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Cayley continuants

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Abstract

In 1858 Cayley considered a particular kind of tridiagonal determinants (or continuants). By a direct inspection of the first cases, he conjectured an identity expressing these determinants in terms of certain other determinants considered by Sylvester in 1854. Then Cayley proved the conjectured identity by induction but, as he wrote, he felt unsatisfied with his proof. The main aim of this paper is to give a straightforward proof of Cayley's identity using the method of formal series. Moreover we use this method and umbral calculus techniques to obtain several other identities.

Cayley continuants appear in several contexts and in particular in enumerative combinatorics. Mittag–Leffler polynomials, Meixner polynomials of the first kind, the falling and the raising factorials are just few instances of these continuants. They can be interpreted in terms of weighted permutations. Moreover, as we prove in this paper, they also appear in the context of Hankel determinants generated by certain Catalan-like numbers.

Keywords: Cayley continuant; Kac matrix; Clement matrix; Mittag–Leffler polynomial; Pidduck polynomial; Meixner polynomial; Delannoy number; Catalan-like path; Lattice path; Hankel determinant; Connection constants; Linearization coefficients

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1. Introduction

In 1854, in a brief communication [30] entirely reported in [21] (Vol. 2, pp. 425–426) and in [31], Sylvester considered the $n \times n$ tridiagonal determinants $H_n(\lambda)$ given by

$$1, \quad |\lambda|, \quad \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix}, \quad \begin{vmatrix} \lambda & 1 & \\ 2 & \lambda & 2 \\ & 1 & \lambda \end{vmatrix}, \quad \begin{vmatrix} \lambda & 1 & & \\ 3 & \lambda & 2 & \\ & 2 & \lambda & 3 \\ & & 1 & \lambda \end{vmatrix}, \quad \dots$$

stating, without proof, that

$$H_n(\lambda) = \prod_{k=1}^{n} (\lambda + n - 2k + 1) = (\lambda - n + 1)(\lambda - n + 3) \cdots (\lambda + n - 1). \tag{1}$$

This identity can be easily obtained by a process due to Mazza (see [21, Vol. 2, p. 442]) which uses only elementary operations on the lines. Other proofs can be found in [31,10].

Notice that for $\lambda = 0$ the matrices underling Sylvester's determinants appear in several contexts as *Clement matrices* or as *Kac matrices*. For instance they appear in numerical analysis [8,18], in the study of roots of random polynomials [10], in connection to the Ehrenfest Urn Model of diffusion [20], in a problem of random walks on a hypercube [17], in linear algebra and representation theory [15,31], in the theory of association schemes and distance regular graphs [6].

Some years after Sylvester's communication, in 1858, Cayley [7] considered the more general determinants $U_n(\theta, x)$ defined by

$$1, \quad |\theta|, \quad \begin{vmatrix} \theta & 1 \\ x & \theta \end{vmatrix}, \quad \begin{vmatrix} \theta & 1 & \\ x & \theta & 2 \\ & x-1 & \theta \end{vmatrix}, \quad \begin{vmatrix} \theta & 1 & & \\ x & \theta & 2 & \\ & x-1 & \theta & 3 \\ & & x-2 & \theta \end{vmatrix}, \dots .$$

All these determinants are particular cases of continuants [21,32]. Therefore, from now on, we will refer to them as *Cayley continuants*.

In his paper Cayley observed that Sylvester's determinants can be obtained setting $\lambda = \theta$ and $x = n - 1$ in his continuants. Then by a direct inspection of the first cases he found an identity that can be rewritten as

$$U_n(\theta, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k k!} (-1)^k p_k(x - n) H_{n-2k}(\theta), \tag{2}$$

where $p_k(x) = (x + 1)(x + 3) \cdots (x + 2k - 1)$, and finally he proved this identity using a rather laborious proof by induction. However he was not satisfied with his proof and wrote: "*I have not been able to find an easier demonstration than the following one, which, it must be admitted, is somewhat complicated*" [7] (see also [21, Vol. 2, p. 430]).

The principal aim of this paper is to give a straightforward proof of Cayley's identity using the method of formal series and to obtain several other identities in the same spirit. The paper is organized in as follows. In the next section we give some basic properties of Cayley continuants, such as a recurrence relation, the exponential generating series and an explicit formula. In Section 3, we give a combinatorial interpretation of Cayley continuants in terms of weighted permutations where the odd cycles have weight θ and the even cycles have weight $-x$. Moreover we show that Cayley continuants admit several well known instances such as the factorial numbers, the Delannoy numbers, the Mittag–Leffler polynomials, the Pidduck polynomials, the Meixner polynomials of the first kind, the falling and the rising factorial powers. In Section 4, we prove Cayley's identity using, as we have already said, the method of formal series. In particular we apply Cauchy's integral theorem to obtain non-trivial series in a single indeterminate as diagonals of bivariate series of simple form. In Section 5, we consider a particular kind of weighted lattice paths and the associated Catalan-like numbers. As well known these numbers can be completely characterized in terms of associated Hankel determinants. We prove that these determinants, in the present case, can be expressed in terms of Cayley continuants. In Sections 6 and 7 we obtain some connection identities for Cayley continuants using Rota's operational method (or umbral calculus). Finally, in Sections 8 and 9, we obtain some other identities concerning Cayley continuants.

2. Basic properties

A *Cayley continuant* is a $n \times n$ tridiagonal determinant

$$U_n(\theta, x) = \begin{vmatrix} \theta & 1 & & & & & \\ x & \theta & 2 & & & & \\ & x-1 & \theta & 3 & & & \\ & & x-2 & \theta & 4 & & \\ & & & \ddots & \ddots & & \\ & & & & \theta & n-1 \\ & & & & x-n+2 & \theta \end{vmatrix}.$$

Expanding the determinant along the last column it is easy to obtain the following recurrence, already known to Cayley [7]:

$$U_{n+2}(\theta, x) = \theta U_{n+1}(\theta, x) - (n+1)(x-n)U_n(\theta, x). \tag{3}$$

Now consider the exponential generating series for Cayley continuants:

$$U(\theta, x; t) = \sum_{n \geqslant 0} U_n(\theta, x)\frac{t^n}{n!}.$$

To obtain a closed form for this series multiply both sides of (3) by $t^{n+1}/(n+1)!$ and sum for $n \geqslant 0$. Since $U_0(\theta, x) = 1$ and $U_1(\theta, x) = \theta$, it follows that recurrence (3) becomes the differential equation

$$(1-t^2)\, U'(\theta, x; t) = (\theta - xt)\, U(\theta, x; t)\,,$$

whose solution, for $U(\theta, x, 0) = 1$, is

$$U(\theta, x; t) = (1 - t^2)^{x/2} \left(\frac{1+t}{1-t}\right)^{\theta/2} = \frac{(1+t)^{(\theta+x)/2}}{(1-t)^{(\theta-x)/2}}. \tag{4}$$

The second form of $U(\theta, x; t)$ in (4) implies the following explicit form for Cayley continuants

$$U_n(\theta, x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\theta + x}{2}\right)^{\underline{k}} \left(\frac{\theta - x}{2}\right)^{\overline{n-k}}, \tag{5}$$

where $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$ are the *falling factorial polynomials* and $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$ are the *rising factorial polynomials*. Another useful property which can be derived from (4) is the identity

$$U_n(-\theta, x) = (-1)^n U_n(\theta, x). \tag{6}$$

Finally, considering θ as an indeterminate and x as a parameter, Cayley continuants form a *section sequence* [22], that is

$$U(\theta_1, x_1; t)\, U(\theta_2, x_2; t) = U(\theta_1 + \theta_2, x_1 + x_2; t)$$

or equivalently

$$\sum_{k=0}^{n} \binom{n}{k} U_k(\theta_1, x_1) U_{n-k}(\theta_2, x_2) = U_n(\theta_1 + \theta_2, x_1 + x_2).$$

3. Combinatorial interpretation and examples

Let S be any set of size n. Let w denote the weight function defined on the set $S!$ of all permutations on S so that each cycle of odd length has weight θ and each cycle of even length has weight $-x$. Finally let $w_n = w(S!) = \sum_{\pi \in S!} w(\pi)$. Then Cayley continuants have the following combinatorial interpretation

$$U_n(\theta, x) = w_n = \sum_{i,j \geqslant 0} u_n(i, j) \theta^i (-x)^j, \tag{7}$$

where $u_n(i, j)$ is the number of n-permutations with i odd cycles and j even cycles. To see this, consider a set S of size $n + 2$ and choose an element $s \in S$. Then observe that in any permutation π of S, s either is a fixed point, or belongs to a 2-cycle, or belongs to a longer cycle. If s is a fixed point, the weight of π is accounted for by θw_{n+1} once x is deleted. If s belongs to a 2-cycle, say (s, s'), the weight of π is accounted by $-(n + 1)x\, w_n$ when (s, s') is deleted. If s belongs to a longer cycle, say $(s, s', s'', \ldots)$, the weight of π is accounted by $n(n + 1)w_n$ once s and s' are deleted. Hence $w_{n+2} = \theta w_{n+1} - (n + 1)(x - n)w_n$ which agrees the form of (3). Finally, since $w_0 = 1$ and $w_1 = \theta$, it follows (7).

This interpretation can be used to obtain combinatorial proofs for other identities concerning Cayley continuants. For instance, the above argument can be generalized in the following way. Let S be a set of size $n+1$ and fix one point $s \in S$. Since any permutation of S can be decomposed in the cycle containing the element s and in the permutation obtained by deleting that cycle, it follows the identity

$$U_{n+1}(\theta, x) = \theta \sum_{k \geqslant 0} n^{\underline{2k}} U_{n-2k}(\theta, x) - x \sum_{k \geqslant 0} n^{\underline{2k+1}} U_{n-2k-1}(\theta, x).$$

Identity (7) can be proved more directly using the theory of weighted species [19,5]. Let **Cay** be the species of w-weighted permutations as defined above. Similarly, let $\mathbf{Cyc}_w$ be the species of w-weighted cycles. Finally let **Exp** be the exponential species, i.e. the uniform species. Since any permutation of species **Cay** is equivalent to a set partition in which each block is endowed with a cycle of weight θ or $-x$ according to the parity of its length, it follows that $\mathbf{Cay} = \mathbf{Exp} \circ \mathbf{Cyc}_w$. Then the cardinality of **Cay** is the exponential series

$$\begin{aligned}\mathbf{Card}(\mathbf{Cay}, t) &= \mathbf{Card}(\mathbf{Exp}; t) \circ \mathbf{Card}(\mathbf{Cyc}_w; t) \\ &= \mathrm{e}^t \circ \left(\theta \ln \sqrt{\frac{1+t}{1-t}} - x \ln \frac{1}{\sqrt{1-t^2}} \right).\end{aligned}$$

Now identity (4) implies $\mathbf{Card}(\mathbf{Cay}, t) = U(\theta, x; t)$ and (7) is reobtained.

We conclude this section with some particular instances of Cayley continuants.

(1) *Mittag–Leffler polynomials* [22, p. 75]: $M_n(\theta) = U_n(2\theta, 0)$,

$$\sum_{n \geqslant 0} M_n(\theta) \frac{t^n}{n!} = \left(\frac{1+t}{1-t} \right)^{\theta} = U(2\theta, 0; t).$$

(2) *Pidduck polynomials* [22, p. 126]: $P_n(\theta) = U_n(2\theta+1, -1)$,

$$\sum_{n \geqslant 0} P_n(\theta) \frac{t^n}{n!} = \frac{1}{1-t} \left(\frac{1+t}{1-t} \right)^{\theta} = U(2\theta+1, -1; t).$$

(3) *Meixner polynomials of the first kind* [22, p. 125,13]: $m_n(\theta; \beta, -1) = U_n(2\theta+\beta, -\beta)$,

$$\sum_{n \geqslant 0} m_n(\theta; \beta, -1) \frac{t^n}{n!} = \frac{1}{(1-t)^{\beta}} \left(\frac{1+t}{1-t} \right)^{\theta} = U(2\theta+\beta, -\beta; t).$$

(4) *Falling and raising factorials* [22, p. 56]: $\theta^{\underline{n}} = U_n(\theta, \theta)$ and $\theta^{\overline{n}} = U_n(\theta, -\theta)$,

$$\sum_{n \geqslant 0} \theta^{\underline{n}} \frac{t^n}{n!} = (1+t)^{\theta} = U(\theta, \theta; t), \qquad \sum_{n \geqslant 0} \theta^{\overline{n}} \frac{t^n}{n!} = \frac{1}{(1-t)^{\theta}} = U(\theta, -\theta; t).$$

Factorial numbers: $U_n(1, -1) = n!$,

$$\sum_{n \geqslant 0} n! \frac{t^n}{n!} = \frac{1}{1-t} = U(1, -1; t).$$

(5) Let a_n be the number of all *permutations with odd cycles* on an n-set [26, sequence #A000246] and let b_n be the number of all *permutations with even cycles* on an n-set. Then

$$\sum_{n\geqslant 0} a_n \frac{t^n}{n!} = \sqrt{\frac{1+t}{1-t}} = U(1,0;t)\,, \qquad \sum_{n\geqslant 0} b_n \frac{t^n}{n!} = \frac{1}{\sqrt{1-t^2}} = U(0,-1;t)$$

and hence $U_n(1,0) = a_n$ and $U_n(0,-1) = b_n$.

(6) Let $d_{n,k}$ be the *Delannoy numbers* [9, p. 81], [28, p. 185], [4,29,26, sequence #A008288]. They are usually defined as the number of lattice paths from $(0,0)$ to (n,k) with horizontal steps $(1,0)$, vertical steps $(0,1)$ and diagonal steps $(1,1)$. From their generating series $(1-t-u-tu)^{-1}$ it follows that

$$\sum_{n\geqslant 0} d_{n,k} t^n = \frac{1}{1-t}\left(\frac{1+t}{1-t}\right)^k.$$

Consider now the polynomials $D_n(x)$ defined by the series

$$D(x;t) = \sum_{n\geqslant 0} D_n(x) t^n = \frac{1}{1-t}\left(\frac{1+t}{1-t}\right)^x.$$

Then $D_n(k) = d_{n,k}$, for every $k \in \mathbb{N}$, and $D_n(x) = P_n(x)/n!$ where the $P_n(x)$'s are the Pidduck polynomials considered in example 2. Hence

$$D_n(x) = \frac{1}{n!} U_n(2x+1,-1)$$

and in particular

$$d_{n,k} = \frac{1}{n!} U_n(2k+1,-1).$$

4. Cayley's identity

As we saw in the introduction, Cayley's identity (2) expresses Cayley continuants in terms of Sylvester's determinants. To prove Cayley's identity we first rewrite (2) in the following equivalent form

$$U_n(\theta, x+n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{2^k k!} (-1)^k\, p_k(x) H_{n-2k}(\theta), \tag{8}$$

where $p_k(x) = (x+1)(x+3)\cdots(x+2k-1)$. Then considering the generating series

$$U^{(0,1)}(\theta, x; t) = \sum_{n\geqslant 0} U_n^{(0,1)}(\theta, x) \frac{t^n}{n!} = \sum_{n\geqslant 0} U_n(\theta, x+n) \frac{t^n}{n!}$$

(the choice of this notation will be explained in Section 8),

$$p(x;t) = \sum_{n\geqslant 0} p_n(x) \frac{t^n}{n!}\,, \qquad H(\theta;t) = \sum_{n\geqslant 0} H_n(\theta) \frac{t^n}{n!}$$

identity (8) becomes

$$U^{(0,1)}(\theta, x; t) = p(x; -t^2/2)H(\theta; t). \tag{9}$$

Our aim now is to prove this identity. To obtain such a proof it is sufficient to have an explicit form for each of the three series appearing in (9). The first series $U^{(0,1)}(\theta, x; t)$ will be obtained as the diagonal series of the following mixed bivariate series

$$\sum_{m,n \geqslant 0} U_n(\theta, x+m)\frac{t^n}{n!}u^m.$$

First, from (4), we find that $U(\theta, x+m; t) = U(\theta, x; t)\,(1-t^2)^{m/2}$, and hence

$$\sum_{m,n \geqslant 0} U_n(\theta, x+m)\frac{t^n}{n!}u^m = \sum_{m \geqslant 0} U(\theta, x+m; t)u^m = \frac{U(\theta, x; t)}{1-\sqrt{1-t^2}u}.$$

Then, by Cauchy's integral theorem [9, p. 42], [16], [28, p. 182], it follows that

$$\begin{aligned} U^{(0,1)}(\theta, x; t) &= \frac{1}{2\pi \mathrm{i}} \oint \frac{U(\theta, x; z)}{1-\sqrt{1-z^2}t/z}\frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi \mathrm{i}} \oint U(\theta, x; z)\frac{z+t\sqrt{1-z^2}}{(1+t^2)z^2-t^2}\,\mathrm{d}z. \end{aligned}$$

Since $z_1 = t/\sqrt{1+t^2}$ is the only pole (of the first order) tending to 0 as $t \mapsto 0$, then from the residue theorem it follows that

$$U^{(0,1)}(\theta, x; t) = \lim_{z \to z_1} \frac{U(\theta, x; z)}{1+t^2}\frac{z+t\sqrt{1-z^2}}{z+t/\sqrt{1+t^2}}$$

that is

$$U^{(0,1)}(\theta, x; t) = \frac{1}{1+t^2}U\left(\theta, x; \frac{t}{\sqrt{1+t^2}}\right).$$

Finally, after some simplifications, we obtain

$$U^{(0,1)}(\theta, x; t) = \frac{1}{1+t^2}\frac{(t+\sqrt{1+t})^{\theta}}{(1+t^2)^{x/2}}. \tag{10}$$

For the second series it is easy to find that

$$p(x; t) = \frac{1}{(1-2t)^{(x+1)/2}} \quad \text{and} \quad p(x; -t^2/2) = \frac{1}{(1+t^2)^{(x+1)/2}}. \tag{11}$$

For the third series we have $H_n(\theta) = U_n(\theta, n-1) = U_n^{(0,1)}(\theta, -1)$. Then

$$H(\theta; t) = U^{(0,1)}(\theta, -1; t) = \frac{(t+\sqrt{1+t^2})^{\theta}}{\sqrt{1+t^2}}. \tag{12}$$

Now, substituting series (10), (11) and (12) in (9), it follows that (9) is identically satisfied. This proves Cayley's identity.

Notice that this proof allows us to obtain immediately the inverse relation of Cayley's identity. Indeed identity (9) implies that

$$H(\theta; t) = (1 + t^2)^{(x+1)/2} U^{(0,1)}(\theta, x; t),$$

which yields the identity

$$H_n(\theta) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{(2k)!}{k!} \left(\frac{x+1}{2}\right)^{\underline{k}} U_{n-2k}(\theta, x + n - 2k).$$

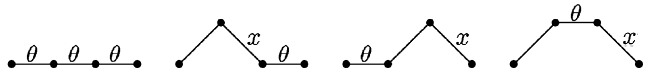
Finally notice also that the exponential generating series for the polynomials at the right-hand side of (1) can be obtained in a very similar way and coincides with (12). This gives another proof of identity (1).

5. Catalan-like paths and Hankel determinants

The aim of this section is to prove that Cayley continuants appear also in the theory of counting of particular weighted lattice paths. First of all we recall some definitions. Let $\sigma = (s_0, s_1, s_2, \dots)$ and $\tau = (t_0, t_1, t_2, \dots)$. A (σ, τ)-*Catalan-like path* of length n [2,3,11,12,14] is a weighted lattice path in $\mathbb{N} \times \mathbb{Z}$ which starts at $(0, 0)$ and ends at $(n, 0)$, has steps $(1, 1)$, $(1, 0)$, $(1, -1)$, never falls below $y = 0$, where the horizontal steps $(n, k) \rightsquigarrow (n + 1, k)$ have weight s_k and the falling steps $(n, k + 1) \rightsquigarrow (n + 1, k)$ have weight t_k. The weight of a path is the product of the weights of all its steps. The *Catalan-like number* $C_n^{\sigma,\tau}$ is the total weight (that is the sum of the weights) of all (σ, τ)-Catalan-like paths of length n. Similarly $C_{n,k}^{\sigma,\tau}$ is the total weight of all (σ, τ)-Catalan-like paths starting at $(0, 0)$ and ending at (n, k). Finally we denote with $(C_{n,k}^{\sigma,\tau})^*$ the total weight of all (σ, τ)-Catalan-like paths starting at $(0, k)$ and ending at $(n, 0)$.

The *Hankel determinant* of order $n + 1$ of a sequence $\{a_n\}_n$ is defined as the determinant $\det[\, a_{i+j} \,]_{i,j=0}^n$. Any sequence $\{a_n\}_n$ with invertible elements is completely characterized by the two sequences $\{\det[\, a_{i+j} \,]_{i,j=0}^n\}_n$ and $\{\det[\, a_{i+j+1} \,]_{i,j=0}^n\}_n$ of Hankel determinants.

Here we will consider the Catalan-like paths with horizontal steps of weight $s_k = \theta$ and falling steps $(n, k + 1) \rightsquigarrow (n + 1, k)$ of weight $t_k = (k + 1)(x - k)$. Let $C_n(\theta, x)$, $C_{n,k}(\theta, x)$ and $C^*_{n,k}(\theta, x)$ be the associated Catalan-like numbers. For instance, for $n = 3$ and $k = 0$ we have the paths



and hence $C_3(\theta, x) = \theta^3 + 3x\theta$. Since each path ends with a rising step, a horizontal step or a falling step, it follows that the coefficients $C_{n,k}(\theta, x)$ satisfy the recurrences

$$C_{n+1,k+1}(\theta, x) = C_{n,k}(\theta, x) + \theta C_{n,k+1}(\theta, x) + (k + 2)(x - k - 1)C_{n,k+2}(\theta, x),$$

$$C_{n+1,0}(\theta, x) = \theta C_{n,0}(\theta, x) + x C_{n,1}(\theta, x)$$

with the initial value $C_{0,0}(\theta, x) = 1$.

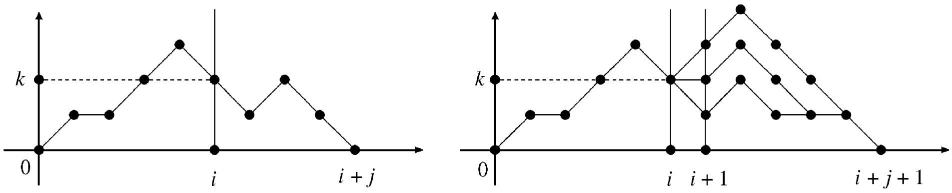


Fig. 1. Catalan-like paths decompositions.

Our aim now is to characterize the Catalan-like numbers $C_n(\theta, x)$ in terms of the Hankel determinants $\det[C_{i+j}(\theta, x)]_{i,j=0}^n$ and $\det[C_{i+j+1}(\theta, x)]_{i,j=0}^n$. We will see that these determinants can be expressed in terms of Cayley continuants.

Consider all the paths from $(0, 0)$ to $(0, i + j)$ (see the first picture in Fig. 1). Since each path touches the line $x = i$ in a point at a certain height k it follows that

$$C_{i+j}(\theta, x) = \sum_{k \geqslant 0} C_{i,k}(\theta, x) C^*_{j,k}(\theta, x).$$

This identity immediately implies that the Hankel matrix generated by the $C_n(\theta, x)$ has the decomposition

$$[C_{i+j}(\theta, x)]_{i,j=0}^n = [C_{i,j}(\theta, x)]_{i,j=0}^n ([C^*_{i,j}(\theta, x)]_{i,j=0}^n)_{\mathrm{T}}$$

where $(-)_{\mathrm{T}}$ stands for transpose. In particular this decomposition yields the identity

$$\det[C_{i+j}(\theta, x)]_{i,j=0}^n = h(n) x^{\underline{\underline{n}}}, \tag{13}$$

where $h(n) = n! \cdot (n-1)! \cdots 2! \cdot 1!$ is the hyperfactorial and $x^{\underline{\underline{n}}} = x^{\underline{n}} \cdot x^{\underline{n-1}} \cdots x^{\underline{1}}$ is the falling hyperfactorial.

Consider now all the paths from $(0, 0)$ to $(0, i + j + 1)$ (see the second picture in Fig. 1). This time each path touches the line $x = i$ at a point (i, k) and then the line $x = i + 1$ at one of the points $(i + 1, k - 1)$, $(i + 1, k)$ or $(i + 1, k + 1)$. This implies the identity

$$C_{i+j+1}(\theta, x) = \sum_{h,k \geqslant 0} C_{i,h}(\theta, x) D_{h,k}(\theta, x) C^*_{j,k}(\theta, x),$$

where $D_{h,k}(\theta, x) = (k+1)(x-k)\delta_{h-1,k} + \theta\delta_{h,k} + \delta_{h+1,k}$. Then the Hankel matrix generated by the $C_{n+1}(\theta, x)$ has the decomposition

$$[C_{i+j+1}(\theta, x)]_{i,j=0}^n = [C_{i,j}(\theta, x)]_{i,j=0}^n [D_{i,j}]_{i,j=0}^n ([C^*_{i,j}(\theta, x)]_{i,j=0}^n)_{\mathrm{T}}.$$

In particular, since $[D_{i,j}(\theta, x)]_{i,j=0}^n = U_n(\theta, x)$, it yields the identity

$$\det[C_{i+j+1}(\theta, x)]_{i,j=0}^n = h(n)\, x^{\underline{\underline{n}}}\, U_n(\theta, x). \tag{14}$$

In conclusion we have the following characterization: the polynomials $C_n(\theta, x)$ form the unique sequence for which identities (13) and (14) hold.

6. A connection identity

Let $C_{n,k}(x)$ be the connection constants such that

$$\theta^n = \sum_{k=0}^{n} C_{n,k}(x) U_k(\theta, x), \tag{15}$$

where the $U_k(\theta, x)$ are considered as polynomials in θ. To obtain a recurrence for these coefficients write (15) as

$$\theta^{n+1} = \theta^n \theta = \sum_{k=0}^{n} C_{n,k}(x) \theta U_k(\theta, x),$$

and then substitute $\theta U_k(\theta, x)$ with the equivalent expression obtained by recurrence (3). It follows that the connection constants $C_{n,k}(x)$ satisfy the recurrences

$$C_{n+1,k+1}(x) = C_{n,k}(x) + (k+2)(x-k-1)C_{n,k+2}(x),$$

$$C_{n+1,0}(x) = xC_{n,1}(x)$$

with the initial value $C_{0,0}(x) = 1$. Hence $C_{n,k}(x) = C_{n,k}(0, x)$, where the $C_{n,k}(\theta, x)$'s are the weights associated to the Catalan-like paths without horizontal steps considered in the preceding section.

Let $C_n(\theta, x) = \sum_{k=0}^{n} C_{n,k}(x)\, \theta^n$ be the polynomials generated by the connection constants $C_{n,k}(x)$ and let

$$C(\theta, x; t) = \sum_{n \geqslant 0} C_n(\theta, x) \frac{\theta^n}{n!}$$

be their exponential generating series. To obtain a closed formula for this series we use Rota's operational method [1,22–24]. Consider the linear operator $L : \mathbb{Z}[x][\theta] \to \mathbb{Z}[x][\theta]$ defined by setting

$$LU_n(\theta, x) = \theta^n \tag{16}$$

and extending by linearity. Then from identity (15) we have

$$L\theta^n = C_n(\theta, x). \tag{17}$$

The operator L can be extended to formal series in a natural way. So identities (16) and (17) become

$$L \frac{(1+t)^{(\theta+x)/2}}{(1-t)^{(\theta-x)/2}} = \mathrm{e}^{\theta t}, \tag{18}$$

$$C(\theta, x; t) = L\mathrm{e}^{\theta t}. \tag{19}$$

From identity (18) we have

$$L\mathrm{e}^{\theta \ln \sqrt{\frac{1+t}{1-t}}} = L\left(\frac{1+t}{1-t}\right)^{\theta/2} = \frac{\mathrm{e}^{\theta t}}{(1-t^2)^{x/2}}.$$

Setting $u = \ln\sqrt{\frac{1+t}{1-t}}$ we have $t = \tanh\, u$ and hence

$$Le^{\theta u} = \frac{e^{\theta \tanh\, u}}{(1-\tanh^2\, u)^{x/2}} = (\cosh u)^x e^{\theta \tanh\, u}.$$

Then (19) becomes

$$C(\theta, x; t) = (\cosh t)^x e^{\theta \tanh\, t}.$$

This result can be generalized to any Sheffer sequence $\{s_n(\theta)\}_{n\in\mathbb{N}}$ [22, p. 17]. In this case there exist two exponential series $g(t)$ and $f(t)$, with $g_0 \neq 0$, $f_0 = 0$ and $f_1 \neq 0$, such that

$$s(\theta; t) = \sum_{n\geqslant 0} s_n(\theta)\frac{t^n}{n!} = g(t)e^{\theta f(t)}.$$

Then, using the same technique, it follows that the exponential series for the polynomials $B_n(\theta, x) = \sum_{k=0}^{n} B_{n,k}(x)\,\theta^n$ generated by the connection constants $B_{n,k}(x)$ such that

$$s_n(\theta) = \sum_{k=0}^{n} B_{n,k}(x)U_k(\theta, x)$$

is given by

$$B(\theta, x; t) = g(t)(\cosh f(t))^x e^{\theta \tanh\, f(t)}.$$

7. Linearization coefficients

In this section, we will determine the *linearization coefficients* [33] for Cayley continuants, that is the connection constants $C_{m,n,k}(x)$ for which

$$U_m(\theta, x)\, U_n(\theta, x) = \sum_{k\geqslant 0} C_{m,n,k}(x)U_k(\theta, x).$$

To obtain these coefficients we will use the following theorem which can be obtained with Rota's operational method. Let $\{s_n(\theta)\}_n$ be a Sheffer sequence with exponential generating series

$$\sum_{n\geqslant 0} s_n(\theta)\frac{t^n}{n!} = g(t)e^{\theta f(t)}.$$

Let $c_{m,n,k}$ be the connection constants such that

$$s_m(\theta)s_n(\theta) = \sum_{k\geqslant 0} c_{m,n,k}\; s_k(\theta).$$

Then consider the polynomials

$$c_{m,n}(\theta) = \sum_{k\geqslant 0} c_{m,n,k}\;\theta^k$$

and their generating series

$$c(\theta; t, u) = \sum_{m,n \geqslant 0} c_{m,n}(\theta) \frac{t^m}{m!} \frac{u^n}{n!}.$$

Then it can be proved that

$$c(\theta; t, u) = g(t) g(u) \frac{e^{\theta \widehat{f}(f(t)+f(u))}}{g(\widehat{f}(f(t) + f(u)))},$$

where $\widehat{f}$ denotes the compositional inverse of f.

Cayley continuants are Sheffer polynomials with respect to θ. Indeed their exponential generating series can be written as

$$U(\theta, x; t) = (1 - t^2)^{x/2} e^{\theta \ln \sqrt{\frac{1+t}{1-t}}} = g(t)\, e^{\theta f(t)},$$

where $g(t) = (1 - t^2)^{x/2}$ and $f(t) = \ln \sqrt{\frac{1+t}{1-t}} = \operatorname{arctanh} t$. Then $\widehat{f}(t) = \tanh t$ and consequently

$$\widehat{f}(f(t) + f(u)) = \frac{t + u}{1 + tu},$$

$$g(\widehat{f}(f(t) + f(u))) = \frac{(1 - t^2)^{x/2}(1 - u^2)^{x/2}}{(1 + tu)^x}.$$

This implies that

$$C(\theta, x; t, u) = (1 + tu)^x e^{\theta \frac{t+u}{1+tu}}. \tag{20}$$

Finally expanding the right-hand side of (20) we obtain

$$C_{m,n}(\theta, x) = \sum_{k \geqslant 0} \binom{x - m - n + 2k}{k} m^{\underline{k}} n^{\underline{k}}\, \theta^{m+n-2k}$$

and consequently

$$U_m(\theta, x)\, U_n(\theta, x) = \sum_{k \geqslant 0} \binom{x - m - n + 2k}{k} m^{\underline{k}} n^{\underline{k}}\, U_{m+n-2k}(\theta, x). \tag{21}$$

See the appendix for some instances of identity (21).

8. Perturbed continuants

To prove Cayley's identity we used an explicit form for the exponential generating series of $U_n^{(0,1)}(\theta, x) = U_n(\theta, x + n)$. This leads us to consider more generally the perturbed continuants $U_n^{(a,b)}(\theta, x) = U_n(\theta + an, x + bn)$ and their exponential generating series

$$U^{(a,b)}(\theta, x; t) = \sum_{n \geqslant 0} U_n^{(a,b)}(\theta, x) \frac{t^n}{n!}. \tag{22}$$

Clearly $U^{(0,0)}(\theta, x; t) = U(\theta, x; t)$. Identity (5) gives an explicit formula for every perturbed continuants. However it could be useful to have an explicit formula also for the series (22). As in Section 4 we can consider the bivariate series

$$\sum_{m,n\geqslant 0} U_m^{(a,b)}(\theta, x)\frac{t^m}{m!}u^n = \frac{U(\theta, x; t)}{1 - U(a, b; t)u}.$$

Then, by Cauchy's integral formula for diagonals, we have

$$U^{(a,b)}(\theta, x; t) = \frac{1}{2\pi \mathrm{i}}\oint \frac{U(\theta, x; z)}{1 - U(a, b; z)t/z}\frac{\mathrm{d}z}{z} = \frac{1}{2\pi \mathrm{i}}\oint \frac{U(\theta, x; z)}{z - U(a, b; z)t}\mathrm{d}z.$$

Explicit calculations are possible when the denominator of the integrand series, after simplification, is a polynomial in z with degree at most 2. This is the case when $(a, b) = (0, 0), (0, 2), (\pm 1, \pm 1), (\pm 2, 0), (\pm 2, 2), (\pm 3, 1)$. For instance we have

$$U^{(1,1)}(\theta, x; t) = \frac{1}{1-t}U\left(\theta, x; \frac{t}{1-t}\right) = \frac{1}{(1-t)^{x+1}(1-2t)^{(\theta-x)/2}},$$

$$\begin{aligned} U^{(2,2)}(\theta, x; t) &= \frac{1}{\sqrt{1-4t}}U\left(\theta, x; \frac{1-2t-\sqrt{1-4t}}{2t}\right) \\ &= \frac{1}{\sqrt{1-4t}}\left(\frac{1-2t-\sqrt{1-4t}}{2t^2}\right)^{x/2}\left(\frac{1}{\sqrt{1-4t}}\right)^{(\theta-x)/2}, \end{aligned}$$

$$\begin{aligned} U^{(1,-1)}(\theta, x; t) &= \frac{1+\sqrt{1-4t}}{2\sqrt{1-4t}}U\left(\theta, x; \frac{1-\sqrt{1-4t}}{2}\right) \\ &= \frac{1+\sqrt{1-4t}}{2\sqrt{1-4t}}\left(\frac{1-t-\sqrt{1-4t}}{t}\right)^{\theta/2}\left(\frac{1+2t+\sqrt{1-4t}}{2}\right)^{x/2}. \end{aligned}$$

However there are also other cases in which a closed form for series (22) can be obtained, as for our starting example $(a, b) = (0, 1)$. It would be interesting to find all the values of a and b for which explicit calculations are possible. Notice that by (6) we have $U_n^{(-a,b)}(\theta, x) = (-1)^n U_n^{(a,b)}(-\theta, x)$ and hence $U^{(-a,b)}(\theta, x; t) = U^{(a,b)}(-\theta, x; -t)$. Then we can assume $a \geqslant 0$.

One reason to have explicit formulas for the series $U_n^{(a,b)}(\theta, x; t)$ is that they can yield identities. For instance all the three examples considered above appear as Riordan transforms of $U(\theta, x; t)$, considered as an ordinary formal power series. To be more precise let us recall that a *Riordan matrix* [25] is an infinite lower triangular matrix $(g(t), f(t)) \sim [r_{n,k}]_{n,k\geqslant 0}$ whose columns have generating series of the form $\sum_{n\geqslant 0} r_{n,k}t^n = g(t)f(t)^k$ for given formal power series $g(t)$ and $f(t)$ with $g_0 = 1$, $f_0 = 0$ and $f_1 \neq 0$. Any Riordan matrix induces a transformation on the algebra of formal series defined by

$$a(t) = \sum_{n\geqslant 0} a_n t^n \mapsto g(t)\,a(f(t)) = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n} r_{n,k}a_k\right)t^n.$$

Then the series considered in the previous examples are the Riordan transform of $U(\theta, x; t)$ with respect to the Riordan matrices $\left(\frac{1}{1-t}, \frac{1}{1-t}\right) \sim \left[\binom{n}{k}\right]_{n,k}$, $\left(\frac{1}{\sqrt{1-4t}}, \frac{1-2t-\sqrt{1-4t}}{2t}\right) \sim \left[\binom{2n}{n+k}\right]_{n,k}$ and $\left(\frac{1+\sqrt{1-4t}}{2\sqrt{1-4t}}, \frac{1-\sqrt{1-4t}}{2}\right) \sim \left[\binom{2n-k}{n-k}\frac{n}{2n-k}\right]_{n,k}$. Hence the following identities:

$$\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{k!}U_k(\theta, x) = U_n(\theta+n, x+n),$$

$$\sum_{k=0}^{n}\binom{2n}{n+k}\frac{n!}{k!}U_k(\theta, x) = U_n(\theta+2n, x+2n),$$

$$\sum_{k=0}^{n}\binom{2n-k}{n-k}\frac{n}{2n-k}\frac{n!}{k!}U_k(\theta, x) = U_n(\theta+n, x-n).$$

9. Final formulas

In this final section we give two other explicit formulas for Cayley continuants. By Cauchy's integral formula we have

$$U_n(\theta, x) = n!\,[t^n]\,\frac{(1+t)^{(\theta+x)/2}}{(1-t)^{(\theta-x)/2}} = \frac{n!}{2\pi\mathrm{i}}\oint \frac{(1+z)^{(\theta+x)/2}}{(1-z)^{(\theta-x)/2}}\frac{\mathrm{d}z}{z^{n+1}}.$$

Let $z = w/(1-sw)$. Then $\mathrm{d}z = \mathrm{d}w/(1-sw)^2$ and

$$\begin{aligned} U_n(\theta, x) &= \frac{n!}{2\pi\mathrm{i}}\oint \frac{(1-(s-1)w)^{(\theta+x)/2}}{(1-sw)^{x-n+1}(1-(s+1)w)^{(\theta-x)/2}}\frac{\mathrm{d}w}{w^{n+1}} \\ &= n!\,[t^n]\,\frac{(1-(s-1)t)^{(\theta+x)/2}}{(1-st)^{x-n+1}(1-(s+1)t)^{(\theta-x)/2}}. \end{aligned}$$

For $s = 1$ we have

$$U_n(\theta, x) = n!\,[t^n]\,\frac{1}{(1-t)^{x-n+1}(1-2t)^{(\theta-x)/2}}$$

that is

$$U_n(\theta, x) = n!\sum_{k=0}^{n}\binom{x-k}{n-k}\left(\!\!\binom{(\theta-x)/2}{k}\!\!\right)2^k,$$

where $\left(\!\binom{x}{n}\!\right) = \frac{x^{\overline{n}}}{n!} = \frac{x(x+1)\cdots(x+n-1)}{n!}$. Similarly, for $s = -1$ we have

$$U_n(\theta, x) = n!\,[t^n]\,\frac{(1+2t)^{(\theta+x)/2}}{(1+t)^{x-n+1}}$$

that is

$$U_n(\theta, x) = n!\sum_{k=0}^{n}\binom{x-k}{n-k}\left(\!\!\binom{(\theta+x)/2}{k}\!\!\right)(-1)^{n-k}\,2^k.$$

Appendix

(1) Using the same notations as in the examples in Section 3 we have the following instances of identity (21):

$$\theta^{\underline{m}}\,\theta^{\underline{n}} = \sum_{k\geqslant 0}\binom{m}{k}\binom{n}{k}k!\,\theta^{\underline{m+n-k}},$$

$$\theta^{\overline{m}}\,\theta^{\overline{n}} = \sum_{k\geqslant 0}\binom{m}{k}\binom{n}{k}(-1)^k k!\,\theta^{\overline{m+n-k}},$$

$$M_m(\theta)\,M_n(\theta) = \sum_{k\geqslant 0}\binom{m+n-2k}{k} m^{\underline{k}}\,n^{\underline{k}}\,M_{m+n-2k}(\theta),$$

$$P_m(\theta)\,P_n(\theta) = \sum_{k\geqslant 0}\left(\!\!\binom{m+n-2k+1}{k}\!\!\right)(-1)^k\,m^{\underline{k}}\,n^{\underline{k}}\,P_{m+n-2k}(\theta),$$

$$m!\,n! = \sum_{k\geqslant 0}\binom{m}{k}\binom{n}{k}(-1)^k\,k!(m+n-k)!,$$

$$a_m\,a_n = \sum_{k\geqslant 0}\left(\!\!\binom{m+n-2k}{k}\!\!\right)(-1)^k\,m^{\underline{k}}\,n^{\underline{k}}\,a_{m+n-2k},$$

$$b_m\,b_n = \sum_{k\geqslant 0}\left(\!\!\binom{m+n-2k+1}{k}\!\!\right)(-1)^k\,m^{\underline{k}}\,n^{\underline{k}}\,b_{m+n-2k},$$

$$D_m(x)\,D_n(x) = \sum_{k\geqslant 0}\binom{m+n-2k}{m-k}\left(\!\!\binom{m+n-2k+1}{k}\!\!\right)(-1)^k\,D_{m+n-2k}(x),$$

$$d_{m,h}\,d_{n,h} = \sum_{k\geqslant 0}\binom{m+n-2k}{m-k}\left(\!\!\binom{m+n-2k+1}{k}\!\!\right)(-1)^k\,d_{m+n-2k,h}.$$

(2) The technique used to prove Cayley's identity also yields the following series:

$$\begin{aligned}
U^{(2,0)}(\theta,x;t) &= \frac{1+t+\sqrt{1-6t+t^2}}{2\sqrt{1-6t+t^2}}\,U\left(\theta,x;\ \frac{1-t-\sqrt{1-6t+t^2}}{2}\right)\\
&= \frac{1+t+\sqrt{1-6t+t^2}}{2\sqrt{1-6t+t^2}}\left(\frac{1-t-\sqrt{1-6t+t^2}}{2t}\right)^{\theta/2}\\
&\quad\times\left(\frac{1+4t-t^2+(1-t)\sqrt{1-6t+t^2}}{2}\right)^{x/2},\\
U^{(0,2)}(\theta,x;t) &= \frac{1}{\sqrt{1+4t^2}}\,U\left(\theta,x;\ \frac{-1+\sqrt{1+4t^2}}{2t}\right)\\
&= \frac{1}{\sqrt{1+4t^2}}\left(2t+\sqrt{1+4t^2}\right)^{\theta/2}\left(\frac{-1+\sqrt{1+4t^2}}{2t^2}\right)^{x/2},
\end{aligned}$$

$$
\begin{aligned}
U^{(3,1)}(\theta, x; t) &= \frac{1+4t+\sqrt{1-8t}}{2(1+t)\sqrt{1-8t}} U\left(\theta, x; \frac{1-2t-\sqrt{1-8t}}{2(1+t)}\right) \\
&= \frac{1+4t+\sqrt{1-8t}}{2(1+t)\sqrt{1-8t}} \left(\frac{1-\sqrt{1-8t}}{4t}\right)^{\theta/2} \\
&\quad \times \left(\frac{1+10t+(1-2t)\sqrt{1-8t}}{2(1+t)^2}\right)^{x/2}.
\end{aligned}
$$

References

[1] M. Aigner, Combinatorial Theory, Springer, New York, 1979.

[2] M. Aigner, Catalan-like numbers and determinants, J. Combin. Theory Ser. A 87 (1999) 33–51.

[3] M. Aigner, Catalan and other numbers: a recurrent theme, in: Algebraic Combinatorics and Computer Science, Springer Italia, Milan, 2001, pp. 347–390.

[4] C. Banderier, S. Schwer, Why Delannoy's numbers?, May 2002, Lattice Paths'02.

[5] F. Bergeron, G. Labelle, P. Leroux, Combinatorial Species and Tree-like Structures, Cambridge University Press, Cambridge, 1998.

[6] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer, Berlin, 1989.

[7] A. Cayley, On the determination of the value of a certain determinant, Quart. J. Math. ii (1858) 163–166 (Collected Mathematics Papers, Vol. 3, Cambridge University press, Cambridge, 1919, pp. 120–123).

[8] P.A. Clement, A class of triple-diagonal matrices for test purposes, SIAM Rev. 1 (1959) 50–52.

[9] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht-Holland, Boston, 1974.

[10] A. Edelman, E. Kostlan, The road from Kac's matrix to Kac's random polynomials, in: J. Lewis (Ed.), Proc. of the Fifth SIAM Conf. on Applied Linear Algebra, SIAM, Philadelpia, 1994, pp. 503–507.

[11] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. 32 (1980) 125–161.

[12] P. Flajolet, F. Guillemin, The formal theory of birth-and-death processes, lattice path combinatorics and continued fractions, Adv. Appl. Probab. 32 (2000) 750–778.

[13] D. Foata, J. Labelle, Modèles combinatoires pour les polynômes de Meixner, Europ. J. Combin. 4 (1983) 305–311.

[14] I.P. Goulden, D.M. Jackson, Combinatorial Enumeration, Wiley, New York, 1983.

[15] P. Hanlon, To the Latimer–Macduffee theorem and beyond!, Linear Algebra Appl. 280 (1998) 21–37.

[16] M.L.J. Hautus, D.A. Klarner, The diagonal of a double power series, Duke Math. J. 38 (1971) 229–235.

[17] F.G. Hess, Alternative solution to the Ehrenfest problem, Amer. Math. Monthly 61 (1954) 323–327.

[18] N. Higham, The Test Matrix Toolbox for Matlab, Numerical Analysis Report No. 237, University of Manchester, England, December 1993.

[19] A. Joyal, Une théorie combinatoire des séries formelles, Adv. Math. 42 (1981) 1–82.

[20] M. Kac, Random walks and the theory of Brownian motion, Amer. Math. Monthly 54 (1947) 369–390.

[21] T. Muir, The Theory of Determinants in the Historical Order of Development, Dover, New York, 1960.

[22] S. Roman, The umbral calculus, Pure and Applied Mathematics, Vol. 111, Academic Press, New York, 1984.

[23] S.M. Roman, G.-C. Rota, The umbral calculus, Adv. Math. 27 (1978) 95–188.

[24] G.-C. Rota, The number of partitions of a set, Amer. Math. Monthly 71 (1964) 498–504.

[25] L.W. Shapiro, S. Getu, W.J. Woan, L.C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.

[26] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences, Published electronically at http://www.research.att.com/∼njas/sequences/.

[28] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.

[29] R.A. Sulanke, Objects counted by the central Delannoy numbers, J. Integer Seq. 6 (1) (2003) Article 03.1.5.
[30] J.J. Sylvester, Théorème sur les déterminants de M. Sylvester, Nouv. Ann. Math. xiii (1854) 305 (Collected Mathematics Papers, vol. 2, Cambridge University Paper, Cambridge, 1908, p. 128.).
[31] O. Taussky, J. Todd, Another look at a matrix of Mark Kac, Linear Algebra Appl. 150 (1991) 341–360.
[32] R. Vein, P. Dale, Determinants and Their Applications in Mathematical Physics, Springer, New York, 1999.
[33] J. Zeng, Weighted derangements and the linearization coefficients of orthogonal sheffer polynomials, Proc. London Math. Soc. 65 (1992) 1–22.