambda}$ over k, and a completely prime ideal $Q \in \mathcal{O}_{\boldsymbol{\lambda}}(k^N)$ such that* $\operatorname{Fract} A/P \cong \operatorname{Fract} \mathcal{O}_{\boldsymbol{\lambda}}(k^N)/Q$.

To complete the question posed by the corollary, one might ask how far the quantum Gel'fand–Kirillov conjecture extends to prime factor algebras. For instance:

Question 6.5. *Find conditions under which we can conclude that for any positive integer n, multiplicatively antisymmetric matrix $\boldsymbol{\lambda} \in M_n(k^\times)$, and completely prime ideal $Q \in \operatorname{spec} \mathcal{O}_{\boldsymbol{\lambda}}(k^n)$, we have*

$$\operatorname{Fract} \mathcal{O}_{\boldsymbol{\lambda}}\big(k^n\big)/Q \cong \operatorname{Fract} \mathcal{O}_{\boldsymbol{p}}\big(K^m\big)$$

for some field extension $K \supseteq k$, integer $m \leqslant n$, and $m \times m$ matrix $\boldsymbol{p}$ over K.

A positive answer in the generic case has been provided in the proof of [10, Theorem 2.1]:

Theorem 6.6 *(Goodearl–Letzter). Let k be a field, $\boldsymbol{\lambda} = (\lambda_{ij})$ a multiplicatively antisymmetric $n \times n$ matrix over $k^\times$, and $\boldsymbol{\Lambda}$ the subgroup of $k^\times$ generated by the λ_{ij}. If $\boldsymbol{\Lambda}$ is torsionfree, then all of the prime ideals Q of $\mathcal{O}_{\boldsymbol{\lambda}}(k^n)$ are completely prime.*

In their proof, they showed that $\operatorname{Fract} \mathcal{O}_{\boldsymbol{\lambda}}(k^n)/Q \cong \operatorname{Fract} \mathcal{O}_{\boldsymbol{p}}(K^m)$, and identified K as the quotient field of a commutative domain embedded in the center of $\mathcal{O}_{\boldsymbol{\lambda}}((k^\times)^n)/Q'$, where Q' is the prime ideal in $\mathcal{O}_{\boldsymbol{\lambda}}((k^\times)^n)$ induced by localization.

The division ring of real quaternions provides an example showing that Question 6.5 needs to have some conditions imposed. Note that

$$\mathbb{H} \cong \mathcal{O}_{\boldsymbol{\lambda}}\big(\mathbb{R}^3\big)/Q, \quad \text{where } \boldsymbol{\lambda} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \text{ and } Q = \big\langle x_1^2+1, x_2^2+1, x_3^2+1\big\rangle.$$

Therefore, we cannot obtain the desired isomorphism of quotient division rings in this case, illustrating the necessity of an extra condition such as the one imposed by Panov in [27, Section 3].

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