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On tree-partition-width

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A B S T R A C T

A *tree-partition* of a graph G is a proper partition of its vertex set into 'bags', such that identifying the vertices in each bag produces a forest. The *width* of a tree-partition is the maximum number of vertices in a bag. The *tree-partition-width* of G is the minimum width of a tree-partition of G. An anonymous referee of the paper [Guoli Ding, Bogdan Oporowski, Some results on tree decomposition of graphs, J. Graph Theory 20 (4) (1995) 481–499] proved that every graph with tree-width $k \geq 3$ and maximum degree $\Delta \geq 1$ has tree-partition-width at most $24k\Delta$. We prove that this bound is within a constant factor of optimal. In particular, for all $k \geq 3$ and for all sufficiently large Δ, we construct a graph with tree-width k, maximum degree Δ, and tree-partition-width at least $(\frac{1}{8} - \epsilon)k\Delta$. Moreover, we slightly improve the upper bound to $\frac{5}{2}(k+1)(\frac{7}{2}\Delta - 1)$ without the restriction that $k \geq 3$.

1. Introduction

A graph[1] H is a *partition* of a graph G if:

- each vertex of H is a set of vertices of G (called a *bag*),
- every vertex of G is in exactly one bag of H, and
- distinct bags A and B are adjacent in H if and only if there is an edge of G with one endpoint in A and the other endpoint in B.

The *width* of a partition is the maximum number of vertices in a bag. Informally speaking, the graph H is obtained from a proper partition of $V(G)$ by identifying the vertices in each part, deleting loops, and replacing parallel edges by a single edge. H is sometimes called the *touching pattern* or *quotient graph* of the partition of $V(G)$.

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[1] All graphs considered are undirected, simple, and finite. Let $V(G)$ and $E(G)$ respectively be the vertex set and edge set of a graph G. Let $\Delta(G)$ be the maximum degree of G.

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If a forest T is a partition of a graph G, then T is a *tree-partition* of G. The *tree-partition-width*[2] of G, denoted by tpw(G), is the minimum width of a tree-partition of G. Tree-partitions were independently introduced by Seese [2] and Halin [3], and have since been widely investigated [4,1,5–8]. Applications of tree-partitions include graph drawing [9–13], graph colouring [14], partitioning graphs into subgraphs with only small components [15], monadic second-order logic [16], and network emulations [17–20]. Planar-partitions and other more general structures have also been studied [21,22,13].

What bounds can be proved on the tree-partition-width of a graph? Let tw(G) denote the tree-width[3] of a graph G. [2] proved the lower bound,

$$2\,\mathrm{tpw}(G) \geq \mathrm{tw}(G) + 1.$$

In general, tree-partition-width is not bounded from above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width [1]. However, tree-partition-width is bounded for graphs of bounded tree-width *and* bounded degree [5,6]. The best known upper bound is due to an anonymous referee of the paper by Ding and Oporowski [5], who proved that

$$\mathrm{tpw}(G) \leq 24\,\mathrm{tw}(G)\,\Delta(G)$$

whenever $\mathrm{tw}(G) \geq 3$ and $\Delta(G) \geq 1$. Using a similar proof, we make the following improvement to this bound without the restriction that $\mathrm{tw}(G) \geq 3$.

Theorem 1. *Every graph G with tree-width $\mathrm{tw}(G) \geq 1$ and maximum degree $\Delta(G) \geq 1$ has tree-partition-width*

$$\mathrm{tpw}(G) < \tfrac{5}{2}\,(\mathrm{tw}(G) + 1)\left(\tfrac{7}{2}\,\Delta(G) - 1\right).$$

Theorem 1 is proved in Section 2. Note that Theorem 1 can be improved in the case of chordal graphs. In particular, a simple extension of a result by Dujmović et al. [11] implies that

$$\mathrm{tpw}(G) \leq \mathrm{tw}(G)\,(\Delta(G) - 1)$$

for every chordal graph G with $\Delta(G) \geq 2$; see [8] for a simple proof. Nevertheless, the following theorem proves that $\mathcal{O}(\mathrm{tw}(G)\,\Delta(G))$ is the best possible upper bound, even for chordal graphs.

Theorem 2. *For every $\epsilon > 0$ and integer $k \geq 3$, for every sufficiently large integer $\Delta \geq \Delta(k, \epsilon)$, for infinitely many values of N, there is a chordal graph G with N vertices, tree-width $\mathrm{tw}(G) \leq k$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width*

$$\mathrm{tpw}(G) \geq \left(\tfrac{1}{8} - \epsilon\right) \mathrm{tw}(G)\,\Delta(G).$$

Theorem 2 is proved in Section 3. Note that Theorem 2 is for $k \geq 3$. For $k = 1$, every tree is a tree-partition of itself with width 1. For $k = 2$, we prove that the upper bound $\mathcal{O}(\Delta(G))$ is again best possible; see Section 4.

2. Upper bound

In this section we prove Theorem 1. The proof relies on the following separator lemma by Robertson and Seymour [25].

2 Tree-partition-width has also been called *strong tree-width* [1,2].

3 A graph is *chordal* if every induced cycle is a triangle. The *tree-width* of a graph G can be defined to be the minimum integer k such that G is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see [23,24] for surveys.

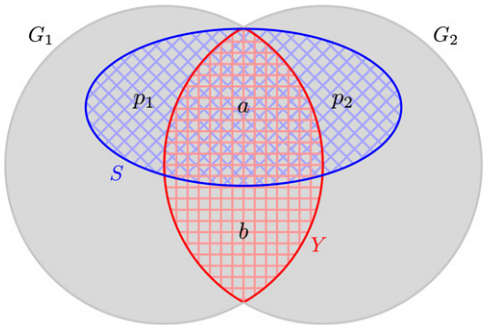

Fig. 1. Illustration of Case 4.

Lemma 1 ([25]). *For every graph G with tree-width at most k, for every set $S \subseteq V(G)$, there are edge-disjoint subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$, $|V(G_1) \cap V(G_2)| \leq k+1$, and $|S - V(G_i)| \leq \frac{2}{3}|S - (V(G_1) \cap V(G_2))|$ for each $i \in \{1, 2\}$.*

Theorem 1 is a corollary of the following stronger result.

Lemma 2. *Let $\alpha := 1 + 1/\sqrt{2}$ and $\gamma := 1 + \sqrt{2}$. Let G be a graph with tree-width at most $k \geq 1$ and maximum degree at most $\Delta \geq 1$. Then G has tree-partition-width*

$$\mathsf{tpw}(G) \leq \gamma(k+1)(3\gamma\Delta - 1).$$

Moreover, for each set $S \subseteq V(G)$ such that

$$(\gamma+1)(k+1) \leq |S| \leq 3(\gamma+1)(k+1)\Delta,$$

there is a tree-partition of G with width at most

$$\gamma(k+1)(3\gamma\Delta - 1),$$

such that S is contained in a single bag containing at most $\alpha|S| - \gamma(k+1)$ vertices.

Proof. We proceed by induction on $|V(G)|$.

Case 1. $|V(G)| < (\gamma+1)(k+1)$: Then no set S is specified, and the tree-partition in which all the vertices are in a single bag satisfies the lemma. Now assume that $|V(G)| \geq (\gamma+1)(k+1)$, and without loss of generality, S is specified.

Case 2. $|V(G) - S| < (\gamma+1)(k+1)$: Then the tree-partition in which S is one bag and $V(G) - S$ is another bag satisfies the lemma. Now assume that $|V(G) - S| \geq (\gamma+1)(k+1)$.

Case 3. $|S| \leq 3(\gamma+1)(k+1)$: Let N be the set of vertices in G that are adjacent to some vertex in S but are not in S. Then $|N| \leq \Delta|S| \leq 3(\gamma+1)(k+1)\Delta$. If $|N| < (\gamma+1)(k+1)$ then add arbitrary vertices from $V(G) - (S \cup N)$ to N until $|N| \geq (\gamma+1)(k+1)$. This is possible since $|V(G) - S| \geq (\gamma+1)(k+1)$.

By induction, there is a tree-partition of $G - S$ with width at most $\gamma(k+1)(3\gamma\Delta - 1)$, such that N is contained in a single bag. Create a new bag only containing S. Since all the neighbours of S are in a single bag, we obtain a tree-partition of G. (S corresponds to a leaf in the touching pattern.) Since $|S| \geq (\gamma+1)(k+1)$, it follows that $|S| \leq \alpha|S| - \gamma(k+1)$ as desired. Now $|S| \leq 3(\gamma+1)(k+1) < \gamma(k+1)(3\gamma\Delta - 1)$. Since the other bags do not change we have the desired tree-partition of G.

Case 4. $|S| \geq 3(\gamma+1)(k+1)$: By Lemma 1, there are edge-disjoint subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$, $|V(G_1) \cap V(G_2)| \leq k+1$, and $|S - V(G_i)| \leq \frac{2}{3}|S - (V(G_1) \cap V(G_2))|$ for each $i \in \{1, 2\}$. Let $Y := V(G_1) \cap V(G_2)$. Let $a := |S \cap Y|$ and $b := |Y - S|$. Thus $a + b \leq k+1$. Let $p_i := |(S \cap V(G_i)) - Y|$. Then $p_1 \leq 2p_2$ and $p_2 \leq 2p_1$. Let $S_i := (S \cap V(G_i)) \cup Y$. Note that $|S_i| = p_i + a + b$ (see Fig. 1).

Now $p_1 + p_2 + a = |S| \geq 3(\gamma+1)(k+1)$. Thus $3p_i + a \geq 3(\gamma+1)(k+1)$ and $3p_i + 3a + 3b \geq 3(\gamma+1)(k+1)$. That is, $|S_i| \geq (\gamma+1)(k+1)$ for each $i \in \{1, 2\}$.

Now $p_1+p_2+a \leq 3(\gamma+1)(k+1)\Delta$. Thus $\frac{3}{2}p_i+a \leq 3(\gamma+1)(k+1)\Delta$ and $p_i \leq 2(\gamma+1)(k+1)\Delta$. Thus $p_i + a + b \leq 2(\gamma + 1)(k + 1)\Delta + (k + 1)$. Hence $|S_i| = p_i + a + b < 3(\gamma + 1)(k + 1)\Delta$.

Thus we can apply induction to the set S_i in the graph G_i for each $i \in \{1, 2\}$. We obtain a tree-partition of G_i with width at most $\gamma(k + 1)(3\gamma\Delta - 1)$, such that S_i is contained in a single bag T_i containing at most $\alpha|S_i| - \gamma(k + 1)$ vertices.

Construct a partition of G by uniting T_1 and T_2. Each vertex of G is in exactly one bag since $V(G_1) \cap V(G_2) = Y \subseteq S_i \subseteq T_i$. Since G_1 and G_2 are edge-disjoint, the touching pattern of this partition of G is obtained by identifying one vertex of the touching pattern of the tree-partition of G_1 with one vertex of the touching pattern of the tree-partition of G_2. Since the touching patterns of the tree-partitions of G_1 and G_2 are forests, the touching pattern of the partition of G is a forest, and we have a tree-partition of G.

Moreover, S is contained in a single bag $T_1 \cup T_2$ and

$$\begin{aligned}
|T_1 \cup T_2| &= |T_1| + |T_2| - |Y| \\
&\leq \alpha|S_1| - \gamma(k+1) + \alpha|S_2| - \gamma(k+1) - (a+b) \\
&= \alpha(p_1+a+b) - \gamma(k+1) + \alpha(p_2+a+b) - \gamma(k+1) - (a+b) \\
&= \alpha(p_1+p_2+a) - 2\gamma(k+1) + (\alpha-1)a + (2\alpha-1)b \\
&\leq \alpha|S| - 2\gamma(k+1) + (2\alpha-1)(a+b) \\
&\leq \alpha|S| - 2\gamma(k+1) + (2\alpha-1)(k+1) \\
&= \alpha|S| - \gamma(k+1).
\end{aligned}$$

Thus $|T_1 \cup T_2| \leq \alpha \cdot 3(\gamma+1)(k+1)\Delta - \gamma(k+1) = \gamma(k+1)(3\gamma\Delta - 1)$. Since the other bags do not change we have the desired tree-partition of G. □

3. General lower bound

The remainder of the paper studies lower bounds on the tree-partition-width. The graphs employed are chordal. We first show that tree-partitions of chordal graphs can be assumed to have certain useful properties.

Lemma 3. *Every chordal graph G has a tree-partition T with width $\mathrm{tpw}(G)$, such that for every independent set S of simplicial*[4] *vertices of G, and for every bag B of T, either $B = \{v\}$ for some vertex $v \in S$, or the induced subgraph $G[B - S]$ is connected.*

Proof. Let T_0 be a tree-partition of a chordal graph G with width $\mathrm{tpw}(G)$. Let T be the partition of G obtained from T_0 by replacing each bag B of T_0 by bags corresponding to the connected components of $G[B]$. Add an edge between bags A and B of T if and only if there is an edge of G between A and B. Then T has width at most $\mathrm{tpw}(G)$.

To prove that T is a forest, suppose on the contrary that T contains an induced cycle C. Since each bag in C induces a connected subgraph of G, G contains an induced cycle D with at least one vertex from each bag in C. Since G is chordal, D is a triangle. Thus C is a triangle, implying that the vertices in D were in distinct bags in T_0 (since the bags of T that replaced each bag of T_0 form an independent set). Hence the bags of T_0 that contain D induce a triangle in T_0, which is the desired contradiction since T_0 is a forest. Hence T is a forest.

Let S be an independent set of simplicial vertices of G. Consider a bag B of T. By construction, $G[B]$ is connected. First suppose that $B \subseteq S$. Since S is an independent set and $G[B]$ is connected, $B = \{v\}$ for some vertex $v \in S$.

Now assume that $B - S \neq \emptyset$. Suppose on the contrary that $G[B - S]$ is disconnected. Thus $B \cap S$ is a cut-set in $G[B]$. Let v and w be vertices in distinct components of $G[B - S]$ such that the distance between v and w in $G[B]$ is minimised. (This is well-defined since $G[B]$ is connected.) Since S is an

4 A vertex is *simplicial* if its neighbourhood is a clique.

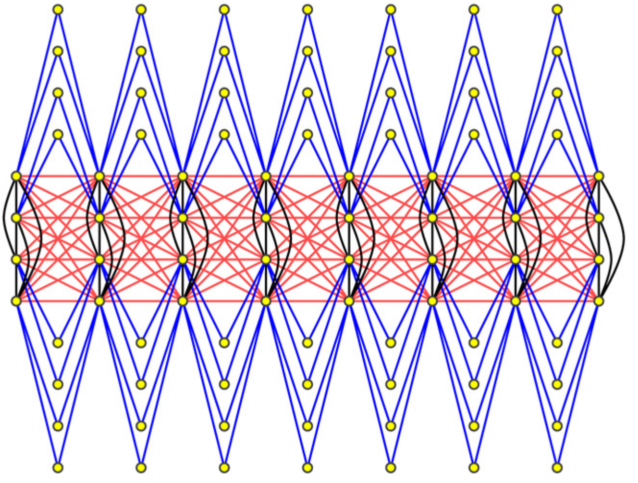

Fig. 2. The graph G with $k = 4$, $\Delta = 15$, and $n = 8$.

independent set, every shortest path between v and w in $G[B]$ has only two edges. That is, v and w have a common neighbour x in $B \cap S$. Since x is simplicial, v and w are adjacent. This contradiction proves that $G[B - S]$ is connected. □

The next lemma is the key component of the proof of Theorem 2. For integers $a < b$, let $[a, b] := \{a, a + 1, \dots, b\}$ and $[b] := [1, b]$.

Lemma 4. *For all integers $k \geq 2$ and $\Delta \geq 3k + 1$, for infinitely many values of N there is a chordal graph G with N vertices, tree-width $\mathrm{tw}(G) = 2k - 1$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width $\mathrm{tpw}(G) > \frac{1}{4}k(\Delta - 3k)$.*

Proof. Let n be an integer with $n > \max\{\frac{1}{2}k(\Delta - 3k), 2\}$. Let H be the graph with vertex set $\{(x, y) : x \in [n], y \in [k]\}$, where distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| \leq 1$. The set of vertices $\{(x, y) : y \in [k]\}$ is the *x-column*. The set of vertices $\{(x, y) : x \in [n]\}$ is the *y-row*. Observe that each column induces a k-vertex clique, and each row induces an n-vertex path.

Let C be an induced cycle in H. If (x, y) is a vertex in C with x minimum then the two neighbours of (x, y) in C are adjacent. Thus C is a triangle. Hence H is chordal. Observe that each pair of consecutive columns form a maximum clique of $2k$ vertices in H. Thus H has tree-width $2k - 1$. Also note that H has maximum degree $3k - 1$.

An edge of H between vertices (x, y) and $(x + 1, y)$ is *horizontal*. As illustrated in Fig. 2, construct a graph G from H as follows. For each horizontal edge vw of H, add $\lceil\frac{1}{2}(\Delta - 3k)\rceil$ new vertices, each adjacent to v and w. Since H is chordal and each new vertex is simplicial, G is chordal. The addition of degree-2 vertices to H does not increase the maximum clique size (since $k \geq 2$). Thus G has clique number $2k$ and tree-width $2k - 1$. Since each vertex of H is incident to at most two horizontal edges, G has maximum degree $3k - 1 + 2\lceil\frac{1}{2}(\Delta - 3k)\rceil \leq \Delta$.

Observe that $V(G) - V(H)$ is an independent set of simplicial vertices in G. By Lemma 3, G has a tree-partition T with width $\mathrm{tpw}(G)$, such that for every bag B of T, either $B = \{v\}$ for some vertex v of $G - H$, or the induced subgraph $H[B]$ is connected. Since G is connected, T is a (connected) tree. Let U be the tree-partition of H induced by T. That is, to obtain U from T delete the vertices of $G - H$ from each bag, and delete empty bags. Since H is connected, U is a (connected) tree. By Lemma 3, each bag of U induces a connected subgraph of H.

Suppose that U only has two bags B and C. Then one of B and C contains at least $\frac{1}{2}nk$ vertices. Since $k \geq 2$, we have $\mathrm{tpw}(G) \geq \frac{1}{2}nk > \frac{1}{4}k(\Delta - 3k)$, as desired. Now assume that U has at least three bags.

Consider a bag B of U. Let $\ell(B)$ be the minimum integer such that some vertex in B is in the $\ell(B)$-column, and let $r(B)$ be the maximum integer such that some vertex in B is in the $r(B)$-column. Since $H[B]$ is connected, there is a path in B from the $\ell(B)$-column to the $r(B)$-column. By the definition of H, for each $x \in [\ell(B), r(B)]$, the x-column contains a vertex in B. Let $I(B)$ be the closed real interval from $\ell(B) - \frac{1}{2}$ to $r(B) + \frac{1}{2}$. Observe that two bags B and C of U are adjacent if and only if $I(B) \cap I(C) \neq \emptyset$. Thus $\{I(B) : B \text{ is a bag of } U\}$ is an interval representation of the tree U. Every tree that is an interval graph is a caterpillar[5]; see [26] for example. Thus U is a caterpillar.

Let $\preceq$ be the relation on the set of non-leaf bags of U defined by $A \preceq B$ if and only if $\ell(A) \leq \ell(B)$ and $r(A) \leq r(B)$. We claim that $\preceq$ is a total order. It is immediate that $\preceq$ is reflexive and transitive. To prove that $\preceq$ is antisymmetric, suppose on the contrary that $A \preceq B$ and $B \preceq A$ for distinct non-leaf bags A and B. Thus $\ell(A) = \ell(B)$ and $r(A) = r(B)$. Since U has at least three bags, there is a third bag C that contains a vertex in the $(\ell(A) - 1)$-column or in the $(r(A) + 1)$-column. Thus $\{A, B, C\}$ induce a triangle in U, which is the desired contradiction. Hence $\preceq$ is antisymmetric. To prove that $\preceq$ is total, suppose on the contrary that $A \not\preceq B$ and $B \not\preceq A$ for distinct non-leaf bags A and B. Now $A \not\preceq B$ implies that $\ell(A) > \ell(B)$ or $r(A) > r(B)$. Without loss of generality, $\ell(A) > \ell(B)$. Thus $B \not\preceq A$ implies that $r(B) > r(A)$. Hence the interval $[\ell(A), r(A)]$ is strictly within the interval $[\ell(B), r(B)]$ at both ends. For each $x \in [\ell(A), r(A)]$, every vertex in the x-column is in $A \cup B$, as otherwise U would contain a triangle (since each column is a clique in H). Moreover, every vertex in the $(\ell(A) - 1)$-column or in the $(r(A) + 1)$-column is in B, as otherwise U would contain a triangle (since the union of consecutive columns is a clique in H). Thus every neighbour of every vertex in A is in B. That is, A is a leaf in U. This contradiction proves that $\preceq$ is a total order on the set of non-leaf bags of U.

Suppose that U has a 4-vertex path (A, B, C, D) as a subgraph.

Thus B and C are non-leaf bags. Without loss of generality, $B \prec C$. If every column contains vertices in both B and C, then B and C and any other bag would induce a triangle in U (since each column induces a clique in H). Thus some column contains a vertex in B but no vertex in C, and some column contains a vertex in C but no vertex in B. Let p be the maximum integer such that some vertex in B is in the p-column, but no vertex in C is in the p-column. Let q be the minimum integer such that some vertex in C is in the q-column, but no vertex in B is in the q-column. Now $p < q$ since $B \prec C$.

We claim that the $(p + 1)$-column contains a vertex in C. If not, then the $(p + 1)$-column contains no vertex in B by the definition of p. Thus $r(B) = p$ since $H[B]$ is connected. Since B is adjacent to C in U, $\ell(C) \leq r(B) + 1 = p + 1$. In particular, the $(p + 1)$-column contains a vertex in C. Since $H[C]$ is connected, for $x \in [p + 1, q]$, each x-column contains a vertex in C. In fact, $\ell(C) = p + 1$ since the p-column contains no vertex in C. By symmetry, for $x \in [p, q - 1]$, each x-column contains a vertex in B, and $r(C) = q - 1$.

The union of the p-column and the $(p + 1)$-column only contains vertices in $B \cup C$, as otherwise U would contain a triangle (since the union of two consecutive columns is a clique in H). By the definition of p, no vertex in the p-column is in C. Thus every vertex in the p-column is in B. By symmetry, every vertex in the q-column is in C. Now for each $y \in [k]$, the vertices $(p, y), (p + 1, y), \ldots, (q, y)$ are all in $B \cup C$, the first vertex (p, y) is in B, and the last vertex (q, y) is in C. Thus $(x, y) \in B$ and $(x + 1, y) \in C$ for some $x \in [p, q - 1]$. That is, in every row of H there is a horizontal edge with one endpoint in B and the other in C.

Thus there are at least k horizontal edges with one endpoint in B and the other in C (now considered to be bags of T). For each such horizontal edge vw, each vertex of $G - H$ adjacent to v and w is in $B \cup C$, as otherwise T would contain a triangle. There are $\lceil \frac{1}{2}(\Delta - 3k) \rceil$ such vertices of $G - H$ for each of the k horizontal edges between B and C. Thus $|B \cup C| \geq \frac{1}{2}k(\Delta - 3k)$. Thus one of B and C has at least $\frac{1}{4}k(\Delta - 3k)$ vertices. Hence $\mathsf{tpw}(G) \geq \frac{1}{4}k(\Delta - 3k)$ as desired.

Now assume that U has no 4-vertex path as a subgraph.

A tree is a star if and only if it has no 4-vertex path as a subgraph. Hence U is a star. Let R be the root bag of U. If R contains a vertex in every column then $|R| \geq n$, implying $\mathsf{tpw}(G) \geq n \geq \frac{1}{4}k(\Delta - 3k)$, as desired. Now assume that for some $x \in [n]$, the x-column of H contains no vertex in R. Let B be a bag

5 A *caterpillar* is a tree such that deleting the leaves gives a path.

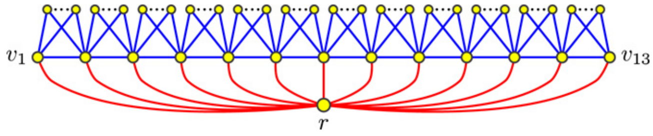

Fig. 3. Illustration for Theorem 3 with $\Delta = 13$.

containing some vertex in the x-column. The x-column induces a clique in H, the only bag in U that is adjacent to B is R, and R contains no vertex in the x-column. Thus every vertex in the x-column is in B. Since R is the only bag in U adjacent to B, there are at least k horizontal edges with one endpoint in B and the other endpoint in R. As in the case when U contained a 4-vertex path, we conclude that $\mathsf{tpw}(G) \geq \frac{1}{4}k(\Delta - 3k)$ as desired. □

Proof of Theorem 2. Let $\ell := \lceil \frac{k}{2} \rceil$. Thus $\ell \geq 2$. By Lemma 4, for each integer $\Delta \geq \Delta(k, \epsilon) := \max\{3\ell + 1, \frac{3\ell}{8\epsilon}\}$, there are infinitely many values of N for which there is a chordal graph G with N vertices, tree-width $\mathsf{tw}(G) = 2\ell - 1 \leq k$, maximum degree $\Delta(G) \leq \Delta$, and tree-partition-width $\mathsf{tpw}(G) > \frac{1}{4}\ell(\Delta - 3\ell)$, which is at least $(\frac{1}{8} - \epsilon)k\Delta$ since $\Delta \geq \frac{3\ell}{8\epsilon}$. □

A *domino* tree decomposition[6] is a tree decomposition in which each vertex appears in at most two bags. The *domino tree-width* of a graph G, denoted by $\mathsf{dtw}(G)$, is the minimum width of a domino tree decomposition of G. Domino tree-width behaves like tree-partition-width in the sense that $\mathsf{dtw}(G) \geq \mathsf{tw}(G)$, and $\mathsf{dtw}(G)$ is bounded for graphs of bounded tree-width and bounded degree [1]. The best upper bound is

$$\mathsf{dtw}(G) \leq (9\,\mathsf{tw}(G) + 7)\ \Delta(G)\ (\Delta(G) + 1) - 1,$$

which is due to Bodlaender [4], who also constructed a graph G with

$$\mathsf{dtw}(G) \geq \frac{1}{12}\,\mathsf{tw}(G)\,\Delta(G) - 2.$$

Tree-partition-width and domino tree-width are related in that every graph G satisfies

$$\mathsf{dtw}(G) \geq \mathsf{tpw}(G) - 1,$$

as observed by Bodlaender and Engelfriet [1]. Thus Theorem 2 provides examples of graphs G with

$$\mathsf{dtw}(G) \geq \left(\frac{1}{8} - \epsilon\right)\,\mathsf{tw}(G)\,\Delta(G).$$

This represents a small constant-factor improvement over the above lower bound by Bodlaender [4].

4. Lower bound for tree-width 2

We now prove a lower bound on the tree-partition-width of graphs with tree-width 2.

Theorem 3. *For all odd $\Delta \geq 11$ there is a chordal graph G with tree-width 2, maximum degree Δ, and tree-partition-width $\mathsf{tpw}(G) \geq \frac{2}{3}(\Delta - 1)$.*

Proof. As illustrated in Fig. 3, let G be the graph with

$$\begin{aligned} V(G) &:= \{r\} \cup \{v_i : i \in [\Delta]\} \cup \left\{w_{i,\ell} : i \in [\Delta - 1], \ell \in \left[\tfrac{1}{2}(\Delta - 3)\right]\right\}, \quad \text{and} \\ E(G) &:= \{rv_i : i \in [\Delta]\} \cup \{v_iv_{i+1} : i \in [\Delta - 1]\} \\ &\quad\ \cup \left\{v_iw_{i,\ell}, v_{i+1}w_{i,\ell} : i \in [\Delta - 1], \ell \in \left[\tfrac{1}{2}(\Delta - 3)\right]\right\}. \end{aligned}$$

Observe that G has maximum degree Δ. Clearly every induced cycle of G is a triangle. Thus G is chordal. Observe that G has no 4-vertex clique. Thus G has tree-width 2.

6 See [27] for an introduction to tree decompositions.

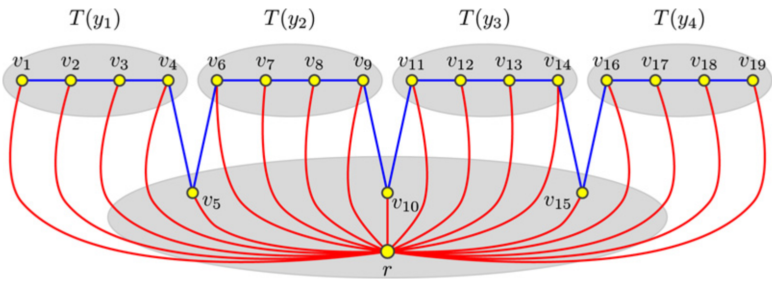

Fig. 4. Illustration for Theorem 3 with $\Delta = 19$ and $d = 4$.

Let T be the tree-partition of G from Lemma 3. Then T has width tpw(G), and every bag induces a connected subgraph of G. Let R be the bag containing r. Let $B_1, \dots, B_d$ be the bags, not including R, that contain some vertex v_i. Thus R is adjacent to each B_j (since r is adjacent to each v_i). Since $\{w_{i,\ell} : i \in [\Delta - 1], \ell \in [\frac{1}{2}(\Delta - 3)]\}$ is an independent set of simplicial vertices, by Lemma 3, for each $j \in [d]$, the vertices $\{v_1, v_2, \dots, v_\Delta\} \cap B_j$ induce a (connected) subpath of G.

First suppose that $d = 0$. Then the $\Delta + 1$ vertices $\{r, v_1, \dots, v_\Delta\}$ are contained in one bag R. Thus $\text{tpw}(G) \geq \Delta + 1 \geq \frac{2}{3}(\Delta - 1)$.

Now suppose that $d = 1$. Thus $\{r, v_1, \dots, v_\Delta\} \subseteq R \cup B_1$. In addition, at least one edge $v_i v_{i+1}$ has one endpoint in R and the other endpoint in B_1. Thus $w_{i,\ell} \in R \cup B_1$ for each $\ell \in [\frac{1}{2}(\Delta - 3)\}]$. Hence $1+\Delta+\frac{1}{2}(\Delta-3)$ vertices are contained in two bags. Thus one bag contains at least $\frac{1}{4}(3\Delta-1)$ vertices, and $\text{tpw}(G) \geq \frac{1}{4}(3\Delta - 1) \geq \frac{2}{3}(\Delta - 1)$.

Finally suppose that $d \geq 2$. Since $\{v_1, v_2, \dots, v_\Delta\} \cap B_j$ induce a subpath in each bag B_j, we can assume that $\{v_1, v_2, \dots, v_\Delta\} \cap B_j = \{v_i : i \in [f(j), g(j)]\}$, where

$$1 \leq f(1) \leq g(1) < f(2) \leq g(2) < \cdots < f(d) \leq g(d) \leq \Delta.$$

Distinct B_j bags are not adjacent (since T is a tree). Thus $v_{f(j)-1} \in R$ for each $j \in [2, d]$. Similarly, $v_{g(j)+1} \in R$ for each $j \in [d-1]$. Thus $w_{f(j)-1,\ell} \in R \cup B_j$ for each $j \in [2, d]$ and $\ell \in [\frac{1}{2}(\Delta - 3)\}]$. Similarly, $w_{g(j),\ell} \in R \cup B_j$ for each $j \in [d-1]$ and $\ell \in [\frac{1}{2}(\Delta - 3)\}]$ (see Fig. 4).

Hence the bags $R, B_1, \dots, B_d$ contain at least

$$1 + \Delta + 2(d - 1) \cdot \tfrac{1}{2}(\Delta - 3)$$

vertices. Therefore one of these bags has at least

$$(1 + \Delta + (d - 1)(\Delta - 3))/(d + 1)$$

vertices, which is at least $\frac{2}{3}(\Delta - 1)$. Hence $\text{tpw}(G) \geq \frac{2}{3}(\Delta - 1)$. □

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