Equation of state and viscosities from a gravity dual of the gluon plasma

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Employing new precision data of the equation of state of the SU(3) Yang–Mills theory (gluon plasma) the dilaton potential of a gravity-dual model is adjusted in the temperature range \((1–10)T_c\) within a bottom-up approach. The ratio of bulk viscosity to shear viscosity follows then as \(\zeta/\eta \approx \pi \Delta v_s^2\) for \(\Delta v_s^2 < 0.2\) and achieves a maximum value of 0.94 at \(\Delta v_s^2 = 0.3\), where \(\Delta v_s^2 = 1/3 - v_s^2\) is the non-conformality measure and \(v_s^2\) is the velocity of sound squared, while the ratio of shear viscosity to entropy density is known as \((4\pi)^{-1}\) for the considered set-up with Hilbert action on the gravity side.

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1. Introduction

With the advent of new precision data [1], which extend previ-
ous lattice QCD gauge theory evaluations [2,3] for the pure gluon plasma to a larger temperature range, a tempting task is to seek
for an appropriate gravity dual model. While such an approach does not necessarily provide new insights in the pure SU(3) Yang–
Mills equation of state above the deconfinement temperature \(T_c\), it however allows to calculate, without additional ingredients, fur-
ther observables, e.g. transport coefficients. (This is in contrast to quasiparticle approaches which require additional input to access trans-
port coefficients [4].) In considering an ansatz of gravity + scalar as framework of effective dual models to pure non-abelian
gauge thermo-field theories within a bottom-up approach one has to adjust either the potential of the dilaton field, or a metric func-
tion, or the dilaton profile.

The improved holographic QCD (IHQCD) model, developed in
[5–8] (for a review cf. [9]) is a particularly successful realization of such a setting. The potential of IHQCD [8] was constructed to
match the \(t'\)Hooft limit Yang–Mills \(\beta\) function to two-loop or-
der (which determines the functional form and two parameters)
in the near-conformal (small \(t'\)Hooft coupling) region, while the zero-temperature (large \(t'\)Hooft coupling) behavior is fixed by de-
manding confinement and a linear glueball spectrum. A potential
smoothly interpolating between the two asymptotic regions was
shown in [8] to well reproduce the \(N_c = 3\) Yang–Mills plasma
equation of state [2], where remaining free parameters were fixed by comparing to the latent heat and scaled pressure from the lat-
tice. Within IHQCD, zero-temperature confining geometries exhibit a
first-order thermodynamic phase transition [7].

A different type of dilaton potentials was considered in [10],
where near the boundary the potential accounts for a massive
scalar field and the spacetime asymptotes to pure \(AdS_5\). The po-
tential parameters were matched to the velocity of sound as sug-
gested by the hadron resonance gas model and the dimension of
\(\text{Tr} F^2\) at a finite scale [11] and reproduce the velocity of sound of
2 + 1 flavor QCD, whereas in [12] the matching to the SU(3) Yang–
Mills equation of state [2] has been accomplished. In IHQCD, the
marginal operator dual to \(\phi\) is \(\text{Tr} F^2\) [5], while in [10,11] and here
the dual operator \(\mathcal{O}\) is interpreted as a relevant deformation of the
boundary theory Lagrangian. In [13], instead of the dilaton poten-
tial, an ansatz for a metric function of the five-dimensional gravity
action is selected and consequences for the boundary theory are
explored (Such an approach suffers however from the conceptual
shortcoming that the dilaton potential and thus the action depend
on the temperature, while, according to the gauge/gravity duality,
the bulk action should be independent of the boundary theory state).

The previous benchmark lattice data [2] (up to \(4.5T_c\)) and fur-
ther SU\((N_c)\) data for \(N_c \leq 8\) [14] (up to \(3.5T_c\)) and \(N_c \leq 6\) [15]
(up to \(4T_c\)) are for \(N_c = 3\) now supplemented and extended up
to \(1000T_c\) [1]. Here we are going to adjust precisely the dilaton
potential to the new lattice data [1] in the temperature range up
to \(4.5T_c\) without dilaton and \(N_c = 3\) in SU(3) and \(N_c = 4,5,6\) in SU(4,5,6) for \(T_c \leq 2.5\) and \(T_c \leq 3.5\), respectively.

The IHQCD results for \(\lambda = \pi^2\) and \(\nu = 1\) are depicted in
Figure 1. Given a temperature \(T\), we have the values of the
minimum \(\eta/\sigma\) (which is the saddle point of the potential) and
the maximum \(\nabla^2 \phi^2\) (which is the barrier of the potential) for
\(\eta = \zeta\). These values are in agreement with the lattice data
[1] and the predictions of other models [16].

The IHQCD model provides a consistent framework for ex-
ploring the dual description of QCD and has been shown to
accurately reproduce the equation of state of QCD plasma in the
low-temperature region. The model is based on the holographic
principle, which connects the gauge theory with the string theory
in the AdS/CFT correspondence. The model is particularly
successful in reproducing the behavior of the gluon-gluon correla-
tors, as well as the pressure and shear viscosity of the plasma.

The IHQCD model can be further improved by incorporating
more recent lattice data and by extending the model to higher
\(N_c\) values. Additionally, the model can be adapted to
include other degrees of freedom, such as quark-gluon plasma
and heavy quarkonia, which are important in the study of high-
energy collisions.

As a final remark, it is worth noting that the IHQCD model
provides a promising avenue for exploring the physics of QCD
plasma in the high-temperature regime, where traditional
gauge theories may fail. The model also offers a unique
framework for studying the phase structure of QCD and
the transition from the hadronic to the quark-gluon plasma
regimes. The IHQCD model is a valuable tool for understanding
the fundamental aspects of QCD and its implications for
particle physics and cosmology.
to $10^7\Gamma$, thus catching the strong-coupling regime, as envisaged as relevant also in [16]. We discard completely a recourse to the \( \beta \) function. Such an approach can be considered as a convenient parameterization of the equation of state. Once the potential is adjusted, it qualifies for further studies, e.g. of transport coefficients. Our goal is accordingly the quantification of the bulk viscosity in the LHC relevant region, in particular near to \( T^* \), and a comparison with results of the quasiparticle model [4].

According to holography, SU(\( N_c \)) Yang–Mills theory at finite \( N_c \) must be described by quantum string theory, which has not yet been completely established. Since in the large-\( N_c \) and large \( t' \)Hoft coupling limits quantum string theory reduces to classical gravity, one presently resorts to a gravitation theory in a five-dimensional space, constructed in such a manner to accommodate certain selected features of the holographically emerging boundary field theory. As in the models [9,10,11] and IHQCD [9] models are different both in the near-boundary region, i.e. at high temperatures and also deep in the bulk, i.e. at low temperatures. When adjusting the potential in an intermediate region suitable for \( (1-10)T \), one would like to know whether it is important to incorporate a certain kind of asymmetries, or whether they have little influence. Put another way, to what extent does a fit to lattice data on \( (1-10)T \) determine the potential? Here, we do not attempt to solve the general problem of computing the potential from a given equation of state, but instead show that various potentials which contain a certain unique relevant section lead to nearly identical equations of state in the corresponding temperature region.

Transport properties of the matter produced in relativistic heavy-ion collisions at RHIC and LHC are important to characterize precisely such novel states of a strongly interacting medium besides the equation of state. The impact of the bulk viscosity on the particle spectra and differential flow phenomena has been recently discussed in [18] and found to be sizeable in [19], in particular for higher-order collective flow harmonics. The bulk viscosity enters also a new soft-photon emission mechanism [20] via the conformal anomaly, thus offering a solution to the photon-\( v_2 \) puzzle (cf. [20] for details and references). Compilations of presently available lattice QCD results of viscosities can be found in [4].

2. The set-up

The action \( S = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left( R + \frac{1}{2} (\partial \phi)^2 - V(\phi) \right) \) (the Hawking–Gibbons term is omitted) leads, with the ansatz for the infinitesimal line element squared \( ds^2 = \exp(2A)(-dt^2 + dx^2 + \exp{2B}h^{-1}dz^2) \) to the field equations quoted in [10] under (25a–25c); the equation of motion (25d) follows from the derivative of (25c) with insertion of (25a–25c). Here, the coordinate transformation \( dz = L\exp(B - A) d\phi \) has been employed to go from the Fefferman–Graham coordinate \( z \) in the infinitesimal line element squared \( dz^2 = \exp(2A)(-dt^2 + dx^2 + \exp{2B}h^{-1}dz^2) \) to a gauged radial coordinate expressed by the dilaton field \( \phi \) which requires the introduction of a length scale \( L \). The metric functions are thus to be understood as \( A(\phi; \phi_1), B(\phi; \phi_1) \) and \( h(\phi; \phi_1) \), and a prime means in the following the derivative with respect to \( \phi \). These equations can be rearranged by defining \( Y_1 = A - A_H, Y_2 = A' + U, Y_3 = A'' + \frac{1}{2}U', Y_4 = B - B_H, \) \( Y_5 = \exp(4A_H - B_H) f^{\phi}_0 \delta \exp(-4A + B), \) where the subscript \( H \) denotes the value of a function at the horizon and \( U \equiv V(3V') \), to change the mixed boundary value problem into an initial value problem, given by

\[ Y_1' = Y_2 - U, \]
\[ Y_2' = Y_3 + \frac{1}{2} U', \]
\[ Y_3' = \frac{1}{2} U'' + Y_3 - \frac{1}{2} U' \left( \frac{Y_3 - Y_1'}{2} \right), \]
\[ Y_4' = 6(Y_3 - \frac{1}{2} U') + 1, \]
\[ Y_5' = \exp(-4Y_1 + Y_4), \]

which is integrated from the horizon \( \phi_H - \epsilon \), to the boundary \( \epsilon \) with the initial values \( Y_i = 0 \) at \( \phi_H - \epsilon \). The limit \( \epsilon \to 0^+ \) has to be taken to obtain the entropy density \( s \) and the temperature \( T \)

\[ G_{SS} = \frac{1}{4} \exp(3A_H), \]
\[ LT = \frac{1}{4\pi} \exp(A_H - B_H) \]

where \( A_H = \log\left( \frac{L}{3\epsilon} \right) \) and \( B_H = \log\left( 1 - \Delta \right) \) (we set \( \Lambda = 1 \)) and \( \Delta = \log\left( \frac{\Delta - 1}{\Delta} \right) \) at \( \phi \to 0^+ \). The boundary asymptotics of \( A \) and \( B \) assume \( L^2 V(\phi) \approx -12 + (\Delta - 1)2\phi^2 \) for small \( \phi \), where \( \Delta \) is the scaling dimension of the conformality-breaking operator of the boundary theory. We consider \( \Delta \leq 4 \), selecting the upper branch of the mass dimension relation \( L^2 M^2 = \Delta - 4 \) and restricting to relevant operators. Hence, the Breitenlohner–Freundeman bound \( L^2 M^2 \geq -4 \) [21] is respected and renormalizability on the gauge theory side is ensured. The quantities \( Y_i(\epsilon) \) depend on the horizon position \( \phi_H \), implying in particular \( s(\phi_H) \) and \( T(\phi_H) \), thus providing the equation of state \( s(T) \) in parametric form.

3. Equation of state

To compare with the lattice results [1] of the relevant thermodynamical quantities (i) sound velocity squared \( v_s^2 = \frac{d\log T}{d\log s} \), (ii) scaled entropy density \( s/T^3 \), (iii) scaled pressure \( p/T^4 \), and

\[ 1 \text{ system (1)-(5) enjoys some redundancy. It can be reduced by introducing } x = 1/4A_H = 1/4(Y_2 - U), Y = h/4b(A' - A_H) + Y_3/4b(Y_2 - U) \text{ which leads to two coupled first-order ODEs for the scalar invariants } X(\phi; \phi_1) \text{ and } Y(\phi; \phi_1) \text{ according to } [7]. \]
(iv) scaled interaction measure $I/T^4 = s/T^3 - 4p/T^4$ (all as functions of $T/T_c$) one must adjust the scale $T_c$ and the 5D Newton's constant $G_5$ (actually, the dimensionless combinations $LT_c$ and $G_5/L^3$ are needed). In the present bottom-up approach, we employ a new potential designed to reproduce the data [1] in the temperature region $1\rightarrow 10T_c$.

$$v_1(\phi) = \frac{V'(\phi)}{V(\phi)} = \begin{cases} \frac{-12M^2}{12} + i_1 \phi^3 \quad &\text{for } \phi \leq \phi_{m1}, \\ \phi' + s_1(\text{erf}(s_2(\phi - s_3)) - 1) \quad &\text{for } \phi \geq \phi_{m1}, \end{cases}$$

as an ansatz and optimize the parameters $\phi_{m1}$, $s_{1,2,3}$ and $\gamma$. Since we are not interested in the high-temperature regime $T > 10T_c$, we choose a simple interpolation from $\phi = 0$ to $\phi = \phi_{m1}$. The latter value is taken as a fit parameter and fixes $L^2M^2$ and $i_1$ by the requirement that $v_1$ be differentiable at $\phi_{m1}$. The critical temperature $LT_c$ is determined by $T_c = T(\phi_{m1}^c)$ with $\phi_{m1}^c$ from the pressure

$$p(\phi_H) = \int_\infty^{\phi_H} d\phi_H s(\phi_H) \frac{dT}{d\phi_H},$$

via $p(\phi_{m1}^c) = 0$. This is the prescription discussed in detail in [7] for the first-order phase transition to a thermal gas configuration at $T < T_c$. According to [6,7] the boundary theory at $T < T_c$ is confining and gapped if $\gamma > \sqrt{2/3}$ and, equivalently, $LT(\phi_H)$ is U shaped, with a global minimum at $\phi_{m1}^c$, implying $T(\phi_{m1}^c) > T(\phi_{m1})$, see Fig. A.3. The construction ensures a minimum free energy for $T < T_c$ (thermal gas with $p = 0$) and $T > T_c$ (large black hole branch which continues in the UV region). In (9), $p(\infty) = 0$ for a “good” IR singularity requires $\gamma < \sqrt{2/3}$.

Our results are exhibited in Fig. 1 for the optimized parameter set

$$\begin{array}{cccccc}
\nu & \phi_{m1} & s_1 & s_2 & s_3 & \gamma & G_5/L^3 \\
\nu_1 & 1.3444 & 0.3954 & 0.6723 & 2.7358 & 0.8222 & 1.1100
\end{array}$$

The velocity of sound is independent of $G_5$ which steers the number of degrees of freedom, thus being important for entropy density, energy density $\epsilon$, pressure and interaction measure. In asymptotically free theories, the $T^4$ term dominates $s$, $\epsilon$ and $p$ at large temperatures; it is subtracted in the interaction measure making it a sensible quantity. (Unlike the HQQCD model our ansatz does not catch pQCD features in the deep UV. That is the reason for our restriction to $T < 10T_c$.) The appearance of a maximum of $1/T^4$ at $T/T_c \approx 1.1$ is related to a turning point of $p/T^4$ as a function of $\log T$. Position and height of $1/T^4$ – the primary quantity in lattice calculations – are sensible characteristics of the equation of state. The dropping of $1/T^4$ at larger temperatures signals the approach towards conformality. (Since in conformal theories $v_2^2 = 1/3$, the quantity $\Delta v_2^2 = 1/3 - v_2^2$ is termed non-conformality measure; also here, the dominating $T^4$ terms at large temperatures drop out.) Inspection of Fig. 1 unravels the nearly perfect description of the lattice data [1]. Note that, by construction, $p/T^4$ always slightly underestimates the lattice data for $T \rightarrow T_c^+$, since $p(\phi_{m1}^c) = 0$, while $p(T_c)/T_c^4$ [lattice] = 0.0222 [1]. We find $\Delta s(T_c)/T_c^4 \approx 1.7$ for the scaled latent heat.

4. Viscosities

Irrespectively of the dilaton potential $V(\phi)$, the present set-up with Hilbert action $R$ for the gravity part delivers $\eta/\epsilon = (4\pi)^{-1}$ [23,24] for the shear viscosity $\eta$, often denoted as KSS value [25]. (See [26] for the original calculation. Inclusion of higher-order curvature corrections can decrease the KSS value [27].) In contrast, the bulk viscosity to entropy density ratio $\xi/\epsilon$ has a pronounced temperature dependence. Following [23] we calculate $\xi$ from the relation

$$\xi \frac{\eta}{\phi_H} \frac{1}{9U(\eta_H)^2 \left[p_{11}(\epsilon)\right]^2},$$

where the asymptotic value $p_{11}(\epsilon)$ of the perturbation $p_{11}$ of the 11-metric coefficient is obtained by integrating
From the horizon $\phi_H - \epsilon$ to the boundary $\epsilon$ with initial conditions $p_{11}(\phi_H - \epsilon) = 1$ and $p'_{11}(\phi_H - \epsilon) = 0$ and $\epsilon \to 0^+$. Equivalently [28], the bulk viscosity can be obtained from the Eling-Oz formula [24]

$$\frac{\zeta}{\eta}|_{\phi_H} = \left(\frac{d \log T}{d \phi_H}\right)^{-2} = \left(\frac{1}{v_T^2} \frac{d \log T}{d \phi_H}\right)^{-2}.$$  

Our results are exhibited in Fig. 2. The scaled bulk viscosity $\zeta/T^3$ has a maximum at 1.05$T_c$ (which is slightly below the maximum of $I/T^4$) and drops rapidly for increasing temperatures, see left panel of Fig. 2. Remarkable is the almost linear section of $\zeta/\eta$ as a function of the non-conformality measure $\Delta v_T^2$ (see right panel), as already suggested in [29] and observed, in particular at high temperatures, in numerous holographic models [30,31]; for further reasoning on such a linear behavior within holography approaches cf. [32]. A non-linear behavior occurs in a small temperature interval $1 \leq T/T_c < 1.05$, i.e. for $\Delta v_T^2 > 0.22$, see right panel of Fig. 2. The maximum value of $\zeta/\eta \approx 0.94$ at $\Delta v_T^2 \approx 0.3$ depends fairly sensitively on the details of the equation of state for $T \to T_c^*.

In order to determine the $\zeta/\eta$ ratio we consider the Buchel bound $\zeta/\eta \geq 2\Delta v_T^2$ [30] and agrees surprisingly well on a qualitative level with the result of [4] in the interval $1.05 < T/T_c < 2$. There, a quasiparticle approach has been employed which needs, beyond the equation-of-state adjustment, further input: In [4] it is the dependence of the relaxation time on the temperature which causes a change from the linear relation $\zeta/\eta \propto \Delta v_T^2$ near $T_c$, i.e. for large values of $\Delta v_T^2$, to a quadratic dependence in the weak-coupling regime [34] at large temperatures corresponding to small values of $\Delta v_T^2$. Note also the shift of the linear section of $\zeta/\eta$ in [4] by a somewhat larger off-set which can cause a descent violation of the Buchel bound, which is not unexpected with respect to [35].

5. Robustness of the bulk viscosity

5.1. Definition of the transition temperature

If one is interested in the thermodynamics of the deconfined phase a theoretically sound determination of $T_c$ can be related to the Hawking–Page transition and to the construction of [7], as strictly applied in section 3. Fitting the data [1], we observe [36] $T_c = (1 + \varepsilon)T_{\text{min}}$ with positive $\varepsilon < 10^{-2}$ and $T_{\text{c}}$ from the pressure loop (see Fig. A.1, inset in left bottom panel). One could be tempted, therefore, to ignore the numerically tiny difference of the proper thermodynamic first-order transition temperature $T_c$ and $T_{\text{min}}$ and to use $T_{\text{min}}$ instead. In fact, then one can easily reproduce the lattice data [1], as shown in [36], e.g. by a potential similar to [11], distorted by polynomial terms,

$$L^2V_{IV}(\phi) = -12\cosh(\gamma \phi) + (6\gamma^2 + \frac{1}{2}\Delta(\Delta - 4))\phi^2 + \sum_{i=2}^5 c_{2i}\phi^{2i},$$

whereby the original Gubser–Nellore potential [10], referred to as $V_1$, follows for $c_{2i} = 0$.

5.2. Generating nearly equal potentials

The scheme of employing the holographic principle here consists of mapping $V(\phi \in [\phi_m, \phi_H]) \Rightarrow T(\phi_H)$, $s(\phi_H) \Rightarrow s(T)$, i.e. the complete non-local potential properties enter the local thermodynamics. Since we are interested in $s/T^3$ as a function of $T/T_c$ in the restricted interval $T = (1 \ldots 10)T_c$, one can ask whether near-boundary properties of $V(\phi)$ are irrelevant. We provide evidence that this is indeed the case, at least for $\varepsilon < 1$, where one can tentatively neglect the difference of $T_{\text{min}}$ and $T_c$, and ignoring the IR behavior. To substantiate this claim, let us consider a special one-parameter potential $V_3(\phi; \phi_0)$ which contains as relevant part the section $V_1(\phi \geq \phi_m)$ where $\phi_m = 0.55$ means a value of $\phi_H$ corresponding to $10T_c$ determined by the potential $V_1$. The relevant section of $V_1$ is now up or down shifted by a parameter $\phi_0$, and $L^2V_{\text{int}}(\phi; \phi_0) = -12 + \frac{1}{2}L^2m_{\text{int}}^2(\phi_0)\phi^2 + b(\phi_0)\phi^4$ is an interpolating section from the boundary $\phi_0$ to the matching point $\phi_m + \phi_0$. The conditions $V_1(\phi_m) = V_{\text{int}}(\phi_m + \phi_0; \phi_0)$, $V_1(\phi_m) = V_{\text{int}}(\phi_m + \phi_0; \phi_0)$, $V_1(\phi_m) = V_{\text{int}}(\phi_m + \phi_0; \phi_0)$ fix $L^2m_{\text{int}}^2$ and $b$. The Breitenlohner–Freundman bound $-4 \leq L^2m_{\text{int}}^2 \leq 0$ restricts the possible values of $\phi_0$ for given $V_{\text{int}}$ and $\phi_m$; in our example, $-0.165 \leq \phi_0 \leq 0.4$. To quote a few numbers, the left-most shift $\phi_0 = -0.165$ yields $L^2m_{\text{int}}^2 = -3.927$, $\Delta = 2.271$, $LT_{\text{min}} = 1.81 \times 10^{-2}$, while the right-most shift $\phi_0 = 0.4$ yields $L^2m_{\text{int}}^2 = -0.098$, $\Delta = 3.975$, $LT_{\text{min}} = 0.002$. 

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2 Here, the boundary position is denoted by $\phi_0$, being at $\phi = 0$ for the potential [14], while in the IHQCD model [5-7] it is at $\phi = -\infty$. Because of this, the approximate symmetry of the equation of state under constant shifts $\phi \to \phi + \phi_0$ discussed here, is exact in IHQCD [9].
3.46 × 10^{19.3} \text{ Despite of a huge variation of } LT_{\text{min}}, \text{ dimensionless thermodynamic quantities } T/T_{\text{min}} \text{ and } s/T^3 \text{ as functions of } \phi_0 - \phi_1 \text{ are within very narrow corridors with relative variations (depending on } \phi_1 - \phi_0 \text{ and parametrically on } \phi_0) \text{ of less than } 4 \times 10^{-2} \text{ for } T/T_{\text{min}} \text{ and } 5 \times 10^{-4} \text{ for } s/T^3. \text{ From the Eling–Oz formula (13), one infers an analogous behavior of } \zeta/\eta \text{ as a function of } \phi_0 - \phi_1, \text{ meaning that the potentials } V_\eta \text{ deliver a nearly unique equation of state and viscosity ratio in the considered temperature interval. We therefore argue that all precise fits of } V(\phi) \text{ to lattice data deliver, up to a linear shift, nearly equivalent potentials in the selected temperature region and, in particular, nearly the same } \zeta/\eta \text{ vs. } \Delta v_2^{-2}. \text{ At the end of this discussion on the role of } T_c \text{ and the conjectured robustness of the bulk viscosity we mention that we are not able to fit precisely (8) with parameters (10) by } V'/V \text{ emerging from the potential (14) with } \gamma > \sqrt{2}/3. \text{ Apparently, (14) and the proper } T_c \text{ definition along (7) with well defined IR behavior seem to fail a precise match to the data (1). In the Appendix we present a potential which accommodates also lattice data below } T_c. \text{ Despite of such ambiguities, we find the bulk viscosity at and above } T_c \text{ as fairly robust, with deviations of at most } 6\% \text{ for } T/T_c \leq 1.02 \text{ and otherwise less than } 2\%, \text{ supposed } T_c \text{ is a proper first-order transition temperature (if not, the bulk viscosity can significantly vary, depending upon the choice of the scale, see also (22)). Within the non-conformal region } 1 \leq T/T_c \leq 10, \text{ where the non-conformality measure is } 0.2 \geq \Delta v_2^2 \geq 0.004 \text{ and the interaction measure is } 2.48 > 1/T^4 > 0.07, \text{ an almost linear dependence } \zeta/\eta \approx \pi \Delta v_2 \text{ on the non-conformality measure } \Delta v_2 \text{ is observed, as already argued in (29) and found within holographic approaches (30,31) and in (4) with a quasi-particle approach to the pure gauge sector of QCD. We mention further that one can identify a relevant section of the potential which determines the equation of state in a selected temperature interval. Shifting, within certain limits that relevant section, the equation of state and the bulk viscosity are marginally modified. Extensions towards including quark degrees of freedom and subsequently non-zero baryon density, i.e. to address full QCD, have been outlined and explored in (37). The Veneziano limit of QCD is investigated in a more string theory inspired setting in (38). Incorporating additional degrees of freedom (which are aimed at mimicking an equal number of quarks and anti-quarks) within the present set-up, one essentially has to lower } G_5/L^3 \text{ in adjusting the extensive and intensive densities. Since the viscosities scale with } L^2/G_5 \text{ (as the entropy density does, too) the corresponding ratios } \zeta/s \text{ and } \zeta/\eta \text{ would stay unchanged, if the same potential would apply and the same behavior of the sound velocity would be used as input. However, as stressed above, } \zeta/\eta \text{ depends rather sensitively on the actual potential } V(\phi) \text{ and its parameters. Since QCD does not display a first-order phase transition at zero baryon density, dedicated separate investigations are required to adjust the dilaton potential to current lattice data. (The results in (23) yield } \zeta/\eta \approx 0.98 \pi \Delta v_2 \text{ for } \Delta v_2 < 0.28 \text{ with a maximum of } \zeta/\eta \approx 0.75 \text{ at } \Delta v_2 \approx 0.26, \text{ i.e. values comparable to the pure glue case.)}

On the gravity side, inclusion of terms beyond the Hilbert action would cause a temperature dependence of the ratio } \eta/s \text{ which is needed to furnish the transition into the weak-coupling regime (40) at large temperatures. It is an open question whether such higher-order curvature corrections also lead to a quadratic dependence of the viscosity ratio on the non-conformality measure (34). In summary, we adjust the dilaton potential exclusively at new lattice data for SU(3) gauge theory thermodynamics and calculate holographically the bulk viscosity. The ratio of the bulk to shear viscosity obeys, in the strong-coupling regime, a linear dependence on the non-conformality measure for temperatures above 1.05T_c, while at T_c it has a maximum of 0.94. Our result, which is based on some fine tuning of the dilaton potential to precision lattice data, agrees well with previous holographic approaches based on former lattice data, such as the IHQCD model, or studies with the GB-Nellore potential types which envisaged qualitatively capturing QCD features. It would be interesting to employ the numerical findings of our holographically motivated guess, even if they are related to the pure gauge theory (with the disclaimers mentioned in the introduction), e.g. in the modelings (18–20) of heavy-ion collisions to elucidate their impact on observables. Our potential(s) may also serve as a suitable background, e.g. for various holographic mesons.}

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Appendix A. Including confined-phase lattice data

The potential \( v_1 \) in (8) can be modified to reproduce also the presently available lattice data in the confined phase:

\[

v_2 = \begin{cases} 
-\frac{1}{2}M_2^2 \phi + i \phi^3 
\quad & \text{for } \phi \leq \phi_m, \\
\gamma + s_1[\tanh(s_1(\phi - s_2)) - 1] + p_1 \phi^2 \phi_2(\phi - \phi_1)^2 
\quad & \text{for } \phi \geq \phi_m.
\end{cases}
\]

This parametrization is inspired by the desired behavior of \( v_2^2 \) as function of \( \phi_H \).\footnote{The parametrization (A.1) is superior to the one given in the appendix of [36], since a better description of \( v_2^2 \) for \( T = T_c \) is accomplished.} Performing a fit to lattice data for \( 0.7 \leq T/T_c \leq 10 \)}
and identifying $T_c$ with $\bar{T}_c$, which is determined by the intersection of the high-temperature and low-temperature branches of the pressure (9) combined with $T(\phi_H)$, we find the parameters

$$
\begin{array}{ccccccccc}
\nu & \phi_m & s_1 & s_2 & \gamma & p_1 & p_2 & p_3 & G_2/L^2 \\
\nu_{2a} & 2.3523 & 0.4452 & 6.9382 & 2.73 & 0.7526 & 0.1707 & 4.6707 & 1.1125 \\
\nu_{2b} & 2.3171 & 0.4259 & 0.5929 & 0.7979 & 0.6982 & 0.1864 & 4.8011 & 1.1176 \\
\end{array}
$$

(A.2)

The resulting equation of state is exhibited in Fig. A1. In the direct vicinity of $T_c$, the model calculation deviates from the lattice data on a 5% level in the high- and low-temperature phases; otherwise the fit is near-perfect. Unlike the potential $\nu_1$ (8) which facilitates a monotonous increase of $LT(\phi_H)$ for $\phi_H > \phi_H^\text{min}$, $LT(\phi_H)$ from $\nu_{2a}$ (with fixed $\gamma = \sqrt{T/3}$) runs to a constant value, while for $\nu_{2b}$ it is dropping, see Fig. A2. That is the potentials $\nu_{2a}$ and $\nu_{2b}$ leave the IR physics of the boundary theory unsettled, which however does not play any role for the description of the lattice data for $T > 0.7T_c$ as seen from Fig. A1. The potential $\nu_{2a}$ can be regarded as the best compromise between two mutually exclusive options: $\nu_1$, a zero-temperature confining and gapped boundary theory and $\nu_{2b}$, a boundary theory with smooth and finite pressure for $0 < T < T_c$; in the classification of [7], the model $\nu_{2a}$ is zero-temperature confining and has a partially discrete spectrum. For both parameter sets (A.2) we find the scaled latent heat $\Delta s(T_c)/T_c^2 \approx 1.3$ which compares well with $\Delta s(T_c)/T_c^2 \approx 1.4$ found in lattice calculations (see [1] and references therein).

The bulk viscosity resulting from the ansatz (A.1) with the parameters (A.2) is exhibited in Fig. A2. The maximum $\zeta/\eta \approx 1$ lies at $\Delta\nu_2 \approx 0.31$. We notice the jump at $T_c$ due to the first-order phase transition; $\zeta/\nu^2$ is rapidly dropping for smaller temperatures; $\zeta/\eta$ vs. $\Delta\nu_2^2$ displays a hook which we would not consider a reliable result since the setting at $T < T_c$ might not be trustworthy. Below $T_c$, in the interval $0.76 \leq T/T_c \leq 0.998$, the viscosity ratio $\zeta/\eta$ violates the Buchel bound (see right panel of Fig. A2). A similar behavior was found in [23] for the potential $V_I$ adjusted to the equation of state of $2 + 1$ flavor QCD.

Fig. A3 summarizes the dependence of the temperature as a function of $\phi_H$. The global minimum for $\nu_1$ (8) is quite shallow (the anticipated U shape becomes better evident when displaying
Fig. A.3. The temperature as a function of $\phi_H$ for the potentials $v_1$ (8) with parameters (10) (solid curve) as well as $v_{2a}$ (dashed curve) and $v_{2b}$ (dotted curve), see (A.1) and (A.2). Light grey portions of the curves denote the unstable/metastable regions of the equation of state, while dots mark the positions of $\phi_H^{\text{min}}$.

$LT$ as a function of $\log \phi_H/\phi_H^{\text{min}}$. The local minima for $v_{2a,b}$ (A.1) are also very shallow. Thus, $T_{\text{min}} \approx T_c$ or $\tilde{T}_c$ follows.

References