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A Central Polynomial of Low Degree for 4 × 4 Matrices

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We have found a central polynomial of degree 13 for the 4 × 4 matrix algebra over a field of characteristic 0. This result agrees with the conjecture that the minimal degree of such polynomials for $n \times n$ matrices is $(n^2 + 3n - 2)/2$. The polynomial has been obtained by explicitly exhibiting an essentially weak polynomial identity of degree 9 for 4 × 4 matrices. © 1994 Academic Press, Inc.

INTRODUCTION

Let $K\langle X\rangle$ be the free associative algebra over a field K of characteristic 0. An element $f(x_1, x_2, \ldots, x_n) \in K\langle X\rangle$ is called a *central polynomial* for a K-algebra R is $f(r_1, r_2, \ldots, r_n)$ lies in the center of R for all $r_1, r_2, \ldots, r_n \in R$, and f is not a polynomial identity for R. The first central polynomials for the matrix algebras $M_n = M_n(K)$ for any n were constructed by Formanek and Razmyslov in [3, 7] with two different methods. The construction of Formanek yields a central polynomial of degree n^2. The original Razmyslov polynomial was of higher degree but Halpin [6] showed that the method of [7] also gives rise to a central polynomial of degree n^2. For a

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long time this value was thought to be the minimal value for the degree of central polynomials for $n \times n$ matrices, and this is in fact the case for $n = 1$ and $n = 2$. But Drensky and Kasparian in [1] found a central polynomial of degree 8 for 3×3 matrices. Formanek in [4] stated the problem to determine the minimal degree of the central polynomials for $M_n(K)$. In [5] he also conjectured that this minimal degree is $(n^2 + 3n - 2)/2$. The main result of this paper is that we determine a central polynomial for 4×4 matrices of degree 13. We have not established that 13 is the minimal degree of the central polynomials for $M_4(K)$. Nevertheless, since $13 = (n^2 + 3n - 2)/2$ for $n = 4$, this agrees with the conjecture. To obtain the central polynomial of degree 13 we have explicitly given the following essentially weak polynomial identity of degree 9 for 4×4 matrices,

$$\begin{aligned} w(x_1, x_2, x_3, x_4, x_5) = {} & s_6(x_1^4, x_1, x_2, x_3, x_4, x_5) \\ & + \sum_{i=2}^{5} x_1 s_6(x_1^2, x_1, x_2, \dots, x_i x_1, \dots, x_5) \\ & + \sum_{2 \le i < j \le 5} s_6(x_1^2, x_1, x_2, \dots, x_i x_1, \dots, x_j x_1, \dots, x_5), \end{aligned}$$

where $s_6(x_1, \dots, x_6)$ is the standard polynomial of degree 6. By the Razmyslov approach in [7] this gives rise to a central polynomial of the right degree. Initially we have found $w(x_1, x_2, x_3, x_4, x_5)$ by computer. It turns out that it has so simple a form that we have been able to prove directly without using any computer that $w(x_1, x_2, x_3, x_4, x_5)$ is an essentially weak polynomial identity for $M_4(K)$.

1. Preliminaries

Let K be a field of characteristic zero. By $K\langle x_1, x_2, \dots, x_m\rangle$ we mean the subalgebra of rank m of $K\langle X\rangle$. We shall also use other variables, e.g., x, $y_1, \dots, y_k$, to denote the free generators. The aim of the paper is to find a new central polynomial for $M_4(K)$, the algebra of 4×4 matrices. We refer to [4, 8] for a background on polynomial identities for matrices. We recall some basic facts.

(α) Let $K[t_1, t_2, \dots, t_{k+1}]$ be the polynomial ring in $k + 1$ commuting variables. To any polynomial

$$g(t_1, \dots, t_{k+1}) = \sum \alpha_p t_1^{p_1} \cdots t_{k+1}^{p_{k+1}} \in K[t_1, \dots, t_{k+1}],$$

we associate the polynomial

$$\phi(g) = \phi(g)(x, y_1, \dots, y_k) = \sum \alpha_p x^{p_1} y_1 x^{p_2} y_2 \cdots x^{p_k} y_k x^{p_{k+1}} \in K\langle x, y_1, \dots, y_k\rangle.$$

Suppose we start with an element $f(x, y_1, \dots, y_k) \in K\langle x, y_1, \dots, y_k\rangle$ which is multilinear in the y_i's. Then f may be written as

$$f(x, y_1, \dots, y_k) = \sum \alpha_{rp} x^{p_1} y_{r_1} x^{p_2} y_{r_2} \cdots x^{p_k} y_{r_k} x^{p_{k+1}} = \sum \phi(g_r)(x, y_{r_1}, \dots, y_{r_k}).$$

(β) Let n be fixed and let $\bar{y}_1 = e_{i_1j_1}, \bar{y}_2 = e_{i_2j_2}, \dots, \bar{y}_k = e_{i_kj_k}$ be matrix units from $M_n(K)$. We relate to the set $\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_k\}$ an oriented graph with n vertices $1, 2, \dots, n$ and edges $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$. In order to decide whether $\phi(g)$ is a polynomial identity or a central polynomial for $M_n(K)$, it is sufficient to set $x = \bar{x} = \rho_1 e_{11} + \cdots + \rho_n e_{nn}$, where $\rho_1, \dots, \rho_n$ are commuting variables and $y_1 = \bar{y}_1 = e_{i_1j_1}, y_2 = \bar{y}_2 = e_{i_2j_2}, \dots, y_k = \bar{y}_k = e_{i_kj_k}$ for all possible $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$. Then $\bar{x}e_{ij} = \rho_i e_{ij}$ and $e_{ij}\bar{x} = \rho_j e_{ij}$, imply that

$$\phi(g)(\bar{x}, e_{i_1j_1}, \dots, e_{i_kj_k}) = \delta g(\rho_{i_1}, \rho_{i_2}, \dots, \rho_{i_k}, \rho_{j_k}) e_{i_1j_k},$$

where δ equals 1 or 0, depending on whether $(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)$ is or is not a path in the graph.

DEFINITION. A polynomial $f(x_1, \dots, x_n) \in K\langle X\rangle$ is called a *weak polynomial identity* for M_k if it is zero when evaluted on all elements of the Lie algebra sl_k of all traceless matrices of M_k. An *essentially weak polynomial identity* is a weak polynomial identity for M_k which is not a polynomial identity for M_k.

Keeping in mind the definition of weak identities and the previous paragraphs, we can make this last observation.

(γ) As in (β), in order to prove that $f(x, y_1, y_2, \dots, y_k)$ is a weak polynomial identity for M_k it is sufficient to consider $x = \bar{x} = \rho_1 e_{11} + \dots + \rho_k e_{kk}$, where $\rho_1, \dots, \rho_k$ are commuting variables satisfying $\rho_1 + \dots + \rho_k = 0$. Let $g_r(t_1, \dots, t_{k+1}) \in K[t_1, \dots, t_{k+1}]$ be such that $f(x, y_1, \dots, y_k) = \sum \phi(g_r)(x, y_{r_1}, \dots, y_{r_k})$ is a polynomial identity for M_{k-1}. Then $f(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) = 0$ if $\bar{y}_1 = e_{i_1j_1}, \dots, \bar{y}_k = e_{i_kj_k}$ and in the related graph all the possible paths go through $k - 1$ vertices only. Hence, if $f(\bar{x}, \bar{y}_1, \dots, \bar{y}_k) \neq 0$ then the graph contains an oriented path $(i_{r_1}, j_{r_1}), (i_{r_2}, j_{r_2}), \dots, (i_{r_k}, j_{r_k})$ going through all the k vertices. Up to a permutation of the indices $1, \dots, n$, this is one of the paths $(1, 2), \dots, (i, i+1), \dots, (j-1, j), (j, i), (i, j+1), \dots, (k-1, k)$, where $i \leq j$. Now, if $g_r(t_1, \dots, t_i, t_{i+1}, \dots, t_j, t_i, t_{j+1}, \dots, t_k)$ is divisible by $t_1 + t_2 + \dots + t_k$ for all $i < j$, this means that $f(\bar{x}, \bar{y}_1, \dots, \bar{y}_k)$ vanishes on

$\bar{x} \in sl_k$ and $\bar{y}_1, \ldots, \bar{y}_k \in M_k$. In particular, $f(x, y_1, \ldots, y_k)$ is a weak polynomial identity for M_k.

2. The Weak Polynomial Identity

In order to obtain a central polynomial for 4×4 matrices, we looked for essentially weak polynomial identities. In the beginning of our search we followed the method used by Drensky and Rashkova in [2]. Using a computer they found in [2] all the weak polynomial identities of degree 8 for M_3. One of these weak identities gives rise to the central polynomial of degree 8 from [1]. We sketch the way to find $w(x_1, \ldots, x_5)$ because it may be used for other computations with polynomial identities. We refer to [2] for details. We looked for a possible weak polynomial identity $w(x_1, \ldots, x_5)$ of degree 9 which is a highest weight vector of an irreducible $GL_m(K)$-submodule W of $K\langle x_1, \ldots, x_m\rangle$ corresponding to the partition $(5, 1^4)$. For each standard tableau of shape $(5, 1^4)$ we associated a highest weight vector $w_i(x_1, \ldots, x_5)$, $i = 1, \ldots, d$, where $d = 70$ is the number of the standard $(5, 1^4)$-tableaux. Then the polynomial $w(x_1, \ldots, x_5)$ can be written uniquely as a linear combination $\sum_{i=1}^{d} \xi_i w_i(x_1, \ldots, x_5)$ with unknown coefficients ξ_i.

To this end we wrote a computer program which calculated all 70 highest weight vectors $w_1, \ldots, w_{70}$ corresponding to the given shape; we then calculated $w_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_5)$ for different values of $\bar{x}_i \in sl_4$. Since the 4^2 entries of the matrix $f(\bar{x}_1, \ldots, \bar{x}_5)$ are equal to 0, we obtain a homogeneous linear system in the 70 unknowns ξ_i. The number of equations equals $4^2 \cdot$ (*number of "experiments"*), where by "experiment" we mean a complete set of replacements of the five variables with elements of sl_4. By making eight experiments we obtained a system which gave a nontrivial solution, which was expressed as a combination of 38 heighest weight vectors; hence it was intractable. So we tried to express this solution as a consequence of the standard identity $s_6(x_1, x_2, x_3, x_4, x_5, x_6)$. We thus looked at consequences of degree 9 of s_6 like

$$f = \sum x_1^a s_6(x_1^{d_1}, x_1^{d_2}, x_1^{b_2} x_2 x_1^{c_2}, \ldots, x_1^{b_5} x_5 x_1^{c_5}), \qquad (1)$$

where a, d_1, and d_2 are fixed, $d_1 > d_2 \geq 1$, and the sum is symmetric under all permutations of the pairs $(b_2, c_2), \ldots, (b_5, c_5)$. Since the degree of the polynomial is 9, the relation $a + \sum_{i=2}^{5} b_i + \sum_{i=2}^{5} c_i + d_1 + d_2 = 5$ holds among the exponents. This allowed us to compute the 14 possible different consequences f_i of kind (1) and to check whether the solution w we had previously found could be expressed as a combination of these. This was the case, and the expression we obtained was particularly simple: we found that $w = f_3 + f_6 + f_{14}$, where f_3 corresponded to the values $d_1 = 4$,

$d_2 = 1$, and all the rest were zero; f_6 is the symmetric sum obtained from the values $a = 1, d_1 = 2, d_2 = 1, c_2 = 1$; and f_{14} from the values $d_1 = 2, d_2 = 1, c_2 = c_3 = 1$.

The aim of this section is therefore to prove the following result.

THEOREM 1. *Let K be a field of characteristic equal to zero. The polynomial*

$$\begin{aligned} w(x_1, x_2, x_3, x_4, x_5) = {} & s_6(x_1^4, x_1, x_2, x_3, x_4, x_5) \\ & + \sum_{i=2}^{5} x_1 s_6(x_1^2, x_1, x_2, \dots, x_i x_1, \dots, x_5) \\ & + \sum_{2\le i<j\le 5} s_6(x_1^2, x_1, x_2, \dots, x_i x_1, \dots, x_j x_1, \dots, x_5) \\ & \in K\langle X\rangle \end{aligned}$$

is an essentially weak polynomial identity for $M_4(K)$.

In order to prove the theorem, we need several lemmas.

LEMMA 1. *Let*

$$\begin{aligned} g(u_1, \dots, u_5; v_1, \dots, v_5) = {} & u_1(-v_2 + v_3 - v_4 + v_5) \\ & + u_2(-v_3 + v_4 - v_5 + v_1) \\ & + u_3(-v_4 + v_5 - v_1 + v_2) \\ & + u_4(-v_5 + v_1 - v_2 + v_3) \\ & + u_5(-v_1 + v_2 - v_3 + v_4) \end{aligned}$$

be a polynomial in the commuting variables u_i, v_i. Then, for any n, if we set

$$\begin{aligned} x_1 &= \bar{x}_1 = \rho_1 e_{11} + \dots + \rho_n e_{nn} \\ y_h &= \bar{y}_h = e_{i_h j_h} \end{aligned}$$

for $h = 2, \dots, 5$, then

$$\begin{aligned} & s_6(\bar{x}_1^{d_1}, \bar{x}_1^{d_2}, \bar{y}_2, \dots, \bar{y}_5) \\ & \quad = \sum \operatorname{sgn}(\sigma) g(\rho_{\sigma(i_2)}^{d_1}, \dots, \rho_{\sigma(i_5)}^{d_1}, \rho_{\sigma(j_5)}^{d_1}; \rho_{\sigma(i_2)}^{d_2}, \dots, \rho_{\sigma(i_5)}^{d_2}, \rho_{\sigma(j_5)}^{d_2}) \\ & \qquad \cdot e_{\sigma(i_2)\sigma(j_2)} \cdots e_{\sigma(i_5)\sigma(j_5)}. \end{aligned}$$

Proof. $s_6(x_1^{d_1}, x_1^{d_2}, y_2, y_3, y_4, y_5) = \sum_{\sigma\in S_6}(-1)^\sigma z_{\sigma(1)} z_{\sigma(2)} \cdots z_{\sigma(6)}$, where $z_1 = x_1^{d_1}, z_2 = x_1^{d_2}, z_3 = y_2, z_4 = y_3, z_5 = y_4, z_6 = y_5$. Now, the values of the coefficients of this polynomial depend on the position of $x_1^{d_1}$ and $x_1^{d_2}$ in the single monomials $y_{\sigma(2)} \cdots x_1^{d_1} \cdots x_1^{d_2} \cdots y_{\sigma(5)}$. If we let ε to be the sign of the permutation of S_6 corresponding to $y_2 \cdots x_1^{d_1} \cdots x_1^{d_2} \cdots y_5$, we can express s_6 as

$$\begin{aligned} s_6(x_1^{d_1}, x_1^{d_2}, y_2, \ldots, y_5) = \sum \operatorname{sgn}(\sigma)\{&\varepsilon(y_{\sigma(2)} \cdots x_1^{d_1} \cdots x_1^{d_2} \cdots y_{\sigma(5)}) \\ &- \varepsilon(y_{\sigma(2)} \cdots x_1^{d_2} \cdots x_1^{d_1} \cdots y_{\sigma(5)})\}, \end{aligned}$$

where σ runs over all permutations of S_4 acting on $\{2, \ldots, 5\}$ and where only positions of $x_1^{d_1}$ and $x_2^{d_2}$ which are not adjacent give a contribution. Now, each summand inside the braces corresponds (in the correspondence stated in point (α) of Section 1) to the polynomial in the commutative variables u_i, v_i, $i = 1, \ldots, 5$,

$$\begin{aligned} g(u_1, u_2, u_3, u_4, u_5; v_1, v_2, v_3, v_4, v_5) = {} & u_1(-v_2 + v_3 - v_4 + v_5) \\ & + u_2(-v_3 + v_4 - v_5 + v_1) \\ & + u_3(-v_4 + v_5 - v_1 + v_2) \\ & + u_4(-v_5 + v_1 - v_2 + v_3) \\ & + u_5(-v_1 + v_2 - v_3 + v_4), \end{aligned}$$

where $u_i = t_i^{d_1}$, $v_i = t_i^{d_2}$, and the coefficient of $u_i v_j$ of g is exactly the coefficient of the monomial $y_{\sigma(2)} \cdots x_1^{d_1} \cdots x_1^{d_2} \cdots y_{\sigma(5)}$, where $x_1^{d_1}$ and $x_1^{d_2}$ are respectively in the ith and in the jth position. If we now replace x_1 by the diagonal matrix $\bar{x}_1 = \rho_1 e_{11} + \ldots + \rho_n e_{nn}$, and the y_h's by the matrix units $e_{i_h j_h} = \bar{y}_h$, then the lemma follows from (β) of Section 1.

If we now set in $s_6(x_1^{d_1}, x_1^{d_2}, y_2, \ldots, y_5)$, $y_k = x_1^{b_k} x_k x_1^{c_k}$, for $k = 2, \ldots, 5$, then, by (β) of Section 1, the following lemma is a simple consequence of Lemma 1.

LEMMA 2. *Let $g(u_1, \ldots, u_5; v_1, \ldots, v_5)$ be the polynomial in commuting variables of Lemma 1. Then if we set $x_1 = \rho_1 e_{11} + \ldots + \rho_n e_{nn}$ and $x_h = e_{i_h j_h}$ for $h = 2, \ldots, 5$, then*

$$\begin{aligned} & x_1^{d_0} s_6(x_1^{d_1}, x_1^{d_2}, x_1^{b_2} x_2 x_1^{c_2}, \ldots, x_1^{b_5} x_5 x_1^{c_5}) \\ &\quad = \sum \operatorname{sgn}(\sigma) g(\rho_{\sigma(i_2)}^{d_1}, \ldots, \rho_{\sigma(i_5)}^{d_1}, \rho_{\sigma(j_5)}^{d_1}; \rho_{\sigma(i_2)}^{d_2}, \ldots, \rho_{\sigma(i_5)}^{d_2}, \rho_{\sigma(j_5)}^{d_2}) \\ &\qquad \times \rho_{\sigma(i_2)}^{d_0} \rho_{\sigma(i_2)}^{b_{\sigma(i_2)}} \rho_{\sigma(i_3)}^{c_{\sigma(i_2)} + b_{\sigma(i_3)}} \rho_{\sigma(i_4)}^{c_{\sigma(i_3)} + b_{\sigma(i_4)}} \rho_{\sigma(i_5)}^{c_{\sigma(i_4)} + b_{\sigma(i_5)}} \rho_{\sigma(j_5)}^{c_{\sigma(j_5)}} \\ &\qquad \cdot e_{\sigma(i_2)\sigma(j_2)} \cdots e_{\sigma(i_5)\sigma(j_5)}, \end{aligned}$$

where $\sigma \in S_4$ acts on $\{2, \ldots, 5\}$.

In the notation of the previous lemmas, the following lemma then holds.

LEMMA 3. $w(x_1, \ldots, x_5) = \sum \operatorname{sgn}(\sigma)\tilde{\omega}(\rho_{\sigma(i_2)}, \ldots, \rho_{\sigma(i_5)}, \rho_{\sigma(j_5)}) \cdot e_{\sigma(i_2)\sigma(j_2)} \cdots e_{\sigma(i_5)\sigma(j_5)}$, *where*

$$\tilde{\omega}(t_1, \ldots, t_5) = g_4(t_1, \ldots, t_5) + g_2(t_1, \ldots, t_5) \cdot e_2(t_1, \ldots, t_5),$$

where $g_4 = g(t_1^4, \ldots, t_5^4; t_1, \ldots, t_5)$, $g_2 = g(t_1^2, \ldots, t_5^2; t_1, \ldots, t_5)$, *and* $e_2(t_1, \ldots, t_5)$ *is the elementary symmetric function of second degree in five variables.*

We are now ready to prove the main result of this section.

Proof of Theorem 1. By the Amitsur–Levitzki theorem, $s_6(x_1, \ldots, x_6)$ is a polynomial identity for M_3. In order to prove that $w(x_1, x_2, \ldots, x_5)$ is an essentially weak polynomial identity for $M_4(K)$, it is sufficient, by (γ) of Section 1, to prove that the polynomial $\tilde{\omega}(t_1, \ldots, t_i, t_{i+1}, \ldots, t_j, t_i, t_{j+1}, \ldots, t_4)$ is divisible by $t_1 + t_2 + t_3 + t_4$ for all $1 \leq i < j \leq 4$, and that $w(x_1, \ldots, x_5)$ is not identically zero on M_4. Since $\tilde{\omega}$ is invariant with respect to cyclic permutations of the variables, it is sufficient to prove that $t_1 + t_2 + t_3 + t_4$ divides $\tilde{\omega}(t_1, t_2, t_3, t_4, t_1)$ and $\tilde{\omega}(t_1, t_2, t_1, t_3, t_4)$ and that $\tilde{\omega}(t_1, t_2, t_3, t_4, t_1)$ is different from zero as a function of t_1, t_2, t_3, t_4. We show that whenever two arguments are the same, $\tilde{\omega}$ is divisible by $\sum_{i=1}^4 t_i$.

Case I. Computation of $\tilde{\omega}(t_1, t_2, t_3, t_4, t_1)$. We compute separately $g_4 = g(t_1^4, \ldots, t_4^4, t_1^4; t_1, \ldots, t_4, t_1)$ and $g_2 \cdot e_2(t_1, \ldots, t_4, t_1)$, where $g_2 = g(t_1^2, \ldots, t_4^2, t_1^2; t_1, \ldots, t_4, t_1)$. In what follows e_2, h_i $(i = 1, 2, 3)$ are respectively the elementary symmetric function and the complete symmetric functions in the variables t_3 and t_4.

Now, $g(u_1, u_2, u_3, u_4, u_1; v_1, \ldots, v_4, v_1) = u_2(-v_3 + v_4) + u_3(-v_4 + v_2) + u_4(-v_2 + v_3)$. Hence,

$$\begin{aligned} g_4 &= t_2^4(-t_3 + t_4) + t_3^4(-t_4 + t_2) + t_4^4(-t_2 + t_3) \\ &= (t_4 - t_3)(t_2 - t_3)(t_2 - t_4)(t_2^2 + t_2 h_1 + h_2), \\ g_2 &= t_2^2(-t_3 + t_4) + t_3^2(-t_4 + t_2) + t_4^2(-t_2 + t_3) \\ &= (t_4 - t_3)(t_2 - t_3)(t_2 - t_4). \end{aligned}$$

From this we obtain by easy calculations

$$\begin{aligned} \tilde{\omega}(t_1, t_2, t_3, t_4, t_1) &= g_4 + g_2 \cdot e_2(t_1, t_2, t_3, t_4, t_1) \\ &= (t_4 - t_3)(t_2 - t_3)(t_2 - t_4)(t_1 + t_2 + t_3 + t_4)^2. \end{aligned}$$

This formula assures us at the same time that $\sum_{i=1}^4 t_i$ divides $\tilde{\omega}(t_1, \ldots, t_4, t_1)$ and that $\tilde{\omega}(t_1, \ldots, t_4, t_1)$ is not identically zero.

Case II. Computation of $\tilde{\omega}(t_1, t_2, t_1, t_3, t_4)$. Again we compute separately $g_4 = g(t_1^4, t_2^4, t_1^4, t_3^4, t_4^4; t_1, t_2, t_1, t_3, t_4)$ and $g_2 = g(t_1^2, t_2^2, t_1^2, t_3^2, t_4^2; t_1, t_2, t_1, t_3, t_4)$:

$$\begin{aligned} &g(u_1, u_2, u_1, u_3, u_4; v_1, v_2, v_1, v_3, v_4) \\ &\quad = 2u_1(v_4 - v_3) + u_2(v_3 - v_4) + u_3(2v_1 - v_2 - v_4) \\ &\qquad + u_4(-2v_1 + v_2 + v_3) \\ &\quad = (2u_1 - u_2)(v_4 - v_3) + u_3(2v_1 - v_2 - v_4) \\ &\qquad + u_4(-2v_1 + v_2 + v_3) \end{aligned}$$

$$\begin{aligned} g_4 &= (2t_1^4 - t_2^4)(t_4 - t_3) + t_3^4(2t_1 - t_2 - t_4) + t_4^4(-2t_1 + t_2 + t_3) \\ &= (t_4 - t_3)(2t_1^4 - 2t_1h_3 - t_2^4 + t_2h_3 + e_2h_2) \\ &= (t_4 - t_3)[2(t_1 - t_3)(t_1 - t_4)(t_1^2 + t_1h_1 + h_2) \\ &\quad - (t_2 - t_3)(t_2 - t_4)(t_2^2 + t_2h_1 + h_2)], \end{aligned}$$

$$\begin{aligned} g_2 &= (2t_1^2 - t_2^2)(t_4 - t_3) + t_3^2(2t_1 - t_2 - t_4) + t_4^2(-2t_1 + t_2 + t_3) \\ &= (t_4 - t_3)[2(t_1^2 - t_1h_1 + e_2) - (t_2^2 - t_2h_1 + e_2)], \end{aligned}$$

$$\begin{aligned} \bar{\omega}(t_1, t_2, t_1, t_3, t_4) &= g_4 + g_2 \cdot e_2(t_1, t_2, t_1, t_3, t_4) \\ &= (t_4 - t_3)\{2(t_1 - t_3)(t_1 - t_4)(t_1^2 + t_1h_1 + h_2) \\ &\quad - (t_2 - t_3)(t_2 - t_4)(t_2^2 + t_2h_1 + h_2) \\ &\quad + [(2(t_1 - t_3)(t_1 - t_4) \\ &\quad - (t_2 - t_3)(t_2 - t_4)][t_1^2 + 2t_1(t_2 + t_3 + t_4) \\ &\quad + t_2t_3 + t_2t_4 + t_3t_4]\} \\ &= (t_4 - t_3)(t_1 + t_2 + t_3 + t_4)[2(t_1^2 - t_1h_1 + e_2)(2t_1 + h_1) \\ &\quad - (t_2^2 - t_2h_1 + e_2)(t_1 + t_2 + h_1)]. \end{aligned}$$

Also in this case it is shown that $t_1 + t_2 + t_3 + t_4$ divides $\bar{\omega}(t_1, t_2, t_1, t_3, t_4)$. The proof of the theorem is completed.

Remark. By (γ) from Section 1, in the proof of Theorem 1 we have established not only that $w(x_1, \ldots, x_5)$ is a weak polynomial identity for M_4 but also that $w(x_1, \ldots, x_5)$ vanishes when replacing x_1 by an element of sl_4 and $x_2, \ldots, x_5$ by elements of M_4.

3. From Weak Polynomial Identities to Central Polynomials

Once we have at our disposal an essentially weak polynomial identity, the way to obtain a central polynomial is indicated by Razmyslov in [7], and we follow his work to show the procedure. Let $f(x_1, x_2, \ldots, x_n)$ be a multilinear polynomial identity. Then we can write f as

$$f(x_1, x_2, \ldots, x_n) = \sum \alpha_{pq} p(x_2, \ldots, x_n) x_1 q(x_2, \ldots, x_n), \tag{2}$$

where p and q are monomials not depending on x_1.

Definition. The polynomial

$$f^*(x_1, x_2, \ldots, x_n) = \sum \alpha_{pq} q(x_2, \ldots, x_n) x_1 p(x_2, \ldots, x_n)$$

is called the *Razmyslov's transform* of f.

The following theorem holds.

THEOREM 2 (Razmyslov). *Let $f(x_1, x_2, \ldots, x_n)$ be multilinear. Then*

(i) *If f is a polynomial identity for M_k then also f^* is a polynomial identity for M_k.*

(ii) *If f is a weak identity of M_k, then f^* is either a weak identity of M_k or central on sl_k.*

(iii) *If f is an essentially weak identity of M_k, then there is a central polynomial of M_k.*

The way to obtain the central polynomial guaranteed by the theorem is to start with a multilinear essentially weak polynomial identity $f(x_1, x_2, \ldots, x_n)$ for M_k such that $f([x_1, x_{n+1}], x_2, \ldots, x_n)$ is an ordinary polynomial identity for M_k. Then, if one writes f in the form (2), the polynomial $f^* = \sum \alpha_{pq} q(x_2, \ldots, x_n) x_1 p(x_2, \ldots, x_n)$ is a central polynomial for M_k.

Let $w(x_1, \ldots, x_5)$ be the essentially weak polynomial identity for 4×4 matrices that we have found in Section 2. By a complete linearization of $w(x_1, \ldots, x_5)$ we mean the multilinear component $w'(x_1, \ldots, x_9)$ of the polynomial $w(x_1 + x_6 + \ldots + x_9, x_2, \ldots, x_5)$.

THEOREM 3. *Let $w'(x_1, x_2, \ldots, x_9)$ be the complete linearization of $w(x_1, \ldots, x_5)$. If we set*

$$f(x_1, x_2, \ldots, x_{13}) = w'(x_1, x_2, x_3, x_4, x_5, [x_6, x_{10}], [x_7, x_{11}], [x_8, x_{12}], [x_9, x_{13}])$$

and write it in the form

$$\sum \alpha_{pq} p(x_2, \ldots, x_{13}) x_1 q(x_2, \ldots, x_{13})$$

then the Razmyslov's transform $f^*(x_1, \ldots, x_{13})$ of f is a central polynomial for M_4.

Proof. It is sufficient to show that $f([x_1, x_{14}], \ldots, x_{13})$ is an ordinary polynomial identity for M_4 and $f(x_1, x_2, \ldots, x_{13})$ is not. By the remark of Section 2, $w(x_1, \ldots, x_5)$ vanishes for $x_1 = \bar{x}_1 \in sl_4$, $x_h = \bar{x}_h \in M_4$, $h = 2, \ldots, 5$. Hence its linearization $w'(x_1, \ldots, x_9)$ vanishes for $x_h = \bar{x}_h \in M_4$, $h = 2, \ldots, 5$, $x_i = \bar{x}_i \in sl_4$, $i = 1, 6, \ldots, 9$. Since the commutators $[\bar{x}_p, \bar{x}_q]$ belong to sl_4 for any $\bar{x}_p, \bar{x}_q \in M_4$, we obtain that $f([x_1, x_{14}], x_2, \ldots, x_{13})$ is a polynomial identity for M_4.

Any diagonal matrix $\bar{x} = \sum_{i=1}^{4} \rho_i e_{ii} \in sl_4$ can be written as a commutator of two matrices, e.g.,

$$\bar{x} = [\rho_1 e_{12} + (\rho_1 + \rho_2) e_{23} + (\rho_1 + \rho_2 + \rho_3) e_{34}, e_{14} + e_{21} + e_{32} + e_{43}].$$

Hence we shall show that $f(x_1, \ldots, x_{13})$ is not a polynomial identity for M_4 if we establish that

$$v(x_1, \ldots, x_5) = w'(1, x_2, \ldots, x_5, \underbrace{x_1, \ldots, x_1}_{4})$$

is not a weak polynomial identity for M_4. Clearly, up to a multiplicative constant (equal to 3!), $v(x_1, \ldots, x_5)$ is equal to the homogeneous component of $w(1 + x_1, x_2, \ldots, x_5)$ of degree 4 in x_1. We shall calculate v for $x_1 = \bar{x}_1 = \sum_{i=1}^{4} \rho_i e_{ii}$, $x_2 = \bar{x}_2 = e_{12}$, $x_3 = \bar{x}_3 = e_{21}$, $x_4 = \bar{x}_4 = e_{13}$, $x_5 = \bar{x}_5 = e_{34}$. As in the proof of Theorem 1 it is easy to see that $v(\bar{x}_1, \ldots, \bar{x}_5)$ is equal to $\bar{v}(\rho_1, \rho_2, \rho_3, \rho_4)e_{14}$, where $\bar{v}(t_1, t_2, t_3, t_4)$ is the homogeneous component of degree 4 of

$$\bar{\omega}(1 + t_1, 1 + t_2, 1 + t_1, 1 + t_3, 1 + t_4) = \sum_{i=1}^{4} \frac{\partial \bar{\omega}(t_1, t_2, t_1, t_3, t_4)}{\partial t_i}$$

Using the expression of $\bar{\omega}(t_1, t_2, t_1, t_3, t_4)$ from Case II of the proof of Theorem 1 we obtain

$$\begin{aligned}\bar{v}(t_1, \ldots, t_4) = {} & (t_1 + t_2 + t_3 + t_4)u(t_1, \ldots, t_4) \\ & + (t_4 - t_3)[2(t_1^2 - t_1 h_1 + e_2)(2t_1 + h_1) \\ & - (t_2^2 - t_2 h_1 + e_2)(t_1 + t_2 + h_1)]\end{aligned}$$

for some polynomial $u(t_1, \ldots, t_4)$. Hence $t_1 + t_2 + t_3 + t_4$ does not divide $\bar{v}(t_1, \ldots, t_4)$ and $v(\bar{x}_1, \ldots, \bar{x}_5) \neq 0$ for suitably choosen ρ_1, ρ_2, ρ_3 and $\rho_4 = -(\rho_1 + \rho_2 + \rho_3)$. This completes the proof of the theorem.

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