Intersection Operation on Union-Preserving Mappings in Completely Distributive Lattices

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1. INTRODUCTION

In his classic paper [1], Zadeh introduced the notion of a fuzzy set. Subsequently, Goguen [2] extended this to the more general notion of an L-fuzzy set. Thereafter the completely distributive lattice [3] became a suitable framework to expound the theory of the L-fuzzy set. Meanwhile, much research has been carried out in the area of fuzzy topology (cf. [6-9]). Some recent articles [4, 5] have considered the uniformities and metrizations on fuzzy sets and obtained rather profound results. In these articles, the importance of the investigation of the intersection operation on union-preserving mappings in completely distributive lattices became apparent. Especially, Hutton’s formula on the intersection operation [4; Lemma 3] is useful. In this paper, by counterexample, we shall show that this formula does not hold for all completely distributive lattices, except the lattice \( \{0, 1\} \). Moreover, under an additional assumption, a proof of this formula is given. We shall also show some properties about the intersection operation which will be needed in our latter work on fuzzy uniformity and fuzzy metrization.

2. PRELIMINARIES

Throughout this paper \((L, \geq, V, \wedge)\) will be a completely distributive lattice which has the least and greatest elements, say 0 and 1, respectively; \(I\) will be an index set (may be infinite).

**Definition 1.** Mapping \(f: L \rightarrow L\) will be called order-preserving iff \(a \geq \beta\) implies \(f(a) \geq f(\beta)\) for \(a, \beta \in L\). Mapping \(f: L \rightarrow L\) will be called union-preserving iff \(f(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} f(a_i)\) for \(a_i \in L\) \((i \in I)\).

Clearly, union-preserving mappings are order-preserving.
DEFINITION 2. Let $\alpha \in L$. A set $B \subseteq L$ is called the minimal set relative to $\alpha$ iff (1) $\sup B = \alpha$, and (2) for every set $A \subseteq L$ satisfying $\sup A = \alpha$ and for every $\beta \in B$, there exists $\gamma \in A$ such that $\beta \leq \gamma$.

LEMMA 1 [4]. For every $\alpha \in L$ there exists a minimal set relative to $\alpha$.

LEMMA 2. Let $f : L \rightarrow L$ be an order-preserving mapping. Suppose $f^* : L \rightarrow L$ is defined by

$$f^*(\alpha) = \bigwedge_{\sup \Gamma = \alpha} \left[ \bigvee_{\gamma \in \Gamma} f(\gamma) \right], \quad \alpha \in L,$$

Then

1. $f^*(\alpha) = \bigvee_{\gamma \in B} f(\gamma)$, where $B$ is minimal set relative to $\alpha$.
2. $f^* : L \rightarrow L$ is a union-preserving mapping.
3. $f^* \leq f$; i.e., for every $\alpha \in L$, $f^*(\alpha) \leq f(\alpha)$.
4. $f^*$ is the greatest $g : L \rightarrow L$ which takes values less than or equal to $f$ and is union-preserving.

Proof. (1) Suppose $\sup \Gamma = \alpha$. By Lemma 1,

$$\bigvee_{\beta \in B} f(\beta) \leq \bigvee_{\gamma \in \Gamma} f(\gamma).$$

Hence

$$f^*(\alpha) = \bigvee_{\beta \in B} f(\beta).$$

(2) Suppose $\gamma \leq \alpha$ ($\alpha, \gamma \in L$). For the minimal set $B$ relative to $\beta$, we have

$$f^*(\alpha) = \bigvee_{\beta \in B} f(\beta) \geq \bigvee_{\beta \in B} f(\gamma \land \beta) \geq f^*(\gamma).$$

Hence for $\alpha = \bigvee_{i \in I} \alpha_i (\alpha_i \in L)$, the inequality

$$f^*(\alpha) \geq \bigvee_{i \in I} f^*(\alpha_i)$$

holds.
The reverse inequality is also true. In fact, let $B_i$ be the minimal set relative to $a_i$ for every $i \in I$. By

$$\bigvee_{i \in I} \bigvee_{\beta_i \in B_i} \beta_i = \bigvee_{i \in I} a_i = a,$$

$$\bigvee_{i \in I} f^*(a_i) = \bigvee_{i \in I} \bigvee_{\beta_i \in B_i} f(\beta_i) \geq f^*(a).$$

Conditions (3) and (4) are trivially satisfied.

**Remark.** The proof of this lemma is based on one given by Hutton [4], where more conditions are assumed to obtain a suitable form for that paper.

**DEFINITION 3.** Let $f_1, f_2 : L \to L$ be union-preserving mappings. Define $f_1 \cap f_2 : L \to L$ by

$$(f_1 \cap f_2)(a) = f_1(a) \lor f_2(a), \quad a \in L,$$

and define

$$f_1 \land f_2 : L \to L$$

by

$$f_1 \land f_2 = (f_1 \cap f_2)^\ast.$$  

$f_1 \land f_2$ is called the intersection of $f_1$ and $f_2$.

Since $f_1 \cap f_2$ is obviously order-preserving, $(f_1 \cap f_2)^\ast$ is well-defined. By Lemma 2, mapping $f_1 \land f_2$ is still union-preserving.

3. **A FORMULA ON INTERSECTION**

Hutton [4, Lemma 3] has given the following formula on intersection:

$$(f_1 \land f_2)(a) = \bigwedge_{a_1 \lor a_2 = a} [f_1(a_1) \lor f_2(a_2)].$$

This formula is rather important and is applied to many cases. But we shall give an example to show that the formula does not hold for all completely distributive lattices except the lattice $\{0, 1\}$.

**Counterexample.** Let $L$ be a completely distributive lattice and
contain an element (denoted by \( b \)) different to 0 and 1. Define \( f_1, f_2: L \rightarrow L \) by
\[
\begin{align*}
f_1(a) &= 1 \quad a \in L, \\
f_2(a) &= a \quad a \in L.
\end{align*}
\]

Clearly, \( f_1 \) and \( f_2 \) are union-preserving mappings and satisfy all conditions of [4, Lemma 3]. But after simple calculation we have
\[
f_1 \land f_2 = (f_1 \cap f_2)^* = f_2 \quad \text{and} \quad \bigwedge_{a_1, a_2 = b} f_1(a_1) \lor f_2(a_2) = 1;
\]
i.e., for \( a = b \) the formula does not hold.

The following statement is similar to that of Hutton's, where a natural assumption is added to.

**THEOREM.** Let \( L \) be completely distributive lattice. Let \( f_1, f_2: L \rightarrow L \) be union-preserving mappings satisfying \( f_1(0) = f_2(0) \). Then for every \( a \in L \),
\[
(f_1 \land f_2)(a) = \bigwedge_{a_1, a_2 = a} [f_1(a_1) \lor f_2(a_2)].
\]

**Proof.** Write simply \( h(a) = \bigwedge_{a_1, a_2 = a} [f_1(a_1) \lor f_2(a_2)] \). For every \( a \in L \),
\[
(f_1 \land f_2)(a) = \bigwedge_{\sup \gamma = a} [f_1(\gamma) \land f_2(\gamma)]
\]
\[
\leq \bigwedge_{a_1, a_2 = a} [(f_1(a_1) \land f_2(a_1)) \lor (f_1(a_2) \land f_2(a_2))]
\]
\[
\leq \bigwedge_{a_1, a_2 = a} [f_1(a_1) \lor f_2(a_2)] = h(a).
\]

Conversely, by Lemma 2, it is sufficient to show that
\[
h \leq f_1 \cap f_2,
\]
and
\[
h: L \rightarrow L \text{ is union-preserving.}
\]

(1) We say that \( h \leq f_1 \cap f_2 \). In fact,
\[
h(a) = \bigwedge_{a_1, a_2 = a} [f_1(a_1) \lor f_2(a_2)]
\]
\[
\leq [f_1(a) \lor f_2(0)] \land [f_1(0) \lor f_2(a)]
\]
\[
= [f_1(a) \land f_1(0)] \lor [f_1(0) \land f_2(a)]
\]
\[
\lor [f_2(0) \land f_1(0)] \lor [f_2(0) \land f_2(a)].
\]
Note that \( f_i(0) = f_j(0) \leq f_i(a) \) for every \( a \in L \) and \( i = 1, 2 \). We have
\[
 h(a) \leq f_i(a) \wedge f_j(a) = (f_i \cap f_j)(a).
\]

(2) To prove that \( h \) is union-preserving. Suppose \( \alpha^i \in L \ (i \in I) \) and \( \bigvee_{i \in I} \alpha^i = a \). We first introduce some notations as follows.

Let
\[
 \Gamma_i = \{(\alpha^i_1, \alpha^i_2) \in \alpha^i_1 \lor \alpha^i_2 = a, \alpha^i_1, \alpha^i_2 \in L\}
\]
for every \( i \in I \). \( F \) denotes the direct product of \( \Gamma_i \ (i \in I) \). Certainly \( F \) can also be defined as set of mappings \( \{\varphi\} \) satisfying \( \varphi(i) \in \Gamma_i \) for every \( i \in I \).
Suppose \( \varphi(i) = (\varphi(i)_1, \varphi(i)_2) \), where \( \varphi(i)_1 \lor \varphi(i)_2 = \alpha^i \). Then for every \( \varphi \in F \), there exists a pair \( (\varphi_1, \varphi_2) \), where
\[
 \varphi_j = \bigvee_{i \in I} \varphi(i)_j \quad (j = 1, 2).
\]

Let
\[
 \mathcal{A} = \{ (\varphi_1, \varphi_2) : \varphi \in F \};
\]
\[
 \mathcal{B} = \{ (\alpha_1, \alpha_2) : \alpha_1 \lor \alpha_2 = a \}.
\]

It can directly be verified that \( \mathcal{A} = \mathcal{B} \).
Furthermore, by completely distributive law
\[
 \bigvee_{i \in I} h(\alpha^i) = \bigvee_{i \in I} \bigwedge_{(\alpha^i_1, \alpha^i_2) \in \Gamma_i} [f_i(\alpha^i_1) \lor f_j(\alpha^i_2)]
\]
\[
 = \bigwedge_{F = \varnothing} \bigvee_{i \in I} [f_i(\varphi(i)_1) \lor f_j(\varphi(i)_2)]
\]
\[
 = \bigwedge_{F = \omega} [f_i(\varphi_1) \lor f_j(\varphi_2)]
\]
\[
 = \bigwedge_{\mathcal{A}} [f_i(\alpha_1) \lor f_j(\alpha_2)]
\]
\[
 = h(a).
\]

**Corollary.** Let \( f, g, k : L \to L \) be union-preserving mapping. Then for every \( a \in L \),
\[
 (f \wedge g) \wedge k)(a) = (f \wedge (g \wedge k))(a)
\]
\[
 = \bigwedge_{a_1 \lor a_2 \lor a_3 = a} f(a_1) \lor g(a_2) \lor k(a_3);
\]
i.e., the associative law for the intersection operations holds.
4. SOME PROPERTIES FOR INTERSECTION OPERATIONS

The following two propositions are needed in theory of fuzzy uniformities and fuzzy metrization.

**Proposition 1.** Let \( f, g, f_1, g_1 : L \rightarrow L \) be union-preserving mappings. Then about the composition of mappings, we have

\[
(f \land g) \circ (f_1 \land g_1) \leq (f \circ f_1) \land (g \circ g_1).
\]

**Proof.** Clearly \( f \land g \leq f \) and \( f_1 \land g_1 \leq f_1 \). Hence \( (f \land g) \circ (f_1 \land g_1) \leq f \circ f_1 \). Similarly, \( (f \land g) \circ (f_1 \land g_1) \leq g \circ g_1 \). Thus \( (f \land g) \circ (f_1 \land g_1) \leq (f \circ f_1) \land (g \circ g_1) \). Note that \( (f \land g) \circ (f_1 \land g_1) \) is a union-preserving mapping, the proposition directly follows by Lemma 2.

**Remark.** In Proposition 1, the equality does not hold in general. For example, let \( L \) be \([0, 1]\), the real unit interval. Define

\[
f(a) = g_1(a) = \min\{ 1, a + \frac{1}{2} \} \quad \forall a \in L,
\]

and

\[
g(a) = f_1(a) = \min\{ 1, a + \frac{1}{2} \} \quad \forall a \in L.
\]

It is easily to show that these mappings are union-preserving and that

\[
(f \land g)(a) = (f_1 \land g_1)(a) = \min\{ 1, a + \frac{1}{2} \} \quad \forall a \in L,
\]

and

\[
(f \circ f_1)(a) = (g \circ g_1)(a) = \min\{ 1, a + \frac{5}{6} \} \quad \forall a \in L.
\]

Hence the equality in the Proposition 1 does not hold.

**Proposition 2.** Let \( L_1 \) and \( L_2 \) be completely distributive lattices. Let the correspondences \( G : L_1 \rightarrow L_2 \) and \( H : L_2 \rightarrow L_1 \) be union-preserving. Then for every union-preserving mapping \( g \) in \( L_1 \), define \( \Omega(g) : L_1 \rightarrow L_1 \) by

\[
\Omega(g)(a) = H(g(G(a))), \quad \forall a \in L_1.
\]

We have

1. \( \Omega(g) \) is union preserving in \( L_1 \);

and

2. \( \Omega(g_1 \land g_2) \leq \Omega(g_1) \land \Omega(g) \) for union-preserving mappings \( (in L_1) g_1 \) and \( g_2 \).
Proof. (1) This follows immediately by $G$, $g$ and $H$ being union-preserving.

(2) By $G$, $g$ and $H$ being order-preserving, it directly follows that $\Omega$ is order-preserving; i.e., $h \leq k$ implies $\Omega(h) \leq \Omega(k)$ for union-preserving mappings $h$ and $k$ (in $L_2$). New $g_1 \land g_2 \leq g_j$ ($j = 1, 2$) implies

$$\Omega(g_1 \land g_2) \leq \Omega(g_1) \cap \Omega(g_2).$$

By Lemma 2 and $\Omega(g_1 \land g_2)$ being union-preserving mapping in $L_1$, we obtain that

$$\Omega(g_1 \land g_2) \leq \Omega(g_1) \land \Omega(g_2).$$


References