Total Positivity and Neville Elimination

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ABSTRACT

Neville elimination is described in terms of Schur complements of matrices and used to improve some well-known characterizations of totally positive and strictly totally positive matrices.

1. INTRODUCTION

Special types of matrices become interesting as soon as they play an important role in various branches of mathematics or other sciences. Totally positive matrices have become increasingly important in approximation theory and other fields. For a comprehensive survey of this subject from an algebraic point of view, complete with historical references, see [1].

A real $n \times m$ matrix $A$ is called totally positive (strictly totally positive) iff all subdeterminants of $A$ are nonnegative (positive). Totally positive (strictly totally positive) matrices will be referred to as TP (STP) matrices. In this paper we are concerned with practical characterizations of these types of matrices. Our approach is related to interpolation techniques, namely the Neville-Aitken technique [4], whose interpretation in the solution of linear systems gives rise to the so-called Neville elimination [5, 8].

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The essence of Neville elimination is to produce zeros in a column of a
matrix by adding to each row an appropriate multiple of the previous one
(instead of using a fixed row with a fixed pivot as in Gaussian elimination).
Eventual reorderings of the rows of the matrix may be necessary, as will be
made precise in Section 2. That section also includes several technical
properties of the elimination procedure, which are also used in subsequent
sections.

In Section 3 Neville elimination is used to prove a characterization of
total positivity of nonsingular matrices (Theorem 3.2) by means of the signs
of some of the minors. This characterization improves Theorem 1.3 of [2],
which requires the signs of a greater number of minors.

A well-known characterization of STP matrices, due to Fekete [3] (see
also [6]), states that a matrix is STP iff all its minors with consecutive rows
and columns are positive. In Section 4 (Theorem 4.1) we prove that one has
only to check minors with consecutive rows and columns that include the
first row or the first column. We call them initial minors. From this result we
deduce a test which considerably reduces the complexity of other tests for
STP which are used [5, 6]. In Section 4 we shall also prove a very simple
characterization (Theorem 4.3) of TP matrices which are STP. The equiva-
lence of condition (3) of Theorem 4.3 for TP matrices with condition (3) of
Theorem 4.1 was proved by K. Metelmann [7] without clarifying the equiva-
lence of either condition with the strict total positivity of the matrix.

Finally, in Section 5 we characterize arbitrary totally positive matrices in
terms of Neville elimination and give further characterizations of totally
positive nonsingular matrices.

2. NOTATION AND PRELIMINARY RESULTS

In general, we shall use similar notations to that of [1]. Given \( k, n \in \mathbb{N} \),
\( k \leq n \), \( Q_{k,n} \) will denote the totality of strictly increasing sequences of \( k \)
natural numbers less than or equal to \( n \):

\[
\alpha = (\alpha_i)_{i=1}^k \in Q_{k,n} \quad \text{if} \quad (1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \ (\leq n)).
\]

For each \( \alpha \in Q_{k,n} \), its dispersion number \( d(\alpha) \) is defined by

\[
d(\alpha) := \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i + 1) = \alpha_k - \alpha_1 - (k - 1),
\]
with the convention \( d(\alpha) = 0 \) for \( \alpha \in Q_{1,n} \). Let us observe that in general \( d(\alpha) = 0 \) means that \( \alpha \) consists of \( k \) consecutive integers.

Let \( n, m, k, l \) be natural numbers with \( k \leq n \) and \( l \leq m \), let \( A \) be a real \( n \times m \) matrix, and let \( \alpha \in Q_{k,n} \) and \( \beta \in Q_{l,m} \). Then \( A[\alpha|\beta] \) is by definition the \( k \times l \) submatrix of \( A \) containing rows numbered by \( \alpha \) and columns numbered by \( \beta \). When \( \alpha = \beta \), \( A[\alpha|\alpha] \) is simply denoted by \( A[\alpha] \).

From now on, we shall frequently use upper triangular \( n \times m \) matrices \( U \) whose nonzero entries are confined to a staircase pattern or upper echelon form, that is, \( n \times m \) matrices \( U \) such that:

1. if the \( k \)th row is zero \( (k < n) \) then the rows below it are zero;
2. if \( u_{ij} \) is the first nonzero entry in the \( i \)th row, then \( u_{hj} = 0 \) \( \forall h \geq i \), and if \( u_{i'j'} \) is the first nonzero entry in the \( i' \)th row \( (i < i' \leq n) \), then \( j' > j \).

For simplicity we shall refer to such a matrix \( U \) as a u.e.f. matrix.

Moreover, we define a row-initial (respectively, column-initial) submatrix of \( A \) as the submatrix of \( A \) formed by consecutive initial rows (columns) and consecutive columns (rows). That is,

\[
A[\alpha|\beta] \quad \text{with} \quad d(\alpha) = 0 = d(\beta) \quad \text{and} \quad \alpha_1 = 1 \quad \text{(respectively,} \quad \beta_1 = 1). 
\]

We call the corresponding determinant a row-initial (respectively, column-initial) minor of \( A \). These determinants play a similar role in the study of STP to that of principal minors for positive definite symmetric matrices.

The main tool in most of our proofs consists of what we call Neville elimination. As we have mentioned in Section 1, it is a procedure to create zeros in a matrix by means of adding to a given row any multiple of the previous one.

More precisely, we describe this elimination method for any \( n \times m \) matrix \( A = (a_{ij})_{1 \leq i \leq n}^{1 \leq j \leq m} \). Let \( \tilde{A}_1 := (a_{ij})_{1 \leq i \leq n}^{1 \leq j \leq m} \) be such that \( \tilde{a}_{ij} = a_{ij} \). If there are zeros in the first column of \( \tilde{A}_1 \), we carry the corresponding rows down to the bottom in such a way that the relative order among them is the same as in \( \tilde{A}_1 \). We denote the new matrix by \( A_1 = (a_{ij}^{'})_{1 \leq i \leq n}^{1 \leq j \leq m} \). If we have not carried any row down to the bottom, then \( A_1 := \tilde{A}_1 \). In both cases, let \( i_1 \) be \( i_1 := 1 \). The method consists in constructing a finite sequence of matrices \( A_k \) such that, for each \( A_k \), the submatrix formed by its \( k - 1 \) initial columns is a u.e.f.-matrix. In fact, if \( A_i = (a_{ij}^{'})_{1 \leq i \leq n}^{1 \leq j \leq m} \), then we introduce zeros in its \( t \)th column below the place \((i_1,t)\), thus forming

\[
\tilde{A}_{i+1} = (\tilde{a}_{ij}^{t+1})_{1 \leq i \leq n}^{1 \leq j \leq m}.
\]
For any \( j \) such that \( 1 \leq j \leq m \), we have

\[
\tilde{a}_{ij}^{t+1} := a_{ij}^t, \quad i = 1, 2, \ldots, i_t,
\]

\[
\tilde{a}_{ij}^{t+1} := a_{ij}^t - \frac{a_{it}^t}{a_{i-1,t}^t} a_{i-1,j}^t \quad \text{if} \quad a_{i-1,t}^t \neq 0, \quad i_t < i < n, \quad (2.1)
\]

\[
\tilde{a}_{ij}^{t+1} := a_{ij}^t \quad \text{if} \quad a_{i-1,t}^t = 0, \quad i_t < i < n.
\]

Observe that with our assumptions, \( a_{i-1,t}^t = 0 \) implies \( a_{it}^t = 0 \). Then we define

\[
i_{t+1} := \begin{cases} i_t & \text{if} \quad a_{i_t,t}^t \left( = \tilde{a}_{i_t,t}^{t+1} \right) = 0, \\ i_t + 1 & \text{if} \quad a_{i_t,t}^t \left( = \tilde{a}_{i_t,t}^{t+1} \right) \neq 0. \end{cases} \quad (2.2)
\]

We shall refer to such indices \( i_t \) of a matrix \( A \) as \( i_t(A) \).

If \( \tilde{A}_{t+1} \) has zeros in the \((t+1)\)th column in the row \( i_{t+1} \) or below it, we will carry these rows down as we have done with \( \tilde{A}_t \). The matrix obtained in this way will be denoted by \( A_{t+1} = (a_{ij}^{t+1})_1 \leq i \leq n \). Of course, if there is no row that has been carried down, then \( A_{t+1} := \tilde{A}_{t+1} \). After a finite number of steps we get \( \tilde{A}_{t-1}, A_{t-1}, \) and

\[
\tilde{A}_t = U \quad (\tilde{t} \leq m + 1), \quad (2.3)
\]

where \( U \) is a u.e.f. matrix. In this process the element

\[
p_{ij} := a_{ij}^t, \quad 1 \leq j \leq m, \quad i_j \leq i \leq n, \quad (2.4)
\]

will be called the \((i, j)\) pivot of the Neville elimination of \( A \).

**Remark 2.1.** We shall frequently consider matrices \( A \) in which the transformation from \( A \) to \( U \) by the Neville elimination does not need row exchanges. Obviously, this is equivalent to \( \tilde{A}_t = A_t \forall t \), that is,

\[
\tilde{a}_{ij}^t = 0 \quad \text{with} \quad i \geq j \Rightarrow \tilde{a}_{kj}^t = 0 \quad \forall k \geq i.
\]

This happens, in particular, when all the pivots are nonzero.
Let us now consider the particular case in which the pivots $p_{ii}$ ($1 \leq i \leq \min(m, n)$) are not zero and the Neville elimination of $A$ can be carried out without row exchanges. In this case

$$i_t = t \quad \forall t$$

in (2.1) and (2.2), and

$$A_{t+1} = \begin{pmatrix}
  a_{11}^1 & a_{12}^1 & \cdots & \cdots & a_{1m}^1 \\
  0 & a_{22}^1 & \cdots & \cdots & a_{2m}^1 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  \vdots & \ddots & 0 & a_{t+1,t+1}^1 & \cdots & a_{t+1,m}^1 \\
  0 & 0 & 0 & a_{n,t+1}^1 & \cdots & a_{nm}^1
\end{pmatrix}$$

The following lemma expresses the pivots as Schur complements of submatrices of $A$ (for the definition of Schur complement, see for example p. 172 of [1]).

**Lemma 2.2.** Let the real $(n+1) \times (m+1)$ matrix $T$ ($n \leq m$) be partitioned as

$$T = \begin{pmatrix}
  L & M \\
  N & S
\end{pmatrix},$$

where $L, M, N, S$ are $n \times m$, $n \times 1$, $1 \times m$, $1 \times 1$ matrices, respectively, with rank $L = n$. Suppose that, by adding to each row a linear combination of the previous ones, $T$ is transformed into

$$\bar{T} = \begin{pmatrix}
  \bar{L} & \bar{M} \\
  0 & \bar{S}
\end{pmatrix}$$

with the same partition as $T$.

If $P$ is a nonsingular submatrix of order $n$ of $L$, then $\bar{S}$ is the Schur complement of $P$ in a submatrix of $T$. Therefore, there exists $(j_1, j_2, \ldots, j_n) \in Q_{n,m}$ such that

$$\bar{S} = \frac{\det T[1, 2, \ldots, n, n + 1 | j_1, j_2, \ldots, j_n, m + 1]}{\det T[1, 2, \ldots, n | j_1, j_2, \ldots, J_n]}.$$
Proof. We have \( \overline{T} = \overline{ET} \), where \( \overline{E} \) is an \((n + 1) \times (n + 1)\) matrix

\[
\overline{E} = \begin{pmatrix} X & 0 \\ Y & 1 \end{pmatrix},
\]

\(X\) being an \(n \times n\) matrix, \(Y\) being a \(1 \times n\) matrix, and \(0\) denoting the \(n \times 1\) zero matrix.

In particular, if \(P = L[1, 2, \ldots, n|j_1, j_2, \ldots, j_n]\),

\[
T[1, 2, \ldots, n, n + 1|j_1, j_2, \ldots, j_n, m + 1] = \begin{pmatrix} P & M \\ R & S \end{pmatrix},
\]

and

\[
\overline{T}[1, 2, \ldots, n, n + 1|j_1, j_2, \ldots, j_n, m + 1] = \begin{pmatrix} \overline{P} & \overline{M} \\ 0 & \overline{S} \end{pmatrix},
\]

then \(\overline{S} = YM + S, 0 = YP + R\). Therefore \(\overline{S} = S - RP^{-1}M\) is the Schur complement of \(P\) in \(T[1, 2, \ldots, n, n + 1|j_1, j_2, \ldots, j_n, m + 1]\) and it can be expressed as in (2.6).

From now on when we say that \(A\) is a matrix "without row exchanges" in Neville elimination, we mean that it is possible to carry out the Neville elimination without exchanging rows.

**Lemma 2.3.** Let \(A\) be an \(n \times m\) matrix without row exchanges in Neville elimination and let \(B\) be a column-initial submatrix of \(A\), that is,

\[
B := A[\alpha|1, 2, \ldots, l]
\]

for some \(\alpha \in Q_{k,n} \), \(1 \leq k \leq n\), with \(d(\alpha) = 0\) and \(1 \leq l \leq m\). Then, denoting \(i'_t := i_t(B)\) (see 2.2)), we have, for each step of the Neville elimination of \(B\),

\[
B_t[i'_t, i'_t + 1, \ldots, \alpha_k|t, t + 1, \ldots, l] - A_t[i'_t, i'_t + 1, \ldots, \alpha_k|t, t + 1, \ldots, l], \quad (2.7)
\]

and \(B\) is a matrix without row exchanges in Neville elimination.

Proof. The equality is obvious for \(t = 1\), and it is easy to prove it by induction, taking (2.1) and (2.2) into account. The last assertion is an immediate consequence of Remark 2.1. \(\blacksquare\)
This result is not true if $B$ is not a column-initial submatrix of $A$, as can be seen for

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$ 

**Lemma 2.4.** Let $A$ be an $n \times m$ ($n \geq 3$) matrix without row exchanges in Neville elimination. If the last row is not a linear combination of the $n - 2$ previous ones, but is a linear combination of the $n - 1$ initial rows, then these $n - 1$ rows are linearly independent.

**Proof.** In this proof we shall use the following fact: a row in the u.e.f. matrix $U$ obtained after the Neville elimination of a matrix $A$ is zero if and only if it is a linear combination of the previous rows.

From the assumptions of the lemma we deduce that the last row of $U$ is zero. We shall prove that the previous one has to be nonzero, and the lemma will follow. In fact, denoting the $i$th row of $A$ ($i = 1, 2, \ldots, n$) by $r_i$, if the $(n - 1)$th row of $U$ were zero, then $r_{n-1}$ would be a linear combination of $r_1, \ldots, r_{n-2}$, and by the assumptions of the lemma we could deduce that it is also a linear combination of $r_2, \ldots, r_{n-2}$. By Lemma 2.3, we could carry out the Neville elimination of $A[2, 3, \ldots, n|1, 2, \ldots, m]$ without row exchanges until we obtained a u.e.f. matrix $V$. Then the penultimate row of $V$ would be zero, but its last row would not be zero, by the assumptions again, and $V$ would not be a u.e.f. matrix.

Let us observe that the lemma does not hold if we replace “Neville elimination” by “Gaussian elimination,” as can be seen from the matrix

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

**Lemma 2.5.** Let $A$ be an $n \times m$ matrix without row exchanges in Neville elimination. If $1 \leq i \leq t$ (see (2.3)), $i_1 \leq i \leq n$, and $t \leq k \leq m$, then either $a'_{ik} (= a'_{ik}) = a_{ik}$ or there exist $l (2 \leq l \leq t)$ and $(j_1, j_2, \ldots, j_{l-1}) \in Q_{l-1, t-1}$ such that

$$\bar{a}_{ik} (= a'_{ik}) = \frac{\det A[i - l + 1, \ldots, i - 1, i|j_1, j_2, \ldots, j_{l-1}, k]}{\det A[i - l + 1, \ldots, i - 1|j_1, j_2, \ldots, j_{l-1}]}.$$
Proof. After \( t - 1 \) elimination steps, \((a_{i_1}, a_{i_2}, \ldots, a_{i_{t-1}}, a_{i_k})\) has been transformed into \((0, \ldots, 0, a_{i_k}')\). If \((a_{i_1}, a_{i_2}, \ldots, a_{i_{t-1}})\) were zero, then \(a_{i_k} = a_{i_k}'\), and the first case of the lemma is proved.

Otherwise, either the vector \(v = (a_{i_1}, \ldots, a_{i_{t-1}})\) is a multiple of the nonzero vector \((a_{i-1,1}, \ldots, a_{i-1,t-1})\) or there exists an \(l\) with \(3 \leq l \leq i_t\) such that \(v\) is a linear combination of the vectors \((a_{k_1}, \ldots, a_{k_{t-1}})\) with \(k = i - l + 1, \ldots, i - l + 2, \ldots, i - 1\) and not with \(k = i - l + 2, \ldots, i - 1\). In both cases (in the second one by Lemma 2.4) there exists an \(l\) with \(2 \leq l \leq i_t\) such that

\[
\text{rank } A[i-l+1, \ldots, i-1, i_{t-1}] = l - 1.
\]

Let us take the submatrix \(Q := A[i-l+1, i-l+2, \ldots, i-1, i_{t-1}, \ldots, k]\); by Lemma 2.3, \(Q\) is a matrix without row exchanges in Neville elimination, and after \(t - 1\) steps, at most, its last row will be \((0, \ldots, 0, a_{i_l}', \ldots, a_{i_k}')\).

Let \(T\) be the matrix

\[
T := A[i-l+1, \ldots, i-1, i_{t-1}, i_{t-2}, \ldots, i_{t-k}].
\]

If we carry out the same operations with the rows of \(T\) as with the rows of \(Q\), the last row of \(T\) becomes \((0, 0, \ldots, 0, a_{i_l}', \ldots, a_{i_k}')\), and we then apply Lemma 2.2.

Let us now consider the particular case in which all the pivots \(p_{ij}\) [with \(i_j = j\ \forall j\) in (2.4)] are nonzero (so that the Neville elimination can be applied without row exchanges).

**Lemma 2.6.** Let \(A\) be an \(n \times m\) matrix. Carrying out the Neville elimination of \(A\), we have that:

1. If all the pivots are nonzero, then

\[
a_{ik}^1 = a_{ik}, \quad 1 \leq i \leq n, \quad 1 \leq k \leq m,
\]

\[
a_{ik}^t = \frac{\det A[i-t+1, i-t+2, \ldots, i-1, i_{t-1}, i_{t-2}, \ldots, k]}{\det A[i-t+1, i-t+2, \ldots, i-1, i_{t-1}, \ldots, t-1]}, \quad (2.8)
\]

\[2 \leq t \leq \min\{n, m\}, \quad t \leq i \leq n, \quad t \leq k \leq m.
\]

2. If \(\det A[\alpha_1, \ldots, k_1, \ldots, k_m] \neq 0\ \forall \alpha \in Q_{k,n}\) with \(d(\alpha) = 0\ and \ 1 \leq k \leq \min(n, m)\), then all the pivots are nonzero.
Proof. (1): From Lemma 2.3 we deduce that the matrix

\[ P := A[i - t + 1, i - t + 2, \ldots, i - t + l - 1, i - t + 2, \ldots, t - 1] \]

is nonsingular, and (2.8) is a consequence of Lemma 2.2.

(2): Carrying out the elimination in the order indicated in the beginning of this section, if the first zero pivot appears in position \((i, k)\), we can write it in the form (2.8) because all the pivots in the previous columns were nonzero. Thus we obtain a contradiction to our hypothesis.

Remark 2.7. Let us observe that the lemma shows that the column-initial minors are nonzero if and only if all the pivots of the Neville elimination are nonzero. Moreover, if those minors are nonzero, Neville elimination can be carried out without row exchanges. This suggests that the column-initial minors play a role in the Neville elimination similar to that of the leading principal minors in the Gaussian elimination.

We finish this section by introducing the concept of complete Neville elimination (CNE) of an \(n \times m\) matrix \(A\), which consists in carrying out the Neville elimination of \(A\) until one arrives at the u.e.f. matrix \(U\), and afterwards proceeding with the Neville elimination of \(U^T\) (the transpose of \(U\)). The last part is equivalent to performing the Neville elimination of \(U\) "by columns." When we say that the CNE of \(A\) is possible without row or column exchanges, we mean that there have not been any row exchanges in the Neville elimination of either \(A\) or \(U^T\).

The \((j, i)\) pivot of the Neville elimination of \(U^T\) will be referred to as the \((i, j)\) pivot of the second phase of the CNE of \(A\), and it will be denoted by \(q_{ij}\). The pivot \(p_{ij}\) will be called the \((i, j)\) pivot of the first phase of the CNE of \(A\).

Let us observe that, after the CNE of \(A\), we get a matrix whose nonzero elements are at the beginning of the main diagonal.

3. A DETERMINANTAL CRITERION FOR INVERTIBLE TP MATRICES

In this section we characterize the nonsingular TP matrices by the sign of some of its minors with consecutive rows. We begin with the following
auxiliary result:

**Lemma 3.1.** Let $A$ be an $n \times m$ matrix such that the Neville elimination from $A$ to $\bar{A}$ is possible without row exchanges, and let us assume that

\[ \bar{a}_{i+1,t}^t > 0, \quad \bar{a}_{i,t}^t = 0, \quad \bar{a}_{ik}^t > 0 \quad \text{with} \quad i \leq t < n, \quad t < k \leq m. \]

Then there exists a submatrix $A[\alpha|\beta]$ of $A$ with $\alpha, \beta \in Q_{k,s}, 1 \leq k \leq s := \min\{n, m\}$, $d(\alpha) = 0$, and $\alpha_1 - \beta_1 \geq i - t$ such that

\[ \det A[\alpha|\beta] < 0, \]

and so $A$ is not TP.

**Proof.** If $(a_{i1}, \ldots, a_{i,t-1}) = (0, 0, \ldots, 0)$, then by our assumptions we have

\[ \begin{pmatrix} a_{it} & a_{ik} \\ a_{i+1,t} & a_{i+1,k} \end{pmatrix} = \begin{pmatrix} 0 & \bar{a}_{ik}^t \\ \bar{a}_{i+1,t}^t & \bar{a}_{i+1,k}^t \end{pmatrix}, \]

and the submatrix $A[i, i + 1|t, k] \ v{\text{already satisfies the conditions of the lemma.}}$

Otherwise, we can reason as in the proof of Lemma 2.5 and deduce that there exists a nonsingular matrix

\[ P := A[i - l + 1, i - l + 2, \ldots, i - 1|j_1, j_2, \ldots, j_{l-1}] \]

[with $2 \leq l \leq i$, and $(j_1, j_2, \ldots, j_{l-1}) \in Q_{l-1, i-1}$] and that after some steps of the Neville elimination of the matrix

\[ M := A[i - l + 1, i - l + 2, \ldots, i - 1, i, i + 1|j_1, j_2, \ldots, j_{l-1}, t, k] \]

we obtain

\[ \bar{M} = \begin{pmatrix} F & D \\ 0 & \bar{a}_{ik}^t \\ \bar{a}_{i+1,t}^t & \bar{a}_{i+1,k}^t \end{pmatrix}, \]
where \( \det F = \det P \) and \( \det M = \det M \). Then, by our assumptions,

\[
\text{sgn} \det M = -\text{sgn} \det P,
\]

and so one of these two minors is negative. It is immediate to check that both matrices satisfy the conditions of the submatrix \( A[\alpha|\beta] \) of the lemma.

**Theorem 3.2.** Let \( A \) be a nonsingular matrix of order \( n \). Then \( A \) is TP if and only if it satisfies, simultaneously, the following conditions:

1. \( \det A[\alpha|\beta] > 0 \) for all \( \alpha, \beta \in Q_{k,n} \) such that \( d(\alpha) = 0 \) and either \( \alpha_1 = 1 \) or \( \alpha_l > \beta_l \) for all \( l \leq k \);
2. \( \det A[1,2,\ldots,k] > 0 \) for all \( k \in \{1,2,\ldots,n\} \).

**Proof.** If \( A \) is TP, it satisfies (1), and also (2) by Corollary 3.8 of [1].

Before proving the converse, we observe that (2) implies that, if we have carried out the Neville elimination of \( A \) in the initial \( j - 1 \) columns \( (2 \leq j \leq n) \) without row exchanges and if the \( r \)th \((r > j)\) row is

\[
\begin{pmatrix}
0, & 0, & \ldots, & 0, & a_{i,r}, & \ldots
\end{pmatrix},
\]

then \( a_{rr} > 0 \). In fact, in that process we have that

\[
\det A[1,2,\ldots,r] = a_{rr}^j \det A[1,2,\ldots,r-1]
\]

and, by (2), \( a_{rr}^j > 0 \).

Assuming now that \( A \) satisfies (1) and (2), we shall prove that \( A \) is a matrix without row exchanges in the Neville elimination.

Let us suppose that we have carried out the Neville elimination of \( A \) without exchanges until we arrive at \( A_{t-1} \) and that row exchanges are necessary in order to get \( A_t \). By Remark 2.1 there exist \( i \) \([t \leq i \leq n]\) by (2.5) and (2) such that \( a_{ii}^t = 0 \) and \( a_{i+1,t}^t \neq 0 \). From (1) and Lemma 2.5 we deduce that \( a_{i+1,t}^t > 0 \) and so \( a_{i+1,t}^t > 0 \).

Let \( a_{ik}^t \) be the first nonzero element in the \( i \)th row of \( A_t \). Obviously, we have \( t < k \leq n \), and, by the observation at the beginning of the proof, we can deduce that \( k < i \) and that, if \( k = i \), \( a_{ii}^t > 0 \). If \( k < i \), by (1) and Lemma 2.5 we get \( a_{ik}^t > 0 \). Then the assumptions of Lemma 3.1 are satisfied, and one can see, in its proof, that the submatrix \( A[\alpha|\beta] \) with negative determinant is one of the following three submatrices: \( A[i,i+1|t,k], A[i-l+1,i-l+2,\ldots,i-1], A[i+1,i+1|j_1,\ldots,j_{i-1},t,k], \) or \( A[i-l+1,i-l+2,\ldots,i-1|j_1,j_2,\ldots,j_{i-1}] \). Since \( k < i \), we have \( i+1 > k \) in the first two cases,
and in the third one, as we also have $t < k$, we get $t < i$ and so $j_{l-1} < i - 1$. In conclusion, there exists a submatrix $A[\alpha|\beta]$ with negative determinant, $d(\alpha) = 0$, and $\alpha_k > \beta_k$ (and so $\alpha_r > \beta_r$ for $1, 2, \ldots, k$), which is a contradiction to (1). Therefore, we have proved that, if $A$ satisfies (1), (2), then it is a matrix without row exchanges in the Neville elimination. Moreover the pivots $p_{it}$ ($i \geq t$) defined in (2.4) are nonnegative. In fact, if $i > t$ it is a consequence of Lemma 2.5 and (1), and if $i = t$ it follows from the observation at the beginning of the proof.

Let us see now that the u.e.f. matrix $U$ (which in this case is upper triangular) obtained from $A$ is TP. By Theorem 1.4 of [2] it is sufficient to show that the determinants of the matrices

$$U[\alpha|\beta], \quad \alpha, \beta \in Q_{k,n}, \quad d(\alpha) = 0, \quad \alpha_1 = 1,$$

are nonnegative for $1 \leq k \leq n$. But it is evident that, under the conditions of (3.1), we have

$$\det U[\alpha|\beta] = \det A[\alpha|\beta]$$

and so by (1) $U$ is TP.

Finally, let us observe that $U$ is the product of a finite number of elementary matrices premultiplying $A$, and that, inverting the process, we obtain $A$ by premultiplying $U$ by elementary matrices which are TP by Theorem 2.3 of [1]. Therefore (by Theorem 3.1 of [1]) $A$ is TP.

**Remark 3.3.** Theorem 1.3 of [2] gives a characterization of nonsingular TP matrices using all the minors formed by consecutive rows. Let us observe that the previous theorem reduces the number of minors to be checked.

In Remarks 6.3 and 6.4 of [2] it has already been pointed out that Theorem 1.3 of [2] cannot be improved upon in certain other ways.

**Remark 3.4.** In Theorem 4.1 we shall obtain a determinantal criterion for STP matrices which will be analogous to the one appearing in [6, p. 85] for triangular matrices. This last criterion may be reformulated as stating that a triangular TP matrix is "strictly totally positive" (meaning that all its minors $A[\alpha|\beta]$ with $\alpha, \beta \in Q_{k,n}$ and $\alpha_l \geq \beta_l$, $\forall l \in \{1, 2, \ldots, k\}$ are positive) if and only if the column-initial minors of $A$ are positive. Curiously, in the case of TP matrices, it is not possible to extend the criterion for triangular nonsingular TP matrices (that is, Theorem 1.4 of [2]) to nonsingular TP matrices, as is seen in the examples for Remark 6.3 of [2].
4. CHARACTERIZATIONS AND TESTS FOR STP MATRICES

Now we shall give a result which improves the classical characterization of STP matrices due to Fekete [3] (another proof thereof can be found in Theorem 2.5 of [1]): an \( n \times m \) matrix is STP if and only if all its minors formed by consecutive rows and columns are strictly positive. The next criterion considerably reduces the number of minors to be checked.

**Theorem 4.1.** Let \( A \) be an \( n \times m \) matrix. Then the following conditions are equivalent:

1. \( A \) is STP.
2. CNE of \( A \) can be carried out without row or column exchanges, and all the pivots are positive.
3. The row-initial and column-initial minors of \( A \) are positive.

**Proof.** (1) \( \Rightarrow \) (2): By Lemma 2.6(2) all the pivots of the Neville elimination of \( A \) (terminating at the u.e.f. matrix \( U \)) are nonzero, and by Lemma 2.6(1) and Remark 2.1, elimination has not required row exchanges and the pivots are positive. Since every row-initial minor of \( A \) has the same value as the corresponding one of \( U \), we may proceed with the Neville elimination of \( U^T \) as we have done with that of \( A \), and in consequence we obtain (2).

(2) \( \Rightarrow \) (3): This is immediate using Lemma 2.6(1).

(3) \( \Rightarrow \) (1): Proceeding as in the first part of this proof, we can deduce that the CNE can be carried out without row or column exchanges and that the pivots are positive. Therefore, the matrix \( D \) finally obtained has only nonzero (positive) elements on the main diagonal, and with the same reasoning as in the last part of the proof of Theorem 3.2 we deduce that \( A \) is TP.

Since the strict total positivity of \( A \) is equivalent to that of its transpose, we may assume that \( n \leq m \). Now, we are going to prove by induction on \( m \) that, if \( A \) is a matrix satisfying (3) (and consequently a TP matrix), then it is also STP.

The property is obvious when \( m = 1 \). Assume it is true with \( m - 1 \) in place of \( m \). Let us suppose that there exists an \( n \times m \) matrix \( A \) which does not satisfy the property. By Fekete's criterion (see Theorem 2.5 of [1]) there is a submatrix \( M \) of \( A \) with consecutive rows and columns having a zero determinant. We shall discuss all the possibilities and we shall always get a contradiction. If \( M \) is of the form

\[
M = A[i, i + 1, \ldots, i + l|m - l, m - l + 1, \ldots, m]
\]
with $0 \leq l \leq m - 1$, $i + l \leq n$, that is, the last column of $A$ is one of the columns of $M$, then we form the submatrix

$$A_1 := A[1, 2, \ldots, i + l| m - l - i + 1, m - l - i + 2, \ldots, m].$$

Since $A_1$ is invertible and TP, by Corollary 3.8 of [1] we have $\det M > 0$, contradicting our assumption.

If $A$ is square ($n = m$) and if

$$M = A[n - l, n - l + 1, \ldots, n| i, i + 1, \ldots, i + l]$$

(4.1)

with $0 \leq l \leq n - 1$, $i + l \leq m = n$, that is, if the last row of $A$ is one of the rows of $M$, we form

$$A_2 := A[n - l - i + 1, n - l - i + 2, \ldots, n| 1, 2, \ldots, i + l]$$

and, proceeding as in the previous case, we obtain $\det M > 0$, a contradiction.

Let us now consider the remaining case. In particular, the last column of $A$ does not appear in $M$. If $M$ is of the form (4.1) and $A$ is not square, then $n \leq m - 1$. Let $A_3$ be the $n \times (m - 1)$ matrix obtained by removing the last column of $A$. Then $M$ is a submatrix of $A_3$, and $A_3$ is a TP matrix satisfying (3). Therefore, by the induction hypothesis, $A_3$ is also STP and consequently $\det M > 0$, obtaining again a contradiction. Finally, if $M$ has elements neither of the last row nor of the last column of $A$, we form the $(n - 1) \times (m - 1)$ submatrix $A_4$ by removing the last row and column of $A$, and by the induction hypothesis, we again obtain a contradiction.

Remark 4.2. In [5] and [8] there is a computational test for $n \times m$ STP matrices based on Neville elimination, with $O(n^4)$ operations (if $n > m$), and another one based on Gaussian elimination with $O(n^5)$ operations. The first one was based on the Fekete's characterization of STP matrices. From the previous theorem we obtain a computational test with $O(n^3)$ operations.

The next theorem gives a criterion for strict total positivity of square TP matrices, reminiscent of Theorem 4.2 of [1] (which shows when a square TP matrix is oscillatory).
THEOREM 4.3. Let $A$ be a TP matrix of order $n$. Then the following conditions are equivalent:

1. $A$ is STP.
2. CNE of $A$ can be carried out without row or column exchanges, and the $(i,n)$ pivots and $(n,i)$ pivots ($1 < i < n$) are positive.
3. For $k = 1, 2, \ldots, n$
   
   $\det A[1, 2, \ldots, k|n-k+1, n-k+2, \ldots, n] > 0$, \hspace{1cm} (4.2)
   
   $\det A[n-k+1, n-k+2, \ldots, n|1, 2, \ldots, k] > 0$. \hspace{1cm} (4.3)

Proof. The implication (1) $\Rightarrow$ (2) has been proved in Theorem 4.1. Let us see now that (2) $\Rightarrow$ (3). By Remark 2.1 all the pivots of the CNE of $A$ are nonzero, and (4.3) is a consequence of Lemma 2.6(1). If $U$ is the u.e.f. matrix obtained from $A$ by Neville elimination, then proceeding analogously with $U^T$, we obtain (4.2).

Finally we shall prove that (3) $\Rightarrow$ (1). Let $A$ be a TP matrix of order $n$ satisfying (3), and let us suppose that there is a submatrix $M$ of $A$ of the type

$$M = A[i, i+1, \ldots, i+l|1, 2, \ldots, l+1]$$ \hspace{1cm} (4.4)

with $i + l < n$ and zero determinant. Then the submatrix of $A$

$$A_1 := A[i, i+1, \ldots, n|1, 2, \ldots, n-i+1]$$

is TP, and nonsingular by (3). Therefore (by Corollary 3.8 of [1]) det $M > 0$: a contradiction. In consequence, all submatrices of the type (4.4) (and those of the type

$$M = A[1, 2, \ldots, l+1|i, i+1, \ldots, i+l]$$

for similar reasons) have positive determinant. Then, by Theorem 4.1, $A$ is STP.

REMARK 4.4. Let us observe that, without requiring $A$ to be TP, (3) does not imply (1), as can be seen from the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 0 & 3 \end{pmatrix}.$$
On the other hand, we remark that both conditions (4.2), (4.3) are necessary. Consider, for example, any lower (respectively, upper) triangular matrix that is not STP but has nonnegative elements.

Finally, let us observe that Theorem 4.3 holds only for square matrices, as the example

\[
B = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}
\]

shows.

5. NEVILLE ELIMINATION AND TP MATRICES

In this section we shall characterize TP (not necessarily invertible) matrices by Neville elimination, and we shall provide additional characterizations of invertible TP matrices.

We shall begin with a definition of a class of matrices which we shall frequently use.

**Definition 5.1.** An \(n \times m\) matrix \(A\) satisfies the condition \(N_t\) if, whenever we have carried some rows down to the bottom in the Neville elimination process from \(A\) to \(\tilde{A}_t\), those rows were zero rows.

A matrix \(A\) satisfies the condition \(N\) if, whenever we have carried some rows to the bottom in the Neville elimination of \(A\), those rows were zero rows, and the same condition is satisfied in the Neville elimination of \(U^T\) (where \(U\) is the u.e.f. matrix obtained from \(A\) by the Neville elimination).

**Lemma 5.2.** Let \(A\) be an \(n \times m\) matrix satisfying the condition \(N_t\). If \(1 \leq t \leq i\) (see (2.3)), \(i \leq i \leq n\), and \(t \leq k \leq m\), then either there exists an \(h\) such that \(1 \leq h \leq n\) and \(a'_{ik} = a_{hk}\) or there exist submatrices \(B\) and \(C\) of \(A\) such that \(a'_{ik} = \det B / \det C\).

**Proof.** With our assumptions, we may observe the following facts in the Neville elimination process. While a row is not transformed into a zero row, it may go up in the matrix because other rows above it have become zero rows and so they have been carried down to the bottom. When a row has been transformed into a zero row (which means that it was a linear combination of the previous ones), it goes down to the last place; later, it can go up, but it will not go down again.
Let us construct a matrix $D'$ in the following way. We reorder the rows of $A$, carrying down to the bottom those rows of $A$ which go down in the Neville elimination process from $A$ to $A'$. Taking into account the above observations, we easily obtain the lemma by applying Lemma 2.5 to $D'$. 

With analogous reasoning and by adapting the proof of Lemma 3.1 we obtain the following

**Lemma 5.3.** Let $A$ be an $n \times m$ matrix satisfying the condition $N$. If $a_{i+1,t} > 0$, $a_{it} = 0$, $a_{ik} > 0$ (with $i_t \leq i < n$, $t < k \leq m$), then there is a submatrix $A[\alpha|\beta]$ with $\alpha, \beta \in Q_{k,s}$ ($1 \leq k \leq s = \min(n,m)$) such that $\det A[\alpha|\beta] < 0$, and so $A$ is not TP.

By these lemmas, it is possible to characterize TP matrices by Neville elimination. Let us remark that, with elimination processes close to this one (although with different proofs and terminology), several authors have characterized these matrices in various ways: see, for example, the main theorem of [10], p. 125 of [9], or the proof of Theorem 1.1 of [2].

**Theorem 5.4.** Let $A$ be an $n \times m$ matrix $A$. Then $A$ is TP if and only if it satisfies the condition $N$ and all the pivots are nonnegative.

**Proof.** These conditions are sufficient because, by our assumptions, if $A_t$ is TP then $A_{t-1}$ is also TP (in the Neville elimination of $A$). In fact, transforming $A_{t-1}$ into $A_t$, we perform operations such that, by Theorems 2.3 and 3.1 of [1], $A_{t-1}$ is TP when $A_t$ is TP. Moreover, the transformation of $A_t$ into $A_t$ consists, at most, of changing the place of a zero row (keeping the relative order of the rest). So $A_t$ is TP iff $A_t$ is TP.

Let us now see that the conditions are necessary. Let $A$ be a TP matrix, and let us suppose that the first time when a nonzero row is carried down to the bottom in the Neville elimination of $A$ is during the transformation of $A_t$ into $A_t$. Then, by Lemma 5.2, we observe that the assumptions of Lemma 5.3 are satisfied and $A$ would not be TP. Therefore, again by Lemma 5.2, we can carry out the Neville elimination of $A$ without row exchanges until we terminate at the u.e.f. matrix $U$. Moreover the pivots are nonnegative. If we show that $U$ is TP, the theorem will be proved by applying the same reasoning to $U^T$. But this is a consequence of Corollary 3.4 of [1] and of the fact that the total positivity holds when zero rows go down.

In this proof, the total positivity of $U$ can also be deduced without using Corollary 3.4 of [1]. In fact, in the Neville process we can find matrices $H, K$ which are products of elementary matrices, such that $HA = U$, $KA^T = V$ (with
U, V u.e.f. matrices), and the matrix $UK^T (= HAK^T = HV^T)$ is an $n \times m$ diagonal nonnegative matrix. As a consequence, $UK^T$ is TP. Now, using the same reasoning as in the beginning of the proof, one can deduce that $U$ is TP.

In remark 4.2 of [2] there is a test using $O(n^3)$ operations to prove the total positivity of a matrix $A$. With the previous theorem we can provide a similar test.

Let us observe that the complexity of these tests is the same as that of the Gaussian elimination to transform a matrix into diagonal form. However, Gaussian elimination does not provide a test for total positivity. Gaussian elimination is not so suitable as Neville elimination for certain problems related to total positivity, because the positive elementary matrices appearing in Neville elimination (except the permutation matrices) are TP, while this is not true in Gaussian elimination, as we can see with the matrix

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 
\end{pmatrix}
$$

In the case of nonsingular matrices, the previous theorem can be expressed in the following way:

**COROLLARY 5.5.** Let $A$ be a nonsingular matrix. Then $A$ is TP if and only if there are no row or column exchanges in the CNE of $A$ and the pivots are nonnegative.

Finally, we shall characterize the nonsingular TP matrices by their inverses. When we say bidiagonal matrices we refer to matrices which are, simultaneously, tridiagonal matrices (that is, Jacobi matrices) and triangular matrices.

**PROPOSITION 5.6.** Let $A$ be a nonsingular matrix of order $n$. Then $A$ is TP if and only if $A^{-1}$ is a product of $2n - 1$ bidiagonal matrices with positive elements on (and only on) the main diagonal.

**Proof.** By Theorem 2.3 and 3.1 of [1], if $A^{-1}$ has the above form then $A$ is TP.

Conversely, if $A$ is a nonsingular TP matrix, we observe that the elementary matrices appearing in the CNE of $A$ (see Corollary 5.5) are of the
type

\[
E_{i,i-1}(m_{i,i-1}) = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & 0 & 1 & 0 & \cdots & 0 \\
& & & & -m_{i,i-1} & 1 & 0 \\
& & & & & 0 & 1 & \ddots \\
& & & & & & \ddots & \ddots & \cdots & 0 \\
0 & & & & & & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}
\]

(with \(m_{i,i-1} > 0\)), or their transposes. We may group the factors, for example in the form

\[
E_{21}(m_{21})E_{32}(m_{32}) \cdots E_{n,n-1}(m_{n,n-1})
\]

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
-m_{21} & 1 & 0 & \cdots & 0 \\
0 & -m_{32} & 1 & \ddots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & 0 & -m_{n,n-1} & 1
\end{pmatrix}
\]

and then \(A^{-1}\) is a product of bidiagonal matrices as in the proposition.

This characterization is reminiscent of Corollary 3.6 of [1], which characterizes a nonsingular TP matrix as a product of Jacobi TP matrices.

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