

ACADEMIC
PRESSAvailable at
WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®

Journal of Approximation Theory 123 (2003) 232–255

JOURNAL OF
**Approximation
Theory**<http://www.elsevier.com/locate/jat>

q -Bernstein polynomials and their iterates

Sofiya Ostrovska*

Department of Mathematics, Atılım University, 06836 Incek, Ankara, Turkey

Received 11 July 2002; accepted in revised form 12 May 2003

Communicated by Dany Leviatan

Abstract

Let $B_n(f, q; x)$, $n = 1, 2, \dots$ be q -Bernstein polynomials of a function $f: [0, 1] \rightarrow \mathbf{C}$. The polynomials $B_n(f, 1; x)$ are classical Bernstein polynomials. For $q \neq 1$ the properties of q -Bernstein polynomials differ essentially from those in the classical case. This paper deals with approximating properties of q -Bernstein polynomials in the case $q > 1$ with respect to both n and q . Some estimates on the rate of convergence are given. In particular, it is proved that for a function f analytic in $\{z: |z| < q + \varepsilon\}$ the rate of convergence of $\{B_n(f, q; x)\}$ to $f(x)$ in the norm of $C[0, 1]$ has the order q^{-n} (versus $1/n$ for the classical Bernstein polynomials). Also iterates of q -Bernstein polynomials $\{B_{j_n}^{j_n}(f, q; x)\}$, where both $n \rightarrow \infty$ and $j_n \rightarrow \infty$, are studied. It is shown that for $q \in (0, 1)$ the asymptotic behavior of such iterates is quite different from the classical case. In particular, the limit does not depend on the rate of $j_n \rightarrow \infty$.

© 2003 Elsevier Science (USA). All rights reserved.

Keywords: q -Bernstein polynomials; q -Integers; q -Binomial coefficients; Convergence; Iterates

1. Introduction

In 1912 Bernstein [2] found his famous proof of the Weierstrass Approximation Theorem. Using probability theory he defined polynomials called nowadays *Bernstein polynomials* as follows.

Definition (Bernstein [2]). Let $f: [0, 1] \rightarrow \mathbf{R}$. The *Bernstein polynomial* of f is

$$B_n(f; x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \quad n = 1, 2, \dots$$

*Fax: +90-312-586-80-91.

E-mail address: ostrovskaofiya@yahoo.com.

Bernstein proved that if $f \in C[0, 1]$, then the sequence $\{B_n(f; x)\}$ converges uniformly to $f(x)$ on $[0, 1]$.

Later it was found that Bernstein polynomials possess many remarkable properties, which made them an area of intensive research. A systematic treatment of the theory of Bernstein polynomials as it was until 1990s is presented, for example, in [8,19]. New papers are constantly coming out (cf. e.g. [3]), and new applications and generalizations are being discovered (cf. e.g. [7,13]). A generalization of Bernstein polynomials involving q -integers was proposed by Lupaş in 1987 (cf. [9]). However, the q -analogue of the Bernstein operator considered by Lupaş gives rational functions rather than polynomials.

Generalized Bernstein polynomials based on the q -integers, or q -Bernstein polynomials were introduced by Phillips in 1997. In the case $q = 1$ these polynomials coincide with the classical ones. For $q \neq 1$ one gets a new class of polynomials having interesting properties. q -Bernstein polynomials have been studied by Phillips et al. ([4,11,12,14–17]), who obtained a great number of results related to various properties of these polynomials.

It should be mentioned that results of these papers deal mostly with the case $q \in (0, 1)$. This is because in this case q -Bernstein polynomials generate *positive* linear operators $B_{n,q}: f \mapsto B_n(f, q; x)$; the fact that is used in investigation significantly. The case $q \in (1, \infty)$, where positivity fails, has not been studied in detail. However, the results of this paper show that in this case approximating properties of q -Bernstein polynomials may be better than in the case $q \leq 1$.

In Sections 3 and 4, we discuss convergence properties of q -Bernstein polynomials with respect to both n and q in the case $q > 1$.

In Sections 5 and 6, we study the rate of approximation of analytic functions by q -Bernstein polynomials in the case $q > 1$. In particular, for entire functions the rate of convergence has the order q^{-n} ($q > 1$) versus $1/n$ for the classical polynomials. We also discuss approximation by q -Bernstein polynomials in case the value of parameter q varies.

It should be emphasized that the results of the paper are the first ones showing that approximation properties of q -Bernstein polynomials can be *better* than of the classical ones.

Sections 7–9 are dedicated to iterates of the q -Bernstein operator. By the definition the k th iterate of $B_{n,q}$ is

$$B_{n,q}^1 := B_{n,q}, \quad B_{n,q}^k := B_{n,q}(B_{n,q}^{k-1}), \quad k = 2, 3, \dots$$

Iterates of the classical Bernstein operator $B_n := B_{n,1}$ have been studied in many papers starting from [6]. In [6], Kelisky and Rivlin studied the convergence of the iterates $B_n^k(f)$ as $k \rightarrow \infty$ if n is fixed, and of the iterates $B_n^{j_n}(p)$ as $n \rightarrow \infty$, where p is a polynomial and $\{j_n/n\} \rightarrow \alpha$, $0 \leq \alpha \leq \infty$. They proved that in both cases the iterates are convergent, and found an explicit formula of the limit function. From a different point of view the iterates of the Bernstein operator were studied by Micchelli [10], who considered them using semigroup methods. Recently, Cooper and Waldron [3]

investigated iterates of the Bernstein operator using properties of eigenvalues and eigenvectors of the operator. In [3] one can also find other references on the subject.

Iterates of the q -Bernstein operator $B_{n,q}^k$ with fixed n and $k \rightarrow \infty$ were considered in [12], where it was proved that these iterates have the same behavior as in the classical case $q = 1$.

In this paper we consider iterates of the q -Bernstein operator of the form $B_{n,q}^{j_n}$, where both n and j_n tend to infinity. We consider in detail the behavior of iterates of the q -Bernstein operator for $q \in (0, 1)$. Our results show that in this case the behavior of iterates is essentially different from the classical case $q = 1$ considered by Kelisky and Rivlin [6, Theorem 2]. In particular, the limit does not depend on the rate of $j_n \rightarrow \infty$ (cf. Theorem 8). For $q \in (1, \infty)$ the situation is very similar to the classical case. Corresponding results and their proofs can be obtained by almost verbatim extension of reasoning given in [3, Theorems 4.1, 4.20, Corollary 5.15]. Therefore, we present them without proofs.

To formulate our results we need the following definitions.

Let $q > 0$. For any $n = 0, 1, 2, \dots$ the q -integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \dots + q^{n-1} \quad (n = 1, 2, \dots), \quad [0]_q := 0$$

and the q -factorial $[n]_q!$ by

$$[n]_q! := [1]_q [2]_q \dots [n]_q \quad (n = 1, 2, \dots), \quad [0]_q! := 1.$$

For integers $0 \leq k \leq n$ the q -binomial, or the Gaussian coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

In the sequel we always assume that $f : [0, 1] \rightarrow \mathbf{C}$. We denote by $C[0, 1]$ (or $C^n[0, 1]$, $1 \leq n \leq \infty$) the space of all continuous (correspondingly, n times continuously differentiable) complex-valued functions on $[0, 1]$ equipped with the uniform norm. The expression $g_n(x) \rightrightarrows g(x)$ means uniform convergence of a sequence $\{g_n(x)\}$ to $g(x)$.

Definition (Phillips [14]). Let $f : [0, 1] \rightarrow \mathbf{C}$, $q > 0$. The q -Bernstein polynomial of f is

$$B_n(f, q; x) := \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1 - q^s x), \quad n = 1, 2, \dots \quad (1)$$

(From here on an empty product is taken to be equal 1.)

Note that for $q = 1$, the polynomials $B_n(f, 1; x)$ are classical Bernstein polynomials. Recall that the famous theorem of Bernstein states:

Theorem (Bernstein [2]). *If $f \in C[0, 1]$, then*

$$B_n(f, 1; x) \rightrightarrows f(x) \quad \text{for } x \in [0, 1] \text{ as } n \rightarrow \infty.$$

For $q \in (0, 1)$ convergence of the sequence $\{B_n(f, q; x)\}$ was investigated in [5].

Theorem (Il'inskii and Ostrovska [5]). *Given $q \in (0, 1)$ and $f \in C[0, 1]$, there exists a continuous function $B_\infty(f, q; x)$ such that*

$$B_n(f, q; x) \rightrightarrows B_\infty(f, q; x) \quad \text{for } x \in [0, 1] \text{ as } n \rightarrow \infty. \tag{2}$$

An explicit formula for $B_\infty(f, q; x)$ is given by (16). It follows from (16) that the equality $B_\infty(f, q; x) = f(x)$ holds if and only if $f(x) = ax + b$, i.e. $f(x)$ is a linear function.

Therefore, in the case $q \in (0, 1)$ the sequence $\{B_n(f, q; x)\}$ is not an approximating sequence for a function f unless f is linear. This is in contrast to the case $q = 1$, when the sequence $\{B_n(f, 1; x)\}$ approximates f for any $f \in C[0, 1]$.

In this paper we show that in the case $q > 1$ approximating properties of the sequence $\{B_n(f, q; x)\}$ are in some sense intermediate between the cases mentioned above. We prove that for $q > 1$ the sequence $\{B_n(f, q; x)\}$ is approximating for functions analytic in a suitable domain, and, moreover, we may achieve a fast rate of convergence. At the same time the sequence may be divergent for some infinitely differentiable functions. We also discuss approximating properties of q -Bernstein polynomials related to the dependence on the value of q .

Equality (1) defines the linear operator

$$B_{n,q} : f \mapsto B_n(f, q; x),$$

which is called the q -Bernstein operator. Clearly,

$$B_{n,q} : C[0, 1] \rightarrow \mathcal{P}_n,$$

where \mathcal{P}_n denotes the set of polynomials of degree $\leq n$. To study iterates of q -Bernstein polynomials it is convenient to present them in the form of linear operators, i.e. $B_n^k(f, q, x) = B_{n,q}^k(f)$. In the sequel, we use polynomial and operator notation interchangeably. We prove that for $q \in (0, 1)$ and any function $f \in C[0, 1]$ the sequence $\{B_n^{j_n}(f, q, x)\}$, where $n \rightarrow \infty$ and $j_n \rightarrow \infty$, converges uniformly to the linear function interpolating f at 0 and 1 regardless the rate of $j_n \rightarrow \infty$.

For $q \in (0, 1)$ the limit function appeared in (2) defines a linear operator on $C[0, 1]$

$$B_{\infty,q} : f \mapsto B_\infty(f, q; x).$$

It was observed in [5] that $B_{\infty,q}(C[0, 1]) \neq C[0, 1]$. We also consider the behavior of the iterates of $B_{\infty,q}$.

2. Preliminaries

In this section we state some general properties of q -Bernstein polynomials which will be used throughout the paper.

It follows directly from the definition that q -Bernstein polynomials possess the *end-point interpolation* property, i.e.

$$B_n(f, q; 0) = f(0), \quad B_n(f, q; 1) = f(1) \text{ for all } q > 0 \text{ and all } n = 1, 2, \dots \quad (3)$$

The following representation of q -Bernstein polynomials, called the *q -difference form*, was obtained in [15, Theorem 1, formula (12)]:

$$B_n(f, q; x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{D}^k f_0 x^k, \quad (4)$$

where $\mathcal{D}^k f_0$ is expressed as

$$\mathcal{D}^k f_0 = \frac{[k]_q!}{[n]_q^k} q^{k(k-1)/2} f \left[0; \frac{1}{[n]_q}; \dots; \frac{[k]_q}{[n]_q} \right]. \quad (5)$$

By $f[x_0; x_1; \dots; x_k]$ we denote the usual divided difference, i.e.

$$f[x_0] = f(x_0), \quad f[x_0; x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \dots,$$

$$f[x_0; x_1; \dots; x_j] = \frac{f[x_1; \dots; x_j] - f[x_0; \dots; x_{j-1}]}{x_j - x_0}.$$

Using (4) and (5), we write

$$B_n(f, q; x) = \sum_{k=0}^n \lambda_{k,q}^{(n)} f \left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q} \right] x^k, \quad (6)$$

where

$$\lambda_{k,q}^{(n)} := \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!}{[n]_q^k} q^{k(k-1)/2} = \left(1 - \frac{1}{[n]_q} \right) \dots \left(1 - \frac{[k-1]_q}{[n]_q} \right). \quad (7)$$

In Section 8 (Lemma 5) we show that $\lambda_{k,q}^{(n)}$ are eigenvalues of the q -Bernstein operator $B_{n,q}$. Note that

$$\lambda_{0,q}^{(n)} = \lambda_{1,q}^{(n)} = 1, \quad (8)$$

and it is clear from (7) that

$$0 \leq \lambda_{k,q}^{(n)} \leq 1, \quad k = 0, 1, \dots, n. \quad (9)$$

Therefore,

$$|B_n(f, q; x)| \leq \sum_{k=0}^n \left| f \left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q} \right] \right| |x|^k. \quad (10)$$

This estimate will be used in the sequel.

It follows immediately from (6) and (8) that q -Bernstein polynomials leave invariant linear functions, that is

$$B_n(ax + b, q; x) = ax + b \text{ for all } q > 0 \text{ and all } n = 1, 2, \dots \quad (11)$$

If f is a polynomial of degree m , then all its divided differences of order $> m$ vanish, and (6) implies that $B_n(f, q; x)$ is a polynomial of degree $\min(m, n)$. In other words, this means that the q -Bernstein operator is degree reducing.

We set

$$p_{nk}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1 - q^s x), \quad k = 0, 1, \dots, n; \quad n = 1, 2, \dots \quad (12)$$

Taking $a = 0, b = 1$ in (11), we conclude that

$$\sum_{k=0}^n p_{nk}(q; x) = 1; \quad \text{for all } q > 0 \text{ and all } n = 1, 2, \dots \quad (13)$$

Obviously,

$$B_n(f, q; x) = \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q; x).$$

The behavior of the sequence $\{B_n(f, q; x)\}$ for $q \in (0, 1)$ and $n \rightarrow \infty$ is described in [5] as follows.

Consider the entire functions

$$p_{\infty k}(q; x) := \frac{x^k}{(1 - q)^k [k]_q!} \prod_{s=0}^{\infty} (1 - q^s x), \quad k = 0, 1, \dots \quad (14)$$

By Euler’s identity (cf. [1, Chapter 2, Corollory 2.2]) we have

$$\sum_{k=0}^{\infty} p_{\infty k}(q; x) = 1 \quad \text{for all } x \in [0, 1). \quad (15)$$

Clearly, for $q \in (0, 1)$ we have

$$\lim_{n \rightarrow \infty} \frac{[k]_q}{[n]_q} = 1 - q^k \quad \text{for all } k = 0, 1, \dots$$

For $f : [0, 1] \rightarrow \mathbf{C}, q \in (0, 1)$ we set

$$B_{\infty}(f, q; x) = \begin{cases} \sum_{k=0}^{\infty} f(1 - q^k) p_{\infty k}(q; x) & \text{if } x \in [0, 1), \\ f(1) & \text{if } x = 1. \end{cases} \quad (16)$$

It can be readily seen that the function $B_{\infty}(f, q; x)$ is well defined on $[0, 1]$ whenever a function $f(x)$ is bounded on the interval. We note that (16) gives the limit function defined in (2). It follows from (2) and (11) that

$$B_{\infty}(at + b, q, x) = ax + b. \quad (17)$$

In the following section we investigate the behavior of the sequence $\{B_n(f, q; x)\}$ in the case $q > 1$.

3. Convergence of q -Bernstein polynomials in the case $q > 1$

Our main result on convergence is the following theorem.

Theorem 1. *Let $q \in (1, \infty)$, and let f be a function analytic in an ε -neighborhood of $[0, 1]$. Then for any compact set $K \subset D_\varepsilon := \{z: |z| < \varepsilon\}$,*

$$B_n(f, q; z) \rightrightarrows f(z) \quad \text{for } z \in K \text{ as } n \rightarrow \infty.$$

Corollary 1. *If f is a function analytic in a disk D_R , $R > 1$, then for any compact set $K \subset D_{R-1}$,*

$$B_n(f, q; z) \rightrightarrows f(z) \quad \text{for } z \in K \text{ as } n \rightarrow \infty.$$

In particular, if $R > 2$, then $B_n(f, q; x) \rightrightarrows f(x)$ for $x \in [0, 1]$ as $n \rightarrow \infty$.

Corollary 2. *If f is an entire function, then for any compact set $K \subset \mathbf{C}$,*

$$B_n(f, q; z) \rightrightarrows f(z) \quad \text{for } z \in K \text{ as } n \rightarrow \infty.$$

Remark. A particular case f being a polynomial and $K = [0, 1]$ was considered in [12].

The condition of analyticity is essential for convergence, and it cannot be dropped completely as the following theorem shows.

Theorem 2. *Let $q \in (1, \infty)$.*

- (i) *There exists $f \in C^\infty[0, 1]$ such that $\{B_n(f, q; x)\}$ does not converge to any finite function on $[0, 1]$.*
- (ii) *There exists $f \in C^\infty[0, 1]$ such that $\{B_n(f, q; x)\}$ converges to a finite discontinuous function on $[0, 1]$.*
- (iii) *There exists $f \in C^\infty[0, 1]$ such that $\{B_n(f, q; x)\}$ converges uniformly on $[0, 1]$ to $g(x) \neq f(x)$.*

The following theorem describes the behavior of the polynomials $B_n(f, q; x)$ as $q \rightarrow +\infty$ under certain smoothness conditions for f .

Theorem 3. *Let $f \in C^{n-1}[0, 1]$. Then for any compact set $K \subset \mathbf{C}$,*

$$B_n(f, q; z) \rightrightarrows B_n(f, \infty; z) := \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k + z^n \left\{ f(1) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \right\}$$

for $z \in K$ as $q \rightarrow +\infty$.

Corollary 3. *If f is analytic in a disk D_R , $R > 1$, then*

$$B_n(f, \infty; z) \rightrightarrows f(z) \text{ for } |z| \leq 1 \text{ as } n \rightarrow \infty.$$

That is, quite unexpectedly, we get good approximating properties of the sequence $\{B_n(f, q; x)\}$ taking the value of q infinite. The corollary below can be derived from Theorem 3 immediately.

Corollary 4. *If p is a polynomial of degree $\leq n$, then*

$$B_n(p, \infty; z) = p(z).$$

Therefore, we may approximate $p(x)$ with its q -Bernstein polynomials of the same degree n taking the limit with respect to q .

4. Proofs of Theorems 1–3

We need the following lemma, which is also of interest for its own sake.

Lemma 1. *Let $q \in (1, \infty)$. If $f \in C[0, 1]$, then*

$$\lim_{n \rightarrow \infty} B_n\left(f, q; \frac{1}{q^m}\right) = f\left(\frac{1}{q^m}\right) \text{ for all } m = 0, 1, 2, \dots$$

Proof. Let the polynomials $p_{nk}(q; x)$ be defined by (12). Obviously,

$$B_n(f, q; x) = \sum_{k=0}^n f\left(\frac{[n-k]_q}{[n]_q}\right) p_{n,n-k}(q; x).$$

We note that

$$p_{n,n-k}\left(q; \frac{1}{q^m}\right) = 0 \text{ for } m < k \leq n$$

and

$$\begin{aligned} p_{n,n-k}\left(q; \frac{1}{q^m}\right) &= \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{q^{m(n-k)}} \left(1 - \frac{1}{q^m}\right) \dots \left(1 - \frac{q^{k-1}}{q^m}\right) \\ &= O(q^{n(k-m)}) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for } k < m. \end{aligned}$$

For $k = m$ we have

$$\lim_{n \rightarrow \infty} p_{n,n-m}\left(q; \frac{1}{q^m}\right) = \lim_{n \rightarrow \infty} \begin{bmatrix} n \\ n-m \end{bmatrix}_q \frac{1}{q^{m(n-m)}} \left(1 - \frac{1}{q^m}\right) \dots \left(1 - \frac{1}{q}\right) = 1.$$

Since $f \in C[0, 1]$ and

$$\lim_{n \rightarrow \infty} \frac{[n-m]_q}{[n]_q} = \frac{1}{q^m},$$

it follows that

$$\lim_{n \rightarrow \infty} B_n \left(f, q; \frac{1}{q^m} \right) = \lim_{n \rightarrow \infty} f \left(\frac{[n-m]_q}{[n]_q} \right) p_{n,n-m} \left(q; \frac{1}{q^m} \right) = f \left(\frac{1}{q^m} \right). \quad \square$$

Proof of Theorem 1. Let f be analytic in an ε -neighborhood U_ε of $[0, 1]$. Take any compact set $K \subset D_\varepsilon$. Then for some $\varepsilon_1 \in (0, \varepsilon)$ we have $|z| \leq \varepsilon_1$ for all $z \in K$.

Let us choose a contour L in U_ε in such a way that the distance between L and $[0, 1]$ equals ρ , $0 < \varepsilon_1 < \rho < \varepsilon$.

Since (cf. [8, Chapter II, Section 2.7])

$$f \left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q} \right] = \frac{1}{2\pi i} \int_L \frac{f(\zeta) d\zeta}{\zeta \left(\zeta - \frac{1}{[n]_q} \right) \dots \left(\zeta - \frac{[k]_q}{[n]_q} \right)}$$

and $|\zeta - x| \geq \rho$ for all $\zeta \in L$ and $x \in [0, 1]$, it follows that

$$\left| f \left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q} \right] \right| \leq \frac{l}{2\pi} \cdot \frac{M_L}{\rho^{k+1}}, \tag{18}$$

where l is the length of L , and $M_L = \max_{\zeta \in L} |f(\zeta)|$. Substituting (18) into (10), we obtain

$$|B_n(f, q; z)| \leq \frac{lM_L}{2\pi\rho} \sum_{k=0}^n \frac{|z|^k}{\rho^k}.$$

If $z \in D_{\varepsilon_1}$, then $|z| < \varepsilon_1 < \rho$, so

$$\sum_{k=0}^n \frac{|z|^k}{\rho^k} \leq \sum_{k=0}^n \left(\frac{\varepsilon_1}{\rho} \right)^k < \sum_{k=0}^{\infty} \left(\frac{\varepsilon_1}{\rho} \right)^k = \frac{1}{1 - \frac{\varepsilon_1}{\rho}},$$

and hence the sequence $\{B_n(f, q; z)\}$ is uniformly bounded in the disk D_{ε_1} . Besides, by Lemma 1 the sequence converges to the function f analytic in D_{ε_1} on the set $\{1/q^m\}_0^\infty$ having an accumulation point in D_{ε_1} . By the Vitali Theorem (cf. e.g. [18, Chapter V, Section 5.2]) the sequence converges to f on any compact set in D_{ε_1} , and thus on K . \square

Proof of Theorem 2. Consider the polynomials $p_{nk}(q; x)$ defined by (12).

Specifically, we have

$$p_{nn}(q; x) = x^n$$

and

$$p_{n,n-1}(q; x) = [n]_q x^{n-1} (1-x) = \frac{q^n - 1}{q - 1} x^{n-1} (1-x).$$

Obviously,

$$\lim_{n \rightarrow \infty} p_{nn}(q; x) = \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x = 1 \end{cases}$$

and

$$\lim_{n \rightarrow \infty} p_{n,n-1}(q; x) = \begin{cases} 0 & \text{for } 0 \leq x < 1/q \text{ and } x = 1, \\ 1 & \text{for } x = 1/q \\ \infty & \text{for } 1/q < x < 1. \end{cases} \tag{19}$$

(i) Consider a function $\varphi \in C^\infty [0, 1]$ satisfying

$$\varphi(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 1/q^2, \\ 1 & \text{for } a \leq x \leq 1/q, \\ 0 & \text{for } x = 1, \end{cases}$$

where $a \in (1/q^2, 1/q)$. Since

$$\frac{[n-k]_q}{[n]_q} \uparrow \frac{1}{q^k} \text{ as } n \rightarrow \infty,$$

we obtain that

$$\varphi\left(\frac{[n-k]_q}{[n]_q}\right) = 0 \text{ for } k \neq 1 \text{ and sufficiently large } n.$$

Therefore

$$B_n(\varphi, q; x) = \varphi\left(\frac{[n-1]_q}{[n]_q}\right) p_{n,n-1}(q; x) = p_{n,n-1}(q; x)$$

for n large enough.

Let $g(x)$ be an entire function. We set

$$f(x) := g(x) + \varphi(x).$$

Then $B_n(f, q; x) = B_n(g, q; x) + B_n(\varphi, q; x)$. By Theorem 1, $B_n(g, q; x) \rightrightarrows g(x)$ on $[0, 1]$. Hence

$$\lim_{n \rightarrow \infty} B_n(f, q; x) = g(x) + \lim_{n \rightarrow \infty} p_{n,n-1}(q; x).$$

By (19) the limit is infinite for $x \in (1/q, 1)$.

(ii) In this case we take $\varphi \in C^\infty [0, 1]$ to satisfy

$$\varphi(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq 1/q, \\ 1 & \text{for } x = 1. \end{cases}$$

Similar to (i) we take an entire function $g(x)$ and set $f(x) := g(x) + \varphi(x)$. Since $B_n(g, q; x) \rightrightarrows g(x)$ and $B_n(\varphi, q; x) = p_{n,n-1}(q; x) = x^n$, we are done.

(iii) Consider $0 \neq \varphi(x) \in C^\infty [0, 1]$ such that $\varphi(x) = 0$ for $x \in [0, 1/q] \cup \{1\}$. Obviously, $B_n(\varphi, q; x) \equiv 0$ for all $n = 1, 2, \dots$. For any entire function $g(x)$ we set as above $f(x) := g(x) + \varphi(x)$ and get $B_n(f, q; x) \rightrightarrows g(x) \neq f(x)$. \square

Proof of Theorem 3. Using (6) and (7) we write

$$B_n(f, q; z) = \sum_{k=0}^n \left(1 - \frac{1}{[n]_q}\right) \cdots \left(1 - \frac{[k-1]_q}{[n]_q}\right) f \left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q}\right] z^k.$$

Note that for $j < n$,

$$\lim_{q \rightarrow +\infty} \frac{[j]_q}{[n]_q} = 0,$$

so all factors in the parentheses tend to 1 as $q \rightarrow +\infty$.

Now, since $f \in C^{n-1}[0, 1]$, for $k \leq n - 1$ we get

$$\lim_{q \rightarrow +\infty} f \left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q}\right] = \frac{f^{(k)}(0)}{k!}.$$

This allows us to evaluate the limit of the coefficients of $1, z, \dots, z^{n-1}$ in $B_n(f, q; x)$ as $q \rightarrow +\infty$. To find the limit of the coefficient of z^n we must evaluate

$$\lim_{q \rightarrow +\infty} f \left[0, \frac{1}{[n]_q}, \dots, \frac{[n-1]_q}{[n]_q}, 1\right].$$

We will use the following lemma.

Lemma 2. Let $f \in C^m[0, 1]$ and $0 \leq x_0 < x_1 < \dots < x_m < 1$. Then

$$\lim_{x_m \rightarrow 0} f[x_0, x_1, \dots, x_m, 1] = f(1) - \sum_{k=0}^m \frac{f^{(k)}(0)}{k!}.$$

Proof. We prove the lemma by induction on m .

For $m = 0$, we have

$$f[x_0, 1] = \frac{f(1) - f(x_0)}{1 - x_0}$$

and, clearly,

$$\lim_{x_0 \rightarrow 0} f[x_0, 1] = f(1) - f(0).$$

Assume that the statement is true if the number of points x_i does not exceed m .

Consider the divided difference with $(m + 1)$ points x_i :

$$f[x_0, x_1, \dots, x_m, 1] = \frac{f[x_1, \dots, x_m, 1] - f[x_0, x_1, \dots, x_m]}{1 - x_m}.$$

By the induction assumption we have

$$\lim_{x_m \rightarrow 0} f[x_1, \dots, x_m, 1] = f(1) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!}.$$

On the other hand, since $f \in C^m[0, 1]$, we get

$$\lim_{x_m \rightarrow 0} f[x_0, x_1, \dots, x_m] = \frac{f^{(m)}(0)}{m!}.$$

Thus,

$$\lim_{x_m \rightarrow 0} f[x_0, x_1, \dots, x_m, 1] = f(1) - \sum_{k=0}^m \frac{f^{(k)}(0)}{k!}. \quad \square$$

Applying Lemma 2 we obtain

$$\lim_{q \rightarrow +\infty} f \left[0, \frac{1}{[n]_q}, \dots, \frac{[n-1]_q}{[n]_q}, 1 \right] = f(1) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}.$$

Finally, we get for $f \in C^{n-1}[0, 1]$,

$$\lim_{q \rightarrow +\infty} B_n(f, q; z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k + z^n \left\{ f(1) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} \right\}. \quad \square$$

5. Rate of convergence of q -Bernstein polynomials in the case $q > 1$

The following is a Voronovskaya-type theorem for monomials. It shows that in the case $q > 1$ the polynomials $B_n(t^m, q; z)$ converge to z^m essentially faster than the classical ones.

Theorem 4. *Let $q \geq 1$ be fixed. Then for any $z \in \mathbf{C}$,*

$$\lim_{n \rightarrow \infty} [n]_q \{ B_n(t^m, q; z) - z^m \} = (1 + [2]_q + \dots + [m-1]_q)(z^{m-1} - z^m).$$

(From here on an empty sum is taken to be equal 0.)

The following theorem provides a uniform estimate of the difference between z^m and its q -Bernstein polynomial in a circle of radius $R > 1$.

Theorem 5. *Let $q \geq 1$ be fixed. Then for $R > 1$ and all $m = 1, 2, \dots; n = 1, 2, \dots$ we have*

$$|B_n(t^m, q; z) - z^m| \leq 2 \frac{(m-1)[m-1]_q}{[n]_q} R^m \quad \text{for } |z| \leq R.$$

Corollary 5. *Let $q \geq 1$ be fixed. Then for any compact set $K \subset \mathbf{C}$,*

$$B_n(t^m, q; z) \rightrightarrows z^m \quad \text{for } z \in K \text{ as } n \rightarrow \infty,$$

and

$$|B_n(t^m, q; z) - z^m| \leq \frac{C_{m,q,K}}{[n]_q} \quad \text{for all } n = 1, 2, \dots$$

If we consider q -Bernstein polynomials of z^m in the closed unit disk $\{z: |z| \leq 1\}$, we can get a more particular estimate.

Corollary 6. *For all $m = 0, 1, 2, \dots, n = 1, 2, \dots$, we have*

$$|B_n(t^m, q; z) - z^m| \leq 2 \frac{mq^m}{[n]_q(q-1)} \quad \text{for } |z| \leq 1.$$

Using the latter estimate we obtain the following statement, which shows that for a wide class of analytic functions their q -Bernstein polynomials provide exponentially fast approximation in the closed unit disk, and, in particular, on the interval $[0, 1]$.

Theorem 6. *Let $q \geq 1$ be fixed. If a function $f(z)$ is analytic in a disk $D_R := \{z: |z| < R\}$, $R > q$, then*

$$|B_n(f, q; z) - f(z)| \leq \frac{C_{f,q}}{[n]_q} \quad \text{for } |z| \leq 1 \text{ and all } n = 1, 2, \dots .$$

That is, if a function is analytic in a disk of radius $R > q$, then its q -Bernstein polynomials form an approximating sequence on $[0, 1]$ with the rate of convergence of order q^{-n} . Therefore, in the case $q > 1$ approximation of an analytic function with q -Bernstein polynomials is essentially faster than with the classical ones.

It turns out that in the case $q \geq 1$, q -Bernstein polynomials of an analytic function form an approximating sequence in the closed unit disk $\{z: |z| \leq 1\}$ even if we do not keep the value of q fixed.

Theorem 7. *If a function $f(z)$ is analytic in a disk D_R , $R > 1$, then for all $q \geq 1$ the following estimate holds uniformly with respect to q :*

$$|B_n(f, q; z) - f(z)| \leq \frac{C_f}{n} \quad \text{for } |z| \leq 1 \text{ and all } n = 1, 2, \dots .$$

The following corollary can be regarded as an analogue for $q_n \geq 1$ of Phillips' convergence theorem [15, Theorem 2].

Corollary 7. *If a function $f(z)$ is analytic in a disk D_R , $R > 1$, then for any sequence $\{q_n\}$, $q_n \geq 1$ we have*

$$B_n(f, q_n, z) \rightrightarrows f(z) \quad \text{for } |z| \leq 1 \text{ as } n \rightarrow \infty .$$

6. Proofs of Theorems 4–7

The following lemma is needed for the sequel.

Lemma 3. Let $f = t^m$, $m \geq 1$. Then

$$B_n(t^m, q; z) = \alpha_1 z + \dots + \alpha_j z^j, \quad j = \min(m, n), \tag{20}$$

where

- (i) all $\alpha_i \geq 0$ ($i = 1, \dots, j$).
 - (ii) $\alpha_1 + \dots + \alpha_j = 1$.
- Besides, for $n \geq m$ we have
- (iii)

$$\alpha_i \leq \frac{C_{i,m}}{[n]_q^{m-i}}, \quad i = 1, \dots, m$$

- (iv)

$$\alpha_m = \lambda_{m,q}^{(n)}, \quad \alpha_{m-1} = \lambda_{m-1,q}^{(n)} \frac{1 + [2]_q + \dots + [m-1]_q}{[n]_q}.$$

Proof. It was already noticed in the Preliminaries that $B_n(t^m, q; z)$ is a polynomial of degree $\min(m, n)$. The end-point interpolation property (3) implies that for $m \geq 1$, the free term of $B_n(t^m, q; z)$ equals 0. Therefore, (20) is justified.

(i) Representation (6) of q -Bernstein polynomials gives the following values of the coefficients in (20):

$$\alpha_i = \lambda_{i,q}^{(n)} f \left[0, \frac{1}{[n]_q}, \dots, \frac{[i]_q}{[n]_q} \right], \quad i = 1, \dots, m, \tag{21}$$

where $0 \leq \lambda_{i,q}^{(n)} \leq 1$ are given by (7).

Since for $f = t^m$:

$$f \left[0, \frac{1}{[n]_q}, \dots, \frac{[i]_q}{[n]_q} \right] \geq 0,$$

the statement is proved.

(ii) This follows readily from (3), if we put $x = 1$ in (20).

(iii) Using (21) and (9), we get

$$\alpha_i \leq f \left[0, \frac{1}{[n]_q}, \dots, \frac{[i]_q}{[n]_q} \right] = \frac{f^{(i)}(\xi_i)}{i!}, \quad \text{where } \xi_i \in \left(0, \frac{[i]_q}{[n]_q} \right).$$

Hence

$$\alpha_i \leq \binom{m}{i} \zeta_i^{m-i} \leq \binom{m}{i} \left(\frac{[i]_q}{[n]_q} \right)^{m-i} =: \frac{C_{m,i}}{[n]_q^{m-i}},$$

as required.

(iv) Obviously,

$$f \left[0, \frac{1}{[n]_q}, \dots, \frac{[m]_q}{[n]_q} \right] = 1,$$

and, therefore $\alpha_m = \lambda_{m,q}^{(n)}$.

To calculate

$$f \left[0, \frac{1}{[n]_q}, \dots, \frac{[m-1]_q}{[n]_q} \right],$$

we use the representation (cf. [8, Chapter II, Section 2.7]):

$$f \left[0, \frac{1}{[n]_q}, \dots, \frac{[k]_q}{[n]_q} \right] = \frac{1}{2\pi i} \int_L \frac{f(\zeta) d\zeta}{\zeta \left(\zeta - \frac{1}{[n]_q} \right) \dots \left(\zeta - \frac{[k]_q}{[n]_q} \right)},$$

where L is a contour around $[0,1]$. Hence for $f(\zeta) = \zeta^m$ we get

$$f \left[0, \frac{1}{[n]_q}, \dots, \frac{[m-1]_q}{[n]_q} \right] = \frac{1}{2\pi i} \int_L \frac{\zeta^{m-1} d\zeta}{\left(\zeta - \frac{1}{[n]_q} \right) \dots \left(\zeta - \frac{[m-1]_q}{[n]_q} \right)}.$$

Direct calculation of the integral implies

$$f \left[0, \frac{1}{[n]_q}, \dots, \frac{[m-1]_q}{[n]_q} \right] = \frac{1 + [2]_q + \dots + [m-1]_q}{[n]_q}$$

and (iv) is proved. \square

Proof of Theorem 4. For $m = 0, 1$ there is nothing to prove, because by (11) q -Bernstein polynomials leave invariant linear functions.

For $m \geq 2$ using (iii) and (iv) of Lemma 3, we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n]_q \{B_n(t^m, q; z) - z^m\} \\ &= \lim_{n \rightarrow \infty} [n]_q \{\alpha_m z^m + \alpha_{m-1} z^{m-1} - z^m\} \\ &= \lim_{n \rightarrow \infty} [n]_q \left\{ (\lambda_{m,q}^{(n)} - 1) z^m + \frac{1 + [2]_q + \dots + [m-1]_q}{[n]_q} z^{m-1} \right\} \\ &= (1 + [2]_q + \dots + [m-1]_q) z^{m-1} \\ &\quad + z^m \lim_{n \rightarrow \infty} [n]_q \left\{ \left(1 - \frac{1}{[n]_q}\right) \dots \left(1 - \frac{[m-1]_q}{[n]_q}\right) - 1 \right\} \\ &= (1 + [2]_q + \dots + [m-1]_q) (z^{m-1} - z^m). \quad \square \end{aligned}$$

Proof of Theorem 5. For $m = 0, 1$ the statement is obvious.

First we consider the case $n \geq m \geq 2$. Applying Lemma 3, we get for $|z| \leq R, R > 1$:

$$\begin{aligned} |B_n(t^m, q; z) - z^m| &= \left| \sum_{k=1}^{m-1} \alpha_k z^k + (1 - \lambda_{m,q}^{(n)}) z^m \right| \\ &\leq \left(\sum_{k=1}^{m-1} \alpha_k + (1 - \lambda_{m,q}^{(n)}) \right) R^m = 2(1 - \lambda_{m,q}^{(n)}) R^m. \end{aligned} \tag{22}$$

Now, by (7)

$$\begin{aligned} 1 - \lambda_{m,q}^{(n)} &= 1 - \left(1 - \frac{1}{[n]_q}\right) \dots \left(1 - \frac{[m-1]_q}{[n]_q}\right) \\ &\leq 1 - \left(1 - \frac{[m-1]_q}{[n]_q}\right)^{m-1} \leq (m-1) \frac{[m-1]_q}{[n]_q}. \end{aligned}$$

Using (22), we get that for $n \geq m$,

$$|B_n(t^m, q; z) - z^m| \leq 2(m-1) \frac{[m-1]_q}{[n]_q} R^m.$$

To complete the proof, we note that statements (i) and (ii) of Lemma 3 yield that $|B_n(t^m, q; z) - z^m| \leq 2R^m$. Therefore, the estimate is also true for $n < m$. \square

Proof of Theorem 6. Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be a function analytic in a disk $D_R, R > q$. Evidently,

$$B_n(f, q; z) = \sum_{m=0}^{\infty} a_m B_n(t^m, q; z) \quad \text{for } |z| \leq R.$$

Hence applying Corollary 6 of Theorem 5, we have for $|z| \leq 1$:

$$|B_n(f, q; z) - f(z)| \leq \sum_{m=0}^{\infty} |a_m| |B_n(t^m, q; z) - z^m| \leq \sum_{m=0}^{\infty} \frac{2|a_m|mq^m}{[n]_q(q-1)} =: \frac{C_{f,q}}{[n]_q},$$

because $\sum_{m=0}^{\infty} |a_m|mq^m < \infty$. \square

Proof of Theorem 7. First, we prove that for all $q \geq 1$ and all $m = 0, 1, \dots, n = 1, 2, \dots$ the following estimate holds uniformly with respect to q :

$$|B_n(t^m, q; z) - z^m| \leq \frac{2m^2}{n} \quad \text{for } |z| \leq 1.$$

If $n < m$, the inequality is true, because $|B_n(t^m, q; z) - z^m| \leq 2$ for $|z| \leq 1$. For $n \geq m$, we have by (22)

$$|B_n(t^m, q; z) - z^m| \leq 2(1 - \lambda_{m,q}^{(n)}) \quad \text{for } |z| \leq 1.$$

If $q \geq 1$, then

$$\frac{[j]_q}{[n]_q} \leq \frac{j}{n} \quad \text{for } j = 0, 1, \dots, n,$$

and hence

$$\lambda_{m,q}^{(n)} = \left(1 - \frac{1}{[n]_q}\right) \cdots \left(1 - \frac{[m-1]_q}{[n]_q}\right) \geq \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right) = \lambda_{m,1}^{(n)}.$$

Therefore, for all $q \geq 1$ we get

$$\begin{aligned} |B_n(t^m, q; z) - z^m| &\leq 2(1 - \lambda_{m,1}^{(n)}) \\ &\leq 2 \left[1 - \left(1 - \frac{m-1}{n}\right)^{m-1}\right] \leq 2 \frac{(m-1)^2}{n} \leq 2 \frac{m^2}{n} \quad \text{for } |z| \leq 1. \end{aligned}$$

Now, let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be a function analytic in a disk $D_R, R > 1$. Then for any $q \geq 1$,

$$|B_n(f, q; z) - f(z)| \leq \sum_{m=0}^{\infty} |a_m| |B_n(t^m, q; z) - z^m| \leq \sum_{m=0}^{\infty} 2 \frac{|a_m|m^2}{n} =: \frac{C_f}{n},$$

since $\sum_{m=0}^{\infty} |a_m|m^2 < \infty$. Clearly, C_f does not depend on q . \square

Remark. The statement remains true if $f(z) = \sum_{m=0}^{\infty} a_m z^m$ is a function analytic in the open unit disk and $\sum_{m=0}^{\infty} |a_m|m^2 < \infty$.

7. Iterates of q -Bernstein polynomials

We recall that the q -Bernstein operator $B_{n,q} : C[0, 1] \rightarrow \mathcal{P}_n$ is defined by

$$B_{n,q} : f \mapsto B_n(f, q; x) = \sum_{k=0}^n f \left(\frac{[k]_q}{[n]_q} \right) p_{nk}(q; x),$$

where $p_{nk}(q; x)$ are given by (12).

For $q \in (0, 1)$ equality (16) defines the limit operator $B_{\infty,q}$ on $C[0, 1]$ as

$$B_{\infty,q} : f \mapsto B_{\infty}(f, q; x) = \begin{cases} \sum_{k=0}^{\infty} f(1 - q^k) p_{\infty k}(q; x) & \text{if } x \in [0, 1), \\ f(1) & \text{if } x = 1, \end{cases} \tag{23}$$

where entire functions $p_{\infty,k}$ are given by (14).

It can be readily seen that for $q \in (0, 1)$, both polynomials $p_{nk}(q; x)$ and entire functions $p_{\infty k}(q; x)$ are non-negative on the interval $[0, 1]$. Therefore, we get from (13) and (15) that

$$\|B_{n,q}\| = \|B_{\infty,q}\| = 1 \quad \text{for } q \in (0, 1). \tag{24}$$

By L we denote the operator of linear interpolation at 0 and 1, i.e.,

$$L(f; x) := (1 - x)f(0) + xf(1).$$

Theorem 8. *Let $q \in (0, 1)$ and $\{j_n\}$ be a sequence of positive integers such that $j_n \rightarrow \infty$. Then for any $f \in C[0, 1]$,*

$$B_{j_n}^{j_n}(f, q; x) \rightrightarrows L(f; x) \quad \text{for } x \in [0, 1] \text{ as } n \rightarrow \infty.$$

The following theorem describes the behavior of iterates of the limit operator $B_{\infty,q}$.

Theorem 9. *Let $q \in (0, 1)$, and the operator $B_{\infty,q}$ be defined by (23). If $\{j_n\}$ is a sequence of positive integers such that $j_n \rightarrow \infty$, then for any $f \in C[0, 1]$,*

$$B_{\infty}^{j_n}(f, q; x) \rightrightarrows L(f; x) \quad \text{for } x \in [0, 1] \text{ as } n \rightarrow \infty.$$

The statement below (proved in [5]) follows from Theorem 9 immediately.

Corollary 8. *Let $q \in (0, 1)$. Then $B_{\infty,q}(f) = f$ if and only if $f = L(f)$, i.e. f is a linear function.*

For $q \in (1, \infty)$ we restrict ourselves to the case when f is a polynomial. This is because in contrast to the case $q \in (0, 1]$, the sequence $\{B_n(f, q; x)\}$ may be divergent even for an infinitely differentiable function f (cf. Theorem 2.) However, behavior of the operators $B_{n,q}$ for $q \in (1, \infty)$ on the space of polynomials $\mathcal{P} = \bigcup_{m=0}^{\infty} \mathcal{P}_m$ is rather

similar to the classical case. In particular, for any $p \in \mathcal{P}$ the sequence $\{B_n(p, q; x)\}$ converges to p uniformly on $[0, 1]$. Behavior of iterates $B_{n,q}^{j_n}$ on \mathcal{P} resembles the situation with $q = 1$. More precisely, the following statement holds.

Theorem 10. *Let $q \in (1, \infty)$ and $\{j_n\}$ be a sequence of positive integers such that $j_n/[n]_q \rightarrow t$ as $n \rightarrow \infty$. Then for any polynomial p and any $0 \leq t \leq \infty$ the sequence $\{B_n^{j_n}(p, q; x)\}$ converges uniformly on $[0, 1]$. In particular, for $t = 0$,*

$$B_n^{j_n}(p, q; x) \rightrightarrows p(x) \quad \text{for } x \in [0, 1],$$

and for $t = \infty$,

$$B_n^{j_n}(p, q; x) \rightrightarrows L(p; x) \quad \text{for } x \in [0, 1].$$

We omit the proof of Theorem 10 since it repeats verbatim the reasoning of [3], where the classical case $q = 1$ was considered.

8. Some auxiliary results

Lemma 4. *For all $q > 0$ the following identity holds:*

$$B_n(t^m, q; x) = \frac{x}{[n]_q^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} ([n]_q - 1)^j B_{n-1}(t^j, q; x),$$

$$n = 2, 3, \dots; \quad m = 1, 2, \dots \quad (25)$$

Proof. Let $p_{n,k}(q; x)$ be defined by (12). Then

$$\begin{aligned} B_n(t^m, q; x) &= \sum_{k=0}^n \left(\frac{[k]_q}{[n]_q} \right)^m p_{nk}(q; x) \\ &= \sum_{k=1}^n \left(\frac{[k]_q}{[n]_q} \right)^{m-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) \\ &= \frac{x}{[n]_q^{m-1}} \sum_{k=0}^{n-1} [k+1]_q^{m-1} p_{n-1,k}(q; x) \\ &= \frac{x}{[n]_q^{m-1}} \sum_{k=0}^{n-1} (1 + q[k]_q)^{m-1} p_{n-1,k}(q; x) \\ &= \frac{x}{[n]_q^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} (q[n-1]_q)^j \left(\sum_{k=0}^{n-1} \left(\frac{[k]_q}{[n-1]_q} \right)^j p_{n-1,k}(q; x) \right) \\ &= \frac{x}{[n]_q^{m-1}} \sum_{j=0}^{m-1} \binom{m-1}{j} ([n]_q - 1)^j B_{n-1}(t^j, q; x). \quad \square \end{aligned}$$

Lemma 5. For all $q > 0$ the operator $B_{n,q}$ has $(n + 1)$ linearly independent monic eigenvectors $p_m^{(n)}(x)$, $\deg p_m^{(n)}(x) = m$, $(m = 0, 1, \dots, n)$, corresponding to the eigenvalues

$$\lambda_{0,q}^{(n)} = \lambda_{1,q}^{(n)} = 1,$$

$$\lambda_{m,q}^{(n)} = \left(1 - \frac{[1]_q}{[n]_q}\right) \left(1 - \frac{[2]_q}{[n]_q}\right) \dots \left(1 - \frac{[m-1]_q}{[n]_q}\right), \quad \text{for } m = 2, \dots, n. \tag{26}$$

Remark. For $q = 1$, (26) coincides with formula (2.5) in [3].

Proof. For $m = 0, 1$ the statement is obvious due to (11). For $n \geq m \geq 2$, using Lemma 3 we write

$$B_n(t^m, q; x) = \lambda_{m,q}^{(n)} x^m + P_{m-1}^{(n)}(x), \tag{27}$$

where $P_{m-1}^{(n)}(x) \in \mathcal{P}_{m-1}$ and $\lambda_{m,q}^{(n)}$ are given by (26).

To find an eigenvector $p_m^{(n)} \in \mathcal{P}_m$ of the operator $B_{n,q}$, we write $p_m^{(n)} = x^m + a_{m-1}x^{m-1} + \dots + a_1x$ and solve a linear system in unknowns a_1, \dots, a_{m-1} :

$$B_{n,q}(x^m + a_{m-1}x^{m-1} + \dots + a_1x) = \lambda_{m,q}^{(n)}(x^m + a_{m-1}x^{m-1} + \dots + a_1x).$$

After we apply $B_{n,q}$ in the left-hand side and equate the coefficients of x^s ($s = 1, \dots, m - 1$), we get a triangular system whose determinant equals

$$(\lambda_{m-1,q}^{(n)} - \lambda_{m,q}^{(n)})(\lambda_{m-2,q}^{(n)} - \lambda_{m,q}^{(n)}) \dots (\lambda_{1,q}^{(n)} - \lambda_{m,q}^{(n)}) \neq 0.$$

Hence there exists a unique monic polynomial of degree $2 \leq m \leq n$ which is an eigenvector of $B_{n,q}$ with the eigenvalue $\lambda_{m,q}^{(n)}$. \square

Corollary 9. For $2 \leq m \leq n$, the operator $\lambda_{m,q}^{(n)}I - B_{n,q}$, where I is the identity operator, is invertible on \mathcal{P}_{m-1} .

Lemma 6. The following equality holds:

$$\lim_{n \rightarrow \infty} \lambda_{m,q}^{(n)} = \begin{cases} q^{\frac{m(m-1)}{2}} & (m = 0, 1, 2, \dots) \quad \text{if } q \in (0, 1), \\ 1 & \text{if } q \in [1, \infty). \end{cases}$$

Proof. The statement follows from formula (26) after we notice that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{[j]_q}{[n]_q}\right) = \begin{cases} q^j & (j = 0, 1, 2, \dots) \quad \text{if } q \in (0, 1), \\ 1 & \text{if } q \in [1, \infty). \end{cases} \quad \square$$

Lemma 7. Let $q \in (0, 1)$. Then for every $m = 0, 1, 2, \dots$ the operator $B_{\infty, q}$ has an eigenvector $p_m(x)$ which is a monic polynomial of degree m , corresponding to the eigenvalue $\lambda_{m, q} = q^{m(m-1)/2}$.

Proof. For $m = 0, 1$ the statement follows immediately from (17).

Taking the limit as $n \rightarrow \infty$ in (25) and noting that for $q \in (0, 1)$ one has

$$\frac{([n]_q - 1)^j}{[n]_q^{m-1}} \rightarrow q^j (1 - q)^{m-j-1},$$

we get

$$B_{\infty}(t^m, q; x) = x \sum_{j=0}^{m-1} \binom{m-1}{j} q^j (1 - q)^{m-j-1} B_{\infty}(t^j, q, x).$$

Hence the coefficient $\lambda_{m, q}$ of x^m in $B_{\infty}(t^m, q; x)$ equals $q^{m-1} \lambda_{m-1, q}$, and recursively,

$$\lambda_{m, q} = q^{m-1} q^{m-2} \dots q \lambda_{1, q} = q^{m(m-1)/2}.$$

We have shown that

$$B_{\infty}(t^m, q; x) = \lambda_{m, q} x^m + Q_{m-1}, \quad Q_{m-1} \in \mathcal{P}_{m-1}.$$

The statement now follows from considering the equations

$$B_{\infty, q}(p_m(x)) = \lambda_{m, q} p_m(x), \quad m = 2, 3, \dots \quad \square$$

Corollary 10. For $m \geq 2$, the operator $\lambda_{m, q} I - B_{\infty, q}$ is invertible on \mathcal{P}_{m-1} .

9. Proofs of Theorems 8 and 9

In this section \rightrightarrows means uniform convergence on $[0, 1]$.

Proof of Theorem 8. Because of (3) it suffices to prove that $B_{n, q}^{j_n}(f) \rightrightarrows ax + b$ for some a and b as $n \rightarrow \infty$.

(1) First we consider the case $f = x^m$.

We will use induction on m . For $m = 0, 1$ the statement is obvious due to (11).

Assume that $B_{n, q}^{j_n}(x^t) \rightrightarrows \varphi_t \in \mathcal{P}_1$ for $t = 0, 1, \dots, m - 1$. Consider

$$B_{n, q}(x^m) = \lambda_{m, q}^{(n)} x^m + P_{m-1}^{(n)}, \tag{28}$$

where $\lambda_{m, q}^{(n)}$ is given by (26), and $P_{m-1}^{(n)} \in \mathcal{P}_{m-1}$. Then

$$B_{n, q}^{j_n}(x^m) = (\lambda_{m, q}^{(n)})^{j_n} x^m + [(\lambda_{m, q}^{(n)})^{j_n-1} I + (\lambda_{m, q}^{(n)})^{j_n-2} B_{n, q} + \dots + B_{n, q}^{j_n-1}](P_{m-1}^{(n)}),$$

where I denotes the identity operator. It follows from Lemma 6 that

$$(\lambda_{m,q}^{(n)})^{j_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

The expression in the brackets is a linear operator on the space \mathcal{P}_{m-1} .

Consider the sequence of polynomials in \mathcal{P}_{m-1} ,

$$y_{m-1}^{(n)} := [(\lambda_{m,q}^{(n)})^{j_n-1} I + (\lambda_{m,q}^{(n)})^{j_n-2} B_{n,q} + \dots + B_{n,q}^{j_n-1}](P_{m-1}^{(n)}). \tag{29}$$

Then

$$(\lambda_{m,q}^{(n)} I - B_{n,q}) y_{m-1}^{(n)} = (\lambda_{m,q}^{(n)})^{j_n} P_{m-1}^{(n)} - B_{n,q}^{j_n} P_{m-1}^{(n)}.$$

It follows from (24) and (28) that $\|P_{m-1}^{(n)}\| \leq 2$. Since $(\lambda_{m,q}^{(n)})^{j_n} \rightarrow 0$ as $n \rightarrow \infty$, we have

$$(\lambda_{m,q}^{(n)})^{j_n} P_{m-1}^{(n)} \rightrightarrows 0 \quad \text{as } n \rightarrow \infty.$$

It can be readily seen from (28) and Lemma 6 that

$$P_{m-1}^{(n)}(x) \rightrightarrows B_{\infty,q}(x^m) - q^{m(m-1)/2} x^m =: Q_{m-1}(x) \in \mathcal{P}_{m-1} \quad \text{as } n \rightarrow \infty,$$

i.e.

$$P_{m-1}^{(n)}(x) = Q_{m-1}(x) + \delta_n(x),$$

where $Q_{m-1} \in \mathcal{P}_{m-1}$, and $\delta_n(x) \rightrightarrows 0$ as $n \rightarrow \infty$.

Thus,

$$B_{n,q}^{j_n}(P_{m-1}^{(n)}) = B_{n,q}^{j_n}(Q_{m-1}) + B_{n,q}^{j_n}(\delta_n),$$

where $\|B_{n,q}^{j_n}(\delta_n)\| \leq \|\delta_n\|$, because of (24). This means that $B_{n,q}^{j_n}(\delta_n) \rightrightarrows 0$ as $n \rightarrow \infty$.

By the induction assumption

$$B_{n,q}^{j_n}(Q_{m-1}) \rightrightarrows cx + d \in \mathcal{P}_1 \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$(\lambda_{m,q}^{(n)} I - B_{n,q}) y_{m-1}^{(n)} \rightrightarrows cx + d \quad \text{as } n \rightarrow \infty$$

or

$$(\lambda_{m,q}^{(n)} I - B_{n,q}) y_{m-1}^{(n)} = cx + d + \omega_n(x),$$

where $\omega_n(x) \rightrightarrows 0$ as $n \rightarrow \infty$.

By Corollary 8 the operators $\lambda_{m,q}^{(n)} I - B_{n,q}$ are invertible on \mathcal{P}_{m-1} for $n \geq m$ and

$$\lim_{n \rightarrow \infty} (\lambda_{m,q}^{(n)} I - B_{n,q}) = q^{\frac{m(m-1)}{2}} I - B_{\infty,q} =: A_{\infty,q},$$

where by Corollary 10 $A_{\infty,q}$ is also invertible on \mathcal{P}_{m-1} . Hence

$$(\lambda_{m,q}^{(n)} I - B_{n,q})^{-1} \rightarrow A_{\infty,q}^{-1} \quad \text{as } n \rightarrow \infty$$

and it follows that

$$\|(\lambda_{m,q}^{(n)} I - B_{n,q})^{-1}\| \leq M \quad \text{for some } M > 0.$$

Therefore,

$$y_{m-1}^{(n)} = (\lambda_{m,q}^{(n)}I - B_{n,q})^{-1}(cx + d) + (\lambda_{m,q}^{(n)}I - B_{n,q})^{-1}(\omega_n).$$

Since $\|(\lambda_{m,q}^{(n)}I - B_{n,q})^{-1}(\omega_n)\| \leq M\|\omega_n\| \rightarrow 0$ as $n \rightarrow \infty$, and $(\lambda_{m,q}^{(n)}I - B_{n,q})^{-1} \rightarrow A_{\infty,q}$ as $n \rightarrow \infty$, we conclude that

$$y_{m-1}^{(n)} \rightrightarrows A_{\infty,q}^{-1}(cx + d) := ax + b \in \mathcal{P}_1.$$

Thus, $B_{n,q}^{j_n}(x^m) \rightrightarrows ax + b$.

The induction is completed and it follows that for any polynomial p ,

$$B_n^{j_n}(p, q; x) \rightrightarrows L(p; x) \quad \text{for } x \in [0, 1] \text{ as } n \rightarrow \infty.$$

(2) Let $f \in C[0, 1]$, and let $\varepsilon > 0$ be given. Then $f(x) = p(x) + \delta(x)$, where $p \in \mathcal{P}$, and $\|\delta(x)\| < \varepsilon$. We have

$$B_{n,q}^{j_n}(f) = B_{n,q}^{j_n}(p) + B_{n,q}^{j_n}(\delta).$$

Since $B_{n,q}^{j_n}(p) \rightrightarrows L(p)$, there exists $n_0 \in \mathbf{N}$ such that $\|B_{n,q}^{j_n}(p) - L(p)\| < \varepsilon$ for all $n > n_0$. Obviously, $\|L(\delta)\| \leq \|\delta\| < \varepsilon$, and finally we obtain

$$\|B_{n,q}^{j_n}(f) - L(f)\| \leq \|B_{n,q}^{j_n}(p) - L(p)\| + \|B_{n,q}^{j_n}(\delta)\| + \|\delta\| < 3\varepsilon \quad \text{for all } n > n_0.$$

Thus, $B_n^{j_n}(f, q; x) \rightrightarrows L(f; x)$ for $x \in [0, 1]$ as $n \rightarrow \infty$. \square

Proof of Theorem 9. (1) First we prove the statement in the case $f \in \mathcal{P}_m$. For $f \in \mathcal{P}_m$ by Lemma 7 we have

$$f = \alpha_0 p_0 + \alpha_1 p_1 + \dots + \alpha_m p_m,$$

where p_0, p_1, \dots, p_m are eigenvectors of $B_{\infty,q}$ corresponding to the eigenvalues $\lambda_{0,q}, \lambda_{1,q}, \dots, \lambda_{m,q}$. Obviously,

$$B_{\infty,q}^{j_n}(f) = \alpha_0 \lambda_{0,q}^{j_n} p_0 + \alpha_1 \lambda_{1,q}^{j_n} p_1 + \dots + \alpha_m \lambda_{m,q}^{j_n} p_m.$$

Since $\lambda_{0,q} = \lambda_{1,q} = 1$, $\lambda_{i,q} \in (0, 1)$ for $i \geq 2$, we obtain

$$B_{\infty,q}^{j_n}(f) \rightrightarrows \alpha_0 p_0 + \alpha_1 p_1 \in \mathcal{P}_1.$$

Taking into account (3), we derive the statement.

(2) For $f \in C[0, 1]$, the statement follows from the density of the set of polynomials in $C[0, 1]$ and the fact that $\|B_{\infty,q}\| = 1$ (cf. (24)). \square

Acknowledgment

The author is grateful to Prof. A. Il'inskii for his thorough reading of the manuscript and valuable comments.

References

- [1] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976.
- [2] S.N. Bernstein, Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités, *Comm. Soc. Math. Charkow Sér. 2 t. 13* (1912) 1–2.
- [3] S. Cooper, S. Waldron, The eigenstructure of the Bernstein operator, *J. Approx. Theory* 105 (2000) 133–165.
- [4] T.N.T. Goodman, H. Oruç, G.M. Phillips, Convexity and generalized Bernstein polynomials, *Proc. Edinburgh Math. Soc.* 42 (1) (1999) 179–190.
- [5] A. Il'inskii, S. Ostrovska, Convergence of generalized Bernstein polynomials, *J. Approx. Theory* 116 (2002) 100–112.
- [6] R.P. Kelisky, T.J. Rivlin, Iterates of Bernstein polynomials, *Pacific J. Math.* 21 (1967) 511–520.
- [7] X. Li, P. Mikusiński, H. Sherwood, M.D. Taylor, On approximation of copulas, in: V. Benes, J. Stepan (Eds.), *Distributions with Given Marginals and Moment Problem*, Kluwer Academic Publishers, Dordrecht, 1997.
- [8] G.G. Lorentz, *Bernstein Polynomials*, Chelsea, New York, 1986.
- [9] A. Lupaş, A q -analogue of the Bernstein operator, University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus, preprint No. 9, 1987.
- [10] C. Micchelli, The saturation class and iterates of the Bernstein polynomials, *J. Approx. Theory* 8 (1973) 1–18.
- [11] H. Oruç, G.M. Phillips, A generalization of Bernstein polynomials, *Proc. Edinburgh Math. Soc.* 42 (2) (1999) 403–413.
- [12] H. Oruç, N. Tuncer, On the convergence and iterates of q -Bernstein polynomials, *J. Approx. Theory* 117 (2002) 301–313.
- [13] S. Petrone, Random Bernstein polynomials, *Scand. J. Statist.* 26 (3) (1999) 373–393.
- [14] G.M. Phillips, On generalized Bernstein polynomials, in: D.F. Griffiths, G.A. Watson (Eds.), *Numerical Analysis: A.R. Mitchell 75th Birthday Volume*, World Science, Singapore, 1996, pp. 263–269.
- [15] G.M. Phillips, Bernstein polynomials based on the q -integers, *Ann. Numer. Math.* 4 (1997) 511–518.
- [16] G.M. Phillips, A de Casteljaou algorithm for generalized Bernstein polynomials, *BIT* 37 (1) (1997) 232–236.
- [17] G.M. Phillips, A generalization of the Bernstein polynomials based on the q -integers, *ANZIAM J.* 42 (2000) 79–86.
- [18] E.C. Titchmarsh, *Theory of Functions*, Oxford University Press, Oxford, 1986.
- [19] V.S. Videnskii, *Bernstein Polynomials*, Leningrad State Pedagogical University, Leningrad, 1990 (Russian).