HISTORIA MATHEMATICA 20 (1993), 68–88

The Development and Understanding of the Concept of Quotient Group

JULIA NICHOLSON

Merton College, Oxford University, Oxford OX1 4JD, United Kingdom

This paper describes the way in which the concept of quotient group was discovered and developed during the 19th century, and examines possible reasons for this development. The contributions of seven mathematicians in particular are discussed: Galois, Betti, Jordan, Dedekind, Dyck, Frobenius, and Hölder. The important link between the development of this concept and the abstraction of group theory is considered. © 1993 Academic Press, Inc.

Cet article decrit la découverte et le développement du concept du groupe quotient au cours du 19ième siècle, et examine les raisons possibles de ce développement. Les contributions de sept mathématiciens seront discutées en particulier: Galois, Betti, Jordan, Dedekind, Dyck, Frobenius, et Hölder. Le lien important entre le développement de ce concept et l'abstraction de la théorie des groupes sera considerée. © 1993 Academic Press, Inc.

Diese Arbeit stellt die Entdeckung beziehungsweise die Entwicklung des Begriffs der Faktorgruppe während des 19ten Jahrhunderts dar und untersucht die möglichen Gründe für diese Entwicklung. Die Beiträge von sieben Mathematikern werden vor allem diskutiert: Galois, Betti, Jordan, Dedekind, Dyck, Frobenius und Hölder. Die wichtige Verbindung zwischen der Entwicklung dieses Begriffs und der Abstraktion der Gruppentheorie wird betrachtet. © 1993 Academic Press, Inc.

AMS subject classifications: 01A55, 20-03
KEY WORDS: quotient group, group theory, Galois theory

1. INTRODUCTION

Nowadays one can hardly conceive any more of a group theory without factor groups ...
B. L. van der Waerden, from his obituary of Otto Hölder [1939]

Although the concept of quotient group is now considered to be fundamental to the study of groups, it is a concept which was unknown to early group theorists. It emerged relatively late in the history of the subject: toward the end of the 19th century. The main reason for this delay is that in order to give a recognizably modern definition of a quotient group, it is necessary to think of groups in an abstract way. Therefore the development of the concept of quotient group is closely linked with the abstraction of group theory. This process of abstraction took place mainly during the period 1870–1890 and was carried out almost exclusively by German mathematicians. Thus by 1890 the development and understanding of the concept of quotient group had largely been completed.

The contributions of seven mathematicians to the evolution of this concept are considered here. An early understanding of the concept within Galois theory and permutation group theory can be found in researches of Galois, Betti, and Jordan

0315-0860/93 $5.00

(work which falls within the period 1829–1873). Some studies by Dedekind during the 1850s reveal his remarkable grasp of abstract group theory and his clear understanding of the concept of quotient group. In the work of Dyck, Frobenius, and finally Hölder, the concept was explored during the 1880s within abstract group theory and after this time it rapidly gained acceptance in the mathematical community.

What is a quotient group? Is there only one 'correct' definition? If there is, it should be possible to trace the development of the concept with this definition in mind and to measure early attempts at a definition against it. There is certainly a *standard* modern definition, employed by such textbooks as [Rose 1978, 42–43], [Herstein 1975, 51–52] and [Macdonald 1968, 56–57]:

> For a group G, the quotient group G/H is the set of cosets Hx ($x \in G$) of the normal subgroup H of G, with multiplication given by $Hx_1Hx_2 = Hx_1x_2$ ($x_1, x_2 \in G$).

This definition could have been given in terms of left cosets xH, as it is in Rose, but such a distinction is redundant since a normal subgroup H of G is a subgroup for which $Hx = xH$ for all $x \in G$.

Why has this become the standard definition? In what ways does it encapsulate the concept of quotient group well? I suggest that it does so in two ways. First, this definition makes use only of the elements of the group G itself, with these elements being combined in a particular way. We do not have to use any concepts 'outside' the group. Second, the definition does not depend on representing G in any one way: it is 'abstract' and can be applied to any group. The disadvantage of this definition is that it is not easy to generalize to other algebraic structures. However, a moment's thought reveals that the basic idea behind this definition is that of equivalence. For any algebraic structure, if we can separate its elements into equivalence classes and produce a well-defined 'multiplication' of these classes, we have formed a quotient structure.

What must one understand in order to give a definition of quotient group which satisfies the two criteria mentioned above? For the first, we must recognize that the elements of a quotient group are not of the same type as the elements of the original group. In the above definition they are *sets* of the original elements: these sets are the normal subgroup H and its cosets. For the second, we must (obviously) understand the abstract notion of a group.

By the end of the last century, the 'standard' definition stated above had been formulated and was in use alongside earlier, more permutation-theoretic, attempts at a definition. My intention is to survey the development of the concept up to that time in the light of the two criteria for definition mentioned above. I am not, however, dismissing the insights of those mathematicians who chose to express the concept in other ways and who therefore do not meet these criteria, either through their own choice or through the constraints of the methods available to them. First let us look at the prehistory of this development.

2. IMPLICIT USE OF THE CONCEPT: GALOIS

With the benefit of hindsight we can see that the concept of quotient group was present on many occasions before an explicit definition was given. One idea which can now be understood in terms of quotient groups is found in the Galois theory of algebraic equations. In his original explanation of the theory, Galois made explicit for the first time the concepts of group ('le groupe') and normality of a subgroup ('décomposition propre'). He discussed how a given equation can be solved by investigating the structure of its associated group. He used 'le groupe' to refer to a set of arrangements of the roots of the equation rather than a set of permutations of these arrangements, to which it later came to refer. He did, however, understand that it is the permutations which have the 'group structure.' Thus he was able to write [1832, 409/26/175]:

> ... quand le groupe d'une équation est susceptible d'une décomposition propre, en sorte qu'il se partage en M groupes de N permutations, on pourra résoudre l'équation donnée au moyen de deux équations: l'une aura un groupe de M permutations, l'autre un de N permutations. [1]

In modern terms, this remark states that if the group G of an equation has a normal subgroup H, the equation can be solved by means of two equations whose groups we know as G/H and H. Since Galois had no concept of quotient group, the group that would now be called G/H was to him the group associated with his 'auxiliary equation.' This remarkable insight lies at the heart of Galois theory; the passage quoted above is a particularly clear formulation of Galois' Proposition III in the Premier Mémoire [1846, 425/41/57]. Galois' thoughts were centered on finding a method for deciding on the solvability of an equation by radicals, so he did not investigate in what way the group of such an auxiliary equation arises from the group of the original equation.

3. EARLY UNDERSTANDING OF THE CONCEPT: BETTI

The first extensive commentary on Galois' writings to be published was attempted by Betti [1852]. The two sections of this paper are devoted to the theory of substitutions and of substitution groups and to Galois' theory of equations. Betti was aware at this time of the advances in substitution group theory as set out in Serret's textbook [1849] but did not mention Cauchy's work of 1845 and 1846 on this subject.

Thus Betti was seeking to explain Galois' work from the viewpoint of substitution group theory. Central to his approach was an investigation of the way in which the group of an auxiliary equation is related to the group of the original equation. The first half of his paper was aimed (in part) at the treatment of this question from a purely group-theoretic viewpoint. That is, Betti was searching for a way to explain the fact that a normal subgroup of a group gives rise to another group—which we now understand as a quotient group—and he was seeking to do this within substitution group theory. Although he made considerable progress, he did not seem to understand the problem in full generality. At first his method treats only groups that are split extensions; that is, where any quotient group is isomorphic to a subgroup. Later he realized that even when this is not the case, it is still possible to develop the theory.

The difficulties in understanding Betti's paper are compounded by his own notation and terminology, which are often highly suggestive and open to over-interpretation. Following Galois, Betti used the term *group* for a set of arrangements (and for him they are arrangements of arbitrary 'quantities,' not just of the roots of an equation). His ideas were all expressed using these groups of arrangements. However, he clearly understood the group structure which exists on the permutations of these arrangements. I use the word 'group' for the substitutions and not for the arrangements and explain his ideas in terms of groups of substitutions.

Early in the paper Betti introduced the idea of conjugation of substitutions [1852, 55–58/36–38], which in his terminology is called *derivation*. After a brief explanation of what is meant by a *group of arrangements* and its *substitutions* he explored conjugation of groups. He considered two groups $\hat{G}$ and $\hat{\Gamma}$, both acting on the same collection of 'quantities'; the group $\hat{G}$ is conjugated by the substitutions of $\hat{\Gamma}$. So he restricted the theory to the situation when the 'conjugating' substitutions form a group.

Betti considered two cases: the first occurs when none of the (nonidentity) substitutions of $\hat{\Gamma}$ normalizes $\hat{G}$ and the second when $\hat{\Gamma}$ is contained in the normalizer of $\hat{G}$. No intermediate case was discussed. In both cases Betti constructed the 'product' $\hat{G}\hat{\Gamma} := \{\theta_j\psi_i | \theta_j \in \hat{G}, \psi_i \in \hat{\Gamma}\}$ and considered what would happen if the collection of substitutions in $\hat{G}\hat{\Gamma}$ formed a group $\hat{H}$. In the second case a group is certainly formed and $\hat{G}$ becomes a normal subgroup of $\hat{H}$. In fact $\hat{H}$ is the split extension of $\hat{G}$ by $\hat{\Gamma}$ if $\hat{G} \cap \hat{\Gamma} = \{1\}$. Betti specified here that $\hat{G} \cap \hat{\Gamma} = \{1\}$ and later dealt with the situation when the substitutions of these groups are not distinct. He referred to $\hat{\Gamma}$ as a *multiplier* of $\hat{G}$ and as a *divisor* of $\hat{H}$. His choice of these terms must reflect to some extent the way in which he saw the relationship between $\hat{\Gamma}$ and $\hat{H}$. We know that in this split extension $\hat{\Gamma}$ is isomorphic to the quotient group $\hat{H}/\hat{G}$. But $\hat{\Gamma}$ is also a subgroup of $\hat{H}$. How did Betti conceive the relationship?

Betti developed the theory further to form a group which we know as the image of the permutation representation of $\hat{H}$ on the right cosets of $\hat{G}$. The name Betti gave to it (he defined it as a collection K of arrangements) was the *group of arrangements on the conjugates*. When the substitutions of $\hat{\Gamma}$ normalize $\hat{G}$ (and Betti specified that the substitutions of these two groups are distinct), the image $\hat{K}$ of this permutation representation is (isomorphic to) the quotient group $\hat{H}/\hat{G}$ and thus is isomorphic to $\hat{\Gamma}$. Betti's confusion in developing his theory only for the case when every quotient group of a given group is isomorphic to a subgroup of that group could therefore have been avoided, had he made use of $\hat{K}$ rather than $\hat{\Gamma}$ when investigating the relationship between a group $\hat{H}$ and a normal subgroup $\hat{G}$.

A few pages later on in his paper Betti encountered the concept of quotient group again. He considered [1852, 64–65/43] what would happen if one were to form the product of two groups whose intersection is not trivial. He changed his notation so that the group $\hat{\Gamma}$ became $\hat{\Gamma}_n$. Betti asserted (in modern terms) that if one substitution of $\hat{\Gamma}_n$ is the product of another one by a substitution of $\hat{G}$ ($\gamma_1 = g\gamma_2$ for $g \in \hat{G}$ and $\gamma_1, \gamma_2 \in \hat{\Gamma}$), then γ_1 and γ_2 are in the same coset of $\hat{G}$. Betti wrote that these two substitutions of $\hat{\Gamma}_n$ have to be considered as equal to each other in $\hat{\Gamma}_n$. In effect he was defining an equivalence relation on the substitutions of $\hat{\Gamma}_n$. Thus the order of $\hat{\Gamma}_n$ is given by the number of its substitutions which are in different cosets of $\hat{G}$. In modern notation the order of $\hat{\Gamma}_n$ is given by $[\hat{\Gamma}_n : \hat{G} \cap \hat{\Gamma}_n]$. Nowadays we would think of the equivalence classes defined here as elements of the quotient group $\hat{\Gamma}_n/(\hat{G} \cap \hat{\Gamma}_n)$. In fact this was the approach that Jordan took to introduce the concept of quotient group 20 years after Betti's work. Betti, however, did not seem to think that he was dealing with a new group but rather including an extra condition in the definition of $\hat{\Gamma}_n$.

Betti then explained that (in modern terms) if we choose a set of coset representatives for the cosets of $\hat{G}$ (these cosets are given by multiplying by the substitutions of $\hat{\Gamma}_n$), we may not be able to choose the representatives to form a group. It seems that Betti was acknowledging here that not all extensions split. He realized, however, that this did not affect the theory he had developed.

So Betti seems to have understood some but not all of the ideas behind that of quotient group. He realized that in special cases one can 'divide' one group by another and that the result of this division will be a third group. However, he did not realize that, unlike division in the real numbers where the result also lies in the real numbers, the result of this division is a group which acts in a different way. It cannot be compared with the original two groups (as groups of substitutions acting on a set of arrangements) in a simple manner.

Mammone in his paper [1989] has shown (Sect. 2) how Betti introduced the set K, his *group of arrangements on the conjugates*. Mammone explains how this gives rise to a quotient group. At the end of his paper Mammone considers Betti's formulation of Galois' Proposition III. Betti had realized that under the hypothesis of Proposition III the Galois group of an equation is built up from smaller groups. But Betti again oversimplified and dealt only with the case when the group is a split extension—a 'product' in his terminology. Mammone rightly points out that this nevertheless shows us Betti's concern to establish the relationship between the group of the given equation and that of the auxiliary equation.

In a later paper [1855], Betti began to tidy up and modify the ideas in [1852]. He no longer insisted that the 'conjugating' substitutions form a group but explained this no further. It does not seem that these papers exerted much influence on the subsequent development of Galois theory, and even less on that of substitution group theory. The scope of Betti's work was certainly known to Jordan, who said in the preface to his book, *Traité des substitutions et des équations algébriques* [1870], that Betti was the first to establish rigorously the complete sequence of Galois' theorems. Jordan referred to Betti's paper on the subject as a 'Mémoire important.' However, there is no evidence that Jordan had read or understood Betti's work in any great detail, and it appears that Jordan did not consider the *mémoire* important enough for him to be influenced by it in his approach to the theory.

4. A SYSTEMATIC APPROACH TO THE CONCEPT: JORDAN

So we come to the next mathematician in the story, Camille Jordan, to whom several commentators have attributed the first explicit definition of a quotient group. In particular Gaston Julia, who wrote the preface to Jordan's *Oeuvres* [1961], says on page V:

> Bien que, comme ses contemporains, Jordan ne considère guère que des groupes de permutations . . . , c'est lui cependant qui dégage la notion "abstraite" de groupe quotient. [2]

It is true that Jordan only considered groups as groups of substitutions or transformations, even in work as late as 1917, although he was no doubt familiar with developments in abstract group theory. It is also true that he defined a group which is certainly isomorphic to that given by our 'standard' definition. And in one sense Jordan did 'bring out the "abstract" notion of quotient group,' in that he understood how the 'abstract' concept works within permutation group theory. But his definition, having been formulated in the context of his research into that theory, is not abstract (as we now understand abstraction), although in his day it would have been considered far more so.

Dieudonné produced some notes on Jordan's work in finite group theory which were published at the beginning of the *Oeuvres* [1961]. He discussed the theorem of Jordan's which Hölder later extended in [1889] to become what is now known as the Jordan–Hölder Theorem. The theorem states that for a finite group G, any two composition series for G have the same length and their composition factors, apart from the order in which they occur, are isomorphic. Jordan proved that the *orders* of these composition factors (thought of as the ratios of the orders of successive groups in a composition series) are the same for any two composition series. The proof of this theorem first appeared in Jordan's *Traité* [1870]. Dieudonné wrote [Jordan 1961, XVIII]:

> A ce moment, la notion de groupe quotient n'est pas encore conçue clairement, bien que Jordan utilise couramment, dès ses premiers travaux, le calcul "modulo" un sous-groupe distingué; ce n'est qu'en 1873 [1873a] qu'il définit explicitement la notion de groupe quotient et considère les groupes quotients successifs d'une suite de composition; mais on sait que c'est seulement avec Hölder (en 1889) que le théorème d'invariance prendra sa forme définitive. [3]

It will be seen that this idea of calculating 'modulo' a normal subgroup is indeed the idea which gave rise to Jordan's concept of quotient group. In his approach there is an obvious parallel with Gauss' work on arithmetic congruences in 1801 which Jordan no doubt had in mind. Jordan employed the symbol $\equiv$ which Gauss had introduced to denote congruence. Dieudonné commented on the Jordan–Hölder Theorem that it would assume its definitive form only with Hölder. This could also be said of the concept of quotient group: in both cases Jordan's ideas were early formulations which later gave way to the accepted 'standard' forms.

Jordan worked to develop Galois' ideas in the 1860s and the concept of quotient group thus appeared implicitly in his research as the group of an auxiliary equation. He understood and explained the methods of Galois theory more clearly than earlier mathematicians and developed the theory of substitution groups. However, he made no attempt to produce a quotient group explicitly. It was only later, in 1873, when Jordan was extending some results of Mathieu on the limit of transitivity of groups, by way of Sylow's Theorems, that he used the idea of congruence of group elements to produce a quotient group structure [1873a, 45–47/370–372]. He remarked that the developments and definitions he would give seemed to him to make the proofs of several important propositions considerably simpler. The first definition is:

> Deux substitutions s et t, permutables à un groupe H, sont dites *congrues suivant le groupe* H, si l'on a une égalité de la forme
>
> $$s = th,$$
>
> h étant une substitution de H.
>
> On peut exprimer cette relation par une formule analogue à celle des congruences ordinaires
>
> $$s \equiv t \quad \bmod H. \text{ [4]}$$

The phrase 'permutables à un groupe H' means that $sH = Hs$ and $tH = Ht$, that is, s and t are in the normalizer of H. The 'congruences ordinaires' are congruences in the integers, as explained by Gauss.

Jordan then showed that congruences can be multiplied together member by member; that is, if $s \equiv t \pmod{H}$ and $s' \equiv t' \pmod{H}$, then $ss' \equiv tt' \pmod{H}$. This proves that multiplication in his quotient group structure is well-defined. Next he gave his definition of this new structure:

> On dira qu'une suite de substitutions $s_1, s_2, \ldots$ (toutes permutables à un même groupe H) forme un *groupe suivant le module* H, si l'on a pour toutes valeurs de α et β une relation de la forme
>
> $$s_\alpha s_\beta \equiv s_\gamma \quad \bmod H. \text{ [5]}$$

So the structure is defined by imposing the condition of closure on its elements. The order of the group is defined to be the number of different substitutions in it which are not congruent mod H. If G is the group generated by $s_1, s_2, \ldots$ in the normal way, then he denotes by $\frac{G}{H}$ the group that they form mod H. Since H is not necessarily contained in G, the order of G is given by $|G| = \left|\frac{G}{H}\right| |G \cap H|$, using modern notation.

Thus Jordan's quotient group structure $\frac{G}{H}$ consists of congruence classes of the elements $s_1, s_2, \ldots$ from which G is formed. His notation was organized so that, in effect, the quotient group consists of one representative from each conjugacy class, although these representatives do not necessarily form a group themselves.

In this way he was able to remain within the limits of substitution group theory while embracing a concept which cannot easily be adequately described in this theory. He did exactly what Betti was unable to grasp that he should do: Jordan took a *set* of substitutions in the normalizer of H and combined them to form a group of a different type, a group formed 'with respect to' H. He went on to say that all the principal definitions relating to 'ordinary' groups (meaning groups of substitutions) can be carried over to groups 'formed with respect to the modulo H': commutativity, membership of a normalizer, conjugation of elements, isomorphism, and homomorphism.

Jordan made use of these new quotient groups in another paper of 1873, "Mémoire sur les groupes primitifs" [1873b]. In this long paper, he linked the degree of certain primitive substitution groups with properties of a particular substitution in each group. It seems unlikely that the references to his quotient groups, which are embedded deep in this paper, would have had much influence on Jordan's contemporaries. This paper appears to be the last in which Jordan used the new concept. Nevertheless, the first of these two papers marks significant progress in its development. The paper was cited several times by Frobenius in his later work on quotient groups (see [1887a, 180/302; 1887b, 273/304; 1895, 86/637]) and also by Burkhardt in an encyclopedia article entitled "Endliche Discrete Gruppen" [1898, 219].

5. ABSTRACTION AND THE CONCEPT OF EQUIVALENCE

After these papers of Jordan's the development becomes less easy to trace. The concept of quotient group began to be approached from new and different angles. This process happened as more abstract ideas were introduced into the study of groups. The old theories were gradually being reformulated and extended by means of these new ideas. As these great changes were taking place, the usefulness of the concept of equivalence was more and more clearly recognized. This concept was certainly known to Dedekind, Kronecker, Frobenius, Cantor, and Hölder, and also to Frege, whose approach was somewhat more philosophical. Here I intend only to give a few indications of the way in which equivalence appeared in their work.

Frege's book *Die Grundlagen der Arithmetik* [1884] includes a detailed discussion of the equality of numbers (Sects. 62–69), which he defined by means of one–one correspondence. The definition depends on the fact that one–one correspondence is an equivalence relation and, although Frege did not give an explicit statement of the concept, he certainly recognized its importance.

Dedekind came upon the concept of equivalence as a result of his investigations into the foundations of analysis and therefore into the real number system. In the introduction to *Stetigkeit und irrationale Zahlen* [1872] (in which he explained the 'Dedekind cut' [6]), he remarked that his attention was first directed towards these matters in the autumn of 1858. Not until 1888 did he publish the results of his study into the nature and meaning of numbers (*Was sind und was sollen die Zahlen?*). In Sections II and III of this essay, he developed the theory of functions and the idea of two systems of elements being *similar* (when there is a bijection between them). In this way, he stated, we can separate all systems into *classes*

> ... by putting into a determinate class all systems Q, R, S, ..., and only those, that are similar to a determinate system R, the *representative* of the class; ... the class is not changed by taking as representative any other system belonging to it. [1888, 55]

So he showed that similarity is an equivalence relation and formed its equivalence classes.

The ideas expressed in Dedekind's essay do not seem to have been universally understood, not because they were unclear but because, in the case of equivalence, they were unfamiliar. J. van Heijenoort includes in his book [1967] a description of the correspondence which ensued between Dedekind and an *Oberlehrer* in Hamburg named Hans Keferstein as a result of this essay. Keferstein had published a paper in 1890 in which he made some suggestions for amending Dedekind's essay. One of his suggestions came about because he confused the equivalence relation between two sets with their identity.

Cantor also discovered the concept of equivalence because of his work on transfinite numbers begun in 1870 and finally summed up in two memoirs [1895 & 1897]. He made use of one–one correspondence to define 'equivalence' of sets and by means of this concept to define cardinal numbers [7].

It may be that several of these mathematicians came across the idea of equivalence independently. However, there was much communication between them and they frequently refer to one another's work in their writings, and therefore the development of these techniques cannot be attributed to any one person. Nevertheless the result of their investigations was to produce a mathematical climate in which the concepts of equivalence and equivalence classes and the possibility of definition by means of these concepts could be freely employed. Thus, as will be seen, this development in the understanding of the role of equivalence is reflected in the development of the concept of quotient group [8].

6. PIONEER OF THE CONCEPT: DEDEKIND

Dedekind appears to have understood the role of equivalence at a much earlier period, in particular in his work during the years 1855–1858. He explored the theory of groups but his writings remained unpublished until after his death in 1916. They were discovered in his *Nachlass* and included in his *Gesammelte Mathematische Werke* (as [1932]). During his lifetime he had done no more than mention this work to Frobenius in a letter of 1895. Dedekind explored the concept of homomorphism in a section entitled "Äquivalenz von Gruppen." He formed a homomorphic image M_1 of a group M by letting each element θ of M 'correspond' to an element θ_1 of M_1, with certain conditions which we now recognize as the conditions for homomorphism. He proved that M_1 is a group and that those elements of M which 'correspond' to the identity in M_1 form a subgroup N of M. He went on to discover the concept of quotient group:

He expressed M in terms of N and its cosets and stated that a 'composition' of cosets can be defined and that in this way the cosets (he referred to them simply as *Komplexe,* that is, 'sets') form a group. There is a correspondence between the cosets and the elements of M_1 such that to each coset corresponds one element of M_1, and to each element of M_1 corresponds one coset. (We would now say that the group M_1 and the group of cosets are isomorphic.) Dedekind gave no name either to the concept of homomorphism or to that of quotient group.

It is also known that during the years 1856–1858 Dedekind lectured on algebra at Göttingen. A manuscript published recently for the first time in a book commemorating the 150th anniversary of his birth [Scharlau 1981, 59–100] contains what is almost certainly the text of these lectures or his writings immediately following them. (See W. Purkert's interesting article [1976] for the evidence supporting this.) Here Dedekind treated the concept of quotient group in much the same way as above but without the parallel development of the concept of homomorphism. The original lectures attracted a total audience of four and appear to have had little influence. In his explanation of this manuscript (see [1981, 107–108]), Scharlau records that one of those who attended the lectures, P. Bachmann, later told Dedekind that he had understood very little of them at the time!

Emmy Noether commented in her note on the 1855–1858 manuscript that we can see how completely Dedekind was in possession of concepts and methods of abstract group theory, even as early as the 1850s. We might also apply her well-known motto "Es steht alles schon bei Dedekind" ("It is already all there in Dedekind").

7. THE INFLUENCES OF ABSTRACTION: DYCK AND FROBENIUS

We return now to the later period, to the year 1882, when a paper by Dyck, "Gruppentheoretische Studien," appeared in *Mathematische Annalen*. This paper has been discussed by Chandler and Magnus [1982]. It begins with the abstract construction of the free group, G say, on elements $A_1, A_2, \ldots, A_m$. In Section 4 Dyck was concerned with exploring the way in which any group $\overline{G}$, generated by elements $\overline{A}_1, \ldots, \overline{A}_m$ (which are defined by 'some predetermined process'), is related to the original free group G. Dyck's notation here already suggests a hidden assumption that $\overline{G}$ is a homomorphic image of G under the homomorphism taking $A_i \rightarrow \overline{A}_i$. Indeed, a few sentences later Dyck proceeded to show that this is indeed so and that there are two cases to consider.

The first case is when the groups are isomorphic; the second occurs when any one element of $\overline{G}$ corresponds to infinitely many elements of G. In the second case, he found the elements of G which correspond to the identity in $\overline{G}$ and showed that they form a normal subgroup H of G. He investigated the structure of this normal subgroup and then stated that the relationship between G and $\overline{G}$ can now be set out [1882, 14]:

> Die isomorphe Zuordnung der Gruppen G und $\overline{G}$ spaltet die Gruppe G in zwei Factoren: In die Gruppe der Substitutionen, welche in den Substitutionen $\overline{A}_i$ geschrieben *verschieden* sind, d. h. die Gruppe $\overline{G}$ selbst—und in die Gruppe H derjenigen Substitutionen, welche in den Substitutionen von $\overline{G}$ geschrieben *der Identität äquivalent* sind. Die letztere Gruppe H ist dabei in G ausgezeichnet enthalten und folgt aus ihr "durch Adjunction von $\overline{G}$." [9]

The phrase 'durch Adjunction von $\overline{G}$' seems to have been borrowed from Galois theory, since one reduces the Galois group of an equation to a normal subgroup by *adjoining* elements to the field, and these elements are the roots of an equation whose Galois group is $\overline{G}$. So the group G can be thought of as having two factors: its normal subgroup H and the homomorphic image $\overline{G}$ which now, of course, we also know as the quotient group G/H.

Later in 1882 Dyck wrote a further paper on group theory, "Gruppentheoretische Studien II" [1883], based in part on some lectures he had recently given. As in [1882], he stressed that it is the 'abstract' properties of any group that are important. In this paper he investigated the concepts of primitivity and transitivity. He found it convenient [1883, 84–86] to consider the regular permutation representation of a group G of order N, and he investigated the relationship between subgroups and imprimitivity: that if the group has a subgroup then the regular representation is imprimitive and vice versa. (Kiernan [1971, 133] appears to have misunderstood this section of Dyck's paper—Dyck was not dealing here with quotient groups.)

Next [1883, 90] he took a subgroup G' of G, where G' has order μ and $N = \mu\nu$. He denoted G' and its right cosets by $\mathrm{T}_1, \mathrm{T}_2, \ldots \mathrm{T}_\nu$, respectively. Then he considered how these 'systems' T_k are permuted amongst themselves by the elements of G. So, as Betti did, he formed the representation of G on the right cosets of G'. He stated that this representation is isomorphic to the regular representation of G as long as G' is not normal in G and does not contain any normal subgroup of G. He finally investigated [1883, 96] the case when G' is normal in G and worked out how the table giving the regular representation of G splits into blocks which are permuted as a whole by the elements of G. But this is all he did: there is no explanation of the way in which G is homomorphic to this representation on the cosets of G'.

The fact that Dyck stopped here is rather puzzling, in view of the explanation of homomorphism he had given at the beginning of [1882]. He approached the concept of quotient group in two completely different ways but was not able to connect the two. Had he done so, he might well have realized that one can dispense with the homomorphism and the representation on the cosets and look within the group itself for the concept of quotient group. After all, he said in the introduction to [1883] that he aimed to take an abstract approach in order to discover the extent to which the properties of groups remain invariant, in all their different modes of representation.

This was not the first time Dyck's work touched on quotient groups. Two years previously, in 1880, they appeared implicitly in a paper on Riemann surfaces [1880, 480ff]. He made use of composition series (p. 486) and spoke of the simple groups out of which a composition series is made—that is, the composition factors.

In the late 1870s and the 1880s Frobenius was also led to consider the idea of equivalence of group elements. His joint paper with Stickelberger on abelian groups [1879] contains an early formulation of this idea. In this paper the fundamental theorem of abelian groups was proved using what are in effect equivalence classes with respect to subgroups of a given abelian group (and since it is abelian all subgroups are normal). This viewpoint was borrowed from an article of Kronecker's of 1870. Here appeared the first proof of the fundamental theorem of abelian groups (without, however, a proof of the uniqueness of the decomposition). The article makes no mention of the term group: Kronecker was nevertheless investigating the structure of what is now known as the ideal class group of an algebraic number field (a group that is finite and abelian). In his proof of the fundamental theorem he talked only of a set of 'elements' and a well-defined binary operation on the set, and gave other conditions which are exactly those for an abelian group. He found it necessary to extend the concept of equality of elements (which he called 'equivalence' and denoted by $\sim$) so that [1870, 884/277]

> . . . zwei Elemente θ' und θ'' als "relativ äquivalent" angesehen werden, wenn für irgend eine ganze Zahl k:
>
> $$\theta' \cdot \theta_1^k \sim \theta''$$
>
> ist. [10]

He went on to say that if a complete set of relatively nonequivalent elements are chosen they will satisfy the same conditions as the original set. Thus in effect this new set is formed of coset representatives for the cosets of the subgroup generated by θ_1 and is isomorphic to the quotient group produced by this subgroup. The proof of the fundamental theorem of abelian groups given by Kronecker was repeated almost verbatim, but explicitly in terms of abelian groups, by Netto in his book on the theory of substitutions [1882/1892, 144ff]. For more information on Kronecker's article refer to Wussing [1969, 44–48/1984, 63–67].

Frobenius later developed a new proof of Sylow's Theorems [1887a], which was published in 1887 but is dated 1884. In this short paper Frobenius followed Jordan's approach to the concept of quotient group but in the setting of abstract group theory. Frobenius proved that if the order of a group is divisible by p^ν where p is prime, then the group has a subgroup of order p^ν.

Let $\mathfrak{H}$ be a group of order h, where p^ν (or a higher power of p) divides h. Assume that the theorem is true for groups of order less than h. Form the center of $\mathfrak{H}$—it is a subgroup $\mathfrak{G}$ of $\mathfrak{H}$ whose order g will therefore divide h. We distinguish two cases: either p divides g or it does not.

The first case gives rise to a quotient group, as follows:

Frobenius first proved Cauchy's Theorem for abelian groups: that $\mathfrak{G}$ contains a nonidentity element of order p, which he denoted by P. Now consider two elements of $\mathfrak{H}$ as (relatively) equal if they differ only by a power of P. Here Frobenius cited Kronecker's paper [1870, 884/277] and Jordan's paper [1873a, 46/371]. (The facts that Frobenius needed to cite the papers containing this definition, and that they had both been published over a decade earlier, suggest that the concept had not been influential or widely recognized.) Then the conditions for a group are fulfilled for this broader idea of equality, since each power of P commutes with each element of $\mathfrak{H}$. The 'relatively different' elements of $\mathfrak{H}$ form a group, whose order is h/p. . . .

The proof of the theorem for this case follows in a few lines from Frobenius's inductive hypothesis. In modern terms, Frobenius was forming the quotient of $\mathfrak{H}$ by the cyclic normal subgroup generated by P.

Sylow's own proof of this theorem in [1872] uses an idea from Galois theory to establish what would now be deduced using the concept of quotient group. He took a function y_0 of the letters on which the group G acts and required that y_0 remain invariant only under the substitutions of a subgroup G of order n^α, where n is prime and G has no subgroup of order n^β for $\beta > \alpha$. He considered how the distinct values of this function (corresponding to cosets of the subgroup) are permuted by the substitutions in the normalizer γ of g. We thus obtain a substitution group γ' that is transitive and homomorphic to γ. We would now write

$$\gamma' \cong \frac{\gamma}{g}.$$

This method of Sylow's is equivalent to forming the permutation representation on the cosets of g in its normalizer γ. The articles by Waterhouse [1980], Scharlau [1988], and Casadio and Zappa [1990] provide detailed studies of the discovery of Sylow's theorems and the developments in their proofs.

One further paper should be mentioned here: that written by Capelli not long after Sylow's theorems were published [1878]. Capelli specifically aimed to show the importance of isomorphism (a term which then covered both isomorphism and homomorphism) in the theory of substitution groups. In the course of the paper he proved most of Sylow's theorems (but had obviously not read Sylow's article), investigated properties of groups of prime power order, and gave a new proof of the Jordan–Hölder Theorem. His exploration of the concept of isomorphism brought him to define a permutation representation in the same way as Sylow did using a function fixed by the substitutions of a normal subgroup.

Returning to the work of Frobenius, another paper dealing with congruence of group elements [1887b] was published soon after that on Sylow's theorem. In this paper Frobenius used the ideas of equivalence relations and equivalence classes to define double cosets, again citing Kronecker and Jordan for the concept of equivalence of group elements. He investigated the properties which such cosets possess, including the fact that the number of equivalence classes does not change when a common normal subgroup $\mathfrak{N}$ is factored out. His use of the concept of quotient group here follows Jordan's definition and notation of 1873, except that it is stated that each set of elements congruent mod $\mathfrak{N}$ is to be considered as *one* element. Then these 'complexen Elemente' form a group—the quotient by the normal subgroup $\mathfrak{N}$.

The fact that Frobenius was able to think of each congruence class as *one* element was an important step forward—his understanding of the abstract approach permitted it. Later, in a paper on finite groups [1895, 86/637], Frobenius cited both Jordan and Hölder when making use of quotient groups and attributed the definition to them equally. Frobenius seems to have been feeling his way between the two: his idea obviously followed on from Jordan's and he later recognized in Hölder's paper a satisfactory abstract definition of quotient group. He does not seem to have considered himself responsible for anything but the *development* of the concept.

8. THE 'STANDARD' DEFINITION: HÖLDER

The final stage, then, in this development comes with Hölder's paper "Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen" ("Reduction of an arbitrary algebraic equation to a chain of equations") [1889]. The questions which Hölder wished to answer here are those prompted by taking a fresh look at Galois theory in the light of abstract group theory. Which groups correspond to the 'auxiliary equations'? To what extent are these groups defined? How many are there? The natural way to answer these questions is to employ the concept of quotient group. In the introduction Hölder discussed the simple groups arising from a composition series, which he named 'Factorgruppen,' and noted that this concept of 'Factorgruppe' is "ein bis jetzt nicht hinreichend gewürdigter gruppentheoretischer Begriff" ("a group-theoretic idea that has until now not been adequately appreciated"). He stated that he would set out only the most elementary group-theoretic ideas in the discussion that followed. It seems that Hölder did not consider the concept of quotient group to be either a new or a difficult one.

The first part of the paper is a group-theoretic section. Hölder gave axioms for a finite group and mentioned normal subgroups and composition series. He then talked about 'Factoren der Composition': a 'Factor der Composition' is the index of a group in a composition series in the preceding group of the series, as defined by Jordan. In modern terms these are the orders of the composition factors. Hölder wrote [1889, 30]:

> Diese Theorie von den Factoren der Zusammensetzung [he reverted to the German word] muss aber dahin vertieft werden, dass die Factoren als *Gruppen* aufgefasst werden.
>
> Es wird im nächsten Paragraphen gezeigt werden, dass durch das Verhältniss einer Gruppe zu einer in ihr ausgezeichnet enthaltenen Untergruppe stets eine neue Gruppe von im Allgemeinen anderen Operationen definiert ist. Diese letztere Gruppe ist völlig bestimmt von dem abstracten Standpunkt aus, welcher von dem Inhalt der Operationen absieht . . . [11]

So he came to define the 'Quotient' of a group by a normal subgroup. He showed how the elements of a group G can be divided amongst the cosets of any subgroup H and proved that if H is normal, the multiplication of any two elements from two cosets will always give an element in one and the same coset. In other words, in this case we can define multiplication of cosets and it is well-defined. He continued [1889, 31]:

> Man erhält so neue Operationen, welche gleichfalls eine Gruppe bilden. Diese vollständig bestimmte Gruppe ist es, welche in die Betrachtung eingeführt werden soll. Man könnte sie den *Quotienten* der Gruppen G und H nennen, dieselbe soll im Folgenden mit
>
> $$G|H$$
>
> bezeichnet werden. [12]

The next paragraphs show that this concept can be expressed in terms of equivalence of group elements. We call two elements of G *equivalent* if they can be transformed into one another by multiplication with an element of the normal subgroup H. Then the equivalence classes will form a group. Hölder's explanation of this is very reminiscent of Frobenius's approach in [1887b]: he used the same terms 'aequivalent' and 'Classe.' The definition he gave is therefore that of Kronecker in [1870] or Jordan in [1873a], as cited by Frobenius, but formulated here from an abstract viewpoint. Hölder then showed that there is a homomorphism from G to this quotient group and mentioned Dyck's comments in [1882, 14] on splitting a group into two factors: he saw that one of these can be thought of as a quotient group rather than a homomorphic image. He realized that one obtains the same result whether one sets out with a normal subgroup or a homomorphism. He went on to investigate direct products of groups and led up to a proof of the Jordan–Hölder Theorem.

In the second section of the paper Hölder returned to Galois' theory of algebraic equations, from where we began, and showed how the 'Factorgruppen' are the groups of particular auxiliary equations. Referring to the group of any auxiliary equation and the corresponding quotient group of the Galois group, he remarked [1889, 43] "Vom abstracten Standpunkt aus, sind diese Gruppen also als identisch zu betrachten" ("From an abstract standpoint these groups are therefore to be considered identical").

Thus in this paper the concept of quotient group was systematically and explicitly defined and its previously implicit appearance in Galois theory was recognized. Hölder introduced the terms 'Quotient' and 'Factorgruppe' and the notation $G|H$: the terms have remained but the notation has been combined with Jordan's to produce our modern G/H.

9. WIDER RECOGNITION OF THE CONCEPT: THE 1890s

It was in the 1890s, after this paper of Hölder's, that the notion of quotient group began to be incorporated into monographs and textbooks. Netto's book on the theory of substitutions and its applications, first published in 1882, was later revised by Netto and translated into English by F. N. Cole. This English edition "differs from the German edition in many important particulars," as Netto remarked in an addition to the Preface. He added that he had taken into account "the whole material which has accumulated in the course of time since the first appearance of the book." In particular the English edition includes the concept of quotient group [1892, 95–96] which is defined as a permutation representation on the cosets of a normal subgroup, as one would expect in a book devoted to substitutions. (The German edition does not mention quotient groups—Kiernan is mistaken here [1971, 133].) Netto denoted his quotient group by $G:H$. He did not give reference to any previous papers when stating his definition but later cited Hölder's 1889 paper when defining the 'factor groups' of a group G (that is, the composition factors).

The second volume of Weber's *Lehrbuch der Algebra* [1896] begins with an abstract definition of a group and, after sections on subgroups and normal subgroups, there is a discussion of the properties of subgroups and cosets and the concept of quotient group is presented. It is defined as the group of cosets of a normal subgroup and Weber carefully proved that these cosets do indeed form a group. He went on to prove the Jordan–Hölder Theorem in the same way that Hölder had proved it in 1889, using quotient groups.

In 1897 an English monograph on group theory appeared: Burnside's *Theory of Groups of Finite Order*. Burnside approached the concept of quotient group of a group G by way of 'multiple isomorphism,' that is, homomorphism, and in this he took after Dyck [1882, 11–14]. He considered [1897, 35ff] a group G' to which a homomorphism from G can be defined: the operations of G which correspond to the identity in G' form a normal subgroup Γ of G. Having investigated, as Dyck did, the ways in which G' and Γ are related to and determined by one another, Burnside explained that the relation of 'multiple isomorphism' can be presented in a rather different manner. Following Hölder, he explained how the cosets of a normal subgroup form a group. He then drew these two methods of approach together: the group G' is 'simply isomorphic' (i.e., isomorphic) with this group of cosets. He remarked that the group G' is completely defined when G and Γ are given. He continued [1897, 38]:

> This being so it is natural to use a symbol to denote directly the group thus defined in terms of G and Γ. Herr Hölder [1889] has introduced the symbol
>
> $$\frac{G}{\Gamma}$$
>
> to represent this group; he calls it the *quotient* of G by Γ, and a *factor-group* of G.

Hölder did in fact reserve the name 'Factorgruppen' for the composition factors: perhaps Burnside's misreading of this is the reason why both the name 'quotient group' and the name 'factor group' are now in common usage. J. J. Rotman preserves Hölder's original distinction in his textbook [1984, 21 and 75] but sadly he appears to be in the minority in doing so.

It is interesting to note which papers and mathematicians are referred to by those who made use of quotient groups in the 1890s. In [1887a] and [1887b] Frobenius had cited Jordan but in 1895, after Hölder's definition had appeared, he attributed the definition of quotient group to both Jordan and Hölder. Burkhardt gave only two sentences to the definition of quotient group in the 'Normalteiler' (normal subgroup) section of his encyclopedia article [1898]. He defined it in an unusual way but his choice of approach becomes understandable when we look at his sources: he attributed the discovery of quotient groups to Betti [1852], with the reservation "wo aber alles nicht klar ist" ("where however everything is not clear"). Burkhardt's explanation *is* clear:

Let G be a substitution group of order mn.

The objects it permutes are arranged in mn ways by the action of G.

A normal subgroup H of G, of index m and order n, will split these arrangements into m systems each containing n arrangements.

The operations of H permute the arrangements in each individual system; the remaining operations of G permute the systems as a whole.

The different permutations of the systems form a group of order m.

Burkhardt added that this group, to which he gave no name, is denoted by G/H, and here he referred to Jordan and Hölder. It is remarkable that an encyclopedia article written at the turn of the century should have put such emphasis on substitution groups and pass up a more abstract definition of quotient group at the very time when abstraction was flourishing.

In 1893 Cayley wrote a "Note on the so-called quotient G/H in the theory of groups," in which he cited Hölder's paper of 1889 and referred to a paper of Young's [1893], who, in his turn, attributed the definition of quotient group and the term 'quotient' to Hölder. In a paper of his own [1893, 317], Hölder cited Jordan, saying that quotient groups had already appeared in Jordan's work and that he (Hölder) drew up their theory anew in 1889.

10. CONCLUSIONS

Thus it is clear that we cannot attribute the development of the concept of quotient group to any one person or any one time. Like almost all mathematical ideas the period from its first occurrence in primitive form to its full understanding and acceptance as commonplace is a long one and includes the contributions of many mathematicians. The opinions of modern commentators on the matter seem to favor their cultural and mathematical backgrounds: in Bourbaki it is stated that Jordan introduced the notion [1960, 76], while van der Waerden says that the modern understanding of quotient group is due to Hölder and that Jordan had it implicitly [1985, 121]. Wussing [1969, 183/1984, 244] takes the view that the idea of quotient group was a deduction from the abstract group concept. He then notes that Hölder put an abstract definition of a group at the beginning of his 1889 paper, which suggests that he considers Hölder to have introduced the concept of quotient group.

Returning finally to this link with abstract group theory it does seem that, without the abstract setting for his work, Hölder could not have defined his quotient group as he did. As mentioned before, a recognizably modern definition of a quotient group depends on such an abstract setting. By 1880 the abstraction process was well advanced and yet at this time only Jordan's idea of quotient group was available. One last thread had still to be woven in to complete the picture—that of using equivalence to define new types of objects. Thus, once this technique had received acceptance, it was not at all unnatural for Hölder to define quotient groups as he did in 1889. Sixteen years earlier, in 1873, such ideas would have been far removed from Jordan's understanding. Although it is extremely unlikely that Jordan's substitution-theoretic concept of quotient group contributed actively to the abstraction process for group theory, the need to extend and deepen this concept could certainly have provided one of the forces moving group theory towards abstraction. The evolution of abstract group theory and the development of the concept of quotient group are interdependent, and thus it is that this theory and this concept emerged during the same period and in the work of the same mathematicians.

NOTES

1. . . . when the group of an equation admits a proper decomposition, in such a way that it divides into M groups of N permutations, one will be able to solve the given equation by means of two equations: one of them will have a group of M permutations, the other a group of N permutations.

Galois and others throughout the 19th century often used the word *permutation* for our modern term arrangement and *substitution* for our modern term permutation. From now on I use the terms substitution and arrangement when not quoting from an original work.

2. Although, like his contemporaries, Jordan hardly considers anything but groups of permutations . . . , it is, however, he who brings out the 'abstract' notion of quotient group.

3. At this time there is still no clear conception of the notion of quotient group, although Jordan freely uses, from his earliest works, calculation 'modulo' a normal subgroup; it is only in 1873 [1873a] that he defines explicitly the notion of quotient group and considers the successive quotient groups of a composition series; but we know that the invariance theorem will assume its definitive form only with Hölder (in 1889).

4. Two substitutions s and t which commute with a group H are said to be *congruent with respect to the group H* if one has an identity of the form

$$s = th,$$

h being a substitution in H.

One can express this relation by a formula analogous to that of ordinary congruences

$$s \equiv t \qquad \text{mod } H.$$

5. We will say that a series of substitutions $s_1, s_2, \ldots$ (which all commute with the same group H) form a *group with respect to the modulo H* if for all values of α and β we have a relation of the form

$$s_\alpha s_\beta \equiv s_\gamma \qquad \text{mod } H.$$

6. Dedekind's definition of a 'cut' has to do with a quite different sort of equivalence from that under examination here. He did not define a real number as a pair of sets or cut but rather stated that to each cut there *corresponds* such a number.

7. As with Dedekind and his 'cuts,' this is a different concept from that used by Cantor in his definition of real numbers by Cauchy sequences. There, a symbol is associated with a Cauchy sequence and this symbol is the 'real number.'

8. It is interesting to note the way in which attitudes have changed since the end of the last century. A footnote in Birkhoff and MacLane's textbook on algebra [1953, 153] provides a good example of modern understanding of the concept of equivalence:

> The procedure of treating sets as elements is not sophisticated; thus, one commonly speaks of a "regiment," meaning a certain set of men, or of a "molecule," meaning a certain set of atoms.

9. The relationship of isomorphism between the groups G and $\overline{G}$ splits the group G into two factors: into the group of substitutions which are *different* when written in terms of the substitutions $\overline{A}_i$, that is, the group $\overline{G}$ itself—and into the group H of those substitutions which, when written in terms of the substitutions of $\overline{G}$, are *equivalent to the identity*. The latter group H is then contained as a normal subgroup in G and comes from it "by adjunction of $\overline{G}$."

10. . . . two elements θ' and θ'' will be considered as "relatively equivalent" if

$$\theta' \cdot \theta_1^k \sim \theta''$$

for some integer k.

11. This theory of the composition factors must however be deepened, in the sense that the factors are to be understood as *groups*.

It will be shown in the next paragraphs that through the relationship of a group to a normal subgroup contained in it, a new group of (usually) other operations is always defined. This latter group is completely determined from the abstract standpoint, which disregards the nature of the operations . . .

12. In this way one obtains new operations, which likewise form a group. It is this well-defined group that is to be brought into consideration. One could call it the *quotient* of the groups G and H and it will from now on be denoted by

$$G|H.$$

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