On Minimal Surfaces Bounded by Two Convex Curves in Parallel Planes

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Received October 28, 1985; revised May 12, 1986

The curvature of the intersection of a minimal surface $S$ with parallel planes $\{z = t\}$, between plane parallel convex curves $\Gamma_0$ and $\Gamma_1$ on $S$, takes its minimum on $\Gamma_0 \cup \Gamma_1$. A sharp lower bound for the curvature of $S \cap \{z = t\}$ is derived. Similarly upper and lower bounds for the gradients of these curves and for their distances from a perpendicular axis are derived too, together with a differential inequalities for their lengths which implies an explicit necessary condition for the surface $S$ to exist. © 1987 Academic Press, Inc.

1. INTRODUCTION

In this note we consider some geometric properties of the shape of a minimal surface $S$ in space bounded by two plane convex curves $\Gamma_0, \Gamma_1$ lying in parallel planes. In [11] Shiffmann proved that under this assumption the intersection of $S$ with a plane parallel to the planes of $\Gamma_0, \Gamma_1$ is again a convex curve.

Here we wish to give sharp estimates for the curvatures of these convex curves and for other related functions such as the lengths, the distances from an axis perpendicular to the planes above, and the gradients of these level curves.

More precisely let $\Gamma_0$ and $\Gamma_1$ be two convex plane curves lying in the planes $z = t_0, z = t_1$, respectively. The minimal surface $S$ between $\Gamma_0$ and $\Gamma_1$ can be parameterized as (see [11])

$$x = x(\tau, t), \quad y = y(\tau, t), \quad z = t,$$

where $(\tau, t)$ are conformal parameters on the surface satisfying

$$x_{\tau\tau} + y_{\tau\tau} = 0, \quad x_{tt} + y_{tt} = 0,$$

$$x_\tau^2 + y_\tau^2 = x_t^2 + y_t^2 + 1, \quad x_\tau x_t + y_\tau y_t = 0.$$
Shiffmann's result ([11] page 80) claims:

\[ \Gamma_t = S \cap \{ z = t \} \text{ is a strictly convex curve.} \] (1.2)

We improve this result in Theorem 3.1 by proving that:

the curvature \( K \) of the convex curves \( \Gamma_t \) on \( S \) takes its minimum on \( \Gamma_0 \cup \Gamma_1 \).

Moreover if \( K_0, K_1 \) are the minimum curvatures of \( \Gamma_0 \) and \( \Gamma_1 \), respectively, and \( t_0 = 0, t_1 = 1 \) we prove that for any point \( P \equiv (x, y, z) \) on \( \Gamma_t \),

\[ \log K(P) \geq (1 - t) \log K_0 + t \log K_1. \] (1.4)

The previous bound is improved in Theorem 3.2 by considering the surface of revolution of minimum area \( S^* \) bounded by two parallel circles \( \Gamma_0^*, \Gamma_1^* \), with common axis of symmetry, lying on the same planes as \( \Gamma_0 \) and \( \Gamma_1 \), and with curvatures \( K_0 \) and \( K_1 \).

If \( K^*(t) \) is the curvature of the circle obtained by intersecting \( S^* \) with the plane \( \{ z = t \} \) then we show in Theorem 3.2 that

\[ K^*(t), \text{ for } P \in \Gamma_t. \] (1.5)

In Theorem 2.1 we show a differential inequality for the length \( L(t) \) of the level curves \( \Gamma_t \). This implies, by Theorem 2.2, a sharp upper bound for \( L \), and an explicit necessary condition, in terms of \( |t_0 - t_1| \) and of the length of \( \Gamma_0 \) and \( \Gamma_1 \), for the minimal surface \( S \) to exist.

In Theorem 4.1 we get two differential inequalities for the maximum and minimum distances of the level curves \( \Gamma_t \), from a fixed axis perpendicular to the \( xy \)-plane. In Theorem 4.2 we show sharp upper and lower bounds for the gradient of the level curves \( \Gamma_t \).

If minimal surfaces are considered in the form \( z = u(x, y) \) any previous result can be read for the classical solution \( u \) to the minimal surface equation

\[ (1 + u_y^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + u_x^2) u_{yy} = 0, \] (1.6)

in a ring-like domain \( \Omega = D_0 - D_1 \), where \( D_1 \subset D_0 \), and \( D_0 \) and \( D_1 \) are plane convex bodies; \( \partial D_0, \partial D_1 \) are the projections onto \( \{ z = 0 \} \) of the curves \( \Gamma_0, \Gamma_1 \) described above; so that \( u \) satisfies the following boundary conditions

\[ u = t_0 \text{ on } \partial D_0, \quad u = t_1 \text{ on } \partial D_1. \] (1.7)

A comparison result between the height \( z \) of the solution to (1.6) and the capacity function of \( D_0 - D_1 \) is given in [8].
It would be possible to give an independent proof of (1.2) by using the described estimates for $K$ and by a continuity argument starting from suitable radially symmetric minimal surfaces. We refer to [2] for continuity argument for minimal surfaces; maximum principles for minimal surface with more general boundary conditions can be found in [3].

The principal idea in this paper is to consider the support function $h$ related to the convex curves $\Gamma_t$. We introduce $t$ and the direction $\theta$ of the exterior normal vector to $\Gamma$, as parameters on $S$. In Section 2 we show that $h$ satisfies a partial differential equation in $(\theta, t)$ coordinates. Then by calculus and maximum principle arguments we get the proof of the theorems described above.

2. Support Function

For simplicity we start by considering a given function $u$ with level convex curves in a ring-like domain $\Omega = D_0 - D_1$ with $\partial D_0 = \{u = t_0\}$, $\partial D_1 = \{u = t_1\}$. Let $x_0$ be a fixed point in $D_1$. Let us choose the origin of the coordinates at $x_0$. For any $t$ in $(t_0, t_1)$ let $D_t$ be the convex body bounded by the level curves $\{u = t\}$. Let us consider for any point $(x, y)$ on $\{u = t\}$ the exterior normal vector at $(x, y)$ to $\partial D_t$: $n = (\cos \theta, \sin \theta)$; the distance of the origin from the tangent line (or support line) at $(x, y)$ to $\partial D_t$ orthogonal to $n$ is given by the support function

$$h = x \cos \theta + y \sin \theta.$$  

(2.1)

Since for any $(x, y)$ in $\Omega$, $(\theta, t)$ is uniquely defined and vice versa, we can introduce the frame system coordinates $(\theta, t)$ as new coordinate.

Now it is not difficult to see, by (2.1) and the geometric meaning of $\theta$, that the following formulas hold when $\{u = t\}$ is a strictly convex curve of class $C^2$ (c.f. [9])

$$h_\theta = \frac{\partial h}{\partial \theta} = -x \sin \theta + y \cos \theta,$$  

(2.2)

$$R = \frac{ds}{d\theta} = h + h_{\theta\theta},$$  

(2.3)

where $R$ is the radius of curvature of the plane curve $\{u = t\}$.

Moreover the partial derivatives of $u$ can be rewritten as partial derivatives of $h$ with respect to $\theta$ and $t$. More precisely if $u_n$ is the outward normal derivative and $u_{nn}, u_{ss}$ are the derivatives of $u$ in normal coordinates we have

$$u_n = h_t^{-1},$$  

(2.4)
Now let us consider the solution \( u \) to (1.6), (1.7). The Eq. (1.6) in normal coordinates becomes

\[
U_{,,} + (1 + U_0) U_{,,} = 0.
\]

(2.7)

By (1.2) we can use the support function \( h \) and by (2.4), (2.5), (2.6), we find that \( h \) satisfies the equation

\[
(h + h_{,0}) h_{,0} - h_{,0}^2 = h_{,0}^2 + 1.
\]

(2.8)

This non linear partial differential equation for \( h \) will be the central formula for the proof of any theorem in this note. In the next theorem we give another proof of (2.8), without using (2.4)-(2.6), which holds for a parametric minimal surface \( S \) given by (1.1) and satisfying (1.2). In this case \( h(\cdot, t) \) will be defined as the support function of any convex curve \( \Gamma_t \) with respect to the point \((0, 0, t)\).

**Theorem 2.1.** Let \( S \) be a minimal surface given by (1.1) and satisfying (1.2). Then the support function \( h \) given by (2.1) satisfies the differential equation (2.8).

**Proof.** The proof follows by a variational method argument.

In fact the parameterization (1.1) of the minimal surface equation \( S \) must satisfy the variational equation

\[
\delta A = \delta \iint (x^2_t + y^2_t)^{1/2} (1 + x^2_t + y^2_t)^{1/2} \, dt \, d\tau = 0,
\]

(2.9)

where \( A \) is the area of \( S \).

Now, since

\[
x_t \cos \theta + y_t \sin \theta = 0, \quad x_t x_r + y_t y_r = 0,
\]

by deriving (2.1) with respect to \( t \) we derive that

\[
h_t^2 = x_t^2 + y_t^2.
\]

(2.10)

So by replacing the coordinate \( \theta \) with \( t \) in (2.9) we have, since

\[
(x_t^2 + y_t^2)^{1/2} \, dt = ds = Rd\theta,
\]

\[
\delta \iint R(1 + h_t^2)^{1/2} \, d\theta \, dt = 0.
\]

(2.11)
So by (2.3), (2.11) is the first variation of the functional

\[ G(h) = \int (h + h_{\theta \theta})(1 + h_i^2)^{1/2} \, d\theta \, dt, \]

with Euler equation

\[ -\frac{\partial}{\partial t} [(1 + h_i^2)^{-1/2} R h_i] + (1 + h_i^2)^{1/2} \frac{\partial^2}{\partial \theta^2} [1 + h_i^2]^{1/2} = 0. \tag{2.12} \]

By manipulation (2.12) becomes (2.8).

Let us consider now the length \( L(t) \) of the level curve \( \Gamma_t \). From (2.3) it follows that

\[ L(t) = \int_{\Gamma_t} ds = \int_0^{2\pi} R(\theta, t) \, d\theta = \int_0^{2\pi} h(\theta, t) \, d\theta. \tag{2.13} \]

**Theorem 2.2.** \( L \) satisfies the following differential inequality in \((t_0, t_1)\):

\[ L''L - L'^2 \geq 4\pi^2. \tag{2.14} \]

Equality holds in (2.14) for some \( \eta \) in \((t_0, t_1)\) if and only if all the level curves \( \Gamma_t \) are concentric circles.

**Proof:** We sketch the proof because it is similar to arguments used by the authors in [5]. By (2.13) we get

\[ L'(t) = \int_0^{2\pi} h_i(\theta, t) \, d\theta, \quad L''(t) = \int_0^{2\pi} h_{ii}(\theta, t) \, d\theta. \tag{2.15} \]

So from (2.8) and Schwarz inequality we derive

\[ L'' \geq \int_0^{2\pi} R^{-1}(1 + h_i^2) \, d\theta \geq \left[ \left( \int_0^{2\pi} h_i d\theta \right)^2 + 4\pi^2 \right] \cdot \left( \int_0^{2\pi} R \, d\theta \right)^{-1}. \tag{2.16} \]

By (2.13) and (2.15) we prove (2.14). Moreover equality holds in (2.14) for some \( \eta \) in \((t_0, t_1)\) if and only if

\[ h_{\theta \theta}(\cdot, \eta) \equiv 0 \text{ and } R(\cdot, \eta) \text{ is proportional to } h_i(\cdot, \eta). \]

Therefore \( h_i \) is constant on \( \Gamma_{\eta} \) and \( \Gamma_{\eta} \) is a circle. By uniqueness of interior analytic continuation of minimal surfaces (see [2]) we complete the proof.

Let us consider now two parallel circles \( \Gamma_0, \Gamma_1 \) with common axis of
symmetry, lying in the same planes as \( \Gamma_0 \) and \( \Gamma_1 \), and with the same perimeter as \( \Gamma_0 \) and \( \Gamma_1 \), respectively.

Let \( \mathcal{S} \), if it exists, be the unique surface of revolution of minimum area between \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \). If \( r(t) \) is the radius of the circle \( \mathcal{S} \cap \{ z = t \} \), then \( r \) satisfies the boundary value problem

\[
y'' \cdot y - y'^2 = 1 \quad \text{in} \quad (t_0, t_1),
\]

\[
y(t_0) = (2\pi)^{-1} \cdot L(t_0), \quad y(t_1) = (2\pi)^{-1} \cdot L(t_1).
\]

The solutions to (2.17) can be written in terms of catenaries and in [1, p. 109], one can find necessary conditions on \( |t_0 - t_1| \), \( L(t_0) \), and \( L(t_1) \) for the function \( r \) to exist.

The following theorems shows that the conditions quoted above are also necessary conditions for the minimal surface \( S \) to exist.

**Theorem 2.3.** A necessary condition for the minimal surface \( S \) between \( \Gamma_0 \) and \( \Gamma_1 \) to exist is that there exists the solution \( r(t) \) to (2.17) corresponding to \( \mathcal{S} \). Moreover

\[
L(t) \leq 2\pi r(t) \quad \text{for} \quad t \in [t_0, t_1].
\]

*Proof.* From (2.14) it follows that the function \( v \) defined by

\[
v(t) = (2\pi)^{-1} L(t), \quad t \in [t_0, t_1]
\]

is a lower solution to (2.17).

By considering the transformation \( u = \log y \), (2.17) becomes

\[
u'' = \exp(-2u) \quad \text{in} \quad (t_0, t_1),
\]

and the function \( \psi = \log v \) is a lower solution to the corresponding boundary value problem. Moreover let us observe that \( \psi \) is a convex function and \( \psi \leq \phi \), where \( \phi \) is a linear function with same values at \( t_0 \) and \( t_1 \) as \( \psi \).

Since \( \phi \) is an upper solution to (2.20), standard arguments for lower and upper solution (see for example [4, p. 354]), show that there exist a solution \( u \) to (2.20) with the same value at \( t_0 \) and \( t_1 \) as \( \psi \); moreover

\[
\psi(t) \leq u(t) \leq \phi(t) \quad \text{in} \quad (t_0, t_1).
\]

So \( w \equiv \exp u \) is a solution to (2.17) and the inequality above we get

\[
v(t) \leq w(t) \quad \text{in} \quad (t_0, t_1).
\]
Since $w(t)$ is one of the two catenaries, eventually coincident and solutions to (2.17), we have that (see [1, p. 961]) or

$$w = r,$$

with $r$ the radius function of $\tilde{S}$,

or either

$$w(t) < r(t) \quad \text{in} \quad (t_0, t_1)$$

In all cases by (2.19), (2.21) and the inequality above we obtain (2.18).

3. CURVATURE ESTIMATES

In the sequel if $\phi$ is a real function on $[t_0, t_1]$ we define the generalized second derivative of $\phi$ at any point $t$ in $(t_0, t_1)$ as

$$D^2 \phi(t) = \lim \sup_{\varepsilon \to 0} \frac{\phi(t + \varepsilon) + \phi(t - \varepsilon) - 2\phi(t)}{\varepsilon^2}$$

(3.1)

Let us consider now the function

$$f(t) = \max\{R(P) \mid P \in \Gamma_t\},$$

(3.2)

where $R(P)$ is the radius of curvature at $P$ of $\Gamma_t$.

**THEOREM 3.1.** If $S$ is a minimal surface between two parallel convex curves $\Gamma_0$ and $\Gamma_1$ then (1.3) and (1.4) hold. Moreover the function $f$ defined by (3.2) satisfies the following differential inequality:

$$D^2 (\log f) \geq f^{-2} \quad \text{in} \quad (t_0, t_1).$$

(3.3)

**Proof.** By Theorem 2.1 we can consider the partial differential equation (2.8) verified by $h$. By differentiating (2.8) with respect to $\theta$ and by (2.3) we get

$$h_{tt} R_\theta + R h_{tt\theta} - 2h h_{t\theta} = 0,$$

(3.4)

which can be rewritten as

$$h_{tt} R_\theta + h_{tt\theta} R - 2h h_{t\theta} R_{t\theta} = 0.$$  

(3.5)

By differentiating the previous equation with respect to $\theta$ we get

$$h_{tt} R_{t\theta} + 2R h_{t\theta} + R h_{tt\theta} - 2h h_{t\theta} R_{t\theta} - 2h h_{t\theta} R_{t\theta} R_{t\theta} = 0.$$  

(3.6)
Adding $Rh_{tt}$ to each side by using (2.3), (3.6) is rewritten as

$$h_{t}R_{\theta \theta} - 2h_{\theta t}R_{\theta t} + RR_{tt} + 2h_{\theta t}R_{\theta} - 2h_{\theta \theta t}R_{t} = Rh_{tt}. \tag{3.7}$$

Let us consider now the partial differential operator

$$L = a \frac{\partial^2}{\partial \theta^2} + b \frac{\partial^2}{\partial \theta \partial t} + c \frac{\partial^2}{\partial t^2} \tag{3.8}$$

where $a = h_{tt}, b = h_{\theta t}, c = R$.

Since $R$ is positive, by (2.8) $h_{tt}$ is positive too. Moreover, by (2.8) we have

$$ac - b^2 = 1 + h_{t}^2 > 0,$$

so

$L$ is an elliptic operator. \tag{3.9}

Moreover (3.7) becomes

$$L(R) + 2h_{\theta t}R_{\theta} - 2h_{\theta \theta t}R_{t} - Rh_{tt} = 0. \tag{3.10}$$

So by applying standard maximum principle arguments, we find that $K = R^{-1}$ takes its minimum at the boundary $\Gamma_0 \cup \Gamma_1$ and (1.3) is proved. We prove now (3.3). Let us observe that from (3.8) and (3.10) we derive

$$RL(\log R) = L(R) - (aR_{\theta}^2 - 2bR_{\theta} R_{t} + cR_{t}^2) R^{-1}. \tag{3.11}$$

So by (3.10) and (3.11) we get

$$RL(\log R) = d \frac{\partial}{\partial \theta} (\log R) + 2h_{\theta \theta t} R_{t} + Rh_{tt} - R_{t}^2,$$

where $d$ is a bounded function in $B \times (t_0, t_1)$. By using (2.3) and (2.8) the previous equation becomes

$$RL(\log R) - d \frac{\partial}{\partial \theta} (\log R) = 1 + h_{\theta}^2 + h_{\theta \theta}^2 \geq 1. \tag{3.12}$$

Now by applying a maximum principle argument quoted in Lemma 1, in the appendix for convenience of reader, we get the proof of (3.3). In fact since the gradient $h_{\theta}$ of the level curves $\Gamma_1$ given by 2.10 is bounded, the function $f(t)$ is continuous on $[t_0, t_1]$. Moreover by (3.2), (3.3) we have

$$\phi(t) = \max\{\log R(\theta, t) | \theta \in B\} = \log f(t).$$
So Lemma 1 applied to the function $\phi(t)$ above, with

$$G = \log R, \quad \mathcal{L} = R^{-1}\left(\mathcal{L} - d\frac{\partial}{\partial\theta}\right)$$

$$F = (1 + h_{10}^2 + h_{20}^2) R^{-2}, \quad \psi(x) = (\exp x)^{-2},$$

implies (3.3). Moreover from (3.3) we derive that $D^2(\log f) \geq 0$ which implies that $\log f$ is convex. From that (1.4) follows and the proof is complete.

Let us consider now two parallel circles $\Gamma_0^*, \Gamma_1^*$ with common axis of symmetry, lying in the same planes as $\Gamma_0$ and $\Gamma_1$, and with radii $K_0^{-1}, K_1^{-1}$. Let $S^*$ be the surface of revolution of minimum area between $\Gamma_0^*$ and $\Gamma_1^*$. Since $\Gamma_0^* \supseteq \tilde{\Gamma}_0$, $\Gamma_1^* \supseteq \tilde{\Gamma}_1$ from Theorem 2.3 and from monotonicity arguments (c.f. [7]), we derive that if $S$ exists then $S^*$ exists too. Moreover let $K^*(t)$ be the curvature of the level circles $S^* \cap \{z = t\}$. In the following theorem we derive a sharp lower bound for the curvature $K$ of the curve $\Gamma_i$.

**Theorem 3.2.** If $S$ is the minimal surface between $\Gamma_0$ and $\Gamma_1$, and if $S^*$ is the surface of revolution of minimum area between $\Gamma_0^*$ and $\Gamma_1^*$ then the inequality (1.5) hold.

**Proof.** The radius $R(t) = K^*(t)^{-1}$ is a solution to the differential equation (2.17). So by computation we derive that

$$(\log R)^{\prime \prime} = R^{-\gamma} \text{ in } (t_0, t_1).$$

By applying to (3.3) and to equality above the same arguments used in the proof of Theorem 2.3 we derive that

$$f(t) \leq R(t) \text{ in } (t_0, t_1),$$

which implies (1.5).

4. **Distance and Gradient Estimates**

Nitsche in [6] has proved that if a minimal surface $S$ exists between $\Gamma_0$ and $\Gamma_1$ then the projections of the domains bounded by $\Gamma_0$ and $\Gamma_1$ onto the $xy$-plane, parallel to $\Gamma_0$ and $\Gamma_1$, must have a nonempty intersection $I$. So
we can choose the origin of the coordinates in \( I \) such that \( \{0, 0, t\} \) is in the
domain bounded by \( \Gamma_t \). Let us consider now the functions
\[
\delta(t) = \max\{(x^2 + y^2)^{1/2} | (x, y, t) \in \Gamma_t\}, \tag{4.1}
\]
\[
\sigma(t) = \min\{(x^2 + y^2)^{1/2} | (x, y, t) \in \Gamma_t\}. \tag{4.2}
\]
\( \delta(t) \) and \( \sigma(t) \) are the radii of two concentric circles tangent and bounding
\( \Gamma_t \). Of course estimates on \( \delta(t) \) and on \( \sigma(t) \) are useful to bound the shape
of \( S \).

**Theorem 4.1.** The function \( \delta(t) \) satisfies the differential inequality
\[
D^2(\log \delta) \geq \delta^{-2}, \tag{4.3}
\]
and \( \delta \) is a log convex function; moreover \( \sigma(t) \) satisfies
\[
D^2(\log \sigma) \leq \sigma^{-2}. \tag{4.4}
\]

**Proof.** As in theorem 3.1 we consider the support function \( h(\theta, t) \). First
we show that
\[
\delta(t) = \max\{h(\theta, t): \theta \in B\}, \tag{4.5}
\]
\[
\sigma(t) = \min\{h(\theta, t): \theta \in B\}. \tag{4.6}
\]
In fact from (2.1), (2.2) we have that
\[
x^2 + y^2 = h^2 + h_\theta. \tag{4.7}
\]
Moreover if \( (x^2 + y^2) \) achieves its maximum or minimum at \( (\tilde{x}, \tilde{y}) \) on
\( \{u = t\} \) the support line to \( \Gamma_t \) at \( (\tilde{x}, \tilde{y}) \) is normal to the vector \( (\tilde{x}, \tilde{y}) \). So by
(2.2) \( h_\theta \) is zero at \( (\tilde{x}, \tilde{y}) \). By this fact and (4.7), we derive (4.5) and (4.6).
To prove (4.3) and (4.4) let us consider the operator \( L \) defined by (3.8). We have by (2.3)
\[
L(h) = 2Rh_{\theta\theta} - hh_{\theta\theta} - 2h_{\theta\theta}. \tag{4.8}
\]
So from (2.8) it follows that
\[
L(h) = -hh_{\theta\theta} + 2h_{\theta\theta}^2 + 2. \tag{4.9}
\]
Moreover by computation and (3.8)
\[
hL(\log h) = L(h) - (ah_\theta^2 - 2bh_\theta + ch_\theta^2) h^{-1}.
\]
By the previous inequality and (4.8) we get
\[ hL (\log h) = -hh_{tt} + h_t^2 - h_{\theta \theta} h_t^2 h^{-1} + 2 + dh, \] (4.9)
where \( d \) is bounded function in \( B \times (t_0, t_1) \).
Moreover, (4.9) can be rewritten as
\[ \left( L + h \frac{\partial^2}{\partial t^2} - d \frac{\partial}{\partial \theta} \right) (\log h) = 2h^{-1} - h_{\theta \theta} h_t^2 h^{-2}. \] (4.10)

Now we consider the operator
\[ \mathcal{L} = (h + R)^{-1} \left( L + h \frac{\partial^2}{\partial t^2} - d \frac{\partial}{\partial \theta} \right), \]
and the functions
\[ G = \log h, \quad F = (2h^{-1} - h_{\theta \theta} h_t^2 h^{-2})(R + h)^{-1}. \] (4.11)

Now if \( h \) achieves its maximum at \((\bar{\theta}, t)\) we have that
\[ h_{\theta \theta}(\bar{\theta}, t) \leq 0, \]
and so by (4.11)
\[ F(\bar{\theta}, t) \geq h^{-2}(\bar{\theta}, t) = \psi(\log \delta(t)). \]

Now by applying Lemma 1 to the function \( \phi = \log \delta \) we prove (4.3). The proof of (4.4) is similar. Since \( \delta \) is a continuous function in \([t_0, t_1]\) by (4.3) it follows that \( \log \delta \) is convex.

**Remark.** Similar arguments in the proof of Theorem (2.3) can be applied to (4.3), (4.4) to obtain
\[ \sigma(t) \leq y(t) \leq \delta(t) \quad \text{for} \quad t \in (t_0, t_1) \]
where \( y(t) \) is a catenary function solution to the equation
\[ \frac{d^2}{dt^2} \log y(t) = y^{-2}(t) \]
and satisfying
\[ \sigma(0) \leq y(0) \leq \delta(0), \quad \sigma(1) \leq y(1) \leq \delta(1). \]

In the following theorem we give sharp estimates of the gradients \( h_t \) of the level curves \( \Gamma_t \).
So let us consider the function

\[ \gamma(t) = \max \{ \sinh^{-1}(h_\tau) \} \]  \hspace{1cm} (4.12)

and

\[ \eta(t) = \min \{ \sinh^{-1}(h_\tau) \} \]  \hspace{1cm} (4.13)

where \( z = \sinh^{-1}(y) \) is the inverse function of \( y = \sinh(z) \). By computation one can show directly that if \( S \) is a radial minimal surface then

\[ \sinh^{-1}(h_\tau) \] is constant on the level circles \( \{ u = t \} \),

and

\[ \gamma(t) = \eta(t) \text{ is a linear function of } t. \]

This follows also from the following:

**Theorem 4.2.** If \( S \) is a minimal surface satisfying (1.2) then

\[ \gamma(t) \text{ is a convex function in } [t_0, t_1] \]

and

\[ \eta(t) \text{ is a concave function in } [t_0, t_1]. \]

Moreover \( \gamma \equiv \eta \) if and only if \( S \) is a radial minimal surface.

**Proof.** For simplicity, we set

\[ E(h_\tau) = \sinh^{-1}(h_\tau) \]  \hspace{1cm} (4.14)

By differentiating (2.8) with respect to \( t \) we have

\[ (h_\tau + h_{\delta\theta\tau}) h_{\tau\tau} + Rh_{\tau\tau\tau} - 2h_{\mu\tau} h_{\tau\mu} = 2h_{\tau\tau}. \]  \hspace{1cm} (4.15)

So by considering the operator \( L \) defined by (3.8) we have

\[ L(h_\tau) = h_\tau h_{\tau\tau}. \]  \hspace{1cm} (4.16)

By (4.14) and (4.16) we get

\[ L(E(h_\tau)) = E'(h_\tau) L(h_\tau) + E''(h_\tau) \cdot (ah_{\tau\tau} - 2bh_{\tau\mu} h_{\tau\mu} + ch_{\tau\tau}^2), \]

and since

\[ E'(h_\tau) = (1 + h_\tau^2)^{-1/2}, \quad E''(h_\tau) = - (1 + h_\tau^2)^{-3/2} h_\tau, \]
we derive

\[ L(E(h_t)) = h_t h_{tt}(1 + h_t^2)^{-1/2} - (1 + h_t^2)^{-3/2} h_t h_{tt}^2 \cdot R + d \frac{\partial}{\partial \phi}(h_t), \quad (4.17) \]

where \( d \) is a bounded function in \([0, 2\pi) \times (t_0, t_1)\). Now at the point \((\bar{\phi}, t)\) where \( h_t \) achieves its maximum or minimum on \( \Gamma_t \), we have \( h_{tt}(\bar{\phi}, t) = 0 \). From (2.8) it follows that

\[ R(\bar{\phi}, t) = (1 + h_t^2) h_{tt}^{-1}(\bar{\phi}, t), \]

and by (4.17) we get

\[ L(f(h_t)) |_{(\bar{\phi}, t)} = 0. \]

Now Lemma 1 applies to the functions \( \gamma \) with

\[ L = L - d \frac{\partial}{\partial \phi}, \quad G = f(h_t), \]

\[ F = h_t h_{tt}(1 + h_t^2)^{-3/2} [1 + h_t^2 - Rh_{tt}], \]

\[ \psi \equiv 0. \]

And we get

\[ D^2 \gamma \geq 0 \text{ in } (t_0, t_1). \quad (4.18) \]

Similarly we prove

\[ D^2 \eta \leq 0 \text{ in } (t_0, t_1). \quad (4.19) \]

Since \( \gamma \) and \( \eta \) are continuous on \((t_0, t_1)\), by (4.18) and (4.19) the proof of theorem 4.2 is obtained. \( \blacksquare \)

Remark. In the case that \( S \) is in the form \( z = u(x, y) \) then by (2.4) we have that \( 1/|\nabla u| = |h_t| \). So Theorem (4.2) implies estimates for \(|\nabla u|\) depending only on \( \gamma(t_0), \gamma(t_1), \eta(t_0), \eta(t_1) \); i.e., depending only on \( \max\{|\nabla(x)|: x \in \partial D_i\} \) and on \( \min\{|\nabla u(x)|: x \in \partial D_i\}, i = 0, 1 \).

APPENDIX

For simplicity and convenience of the reader we report in the form used in this paper one lemma related to maximum principle arguments.

Let set \( B \) the quotient set \( B = \mathbb{R}/2\pi \mathbb{Z} \) and let \( Q = B \times (t_0, t_1) \).
Let $F(\theta, t)$ and $G(\theta, t)$ be regular functions in $Q$ such that

$$\mathcal{L}(G(\theta, t)) = F(\theta, t), \quad (\theta, t) \in Q,$$  \hspace{1cm} (5.1)

where $\mathcal{L}$ is an elliptic operator of the form

$$\mathcal{L} = \alpha \frac{\partial^2}{\partial \theta^2} + \beta \frac{\partial^2}{\partial \theta, t} + \lambda \frac{\partial^2}{\partial t^2} + \gamma \frac{\partial}{\partial \theta},$$

with regular coefficients $\alpha, \beta, \gamma$. Let $\psi$ be a given real function on $\mathbb{R}$.

**Lemma 1.** If $\phi(t) = \max\{G(\theta, t) \mid \theta \in B\}$ is a continuous function in $[t_0, t_1]$ and if at any point $(\bar{\theta}, t)$ such that

$$\phi(t) = G(\bar{\theta}, t)$$

we have

$$F(\bar{\theta}, t) \geq \psi(\phi(t)),$$  \hspace{1cm} (5.2)

then $\phi(t)$ satisfies the following differential inequality:

$$D^2\phi(t) \geq \psi(\phi(t)).$$  \hspace{1cm} (5.3)

**Proof.** Indeed the proof follows by standard maximum principle arguments (c.f. [10, p. 131–136]), so we sketch the proof.

Let set the function

$$\Phi(\theta, t) = \phi(t), \quad (\theta, t) \in Q.$$

So by definitions

$$G(\theta, t) \leq \Phi(\theta, t), \quad (\theta, t) \in Q.$$  \hspace{1cm} (5.5)

Now let $t$ be fixed. At any point $(\bar{\theta}, t)$ such that

$$G(\bar{\theta}, t) = \Phi(\bar{\theta}, t),$$

we have

$$G_{\theta}(\bar{\theta}, t) = 0.$$  \hspace{1cm} (5.6)

Let us consider now the generalized elliptic operator

$$\mathcal{\tilde{L}} = \alpha \frac{\partial^2}{\partial \theta^2} + \beta \frac{\partial^2}{\partial \theta, t} + D_{\lambda} + \gamma \frac{\partial}{\partial \theta},$$
where for any real function \( v \) on \( Q \), \( D_\theta v \) is the generalized second derivative of \( v(\theta, \cdot) \) with respect to \( t \), \( \theta \) fixed in \( B \).

By (5.5) and (5.6) we have that

\[
P_G(\theta, t) \leq P\Phi(\theta, t) = D^2\phi(t). \tag{5.8}
\]

Since \( G \) is regular \( \mathcal{L}G = \mathcal{L}G \) and so (5.3) follows from (5.1), (5.2) and (5.8).

**ACKNOWLEDGMENTS**

This research was performed while the author was visiting Cornell University. He wishes to thank Cornell University for its hospitality and Professor L. E. Payne for his advice on the several fruitful discussions about the problems here presented.

**REFERENCES**