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A Fixed Point Theorem for Function Spaces

GIOVANNI VIDOSSICH

Istituto Matematico, Università degli Studi, 56100 Pisa, Italy[1]

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We are interested on the structure of the set of fixed points of nonlinear operators in spaces of continuous functions, i.e., when this set is nonempty, what are its topological properties? In Browder [2] and in Vidossich [8] there are some theorems showing the existence of fixed points of a map f in a Banach space, whenever $I - f$, I the identity, is a closed map. These theorems require that $I - f$ can be approximated by sufficiently regular maps as, e.g., homeomorphisms. Aronszajn [1], Stampacchia [6], Browder-Gupta [3], Browder [2], Vidossich [7, 8] study the connectedness of the set of fixed points. Again, a crucial hypothesis is the existence of an approximation of $I - f$ by sufficiently regular maps, as homeomorphisms, for example.

The aim of this note is to present a simple (particularly, in case of compact mappings) set of conditions on a map f in a space of continuous mappings leading to an approximation of $I - f$ by homeomorphisms. If $I - f$ is a closed map and our conditions are fulfilled by f, then the fixed points of f form a nonempty connected set. Since our conditions are easy to handle, our results may advantageously substitute the theorems of [1–3], [6–8] in the applications, at least to nonlinear operators arising in ordinary differential equations, integral equations of Volterra type, implicit functions.

NOTATION

We shall denote by

$C_u(X, Y)$	the set of all (bounded) continuous mappings $X \to Y$, Y a metric space, equipped with the metric of uniform convergence;
$B(x, \epsilon)$	the closed ball of center x and radius ϵ;
I	the identity map;
$x\,|_A$	the restriction on A of the map x.

[1] Present affiliation: Scuola Normale Superiore, 56100 Pisa, Italy.

1. Results and Comments

The main result of this note is the following theorem

1.1 Theorem. *Let K be a bounded convex subset of a normed space, Y a Banach space, $X = C_u(K, Y)$, $F : X \to X$ continuous and $T = I - F$. Suppose that*

(1) *T is a closed map $X \to X$.*

(2) *$T(X)$ is an equiuniformly continuous set of maps $K \to Y$.*
and that there are $t_0 \in K$ and $y_0 \in Y$ such that

(3) $F(x)(t_0) = y_0$ $(x \in X)$.

(4) *For every $\epsilon > 0$,*

$$x|_{K_\epsilon} = y|_{K_\epsilon} \Rightarrow F(x)|_{K_\epsilon} = F(y)|_{K_\epsilon} \qquad (x, y \in X)$$

where

$$K_\epsilon = B(t_0, \epsilon) \cap K.$$

Then T is a uniform limit of homeomorphisms $X \to X$, and the fixed points of F form a nonempty connected set which is an R_δ whenever it is compact.

whose statement is simpler and stronger (since we have a local version) in case of compact mappings:

1.2 Corollary. *Let K be a compact convex subset of a normed space, Y a closed convex subset of a Banach space Y_0, F a compact map $C_u(K, Y) \to C_u(K, Y)$. If there are $t_0 \in K$ and $y_0 \in Y$ such that the following two conditions*

(i) $F(x)(t_0) = y_0$ $(x \in C(K, Y))$.

(ii) *For every $\epsilon > 0$,*

$$x|_{K_\epsilon} = y|_{K_\epsilon} \Rightarrow F(x)|_{K_\epsilon} = F(y)|_{K_\epsilon} \qquad (x, y \in C(K, Y))$$

where

$$K_\epsilon = B(t_0, \epsilon) \cap K.$$

hold, then the set of fixed points of F is an R_δ.

Recall that, according to [1], an R_δ is the intersection of a decreasing sequence of compact *AE* (metric). Therefore an R_δ is a nonempty, compact, connected set. A *compact map* is a continuous map with relatively compact range. Finally, Condition (2) of Theorem 1.1 means that for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\| t' - t'' \| \leqslant \delta \Rightarrow \| F(x)(t') - F(x)(t'') \| \leqslant \epsilon \qquad (x \in X).$$

Obviously, an equicontinuous family of maps in a compact space is uniformly equicontinuous. Therefore, Conditions (1) and (2) are automatically fulfilled in case K is a compact set and F a compact mapping. This is the reason of the simplest version of Theorem 1.1 for compact operators.

Let K be a compact interval $[t_0, b]$ of $\mathbf{R}$ and Y a subset of a Banach space Z. Suppose that $f : [t_0, b] \times Y \to Z$ and $g : [t_0, b]^2 \times Y \to Z$ are, e.g., contînous and bounded. Then the mapping $F : C(K, Y) \to C(K, Y)$ defined alternatively by

$$F(x)(t) = x_0 + \int_{t_0}^{t} f(\xi, x(\xi))\, d\xi$$

$$F(x)(t) = x_0 + \int_{t_0}^{t} g(t, \xi, x(\xi))\, d\xi$$

satisfies, obviously, the conditions (i) and (ii) of Theorem 1.2. Therefore, Theorems 1.1 and 1.2 are applicable to mappings arising in ordinary differential equations, integral equations of Volterra type, and, as we shall see just below, implicit functions—provided that they are compact or satisfy the conditions (1) and (2) of Theorem 1.1.

Let K be a metric AE(metric). We may assume that the metric of K is bounded. Then K is isometric to a closed subset, say K again, of a bounded convex subset K_0 of a Banach space. Let $r : K_0 \to K$ be a continuous retraction. Obviously, $f \rightsquigarrow f \circ r$ is a continous map $\varphi : C_u(K, Y) \to C_u(K_0, Y)$, while $g \rightsquigarrow g|_K$ is a continuous map $\psi : C_u(K_0, Y) \to C_u(K, Y)$. Then $F_0 = \varphi \circ F \circ \psi$ is a compact map $C_u(K_0, Y) \to C_u(K_0, Y)$ whenever $F : C_u(K, Y) \to C_u(K, Y)$ is, and the fixed points of F are the images by ψ of the fixed points of F_0. Therefore, the fixed points of F form a continuous image of an R_δ whenever F is compact and verifies (i) of Theorem 1.2 as well as

(iii) *For every* $A \subseteq K$,

$$x|_A = y|_A \Rightarrow F(x)|_A = F(y)|_A \qquad (x, y \in C(K, Y)).$$

A type of mapping F satisfying to (iii) is given by implicit functions. Suppose $f : K \times Y \to Y$ is continuous. Then,

$$F(x)(t) = f(t, x(t))$$

clearly verifies (iii). In order that this F satisfies (i), we must assume the existence of $t_0 \in K$ and $y_0 \in Y$ such that

$$f(t_0, y) = y_0 \qquad (y \in Y).$$

By the Dugundji extension theorem [4], every convex subset of a normed space is a metric AE(metric). Therefore the above remark shows that

Theorem 1.1 is applicable to an arbitrary convex subset K of a normed space, provided that (4) is substituted by (iii).

It would be better to prove the above results for some important proper subset of $C(K, Y)$, as the set of Lipschitz maps with the same Lipschitz constant, or the set of functions of class C^p ($p = 1,..., \infty$). An interesting case is suggested by the embeddability of any Banach space into the space of continuous functions on the closed unit ball K of the dual space, K being compact under the weak* topology.

1.3 *Remark.* From [8] it follows that for the mere existence of fixed points for F it is enough to substitute (1) with "$T(X)$ is a closed set", while, for the mere connectedness of their set, (1) may be changed with "T is a closed map $X \to T(X)$".

2. Proofs

Proof of 1.1. It is enough to show that T is a uniform limit of homeomorphisms $X \to X$, since the remaining statements follow from [8; 1.1, 2.2, and 2.4]. Due to the convexity of K, the point

$$\alpha_n(t) = (1 - 1/n \| t - t_0 \|)\, t + t_0/n \| t - t_0 \|$$

belongs to K whenever $t \in K \backslash K_{1/n}$ and $n \in \mathbf{Z}^+$. By this and (3), the equality

$$F_n(x)(t) = \begin{cases} y_0 & \text{if} \quad \| t - t_0 \| \leqslant 1/n \\ F(x)(\alpha_n(t)) & \text{if} \quad \| t - t_0 \| \geqslant 1/n \end{cases}$$

defines a map $F_n(x) : K \to Y$ ($n \in \mathbf{Z}^+$; $x \in X$). Each map $F_n(x)$ is continuous since its restrictions on $K_{1/n}$ and $K' = \{t \in K \mid \| t - t_0 \| \geqslant 1/n\}$ are continuous, and $\{K_{1/n}, K'\}$ is a (locally) finite closed covering of K. Therefore, F_n maps $X \to X (n \in \mathbf{Z}^+)$. For every $n \in \mathbf{Z}^+$ and every $t \in K$, there is $s \in K$ such that

$$\| s - t \| \leqslant 1/n \qquad \text{and} \qquad \| F_n(x)(t) - F(x)(t) \| = \| F(x)(s) - F(x)(t) \| .$$

By this and (2), $\lim_n F_n = F$, hence $\lim_n I - F_n = T$ uniformly on X. Therefore, we have only to prove that every $T_n = I - F_n$ is a homeomorphism $X \to X$. The continuity of F_n, hence of T_n, is a direct consequence of the continuity of F. To prove the injectivity of T_n, suppose $T_n(x) = T_n(y)$ and argue for the equality $x = y$. Since K is bounded, there is $k \in \mathbf{Z}^+$ such that $K \subseteq B(t_0, k/n)$. Define

$$C_i = \{t \in K \mid (i - 1)/n \leqslant \| t - t_0 \| \leqslant i/n\} \qquad (i = 1,..., k).$$

From $T_n(x) = T_n(y)$, it follows $x|_{C_1} = y|_{C_1}$ in view of the definition of F_n. Therefore, from (4) and the definition of F_n, it follows $x|_{C_2} = y|_{C_2}$; hence $x|_{C_3} = y|_{C_3}$ again by (4) and the definition of F_n, and so on, we obtain $x = y$. To prove the surjectivity of T_n, choose $y \in X$ and argue for the existence of $x \in X$ such that $T_n(x) = y$. Define $x_1 : C_1 \to Y$ by $x_1(t) = y(t) + y_0$. Since C_1 is a closed subset of the metric space K, by the Dugundji extension theorem [4] there is a continuous extension $\overline{x_1} : K \to Y$ of x_1. Define $x_2 : C_2 \to Y$ by $x_2(t) = y(t) + F(\overline{x_1})(\alpha_n(t))$. If $t \in C_1 \cap C_2$, then $\alpha_n(t) = t_0$, so that by (3) we have $F(\overline{x_1})(\alpha_n(t)) = y_0$, and hence $x_1(t) = x_2(t)$. Therefore, there is a well-defined map x^2 on $K_{2/n} = C_1 \cup C_2$ agreeing with x_1 on C_1 and with x_2 on C_2. Since $\{C_1, C_2\}$ is a (locally) finite closed covering of $K_{2/n}$, x^2 is continuous. Therefore, $K_{2/n}$ being a closed subset of the metric space K, by the Dugundji extension theorem [4] there is a continuous extension $\overline{x_2} : K \to Y$ of x^2. Define $x_3 : C_3 \to Y$ by $x_3(t) = y(t) + F(\overline{x_2})(\alpha_n(t))$. If $t \in C_2 \cap C_3$, then $\alpha_n(t) \in C_1$, so that by (4) and $\overline{x_2}|_{K_{1/n}} = \overline{x_1}|_{K_{1/n}}$, we have $F(\overline{x_2})(\alpha_n(t)) = F(\overline{x_1})(\alpha_n(t))$, hence $x_3(t) = x_2(t)$. Therefore, there is a well-defined continuous map $x^3 : K_{3/n} \to Y$ agreeing with x_i on C_i ($i = 1,..., 3$). By the Dugundji extension theorem [4], there is a continuous extension $\overline{x_3} : K \to Y$ of x^3. Then we define $x_4 : C_4 \to Y$ by $x_4(t) = y(t) + F(\overline{x_3})(\alpha_n(t))$. If $t \in C_3 \cap C_4$, then $\alpha_n(t) \in C_2$, so that by (4) and $\overline{x_3}|_{K_{2/n}} = \overline{x_2}|_{K_{2/n}}$ we have $F(\overline{x_3})(\alpha_n(t)) = F(\overline{x_2})(\alpha_n(t))$; hence $x_4(t) = x_3(t)$. By repeating this argument for $i = 5,..., k$, we define a continuous map $x : K \to Y$ such that $x|_{C_i} = x_i$ ($i = 1,..., k$). Consequently,

$$x|_{K_{i/n}} = \overline{x_i}|_{K_{i/n}} \qquad (i = 1,..., k),$$

so that by (4) we have

$$x(t) = \begin{cases} y(t) + y_0 & \text{for } t \in K_{1/n} \\ y(t) + F(x)(\alpha_n(t)) & \text{for } t \in K \backslash K_{1/n}, \end{cases}$$

which means $T_n(x) = y$. It remains to check the continuity of $T_n^{-1} : X \to X$. Suppose $\lim_k T_n(x_k) = T_n(x)$, and argue for $\lim_k x_k = x$. From the definition of F_n, it follows that $\lim_k x_k = x$ uniformly on $K_{1/n}$. For every $k \in \mathbf{Z}^+$, put

$$\epsilon_k = \max(1/k, \sup_{t \in K_{1/n}} \| x_k(t) - x(t) \|).$$

Again by the Dugundji extension theorem [4], there is a continuous extension $x_k^1 : K \to B(0, \epsilon_k) \subseteq Y$ of $(x_k - x)|_{K_{1/n}}$. Since $\lim_k \epsilon_k = 0$, $\lim_k x + x_k^1 = x$ in X, so that $\lim_k F(x + x_k^1) = F(x)$ in X by the continuity of F. Since $(x + x_k^1)|_{K_{1/n}} = x_k|_{K_{1/n}}$, from (4) it follows $F(x + x_k^1)|_{K_{1/n}} = F(x_k)|_{K_{1/n}}$. Therefore, $\lim_k F(x_k) = F(x)$ uniformly on $K_{1/n}$. This implies $\lim_k x_k = x$

uniformly on $K_{2/n}$ by the definition of F_n. By repeating this argument we find $\lim_k x_k = x$ uniformly on $K_{i/n}$ $(i = 3,..., k)$. Thus $\lim_k x_k = x$ in X. Q.E.D.

The proof of Theorem 1.1 is much simpler in case F is compact. For, the most difficult part, the surjectivity of T_n, is then an easy consequence of the compactness of F_n, since the map $x \rightsquigarrow F_n(x) + y$ in that case has a fixed point for every $y \in X$. Moreover, in that case also the continuity of T_n^{-1} is trivial, since T_n is a proper, hence closed, map by a well-known property of the compact map.

Proof of 1.2. Put $X = C_u(K, Y_0)$. By the Dugundji extension theorem [4], there is a continuous retraction $r : Y_0 \to Y$. Define $G : X \to X$ by $G(x) = F(r \circ x)$. Since $G(X) \subseteq F(C(K, Y))$, $\overline{G(X)}$ is compact. In order to prove the continuity of G, suppose $\lim_n x_n = x$ in X. Let μ be the fine (= finest compatible) uniformity of Y_0. By a well-known theorem on uniform spaces, $r : \mu Y_0 \to Y$ (here Y is a metric subspace of Y_0) is uniformly continuous. Due to the compactness of K, by [5, Chap. 7, Theorem 11] the topology of X equals the topology of uniform convergence with respect to the uniform space μY_0. Therefore, $\lim_n x_n = x$ uniformly with respect to μY_0. To every $\epsilon > 0$, there corresponds an entourage W of μY_0 such that

$$(y', y'') \in W \Rightarrow \| r(y') - r(y'')\| \leqslant \epsilon.$$

To W there corresponds $n_0 \in \mathbf{Z}^+$ such that

$$(x_n(t), x(t)) \in W \qquad (n \geqslant n_0;\ t \in K).$$

Therefore, $\lim_n r \circ x_n = r \circ x$ in X, so that $\lim_n G(x_n) = G(x)$ by the continuity of F. Therefore, G is a compact map. Then the fixed points of G form an R_δ by Theorem 1.1 (as it was remarked in Section 1, Conditions (1) and (2) of Theorem 1.1 are always fulfilled by compact mappings.) Due to $r(y) = y$ for every $y \in Y$, G and F have the same fixed points. Q.E.D.

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Added in proof: For further comments on this subject and for a simplification of the proof of the surjectivity of the maps T_n in the proof of Theorem 1.1, see author's paper

"Applications of Topology to Analysis: On the topological properties of the set of fixed points of nonlinear operators," *Confer. Sem. Mat. Univ. Bari* (to appear).

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