Coexistence States for Periodic Competitive
Kolmogorov Systems*

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We prove the existence of positive periodic solutions for a class of nonautonomous competitive periodic Kolmogorov systems which generalize the May–Leonard model. A necessary and sufficient condition is also obtained. © 1998 Academic Press

1. INTRODUCTION, STATEMENT OF THE MAIN RESULT, AND APPLICATIONS

This paper deals with the existence of positive periodic solutions for a class of ordinary differential equations which has been considered as relevant for many applications to population dynamics. More precisely, we study the Kolmogorov system

$$x'_i = x_i h_i(t, x_1, \ldots, x_N),$$

where we assume

$$(H_i) \text{ for each } i = 1, \ldots, N, \; h_i: \mathbb{R} \times (\mathbb{R}_+)^N \rightarrow \mathbb{R} \text{ is a continuous function which is } T\text{-periodic } (T > 0) \text{ in the } t\text{-variable.}$$

(Here and in what follows, we denote by $\mathbb{R}_+ := [0, +\infty)$, the set of non-negative real numbers and by $\mathbb{R}^+ := [0, +\infty)$, the set of positive reals.)

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The assumption of a common period for all the functions $h_i(\cdot, x)$ is, perhaps, a strong restriction, which, nevertheless, has been considered by several authors (see, for instance [2, 4, 9–11, 15, 16, 21, 23, 28, 29, 32, 33, 35, 38, 40] and the references therein) and which is justified by many applications describing the behaviour of the $N$-species $x_1, \ldots, x_N$ in a periodic environment due to seasonal fluctuations. In our main application (Theorem 1, below) we also suppose that system (1) is competitive (see [33]), in a weak sense; namely, we assume

\[(H_2) \quad \text{the function } h_i \text{ is nonincreasing in the } x_j \text{-variable for each } i \neq j \text{ (with, } i, j = 1, \ldots, N).\]

With respect to the behaviour of the $i$th component in absence of the other competitors, we will consider some assumptions of logistic growth. Namely, denoting by $e_i$ the $i$th element of the standard orthonormal basis of $\mathbb{R}^N$, we define, for $s \in \mathbb{R}$,

$$h_{i, s}(t, s) := h_i(t, s e_i) = h_i(t, 0, \ldots, 0, s, 0, \ldots, 0)$$

(the variable $s \in \mathbb{R}_+$ is in the $i$th position with respect to the $x_2, \ldots, x_N$-variables) and suppose that

$$\limsup_{s \to +\infty} h_{i, s}(t, s) \leq h_{i, \text{w}}(t), \quad \text{uniformly in } t \in [0, T], \quad (2)$$

with $h_{i, \text{w}} : \mathbb{R} \to \mathbb{R}$ a suitable continuous and $T$-periodic function. It is known (cf. [24, p. 71]) that if the Landesman–Lazer type condition

$$\int_0^T h_i(t, 0) \, dt > 0 > \int_0^T h_{i, \text{w}}(t) \, dt, \quad \forall i = 1, \ldots, N, \quad (3)$$

holds, then, for each $i = 1, \ldots, N$, the equation

$$u' = u h_{i}(t, u) \quad (4)$$

has a positive $T$-periodic solution. Actually, one could find minimal and maximal positive $T$-periodic solutions $\bar{u}_i(\cdot)$ and $\hat{u}_i(\cdot)$ such that any positive $T$-periodic solution $\tilde{u}_i(\cdot)$ of (4) satisfies $\bar{u}_i(t) \leq \tilde{u}_i(t) \leq \hat{u}_i(t), \forall t \in \mathbb{R}$. It is also easy to see that $\bar{u}_i = \hat{u}_i$, that is, the positive $T$-periodic solution is unique, if $h_i(t, \cdot)$ is nonincreasing for all $t$ and decreasing (strictly) for some $t \in \mathbb{R}$ (see, e.g., [29, 39]).

In a previous paper [38], it was proved that if the condition

$$\int_0^T h_i(t, \hat{u}_i(t), \ldots, \hat{u}_{i-1}(t), 0, \hat{u}_{i+1}(t), \ldots, \hat{u}_N(t)) \, dt > 0,$$

$$\forall i = 1, \ldots, N,$$
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holds, then permanence (named also uniform persistence) occurs for system (1); that is, there is a compact attractor contained in the open positive orthant \((\mathbb{R}^+)^N\) which attracts all the trajectories of (1) starting in \((\mathbb{R}^+)^N\).

This, by results on dissipative systems [37], implies the existence of a \(T\)-periodic solution \(\tilde{x}(\cdot) = (\tilde{x}_1(\cdot), \ldots, \tilde{x}_N(\cdot))\), for system (1), with \(\tilde{x}_i(t) > 0\) for all \(i = 1, \ldots, N\) and \(t \in \mathbb{R}\) (see also [38]). Periodic solutions of this kind will be called, according to [23], coexistence states for (1).

The aim of this paper is to provide a result on the existence of coexistence states for (1) in situations when permanence may not occur. With this respect, we recall some classical theorems of Butler et al. [5] and Butler and Waltman [6] (see also [20]), who found sufficient conditions for uniform persistence in dynamical systems under the key hypothesis of nonexistence of cyclic connections on the boundary. This excludes from the applicability of the theorems in [5] and [6] a famous example of three species interaction due to May and Leonard [26], further analyzed by Schuster et al. [31], Hofbauer [18], Freedman and Waltman [13], Wang and Ma [36], and others (see also [19, Chapter 9.5]).

In [26] and [31], the Gause–Lotka–Volterra (GLV) system with constant coefficients in \((\mathbb{R}^+)^3\)

\[
\begin{align*}
x_1' &= x_1(1 - x_1 - \alpha x_2 - \beta x_3) \\
x_2' &= x_2(1 - \beta x_1 - x_2 - \alpha x_3) \\
x_3' &= x_3(1 - \alpha x_1 - \beta x_2 - x_3)
\end{align*}
\]

was considered. If \(0 < \alpha < 1 < \beta\), there are exactly five equilibrium points: \(O = (0,0,0)\), \(e_1 = (1,0,0)\), \(e_2 = (0,1,0)\), \(e_3 = (0,0,1)\), and \(P = (1,1,1)/(1 + \alpha + \beta)\). In this case, it is easy to see that the origin is a repellor, and on the three quadrants \(x_3 = 0\), \(x_2 = 0\) and \(x_1 = 0\) at the boundary of \((\mathbb{R}^+)^3\), there are, respectively, the heteroclinic orbits,

\[
e_2 \xrightarrow{\Gamma_2} e_1, \quad e_1 \xrightarrow{\Gamma_1} e_3, \quad \text{and} \quad e_3 \xrightarrow{\Gamma_3} e_2.
\]

Thus, a cyclic connection \(\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\), of the form \(e_3 \rightarrow e_2 \rightarrow e_1 \rightarrow e_3\), appears at the boundary of \((\mathbb{R}^+)^3\). In other words, we have that when \(x_3 = 0\), \(x_2\) is carried to the extinction by \(x_1\) (a situation that we will describe by the notation \(x_1 \Rightarrow x_2\)), and, cyclically, \(x_3 \Rightarrow x_1\) when \(x_2 = 0\) and \(x_2 \Rightarrow x_3\) when \(x_1 = 0\). The constant coexistence state \(P\) is asymptotically stable for \(\alpha + \beta < 2\) and unstable for \(\alpha + \beta \geq 2\). In this latter case, for \(\alpha + \beta = 2\), there is an attracting limit cycle contained in \((\mathbb{R}^+)^3\), while, for \(\alpha + \beta > 2\), it is the heteroclinic cycle \(\Gamma\) that attracts an open set of points in \((\mathbb{R}^+)^3\). Similar qualitative features were proved in [31] for the
unsymmetric equation

\[
\begin{align*}
  x'_1 &= x_1(1 - x_1 - \alpha_1 x_2 - \beta_1 x_3) \\
  x'_2 &= x_2(1 - \beta_2 x_1 - x_2 - \alpha_2 x_3) \\
  x'_3 &= x_3(1 - \alpha_3 x_1 - \beta_3 x_2 - x_3)
\end{align*}
\]  

(5)

with

\[
0 < \alpha_i < 1 < \beta_i, \quad i = 1, 2, 3.
\]  

(6)

Thus, systems with the cyclic structure described above, represent a class of models in which coexistence states are possible, independently of the presence or lack of persistence. On the other hand, as long as we restrict our consideration to the constant coefficients case, coexistence states will be mainly given by positive equilibria, even if more complicated situations, with limit cycles and Hopf bifurcations may occur (cf. Hofbauer [18]). Indeed, a coexistence state for the autonomous system \( x'_i = x_i h_i (x_1, \ldots, x_N), \) \( i = 1, \ldots, N, \) will be obtained as soon as we are able to guarantee the existence of a point \( Q \in (\mathbb{R}^+)^N \) which is a zero of the vector field \( h = \text{col}(h_1, \ldots, h_N). \) Actually, with respect to some applications to "ecological" models, it will also be interesting for us to show that these equilibria do not accumulate to the boundary of \( (\mathbb{R}^+)^N, \) and (even better) to find a compact set \( \mathcal{K} \subset (\mathbb{R}^+)^N \) containing all the possible coexistence states.

In the case of nonautonomous systems, a natural question which can be raised is whether we can find positive periodic solutions for GLV equations of the May–Leonard type (or even for more general equations), with periodic coefficients, e.g., for systems like

\[
\begin{align*}
  x'_1 &= x_1 (a_1(t) - b_1(t)x_1 - c_1(t)x_2 - d_1(t)x_3) \\
  x'_2 &= x_2 (a_2(t) - b_2(t)x_1 - c_2(t)x_2 - d_2(t)x_3) \\
  x'_3 &= x_3 (a_3(t) - b_3(t)x_1 - c_3(t)x_2 - d_3(t)x_3)
\end{align*}
\]  

(7)

where \( a_i, b_i, c_i, d_i : \mathbb{R} \to \mathbb{R} \) are continuous and \( T \)-periodic functions satisfying some positivity conditions that will be fixed later. Clearly, looking for a proper set of assumptions on system (7) in order to reproduce the boundary behaviour of the May–Leonard model, we should search for conditions implying \( x_i \gg x_{i+1} \) when \( x_{i-1} = 0 \) (with \( i = j \text{ mod } 3 \)), or, at least, the nonexistence of positive periodic solutions contained in the interiors of the faces in the boundary of \( (\mathbb{R}^+_+) \) (cf. [13]). With this in mind, we can take advantage of a theorem of Ahmad [1] who for the competitive
planar system

\[ u' = u(a(t) - b(t)u - c(t)v) \]
\[ v' = v(d(t) - e(t)u - f(t)v) \]

proved that \( u \gg v \), if

\[ a_L f_L > c_M d_M \quad \text{and} \quad b_M d_M \leq a_L e_L, \]

where \( w_L \) and \( w_M \) denote, respectively, the “inf” and the “sup” taken on the whole real line, of a function \( w: \mathbb{R} \to \mathbb{R} \). For further results in this direction, see also [17, p. 121; 34]. Clearly, Eq. (8) can be considered as a model for any of the three \( 2 \times 2 \) reduced subsystems of (7). In [1], the (not necessarily periodic) coefficients in Eq. (8) are assumed to be bounded above and below by positive constants. The case of periodic coefficients considered here allows us to propose a slight improvement for the constraints in the parameters and therefore we assume for the two-dimensional Kolmogorov system

\[ u' = u\mathcal{Z}(t, u, v) \]
\[ v' = v\mathcal{Z}(t, u, v), \]

with \( T \)-periodic dependence in the \( t \)-variable, one of the following conditions:

\( (H^1) \) there are a constant \( m > 0 \) and a \( T \)-periodic continuous map \( \gamma: \mathbb{R} \to \mathbb{R} \), with \( \int_0^T \gamma \geq 0 \), such that, for any \( \epsilon > 0 \), the function

\[ \phi_{\epsilon}(t) := \inf \left\{ m\mathcal{W}(t, u, v) - \mathcal{Z}(t, u, v) \right\} \]

satisfies

\[ \phi_{\epsilon}(t) \geq \gamma(t), \quad \forall t \in [0, T] \quad \text{and} \quad \phi_{\epsilon} \neq \gamma, \]

or

\( (H^2) \) there are a constant \( m > 0 \) and a \( T \)-periodic continuous map \( \gamma: \mathbb{R} \to \mathbb{R} \), with \( \int_0^T \gamma \geq 0 \), such that, for any \( \epsilon > 0 \), the function

\[ \phi_{\epsilon}(t) := \inf \left\{ m\mathcal{W}(t, u, v) - \mathcal{Z}(t, u, v) \right\} \]

satisfies

\[ \phi_{\epsilon}(t) \geq \gamma(t), \quad \forall t \in [0, T] \quad \text{and} \quad \phi_{\epsilon} \neq \gamma. \]
With respect to condition \((H'_3)\), we observe that it can be useful to consider the auxiliary function \(\psi(t, u, v) := m\|\varphi(t, u, v) - \varphi(t, u, v)\|\). Note that if, for a suitable constant \(m > 0\), the function \(\psi\) is nonincreasing in \(u\) and \(v\), for all \(t\), then condition \((H'_3)\) holds provided that, for each \(u > 0\), \(\psi(t, u, 0) \geq \gamma(t)\), with the strict inequality on a set of positive measure. In particular, introducing the notation \(\bar{w} := \int_0^t w(t) \, dt\) (the mean value in a period) for a continuous and \(T\)-periodic function \(w: \mathbb{R} \to \mathbb{R}\), we can find some simple conditions on the periodic coefficients of Eq. (8) in order to have \((H'_3)\) satisfied. In fact, taking \(m = \bar{d}/\bar{a}\) and \(\gamma(t) = d(t) - (\bar{d}a(t)/\bar{a})\), we obtain that \((H'_3)\) holds for Eq. (8), provided that \(\bar{a}, \bar{d} > 0, \bar{d}b(t) \leq \bar{a}e(t)\), for all \(t \in [0, T]\), with the strict inequality on a set of positive measure, and \(\bar{a}f(t) \geq \bar{d}c(t)\), for all \(t \in [0, T]\). Similar remarks can be obviously made with respect to condition \((H'_3)\) which holds for Eq. (8), whenever \(\bar{a}, \bar{d} > 0, \bar{a}f(t) \geq \bar{d}c(t)\), for all \(t \in [0, T]\), with the strict inequality on a set of positive measure, and \(\bar{d}b(t) \leq \bar{a}e(t)\), for all \(t \in [0, T]\).

If we deal now with the Kolmogorov system (1) in \((\mathbb{R}_+)^3\), which we rewrite as

\[
\begin{align*}
x'_1 &= x_1h_1(t, x_1, x_2, x_3) \\
x'_2 &= x_2h_2(t, x_1, x_2, x_3) \\
x'_3 &= x_3h_3(t, x_1, x_2, x_3)
\end{align*}
\]

and assume \((H'_3)\) on each of the three \(2 \times 2\) subsystems

\[
\begin{align*}
x'_1 &= x_1h_1(t, x_1, x_2, 0), & x'_2 &= x_2h_2(t, x_1, x_2, 0), \\
x'_2 &= x_2h_2(t, 0, x_2, x_3), & x'_3 &= x_3h_3(t, 0, x_2, x_3)
\end{align*}
\]

and

\[
\begin{align*}
x'_3 &= x_3h_3(t, x_1, 0, x_3), & x'_1 &= x_1h_1(t, x_1, 0, x_3),
\end{align*}
\]

then we are led to consider the following condition.

\((H''_3)\) For \(i = 1, 2, 3\), there are positive constants \(m_i\) and \(T\)-periodic continuous maps \(\gamma_i: \mathbb{R} \to \mathbb{R}\), with \(\int_0^T \gamma_i \geq 0\), such that, for any \(\varepsilon > 0\), the functions

\[
\begin{align*}
\phi_{\varepsilon, 1}(t) &= \inf_{\varepsilon \leq x_1 \leq 1/\varepsilon} \{m_1h_1(t, x_1, x_2, 0) - h_2(t, x_1, x_2, 0)\}, \\
\phi_{\varepsilon, 2}(t) &= \inf_{\varepsilon \leq x_2 \leq 1/\varepsilon} \{m_2h_2(t, x_2, x_3) - h_3(t, 0, x_2, x_3)\}, \\
\phi_{\varepsilon, 3}(t) &= \inf_{\varepsilon \leq x_3 \leq 1/\varepsilon} \{m_3h_3(t, x_3) - h_1(t, 0, x_3)\},
\end{align*}
\]
and
\[\phi_{e,3}(t) := \inf_{\varepsilon \leq x_3 \leq 1/\varepsilon, 0 \leq x_1 \leq 1/\varepsilon} \{m_3 h_3(t, x_1, 0, x_3) - h_3(t, x_1, 0, x_3)\}\]
satisfy
\[\phi_{e,i}(t) \geq \gamma_i(t), \quad \forall t \in [0, T] \text{ and } \phi_{e,i} \neq \gamma_i.\]

Now, we are in position to state our existence result for system (1) in the case \(N = 3\).

**Theorem 1.** Assume \((H_1), (H_2), (2), \text{ and } (3)\). Then system (10) has at least one \((T\text{-periodic})\) coexistence state if \((H_3)\) holds. Moreover, there is a compact set \(\mathcal{K} \subset \mathbb{R}^3\) containing all the \((T\text{-periodic})\) coexistence states of (10).

It could be noteworthy to compare Theorem 1 to a recent result of Ortega and Tineo [29] who, under the stronger competitiveness assumption that \(h_i(t, x_1, x_2, x_3)\) is strictly decreasing in \(x_j\) for all \(i, j = 1, 2, 3\) and \(\int_0^T h_i(t, R_0) dt < 0\), for some \(R > 0\) and all \(i = 1, 2, 3\) (a condition which corresponds to (2), with the second inequality in (3)), proved that either system (10) has a positive coexistence state or, alternatively,
\[\min\{x_1(t), x_2(t), x_3(t)\} \to 0, \quad \text{as } t \to +\infty,\]for each solution which is positive in all its components. Our result, in view of May–Leonard model and Corollary 2 below, can provide coexistence states for equations when (11) may happen for some solutions. Further investigations in this direction have been pursued in [7] where sharp conditions for the alternative coexistence/extinction are given for the strictly monotone case.

We remark that recently Merino [27] also studied a similar model in the more general setting of reaction diffusion systems with periodic coefficients. The results in [27] require some smoothness in the \(h\)-functions, as well as the fact that \(\partial_t h_i(t, s) < 0\) for \(s > 0\) and \(h_i(t, s) \to -\infty\) as \(s \to +\infty\) (two conditions which are not needed in our case). Thus, the theorems in [27] do not contain our results and, in particular, for the simpler situation described by Eq. (7), we obtain some improvements in the constraints on the coefficients.

We wish now to apply Theorem 1 to the GLV system (7) with continuous and \(T\text{-periodic}\) coefficients. For the validity of \((H_2), (2), \text{ and } (3)\), we have to assume that, for \(i = 1, 2, 3\),
\[\bar{a}_i > 0, b_i(t), c_i(t), d_i(t) \geq 0, \quad \forall t \in [0, T] \text{ and } \bar{b}_i \bar{c}_i \bar{d}_i > 0.\]
Then, we have:

**Corollary 1.** Assume (12) and suppose that the following inequalities hold for all \( t \in [0, T] \),

\[
\begin{align*}
\bar{a}_1 b_2(t) &\geq \bar{a}_2 b_1(t), & \bar{a}_1 c_2(t) &\geq \bar{a}_2 c_1(t), \\
\bar{a}_2 c_3(t) &\geq \bar{a}_3 c_2(t), & \bar{a}_2 d_3(t) &\geq \bar{a}_3 d_2(t), \\
\bar{a}_3 d_1(t) &\geq \bar{a}_1 d_3(t), & \bar{a}_3 b_1(t) &\geq \bar{a}_1 b_3(t),
\end{align*}
\]

and

\[
\bar{a}_i \bar{b}_2 > \bar{a}_i \bar{b}_1, \quad \bar{a}_2 \bar{c}_3 > \bar{a}_3 \bar{c}_2, \quad \bar{a}_3 \bar{d}_1 > \bar{a}_1 \bar{d}_3.
\]

Then, there is at least one coexistence state for Eq. (7).

To illustrate the meaning of the above conditions, let us consider the unsymmetric May–Leonard equation with periodic coefficients

\[
\begin{align*}
x_1' &= p_1(t) x_1(1 - x_1 - \alpha_1(t) x_2 - \beta_1(t) x_3) \\
x_2' &= p_2(t) x_2(1 - \beta_2(t) x_1 - x_2 - \alpha_2(t) x_3) \\
x_3' &= p_3(t) x_3(1 - \alpha_3(t) x_1 - \beta_3(t) x_2 - x_3),
\end{align*}
\]

where \( p_i: \mathbb{R} \to \mathbb{R}^+ \) and \( \alpha_i, \beta_i: \mathbb{R} \to \mathbb{R}_+ \) are continuous and \( T \)-periodic functions, for \( i = 1, 2, 3 \). From a result by de Mottoni and Schiaffino [28, Appendix] it is known that any system of the form (7), with coefficients satisfying (12), can be put into the form (13), where \( p_i(t) \), for \( i = 1, 2, 3 \), is the unique positive \( T \)-periodic solution of the logistic equation for the \( i \)th single species in (7), suppressing all the competitors (that is, with \( x_j = 0 \) for \( j \neq i \)). Hence, applying Corollary 1, we have that Eq. (13) has at least one coexistence state if the conditions

\[
\alpha_1(t) \leq \frac{\bar{p}_1 p_2(t)}{p_1(t),} \quad \alpha_2(t) \leq \frac{\bar{p}_2 p_3(t)}{p_2(t),} \quad \alpha_3(t) \leq \frac{\bar{p}_3 p_1(t)}{p_3(t),}
\]

and

\[
\frac{p_2(t) \bar{p}_2}{\bar{p}_1 p_2(t)} \leq \beta_2(t), \quad \frac{p_2(t) \bar{p}_3}{\bar{p}_2 p_3(t)} \leq \beta_3(t), \quad \frac{p_3(t) \bar{p}_1}{\bar{p}_3 p_1(t)} \leq \beta_3(t)
\]

hold for all \( t \in [0, T] \), where each of the inequalities for \( \beta_i(t) \) is strict on some set \( I_i \subset [0, T] \), of positive measure.

It is also possible to prove (see Remark 2 in Section 2), using a variant of condition \((H^*_2)\) which can be derived from \((H^*_3)\), that the same result
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holds true if we assume the hypothesis concerning the strict inequalities on sets of positive measure for all the \( \alpha_i \)'s (instead of all the \( \beta_i \)'s). Thus, in the special case when \( p_1(t) = p_2(t) = p_3(t) \), so that Eq. (13) takes the form

\[
\begin{align*}
    x'_1 &= x_1(1 - x_1 - \alpha_1(t)x_2 - \beta_1(t)x_3) \\
    x'_2 &= x_2(1 - \beta_2(t)x_1 - x_2 - \alpha_2(t)x_3) \\
    x'_3 &= x_3(1 - \alpha_3(t)x_1 - \beta_3(t)x_2 - x_3)
\end{align*}
\]

(14)

we obtain that the condition

\[
0 \leq \alpha_i(t) \leq 1 \leq \beta_i(t), \quad \forall t \in [0, T], i = 1, 2, 3,
\]

(15)

with

\[
(1 - \overline{\alpha}_1)(1 - \overline{\alpha}_2)(1 - \overline{\alpha}_3) > 0 \quad \text{or} \quad (\overline{\beta}_1 - 1)(\overline{\beta}_2 - 1)(\overline{\beta}_3 - 1) > 0,
\]

(16)

implies the existence of a coexistence state for Eq. (14). These assumptions clearly improve and generalize condition (6) to the case of periodic coefficients. Actually, we can prove even more, and indeed, we have the following:

**Corollary 2.** Let the \( T \)-periodic continuous coefficients \( \alpha_i, \beta_i \) satisfy (15) with \( \overline{\alpha}_i < \overline{\beta}_i, \forall i \). Then, system (14) has at least one coexistence state and all its coexistence states are contained in a compact subset of \((\mathbb{R}^+)\), if and only if (16) holds.

The easy proof of the “only if” part is omitted.

We note that Corollary 2, as well as the more general Theorem 1 and Corollary 1, can be applied to situations which are not covered by other results implying persistence. With this respect, we observe that using the result in [38] for the case of system (14), it is possible to obtain uniform persistence (and hence, the same conclusion of Theorem 1) under the assumption

\[
\alpha_i \geq 0, \beta_i \geq 0 \quad \text{and} \quad \overline{\alpha}_i + \overline{\beta}_i < 1, \forall i = 1, 2, 3,
\]

which is not satisfied when (15) holds.

The proof of Theorem 1 is based on a continuation principle in [8] that we adapt to the special structure of Eq. (1). This is discussed in the next section.
2. AUXILIARY RESULTS AND PROOF OF THEOREM 1

The proof of Theorem 1 will be obtained by means of a continuation lemma based on coincidence degree (cf. [24]). To this end, we first embed Eq. (1) into a one-parameter family of equations of the form

\[ x_i' = h_i^\lambda(t, x_1, \ldots, x_N; \lambda), \quad \lambda \in [0, 1], \]

where, for each \( i = 1, \ldots, N \), \( h_i^\lambda : \mathbb{R} \times (\mathbb{R}_+)^N \times [0, 1] \to \mathbb{R} \) is a continuous function which is \( T \)-periodic (\( T > 0 \)) in the \( r \)-variable and such that

\[ h_i^\lambda(t, x; 1) = h_i(t, x), \quad \forall t \in [0, T], \forall x \in (\mathbb{R}_+)^N. \]

As in [8], we also assume that for \( \lambda = 0 \) the system (17) is autonomous, that is,

\[ h_i^0(t, x; 0) = h_i^0(x), \]

where \( h^0 = (h_1^0, \ldots, h_N^0) : (\mathbb{R}_+)^N \to \mathbb{R}^N \) is a continuous function.

In the sequel, we make use of the following notation.

Given any set \( J = \{ j_1, \ldots, j_r \} \subset \{ 1, \ldots, N \} \) of \( r \geq 1 \) indexes, we define, for any \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \),

\[ x_J := (x_{j_1}, \ldots, x_{j_r}) \in \mathbb{R}^{|J|} = \mathbb{R}^r. \]

Conversely, for any \( y \in \mathbb{R}^r \), we set

\[ y_J := (z_1, \ldots, z_N), \text{ where } z_i = y_i, \text{ for } i \in J \text{ and } z_i = 0, \text{ for } i \notin J. \]

Moreover, if \( g: \mathbb{R}^N \supset \mathcal{O} \to \mathbb{R}^N \) is a continuous function such that, for some open set \( \Omega \subset \mathcal{O} \), \( g^{-1}(0) \cap \Omega \) is compact, then we can define the integer

\[ d(g, \Omega) := \text{deg}(g, \mathcal{O}, 0), \]

where \( \mathcal{O} \) is any open and bounded set with \( g^{-1}(0) \cap \Omega \subset \mathcal{O} \subset \mathcal{O} \subset \Omega \) and “deg” is the Brouwer degree. Note that the definition of \( d(g, \Omega) \) is well-posed, as it is independent of the set \( \mathcal{O} \), by virtue of the excision property of the degree. Then we have:

**Lemma 1.** Assume

\[ (K_0) \quad \forall \lambda \in [0, 1], \exists \lambda \in (1, \ldots, N): \int_0^T h_i^\lambda(t, \lambda) \, dt \neq 0 \]

and suppose that the following conditions hold:

\[ (K_1) \quad \text{there is } M > 0, \text{ such that any positive } T\text{-periodic solution } x(t) \text{ of (17) satisfies } x_i(t) \leq M, \text{ for all } t \in [0, T] \text{ and } i = 1, \ldots, N; \]
(K_2) for each J = \{j_1, \ldots, j_r\} \subset \{1, \ldots, N\}, with 1 \leq r \leq N - 1 and each positive T-periodic solution y(\cdot) of the reduced subsystem

\[ y_i' = y_i h_i^+(t, y^r; \lambda), \quad \lambda \in [0, 1], \]

\[ i = 1, \ldots, r, \]

there is \( v \notin J \) such that \( \int_0^T h_v^+(t, y^r(t); \lambda) \, dt \neq 0. \)

Then, there is a compact set \( \mathcal{K} \subset (\mathbb{R}^+)^N \) such that any positive T-periodic solution \( x(\cdot) \) of (17) satisfies \( x(t) \in \mathcal{K} \), for all \( t \in \mathbb{R} \). If, moreover,

\[ (K_3) \quad d(h^0, (\mathbb{R}^+)^N) \neq 0, \]

then, there exists a coexistence state for equation (1).

We notice that the condition in \( (K_2) \) has to be considered as vacuously satisfied when system (18) has no positive T-periodic solutions.

Lemma 1 is an application to Kolmogorov systems of a continuation theorem in [8]. A version of Lemma 1 for the case of planar systems was recently considered in [3] and applied to various different models for the interaction of two species.

Proof. We prove at first that there is a compact set \( \mathcal{K} \subset (\mathbb{R}^+)^N \) such that any positive T-periodic solution \( x(\cdot) \) of (17) satisfies \( x(t) \in \mathcal{K} \), for all \( t \in \mathbb{R} \).

To see this, assume by contradiction that, for any \( n \in \mathbb{N} \), there is a positive T-periodic solution \( x_n(\cdot) \) of (17), for some \( \lambda = \lambda_n \in [0, 1] \), such that \( x_n(0) \not\in [1/n, n]^N \). By \( (K_1) \), we know that, for each \( n > M \), there is some \( i = i_n \in \{1, \ldots, N\} \) such that \( \min_{0 \leq t \leq T} (x_n)_i < 1/n \). On the other hand, \( (K_2) \) also implies that \( \|x_n\|_\infty \leq M \) and \( \|x'_n\|_\infty \leq ML \), for each \( n \in \mathbb{N} \) and \( i = 1, \ldots, N \), where

\[ L := \max_{i=1, \ldots, N} \left( \sup_{0 \leq t \leq T} \|h_i^+(t, z; \lambda)\|: t \in [0, T], z \in [0, M]^N, \lambda \in [0, 1] \right). \]

Hence, using the Ascoli–Arzelà theorem and passing to a subsequence, we can find a parameter \( \lambda = \lambda_n \in [0, 1] \) and a T-periodic solution \( u(\cdot) \) of (17), for \( \lambda = \lambda_n \), such that \( \min u_k = 0 \) for some index \( k \in \{1, \ldots, N\} \), with

\[ \lim_{n \to +\infty} x_n(t) = u(t), \quad \text{uniformly w.r. to } t \in \mathbb{R}. \]

Now, for an arbitrary index \( i \in \{1, \ldots, N\} \), we consider the ith equation in (17), with \( \lambda = \lambda_n \). Using the T-periodicity of the positive solution \( x_n \), we obtain that

\[ \int_0^T h_i^+(t, x_n(t); \lambda_n) \, dt = \int_0^T \frac{(x'_n)_i(t)}{(x_n)_i(t)} \, dt = 0. \]
Hence, passing to the limit as $n \to \infty$, we obtain
\[
\int_0^T h^\alpha_i(t, u_1(t), \ldots, u_N(t); \lambda) \, dt = 0, \quad \forall i = 1, \ldots, N. \tag{19}
\]

Let $(1, \ldots, N) = J \cup K$ be such that $\min u_j > 0$ for $j \in J$, and $\min u_k = 0$ if $k \in K$. Considering the $k$th equation in (17), for $\lambda = \lambda_n$, we have that $\max(x_n)^{\prime} \leq \min(x_n)^{\prime} \exp \left( \int_0^T h^\alpha_k(t, x_n(t); \lambda_n) \, dt \right) \leq \min(x_n)^{\prime} \exp LT$. Hence, passing to the limit as $n \to \infty$, we find that $\max u_k = 0$ for all $k \in K$.

Using assumption $(K_0)$ we know that there is an index $l = l(\lambda)$ such that
\[
\int_0^T h^\alpha_l(t, 0; \lambda) \, dt \neq 0.
\]
This, with (19), implies that there is some index $j$ with $\min u_j > 0$ and then we can conclude that $J \neq \emptyset$ as well as $K \neq \emptyset$ (this latter fact was already proved above).

Now, let $J = (j_1, \ldots, j_r)$ for some $1 \leq r \leq N - 1$ and observe that $u_j$ is a positive $T$-periodic solution of the reduced subsystem
\[
y_j^{\prime} = y_j h^\alpha_j(t, y_j, \lambda),
\]
\[i = 1, \ldots, r.
\]
Notice also that, consistently with the notation introduced above, we have that $(u_r)^{\prime} = u$, as $u_k = 0$ for $k \notin J$. Hence, according to assumption $(K_3)$, there is an index $\nu \in K$ such that $\int_0^T h^\alpha_{\nu}(t, u(t); \lambda) \, dt \neq 0$. This clearly contradicts condition (19) we have found before. Thus we have proved the initial claim and we conclude that there is a compact set $\mathcal{K} \subset (\mathbb{R}^+)^N$ containing all the positive $T$-periodic solutions (17) for any $\lambda \in [0, 1]$. We observe also that $\mathcal{K}$ contains all the positive zeros of $h^0$ which are, in fact, positive $T$-periodic solutions of (17) for $\lambda = 0$.

We take now $\varepsilon > 0$ sufficiently small, so that $\mathcal{K} \subset ]\varepsilon, \varepsilon^{-1}[^N$, and thus, according to $(K_3)$,
\[
\deg(\text{col}(x, h^\alpha_i)_{1 \leq i \leq N}, ]\varepsilon, \varepsilon^{-1}[^N, 0) = \deg(h^0, ]\varepsilon, \varepsilon^{-1}[^N, 0) = d(h^0, (\mathbb{R}^+)^N) \neq 0.
\]
Consider also the set $\omega_\varepsilon$ in the space $C_T$ of continuous and $T$-periodic functions $x : \mathbb{R} \to \mathbb{R}^N$ (endowed with the sup-norm $|\cdot|_{\infty}$), defined by
\[
\omega_\varepsilon := \{ x \in C_T : \varepsilon < x_i(t) < \varepsilon^{-1}, \forall t \in \mathbb{R}, \forall i = 1, \ldots, N \}.
\]
Note that $\omega_\varepsilon$ is bounded, open with $\omega_\varepsilon \cap \mathbb{R}^N = ]\varepsilon, \varepsilon^{-1}[^N$. 

\[\]
By the choice of $\varepsilon$, we know that for any $\lambda \in [0, 1]$, there is no positive $T$-periodic solution $x$ of (17) with $x \in \partial \omega_\varepsilon$. Hence, a continuation theorem in [8, Theorem 2] can be applied in order to obtain the existence of at least one solution $\tilde{x}$ for Eq. (17) for $\lambda = 1$, with $\tilde{x} \in \omega_\varepsilon$. This concludes the proof of Lemma 1.

**Remark 1.** Adapting to this setting a result on the Leray–Schauder degree for parametrized equations see [22, Théorème Fondamental; 25; or 30, Appendix]; one could also prove that there is a connected set $\Sigma \subset C_T \times [0, 1]$ of solution pairs $(x, \lambda)$ with $x$ a positive $T$-periodic solution of (17), such that for each $\lambda \in [0, 1]$ there is some $(x, \lambda) \in \Sigma$. More precisely, arguing like in [14], it is possible to see that there is a connected branch of solution pairs $(x, \lambda)$ with $x$ a positive $T$-periodic solution of (17), starting, at $\lambda = 0$, from the set of positive zeros of $h^0$ and reaching, at $\lambda = 1$, the set of positive $T$-periodic solutions of Eq. (1).

Looking at $(K_1)$, it appears that it will be easier to check such condition for low-dimensional systems. The case when $N = 2$, has been recently examined in [3]. Here, we propose an application to a three-dimensional system. Now we are in position to prove Theorem 1.

**Proof of Theorem 1.** In order to apply Lemma 1 to system (1) in the competitive case, we choose, for each $i = 1, \ldots, N$,

$$h^*_i(t, z; \lambda) := \lambda h_i(t, z) + (1 - \lambda) \frac{1}{T} \int_0^T h_i(t, z) \, dt,$$

$$\forall t \in \mathbb{R}, z \in (\mathbb{R}^+)^N,$$  \hspace{1cm} (20)

so that

$$h^0(z) = \frac{1}{T} \int_0^T h(t, z) \, dt, \quad z \in (\mathbb{R}^+)^N.$$  

Assuming $(H_2)$ and (2), (3), we can easily check that $(K_0)$ and $(K_1)$ of Lemma 1 are satisfied.

Indeed, from the definition of $h^*_i(t, z; \lambda)$ in (20), we have that

$$\int_0^T h^*_i(t, 0, \lambda) \, dt = \int_0^T h_i(t, 0) \, dt > 0,$$

for each $i$ and $\lambda$, and thus $(K_0)$ follows from the first condition in (3). On the other hand, from (2) and the second condition in (3), we can find a constant $R > 0$ and, for each $i = 1, \ldots, N$, a continuous and $T$-periodic function $\sigma_i \colon \mathbb{R} \to \mathbb{R}$, with $\bar{\sigma}_i := T^{-1} \int_0^T \sigma_i < 0$, such that $h_i(t, s) \leq \sigma_i(t)$, for all $s \geq R$. From this, it follows that $h^*_i(t, s; \lambda) \leq \lambda \sigma_i(t) + (1 - \lambda)\bar{\sigma}_i$. 


for all $t \in \mathbb{R}$ and $s \geq R$. Now, if $x(\cdot)$ is any positive solution of system (17) for some $\lambda \in [0, 1]$, then, using the fact that $h_\lambda^i$ is nonincreasing in the $x_j$-variable for each $j \neq i$, we obtain that $x_j'(t)/x_i(t) \leq h_\lambda^i(t, x_i(t); \lambda) \leq \lambda \sigma(t) + (1 - \lambda) \bar{\sigma}_i$ for any $t$ such that $x_i(t) \geq R$. Hence, from $\int_0^T x_j'/x_i = 0$, we find that there is some $t \in [0, T]$ such that $x_i(t) < R$. If $x_i(t) \leq R$, for all $t \in \mathbb{R}$, we are done, so, suppose that $x_i(t) > R$ for some $t \in \mathbb{R}$. By the $T$-periodicity of $x_i(\cdot)$, we can then find two points $t'$ and $t''$ with $t' < t'' < t' + T$ such that $x_i(t') = R$, $x_i(t'') = \max x_i(t)$ and $x_i(t) \geq R$, for all $t \in [t', t'']$. This implies that $x_i(t'') \leq x_i(t') \exp(\int_{t'}^{t''} (\lambda \sigma(t) + (1 - \lambda) \bar{\sigma}_i) dt) \leq R \exp(\|\sigma\|_2)$. Hence we have proved the validity of property $(K_1)$ with $M := R \exp(\max(\|\sigma\|_2 : i = 1, \ldots, N))$.

Observe that this first step in the proof uses only the (weak) competitiveness assumption $(H_2)$ as well as (2) and (3), and thus it is valid for any dimension. For the remaining part of the proof, we use in a crucial way the restriction on the dimension $N = 3$.

We wish now to check condition $(K_2)$. To this aim, we first observe that there are no $T$-periodic solutions of system (17) having the form $(x_1, x_2, 0), (0, x_2, x_3), (x_1, 0, x_3)$, with $x_i > 0$. We consider only the first case as the other ones are completely similar. So, assume, by contradiction, that a positive $T$-periodic pair $(x_1, x_2)$ exists such that

$$\begin{aligned}
x_1'(t) &= x_1(t) h_1^1(t, x_1(t), x_2(t), 0; \lambda), \\
x_2'(t) &= x_2(t) h_2^2(t, x_1(t), x_2(t), 0; \lambda),
\end{aligned}$$

for some $\lambda \in [0, 1]$. For $m_2$ and $\gamma_1$ given in $(H_3^s)$, we obtain that

$$m_2 h_1^1(t, x_1(t), x_2(t), 0) - h_2^2(t, x_1(t), x_2(t), 0) \geq \gamma_1(t),$$

for all $t \in [0, T]$ with the strict inequality on a set of positive measure and, by the definition of $h^0$, we obtain also that

$$m_1 h_1^1(z_1, z_2, 0) - h_2^2(z_1, z_2, 0) > 0,$$

for any $z_1 > 0$ and $z_2 > 0$, so that

$$m_1 h_1^1(x_1(t), x_2(t), 0) - h_2^2(x_1(t), x_2(t), 0) > 0.$$
for all $t \in [0, T]$. Then, from (21), we have

$$0 = \int_0^T \left( m_1 \frac{x_1''(t)}{x_1(t)} - \frac{x_2'(t)}{x_2(t)} \right) dt$$

$$= \int_0^T \left( m_1 h_1'(t, x_1(t), x_2(t), 0; \lambda) - h_2'(t, x_1(t), x_2(t), 0; \lambda) \right) dt$$

$$= \lambda \int_0^T \left( m_1 h_2(t, x_1(t), x_2(t), 0) - h_2'(t, x_1(t), x_2(t), 0) \right) dt$$

$$+ (1 - \lambda) \int_0^T \left( m_1 h_1(t, x_1(t), x_2(t), 0) - h_2'(t, x_1(t), x_2(t), 0) \right) dt$$

$$> \lambda \int_0^T \gamma_1(t) dt \geq 0$$

and thus we have reached a contradiction. This argument, repeated for the other $2 \times 2$ subsystems, leads to the conclusion that there are no positive $T$-periodic solutions for any of the subsystems (18) with $J = (1, 2)$, $J = (2, 3)$, or $J = (3, 1)$ and hence we have proved the corresponding conditions in $(K_2)$ which are (vacuously) satisfied.

Suppose now that $J = (1)$, $J = (2)$, or $J = (3)$. Again, we shall discuss only the first case, as the treatment of the other ones is completely similar. So, let $u: \mathbb{R} \to \mathbb{R}^+$ be a $T$-periodic solution of

$$x_1' = x_1 h_1'(t, x_1, 0, \lambda)$$

for some $\lambda \in [0, 1]$. Using the first condition in $(H_2^0)$, we have that

$$m_1 h_1(t, x_1(t), 0, 0) \geq \gamma_1(t) + h_2(t, x_1(t), 0, 0),$$

with the strict inequality on a set of positive measure and also

$$m_1 h_1(t, x_1(t), 0, 0) > h_2(t, x_1(t), 0, 0),$$

for all $t \in [0, T]$. Therefore, we obtain that

$$0 = \int_0^T m_1 \frac{x_1''(t)}{x_1(t)} dt = \int_0^T m_1 h_1'(t, x_1(t), 0, \lambda) dt$$

$$= \lambda \int_0^T m_1 h_2(t, x_1(t), 0, 0) dt + (1 - \lambda) \int_0^T m_1 h_2(t, x_1(t), 0, 0) dt$$

$$> \lambda \int_0^T \gamma_1(t) + \lambda \int_0^T h_2(t, x_1(t), 0, 0) dt + (1 - \lambda) \int_0^T h_2(t, x_1(t), 0, 0) dt$$

$$\geq \int_0^T h_2(t, x_1(t), 0, 0; \lambda) dt.$$
Thus we have proved that if $J = (1)$, then the condition on the average is satisfied with $\nu = 2$. Repeating the same argument for the other possible choices of $J$, we can complete the proof of the validity of $(K_2)$. Thus, it remains only to check that the degree condition $(K_3)$ is satisfied. Before doing this, we recall a list of properties of $h^0$ which come from the assumptions on $h$ in Theorem 1.

The map $h^0: (\mathbb{R}_+)^3 \to \mathbb{R}^3$ is continuous with $h^0$ nonincreasing in the $x_i$-variable for $i \neq j$, $1 \leq i, j \leq 3$ and with $h^0(0,0,0) > 0$, for $i = 1, 2, 3$. Furthermore, we have that there are $\delta, R$ with $0 < \delta < R$ such that $h^0(se) > 0$, for $0 \leq s \leq \delta$ and $h^0(se) < 0$, for $s \geq R$. In particular, we find for each $i = 1, 2, 3$ a first (positive) zero $u_i > \delta$ of the real-valued function $s \mapsto h^0(se)$, so that $h^0(u_i e_i) = 0$ and $h^0(se_i) > 0$ for each $s \in [0, u_i]$. Moreover, as a consequence of $(H^3)$, there are three positive constants $M_i$, $i = 1, 2, 3$ such that

\begin{align*}
(A_1) & \quad m_1 h^0(x_1, x_2, 0) > h^0(x_1, x_2, 0), \quad \forall x_1 > 0, \forall x_2 \geq 0, \\
(A_2) & \quad m_2 h^0(0, x_2, x_3) > h^0(0, x_2, x_3), \quad \forall x_2 > 0, \forall x_3 \geq 0, \\
(A_3) & \quad m_3 h^0(x_1, 0, x_3) > h^0(x_1, 0, x_3), \quad \forall x_3 > 0, \forall x_1 \geq 0.
\end{align*}

From $(A_1)$, $(A_2)$, and $(A_3)$, we have immediately that

$h^0(u_1, 0, 0) < 0, \quad h^0(0, u_2, 0) < 0, \quad h^0(0, 0, u_3) < 0$.

Now, we proceed with the computation of the degree, using two homotopies.

Let $\varepsilon > 0$, with $\varepsilon < \min(\delta, 1/R)$, be sufficiently small and such that all the positive zeros of $h^0$ are contained in the open bounded set

$$\Omega_\varepsilon := \{x = (x_1, x_2, x_3): \varepsilon < x_i < \varepsilon^{-1}, i = 1, 2, 3\}.$$ 

(The existence of such a set comes from the first part of the theorem.) For $\mu \in [0, 1]$, we consider the system in $(\mathbb{R}_+)^3$,

\begin{align*}
h^0_1(x_1, \mu x_2, x_3) &= 0 \\
h^0_2(x_1, x_2, \mu x_3) &= 0 \tag{22} \\
h^0_3(\mu x_1, x_2, x_3) &= 0.
\end{align*}

Suppose that $z = (z_1, z_2, z_3) \in (\mathbb{R}_+)^3$ is a solution of (22) for some $\mu \in [0, 1]$. As $h^0_0(0, 0, 0) > 0$, we have that $z \neq 0$. So, suppose that $z_3 > 0$. If, by contradiction, $z_3 = 0$, we can consider, respectively, the second equation in (22), the assumption $(A_1)$, the monotonicity of $h^0_1$ in the second variable, and the first equation in (22), in order to have

$$0 = h^0_2(z_1, z_2, 0)/m_1 < h^0_2(z_1, z_2, 0) \leq h^0_2(z_1, \mu z_2, 0) = 0,$$
which yields to a contradiction. Thus we have that $z > 0$. If, by contradiction, $z = 0$, we can consider, respectively, the first equation in (22), the assumption ($A_3$), the monotonicity of $h_0^3$ in the first variable, and the third equation in (22), in order to have

$$0 = h_0^3(z_1, 0, z_3)/m_3 < h_0^3(z_1, 0, z_3) \leq h_0^3(\mu z_1, 0, z_3) = 0,$$

which yields to a contradiction. Thus we have proved that $z > 0$, for $i = 1, 2, 3$, when $z_1 > 0$. In the same manner, one can check that if $z_2 > 0$ or if $z_3 > 0$, then all the other components are positive.

Moreover, from the first equation in (22) and using the monotonicity in the second and third variables, we have that $0 = h_0^3(z_1, \mu z_2, z_3) \leq h_0^3(z_1, 0, 0)$ and hence we conclude that $z_1 < R$, because $h_0^3(x_1, 0, 0) < 0$ for $x_1 > R$. Similarly, we can see that $z_2 < R$ and $z_3 < R$.

Therefore, we have verified that system (22), in $(\mathbb{R}_+)^3$, has all its solutions contained in $[0, R]^3$ and no solution on the boundary of $[0, R]^3$. Thus we can find an $\varepsilon > 0$ sufficiently small such that all the solutions of (22), for any $\mu \in [0, 1]$, are contained in $\Omega$. Then, by the homotopic invariance of the Brouwer degree, we have that

$$\text{deg}(h^0, \Omega_x, 0) = \text{deg}(g, \Omega_x, 0),$$

where

$$g(x_1, x_2, x_3) := \text{col}(h_0^3(x_2, 0, x_3), h_0^3(x_1, x_2, 0), h_0^3(0, x_2, x_3)).$$

Next, for $\lambda \in [0, 1]$, we consider the equation in $(\mathbb{R}_+)^3$,

$$(1 - \lambda) h_1^3(x_1, 0, x_3) + \lambda (R - x_1 - \eta x_3) = 0,$$

$$(1 - \lambda) h_2^3(x_1, x_2, 0) + \lambda (R - \eta x_1 - x_2) = 0,$$

$$(1 - \lambda) h_3^3(0, x_2, x_3) + \lambda (R - \eta x_2 - x_3) = 0,$$

where $\eta > 1$ is a suitable constant to be fixed later.

Suppose that $z = (z_1, z_2, z_3) \in (\mathbb{R}_+)^3$ is a solution of (23) for some $\lambda \in [0, 1]$ (the case $\lambda = 0$ is already included in the previous step). If, by contradiction, $z_1 \geq R$, we have that $h_1^3(z_1, 0, z_3) \leq h_1^3(z_1, 0, 0) < 0$ and therefore, from the first equation in (23), we obtain that $\lambda = 1$, $z_3 = 0$ and $z_2 = R$. Then, by the third equation in (23), we have $z_2 = R/\eta$ and this yields to a contradiction with the second equation in (23). Hence, we find that $z_1 < R$. Similarly, we can show that $z_2 < R$ and $z_3 < R$.

As $h_0^3(0, 0, 0) > 0$, we have that $z \neq 0$. So, suppose that $z_1 > 0$. We claim that also $z_2 > 0$ and $z_3 > 0$. At first, suppose by contradiction that $z_2 = z_3 = 0$. In this case, from the third equation in (23), we have that $(1 - \lambda) h_0^3(0, 0, 0) + \lambda R = 0$, which is impossible as $h_0^3(0) > 0$. 
Next, suppose by contradiction that \( z_2 > 0 \) and \( z_3 = 0 \). In this case, using \( z_1 < R \), we find that \( \lambda < 1 \) and thus, from the first equation in (23), we have that \( h^{(0)}_{i}(z_1,0,0) < 0 \). This implies that \( z_1 > \overline{a}_1 \). Now, by the monotonicity of \( h^g \) with respect to the first variable, and using condition \((A_1)\), we find that \( h^g_i(z, z_2, 0) \leq h^g_i(\overline{a}_1, z_2, 0) < m_1 h^g_i(\overline{a}_1, z_2, 0) \leq m_1 h^g_i(\overline{a}_1, 0, 0) = 0 \). Hence, from the second equation in (23), we obtain that \((R - \eta z_1 - z_2) > 0\) and then \( z_1 < R/\eta \), with \( z_2 > u_1 > \delta \). Thus we get a contradiction if we have chosen at the beginning

\[ \eta \geq R/\delta. \quad (24) \]

In the same manner, if we suppose by contradiction that \( z_2 = 0 \) and \( z_3 > 0 \), then, using \( z_3 < R \), we find that \( \lambda < 1 \) and from the third equation in (23), we have that \( h^{(0)}_{i}(0,0,z_3) < 0 \). This implies that \( z_3 > \overline{a}_3 \). Now, by the monotonicity of \( h^g \) with respect to the third variable, and using condition \((A_3)\), we find that \( h^g_i(z_1,0,z_3) \leq h^g_i(\overline{a}_1,0,\overline{a}_3) < m_3 h^g_i(\overline{a}_1,0,\overline{a}_3) \leq m_3 h^g_i(0,0,\overline{a}_3) = 0 \). Hence, from the first equation in (23), we obtain that \((R - z_1 - \eta z_3) > 0 \) with \( z_3 > \overline{a}_3 > \delta \). Thus we get again a contradiction if (24) is assumed.

Therefore, under the supplementary assumption (24), we have proved that \( z_i > 0 \), for \( i = 1,2,3 \), when \( z_1 > 0 \). In the same manner, one can check that if \( z_2 > 0 \) or if \( z_3 > 0 \), then all the other components are positive.

We have thus proved that system (23) in \((\mathbb{R}_+)^3\) has all its solutions contained in \([0,R]^3\) and no solution on the boundary of \([0,R]^3\). Hence, we have that for \( \varepsilon > 0 \) sufficiently small, all the solutions of (23) for any \( \varepsilon \in [0,1] \) are contained in \( \Omega_\varepsilon \). By the homotopic invariance of the Brouwer degree, we find that

\[ \deg(g,\Omega_\varepsilon,0) = \deg(R \cdot 1 - \mathcal{M},\Omega_\varepsilon,0) = -\deg(\mathcal{M},\Omega_\varepsilon,R \cdot 1), \]

where \( 1 = \text{col}(1,1,1) = e_1 + e_2 + e_3 \) and

\[ \mathcal{M} = \begin{pmatrix} 1 & 0 & \eta \\ \eta & 1 & 0 \\ 0 & \eta & 1 \end{pmatrix}. \]

Since \( \det \mathcal{M} = 1 + \eta^3 > 0 \) and the equation \( \mathcal{M} x = R \cdot 1 \) has the unique solution \( x = (\eta + 1)^{-1}R \cdot 1 \), which is positive in all its components, we immediately obtain that

\[ \deg(R \cdot 1 - \mathcal{M},\Omega_\varepsilon,0) = -\deg(\mathcal{M},\Omega_\varepsilon,R \cdot 1) = -\text{sgn}(\det \mathcal{M}) = -1. \]
Therefore, we have proved that
\[ d\left(h^0, (\mathbb{R}^+)\right) \neq 0 \]
and the proof of Theorem 1 is completed. \( \blacksquare \)

**Remark 2.** We notice that a variant of Theorem 1 could be obtained, replacing condition \((H^*_3)\) with condition

\((H^*_3^+)\) For \( i = 1, 2, 3 \), there are positive constants \( m_i \) and \( T \)-periodic continuous maps \( \gamma_i : \mathbb{R} \to \mathbb{R} \), with \( \int_0^T \gamma_i \geq 0 \), such that, for any \( \varepsilon > 0 \), the functions

\[
\phi_{\varepsilon, 1}(t) := \inf_{0 \leq x_1 \leq 1/\varepsilon} \inf_{0 \leq x_2 \leq 1/\varepsilon} \{ m_1 h_1(t, x_1, x_2, 0) - h_2(t, x_1, x_2, 0) \},
\]
\[
\phi_{\varepsilon, 2}(t) := \inf_{0 \leq x_2 \leq 1/\varepsilon} \inf_{0 \leq x_3 \leq 1/\varepsilon} \{ m_2 h_2(t, 0, x_2, x_3) - h_3(t, 0, x_2, x_3) \},
\]
and
\[
\phi_{\varepsilon, 3}(t) := \inf_{0 \leq x_3 \leq 1/\varepsilon} \inf_{0 \leq x_1 \leq 1/\varepsilon} \{ m_3 h_3(t, x_1, 0, x_3) - h_1(t, x_1, 0, x_3) \},
\]
satisfy
\[
\phi_{\varepsilon, i}(t) \geq \gamma_i(t), \quad \forall t \in [0, T] \text{ and } \phi_{\varepsilon, i} \neq \gamma_i.
\]

**Remark 3.** It is clear that the cyclic symmetry of the form \( x_1 \gg x_2, x_2 \gg x_3 \), and \( x_3 \gg x_1 \), which is implicit from condition \((H^*_3^+)\) can be replaced by another condition which implies a cyclic symmetry of the form \( x_1 \gg x_3, x_3 \gg x_2 \) and \( x_2 \gg x_1 \). This would yield, when applied to system (13) with \( p_i = p_j \), a condition of the form \( \beta_i(t) \geq 1 \geq \alpha_i(t) \), with the strict inequalities on sets of positive measure for the \( \alpha_i \)'s or for the \( \beta_i \)'s.

**References**


