The Constrained Bilinear Form and the C-Numerical Range

Nam-Kiu Tsing*
Department of Mathematics
University of Hong Kong
Hong Kong

Submitted by Olga Taussky Todd

ABSTRACT

Let V be an n-dimensional unitary space with inner product $(\cdot,\cdot)$ and S the set $\{x \in V : (x,x)=1\}$. For any $A \in \mathrm{Hom}(V,V)$ and $q \in \mathbb{C}$ with $|q| \leqslant 1$, we define

$$W(A:q) = \{(Ax,y) : x, y \in S,\ (x,y)=q\}.$$

If $q=1$, then $W(A:q)$ is just the classical numerical range $\{(Ax,x): x \in S\}$, the convexity of which is well known. Another generalization of the numerical range is the C-numerical range, which is defined to be the set

$$W_C(A) = \{\mathrm{tr}(CU^*AU) : U \text{ unitary}\}$$

where $C \in \mathrm{Hom}(V,V)$. In this note, we prove that $W(A:q)$ is always convex and that $W_C(A)$ is convex for all A if $\mathrm{rank}\, C = 1$ or $n=2$.

1. INTRODUCTION

Let V be an n-dimensional unitary space with inner product $(\cdot,\cdot)$, S the set $\{x \in V : (x,x)=1\}$, and $\mathrm{Hom}(V,V)$ the set of all linear operators on V. For any $A \in \mathrm{Hom}(V,V)$, the numerical range of A is defined to be the compact set

$$W(A) = \{(Ax,x) : x \in S\}.$$

*The author wishes to express his thanks to Dr. Yik-Hoi Au-Yeung for his valuable advice and continuous encouragement.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/84/$3.00

It is well known that $W(A)$ is always convex [5, 10] and in particular is an elliptical disk (possibly degenerate) if $n=2$ [8]. For $q \in \mathbb{C}$ with $|q| \leqslant 1$, Marcus and Andresen [6] considered the set

$$W(A:q)=\{(Ax,y):x,y\in S,(x,y)=q\}$$

and showed that the set generated by rotating $W(A:q)$ about the origin is an annulus. They also gave the inner and outer radii of this annulus for hermitian A. If $q=1$, then $W(A:q)$ is identical with $W(A)$. So $W(A:q)$ is a generalized form of the classical numerical range.

Another generalization of $W(A)$ is the C-numerical range, which is given by Goldberg and Straus [4] and defined as

$$W_C(A)=\{\operatorname{tr}(CU^*AU):U \text{ unitary}\}$$

where $C\in \operatorname{Hom}(V,V)$ is fixed. Westwick [12] (and later Poon [9] with another proof) showed that if C is hermitian, then $W_C(A)$ is convex for all A. It is not true in general that $W_C(A)$ is convex even for normal A and C. For example, if $n=3$ and A, C normal, Au-Yeung and Poon [1] gave a necessary and sufficient condition for the convexity of $W_C(A)$. Au-Yeung and the present author [2] also showed that if C is a normal operator with noncollinear eigenvalues and $A=C^*$, then $W_C(A)$ is not convex. However, if C is normal, the author [11] has proved that $W_C(A)$ is always star-shaped.

In this present note, we show that $W(A:q)$, which equals $W_C(A)$ for some particular C, is always convex, and we use this result to prove the convexity of $W_C(A)$ for $\operatorname{rank} C=1$. We neglect the trivial case $n=1$ and always assume $n\geqslant 2$ in the following.

2. THE CONSTRAINED BILINEAR FORM

The following lemma is given by Das and Embry-Wardrop [3].

LEMMA 1. *Let $n=2$ and $x,z\in S$ be such that $(Ax,x)\neq(Az,z)$.*

(a) *If $W(A)$ is the line segment $[(Ax,x),(Az,z)]$, then*

$$(x,z)=(Ax,z)=(Az,x)=0.$$

(b) *If $(Ax,x)\in \partial W(A)$, i.e. the boundary of $W(A)$, and $(x,z)=0$, then $(Az,z)\in\partial W(A)$ and $\frac{1}{2}[(Ax,x)+(Az,z)]$ is the center of $W(A)$.*

LEMMA 2. *Suppose x and z are orthonormal vectors in V and* $(Ax,x)\in \partial W(A)$. *Then*

$$|(Ax,z)|=|(Az,x)|.$$

Proof. Let $u=\sqrt{1-\epsilon^2}\,x+\epsilon e^{\phi\sqrt{-1}}z$, where $1>\epsilon>0$ and $\phi\in\mathbb{R}$. Then $u\in S$ and

$$\begin{aligned}(Au,u)&=(Ax,x)+\epsilon^2[(Az,z)-(Ax,x)]+\epsilon\sqrt{1-\epsilon^2}\,\zeta_\phi\\&\in W(A),\end{aligned}$$

where $\zeta_\phi=(Ax,e^{\phi\sqrt{-1}}z)+(e^{\phi\sqrt{-1}}Az,x)$. As ϕ varies in $\mathbb{R}$, the locus of ζ_ϕ is an ellipse (possibly degenerate) centered at the origin with $2\big||(Ax,z)|-|(Az,x)|\big|$ as length of minor axis. Suppose this ellipse does not degenerate. Then for ϵ small enough, the locus of (Au,u), as ϕ varies in $\mathbb{R}$, is an ellipse enclosing (Ax,x). But then $(Ax,x)\notin\partial W(A)$ by the convexity of $W(A)$. Thus this ellipse must degenerate, and hence $|(Ax,z)|=|(Az,x)|$. ∎

LEMMA 3. *Let $n=2$ and $W(A)$ be a nondegenerating elliptical disk. Suppose $\eta\in W(A)$.*

(a) *If $\eta\in\partial W(A)$, then there is exactly one (up to scalar multiples) $u\in S$ satisfying $(Au,u)=\eta$.*

(b) *If $\eta\notin\partial W(A)$, then there are exactly two (up to scalar multiples) $u\in S$ satisfying $(Au,u)=\eta$.*

Proof. Let $\eta\in W(A)$ and $x,z\in S$ be such that $[(Ax,x),(Az,z)]$ is a chord of $W(A)$ passing through η and the center of $W(A)$. Since $W(A)$ is nondegenerating, $(Ax,x)\neq(Az,z)$. By Lemma 1(b) we may assume $(x,z)=0$, and hence by Lemma 2, $(Ax,z)=\zeta\theta$ and $(Az,x)=\zeta\bar{\theta}$ for some $\zeta,\theta\in\mathbb{C}$ with $|\theta|=1$. For any $u\in S$, u can be written as $\alpha x+\beta z$ for some $\alpha,\beta\in\mathbb{C}$ with $|\alpha|^2+|\beta|^2=1$. Then

$$\begin{aligned}(Au,u)&=|\alpha|^2(Ax,x)+|\beta|^2(Az,z)+2(\operatorname{Re}\alpha\bar{\beta}\theta)\zeta\\&=(Ax,x)+|\beta|^2[(Az,z)-(Ax,x)]+2(\operatorname{Re}\alpha\bar{\beta}\theta)\zeta.\qquad(1)\end{aligned}$$

ζ cannot be equal to $s[(Az,z)-(Ax,x)]$ for any $s\in\mathbb{R}$. Otherwise (Au,u) would always be on the line joining (Ax,x) and (Az,z), and then $W(A)$ must be degenerate. Suppose $(Au,u)=\eta$ and $\eta\in\partial W(A)$. We may assume without loss of generality that $\eta=(Ax,x)$. Then from (1) and the fact that

$\zeta \neq s[(Az, z)-(Ax, x)]$ for any $s \in \mathbb{R}$, we have

$$|\beta|^2[(Az, z)-(Ax, x)]+2(\operatorname{Re}\alpha\bar{\beta}\theta)\zeta = 0$$
$$\Rightarrow \quad |\beta|^2[(Az, z)-(Ax, x)] = 2(\operatorname{Re}\alpha\bar{\beta}\theta) = 0.$$

Therefore $\beta = 0$ and hence u must be a scalar multiple of x.

Now suppose $(Au, u) = \eta \notin \partial W(A)$. Then $(Au, u) = t(Ax, x)+(1-t)(Az, z)$ for some $1 > t > 0$. From (1), the only possible solutions for u, if α is chosen to be real nonnegative, are $(\alpha, \beta) = (\sqrt{t}, \sqrt{1-t}\,\theta\sqrt{-1})$ and $(\sqrt{t}, -\sqrt{1-t}\,\theta\sqrt{-1})$. Hence (b) is justified also. ∎

LEMMA 4. *Let the line segment* $[(Ax, x),(Az, z)]$ *be a chord of* $W(A)$ *where* $x, z \in S$ *and* $(Ax, x) \neq (Az, z)$. *Then*

$$(x, z)(Az, x) = (z, x)(Ax, z).$$

Proof. The vectors x and z are linearly independent, since $(Ax, x) \neq (Az, z)$. Let $V' = \langle x, z\rangle$, i.e. the subspace spanned by x and z, and $A' = PA|_{V'}$, where P is the orthogonal projection of V onto V'. Then $A' \in \operatorname{Hom}(V', V')$ and $(A'u, v) = (Au, v)$ for all $u, v \in V'$. In particular, $W(A') \subset W(A)$ and so $[(Ax, x),(Az, z)]$ is a chord of $W(A')$ also. If $W(A')$ is degenerating, then $W(A') = [(Ax, x),(Az, z)]$. By Lemma 1(a), $(x, z) = 0$ and the result follows. If $W(A')$ is nondegenerating and $(x, z) \neq 0$, let $w \in V'$ be orthonormal to x. Then $z = \alpha x + \beta w$ for some $\alpha, \beta \in \mathbb{C}\backslash\{0\}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. By Lemma 2, since $(Ax, x) \in \partial W(A)$, we can write $(Ax, w) = \zeta\theta$ and $(Aw, x) = \zeta\bar{\theta}$ for some $\zeta, \theta \in \mathbb{C}$ with $|\theta| = 1$. Hence

$$\begin{aligned}(Az, z) &= (A(\alpha x + \beta w), \alpha x + \beta w)\\ &= |\alpha|^2(Ax, x)+|\beta|^2(Aw, w)+2(\operatorname{Re}\alpha\bar{\beta}\theta)\zeta.\end{aligned}$$

Let $z' = \alpha x + \beta' w$, where $\beta' = \alpha\theta\bar{\beta}/(\overline{\alpha\theta})$. Then z' is a unit vector in V' and $(Az', z') = (Az, z)$ by direct calculation. From Lemma 3(a), since $(Az, z) \in \partial W(A')$, β must equal β'. So $\bar{\alpha}\beta\bar{\theta} = \alpha\bar{\beta}\theta$. Then

$$\begin{aligned}(x, z)(Az, x) &= (x, \alpha x + \beta w)(A(\alpha x + \beta w), x)\\ &= \bar{\alpha}[\alpha(Ax, x)+\beta\bar{\theta}\zeta]\\ &= \alpha[\bar{\alpha}(Ax, x)+\bar{\beta}\theta\zeta]\\ &= (z, x)(Ax, z).\end{aligned}$$

∎

LEMMA 5. *For any $q \in \mathbf{C}$ with $|q| \leqslant 1$,*

$$W(A:q)=\left\{\xi \in \mathbf{C}:|\xi-q(Ax,x)| \leqslant \sqrt{1-|q|^2}\left[\|Ax\|^2-|(Ax,x)|^2\right]^{1/2} \text{ for some } x \in S\right\}$$

where $\|\cdot\|$ denotes the norm induced by the inner product.

Proof. For any pair of $x, y \in S$ satisfying $(x,y)=q$, y can be written as $\bar{q}x+pw$ where w is some vector orthonormal to x and $p=\sqrt{1-|q|^2}$. Then

$$\begin{aligned}(Ax,y)&=(Ax,\bar{q}x+pw)\\&=q(Ax,x)+p(Ax,w).\end{aligned}$$

By considering Ax as a linear combination of vectors in an orthonormal basis containing x and w, and using the Bessel's inequality,

$$\begin{aligned}|(Ax,y)-q(Ax,x)|&=p|(Ax,w)|\\&\leqslant p\left[\|Ax\|^2-|(Ax,x)|^2\right]^{1/2}.\end{aligned}$$

Conversely, suppose $p=\sqrt{1-|q|^2}$ and $\xi \in \mathbf{C}$ satisfies

$$|\xi-q(Ax,x)| \leqslant p\left[\|Ax\|^2-|(Ax,x)|^2\right]^{1/2}$$

for some $x \in S$. Let u be any unit eigenvector of A, and consider a continuous function $f:[0,1]\to S$ satisfying $f(0)=x$ and $f(1)=u$. Since

$$\begin{aligned}p\left[\|Au\|^2-|(Au,u)|^2\right]^{1/2}&=0\\&\leqslant|\xi-q(Au,u)|,\end{aligned}$$

by the continuity of f there exists $v=f(t_0)\in S$ for some $t_0\in[0,1]$ such that

$$|\xi-q(Av,v)|=p\left[\|Av\|^2-|(Av,v)|^2\right]^{1/2}.$$

Then $\xi = q(Av, v) + p\theta[\|Av\|^2 - |(Av, v)|^2]^{1/2}$ for some $\theta \in \mathbb{C}$ with $|\theta| = 1$. Let $z \in S$ be such that $(v, z) = 0$, and define

$$w = \begin{cases} \dfrac{Av - (Av, v)v}{\|Av - (Av, v)v\|} & \text{if } Av \neq (Av, v)v, \\ z & \text{if } Av = (Av, v)v. \end{cases}$$

Then v and w are orthonormal and

$$(Av, w) = \left[\|Av\|^2 - |(Av, v)|^2\right]^{1/2}.$$

Let $y = \bar{q}v + p\bar{\theta}w$. Then $y \in S$, $(v, y) = q$, and

$$\begin{aligned} \xi &= q(Av, v) + p\theta\left[\|Av\|^2 - |(Av, v)|^2\right]^{1/2} \\ &= q(Av, v) + p\theta(Av, w) \\ &= (Av, y) \\ &\in W(A : q). \end{aligned}$$

■

Now we proceed to prove the main result of this section.

THEOREM 1. *Let $q \in \mathbb{C}$ with $|q| \leqslant 1$. Then $W(A : q)$ is convex for all $A \in \mathrm{Hom}(V, V)$.*

Proof. Let $\xi = \lambda\xi_1 + (1 - \lambda)\xi_2$ for some $\xi_1, \xi_2 \in W(A : q)$ with $1 \geqslant \lambda \geqslant 0$. For $i = 1, 2$, by Lemma 5, there exists $x_i \in S$ such that

$$|\xi_i - q(Ax_i, x_i)| \leqslant p\left[\|Ax_i\|^2 - |(Ax_i, x_i)|^2\right]^{1/2}$$

where $p = \sqrt{1 - |q|^2}$. Therefore

$$\begin{aligned} &|\xi - q[\lambda(Ax_1, x_1) + (1 - \lambda)(Ax_2, x_2)]| \\ &\quad = |\lambda[\xi_1 - q(Ax_1, x_1)] + (1 - \lambda)[\xi_2 - q(Ax_2, x_2)]| \\ &\quad \leqslant p\Big\{\lambda\left[\|Ax_1\|^2 - |(Ax_1, x_1)|^2\right]^{1/2} \\ &\qquad + (1 - \lambda)\left[\|Ax_2\|^2 - |(Ax_2, x_2)|^2\right]^{1/2}\Big\}. \end{aligned} \tag{2}$$

Suppose $(Ax_1, x_1) = (Ax_2, x_2)$. We may assume without loss of generality that $\|Ax_1\|^2 - |(Ax_1, x_1)|^2 \geqslant \|Ax_2\|^2 - |(Ax_2, x_2)|^2$. By using (2),

$$\begin{aligned}|\xi - q(Ax_1, x_1)| &= |\xi - q[\lambda(Ax_1, x_1) + (1-\lambda)(Ax_2, x_2)]| \\ &\leqslant p\left[\|Ax_1\|^2 - |(Ax_1, x_1)|^2\right]^{1/2}.\end{aligned}$$

So $\xi \in W(A:q)$ by Lemma 5.

Now suppose $(Ax_1, x_1) \neq (Ax_2, x_2)$. Then x_1, x_2 are linearly independent. Denote $\langle x_1, x_2\rangle$ by V' and $PA|_{V'}$ by A', where P is the orthogonal projection of V onto V'. Let x, z be unit vectors of V' such that $[(Ax, x), (Az, z)]$ is a chord of $W(A')$ passing through (Ax_i, x_i) and

$$(Ax_i, x_i) = t_i(Ax, x) + (1 - t_i)(Az, z) \qquad (i = 1,2) \tag{3}$$

with $0 \leqslant t_i < t_2 \leqslant 1$.

There are two possible cases:

Case 1. *$W(A')$ is the line segment* $[(Ax, x), (Az, z)]$. Then by Lemma 1(a), $(x, z) = (Ax, z) = (Az, x) = 0$. For any unit vector $u \in V'$, $u = \alpha x + \beta z$ for some $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. So

$$\begin{aligned}(Au, u) &= |\alpha|^2(Ax, x) + |\beta|^2(Az, z) + (\alpha Ax, \beta z) + (\beta Az, \alpha x) \\ &= |\alpha|^2(Ax, x) + |\beta|^2(Az, z)\end{aligned} \tag{4}$$

and

$$\begin{aligned}\|Au\|^2 &= |\alpha|^2\|Ax\|^2 + |\beta|^2\|Az\|^2 + 2\operatorname{Re}(\alpha Ax, \beta Az) \\ &\leqslant |\alpha|^2\|Ax\|^2 + |\beta|^2\|Az\|^2 + 2|\alpha\beta\rho|,\end{aligned} \tag{5}$$

where $\rho = (Ax, Az)$. For $i = 1,2$, as $x_i \in V'$, by (3), (4), and (5) we have

$$\|Ax_i\|^2 \leqslant t_i\|Ax\|^2 - (1 - t_i)\|Az\|^2 + 2\sqrt{t_i(1 - t_i)}\,|\rho|.$$

Now define

$$\theta = \begin{cases} \dfrac{\rho}{|\rho|} & \text{if } \rho \neq 0, \\ 1 & \text{if } \rho = 0. \end{cases}$$

For $0 \leqslant t \leqslant 1$, let $u(t) = \sqrt{t}\,x + \sqrt{1 - t}\,\theta z$. Then $\|u(t)\| = 1$ for all $t \in [0,1]$.

Case 2. *$W(A')$ is nondegenerating.* If $(x,z)=0$ then, by Lemma 2, $|(Ax,z)|=|(Az,x)|$, and so $(Ax,z)=-\zeta\phi$ and $(Az,x)=\zeta\bar{\phi}$ for some $\zeta,\phi\in\mathbf{C}$ with $|\phi|=1$. If $(x,z)\neq 0$, we may choose $\phi=(x,z)\sqrt{-1}/|(x,z)|$ instead. For $0\leqslant t\leqslant 1$, consider

$$\begin{aligned} u_1(t)&=\sqrt{t}\,x+\sqrt{1-t}\,\phi z,\\ u_2(t)&=\sqrt{t}\,x-\sqrt{1-t}\,\phi z. \end{aligned} \tag{6}$$

Then by direct computation and with Lemma 4 for the situation $(x,z)\neq 0$, we have, for $j=1,2$ and $t\in[0,1]$,

$$\|u_j(t)\|=1,$$
$$\big(Au_j(t),u_j(t)\big)=t(Ax,x)+(1-t)(Az,z),$$
$$\|Au_1(t)\|^2=t\|Ax\|^2+(1-t)\|Az\|^2+2\sqrt{t(1-t)}\,\rho,$$

and

$$\|Au_2(t)\|=t\|Ax\|^2+(1-t)\|Az\|^2-2\sqrt{t(1-t)}\,\rho,$$

where $\rho=\operatorname{Re}(Ax,\phi Az)$. For $t\in[0,1]$, by Lemma 3, these $u_j(t)$'s are the only unit vectors in V' (up to scalar multiplies) satisfying $(Au,u)=t(Ax,x)+(1-t)(Az,z)$. So for $i=1,2$, x_i must be a scalar multiple of one of the $u_j(t_i)$'s defined in (6). Choose $u(t)=u_1(t)$ or $u(t)=u_2(t)$ according as $\rho\geqslant 0$ or $\rho<0$.

Now for both cases 1 and 2, though $u(t)$ and ρ are defined differently in different cases, we have the same results:

$$\|u(t)\|=1,$$
$$(Au(t),u(t))=t(Ax,x)+(1-t)(Az,z), \tag{7}$$
$$\|Au(t)\|^2=t\|Ax\|^2+(1-t)\|Az\|^2+2\sqrt{t(1-t)}\,|\rho| \tag{8}$$

and for $i=1,2$,

$$(Au(t_i),u(t_i))=(Ax_i,x_i), \tag{9}$$
$$\|Au(t_i)\|^2\geqslant\|Ax_i\|^2. \tag{10}$$

Define

$$r(t)=\left[\|Au(t)\|^2-|(Au(t),u(t))|^2\right]^{1/2},$$

which is a nonnegative function of $t\in[0,1]$. Then by using (7) and (8),

$$\begin{aligned}r^2(t)&=t\|Ax\|^2+(1-t)\|Az\|^2+2\sqrt{t(1-t)}\,|\rho|\\&\quad-t^2|(Ax,x)|^2-(1-t)^2|(Az,z)|^2\\&\quad-2t(1-t)\operatorname{Re}\left[(Ax,x)\overline{(Az,z)}\right].\end{aligned}$$

By routine computation, $(d^2/dt^2)r^2(t)\leqslant 0$ for all $t\in(0,1)$. So if there exists $t_0\in(0,1)$ such that $r(t_0)=0$, by the fact that $r(t)$ is nonnegative we can conclude that $r(t)=0$ for all $t\in[0,1]$. If $r(t)>0$ for all $t\in(0,1)$, then by the identity

$$\frac{d^2}{dt^2}r^2\equiv 2\left[r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\right],$$

$(d^2/dt^2)r(t)\leqslant 0$ for all $t\in(0,1)$. In both situations we have

$$r(t_3)\geqslant\lambda r(t_1)+(1-\lambda)r(t_2)$$

where $t_3=\lambda t_1+(1-\lambda)t_2$. Using (7), (3), (2), (9), and (10),

$$\begin{aligned}|\xi-q(Au(t_3),u(t_3))|&=|\xi-q[\lambda(Ax_1,x_1)+(1-\lambda)(Ax_2,x_2)]|\\&\leqslant p\left\{\lambda\left(\|Ax_1\|^2-|(Ax_1,x_1)|^2\right)^{1/2}\right.\\&\qquad\left.+(1-\lambda)\left[\|Ax_2\|^2-|(Ax_2,x_2)|^2\right]^{1/2}\right\}\\&\leqslant p[\lambda r(t_1)+(1-\lambda)r(t_2)]\\&\leqslant pr(t_3)\\&=p\left[\|Au(t_3)\|^2-|(Au(t_3),u(t_3))|^2\right]^{1/2}.\end{aligned}$$

So $\xi\in W(A:q)$ by Lemma 5. ∎

REMARK 1. Suppose A is hermitian with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. Then by Lemma 5 and some calculations, it can be shown that $W(A:q)$ is equal to the set

$$\Big\{\xi \in \mathbf{C} : |\xi - q[t\lambda_1 + (1-t)\lambda_n]| \leqslant (1-|q|^2)^{1/2}\sqrt{t(1-t)}\,(\lambda_1 - \lambda_n),\ 0 \leqslant t \leqslant 1\Big\},$$

which is an elliptical disk with foci $q\lambda_1$ and $q\lambda_n$ and eccentricity $|q|$ if $q \neq 0$. For the case $q = 0$, $W(A:q)$ is merely a circular disk with origin as center and radius $\frac{1}{2}(\lambda_1 - \lambda_n)$. Hence the results obtained in [6] on the set $\{(Ax, y) : x, y \in S,\ |(x,y)| = |q|\}$ follow directly.

REMARK 2. Marcus and Sandy defined the G-bilinear range $W(A:G)$ where G is an $r \times 3r$ matrix and gave a sufficient condition for its convexity (for details see [7]). If $r = 1$, then, by our Theorem 1, $W(A:G)$ is always convex. So we see that the mentioned sufficient condition is not a necessary one even for $r = 1$.

3. THE C-NUMERICAL RANGE

Let $e = (e_1, \dots, e_n)$ be a fixed ordered orthonormal basis of V, and let $[x]_e$ denote the column n-tuple of coefficients of any $x \in V$ with respect to e. Suppose $x_0, y_0 \in V$ are such that $a = [x_0]_e$, $b = [y_0]_e$, and $B \in \mathrm{Hom}(V,V)$ has the matrix representation $[B]_e^e$ with respect to e equal to ab^* (here * denotes the conjugate transpose). The following is due to Marcus and Sandy [7], who proved a more general result.

LEMMA 6. *In the preceding notation*

$$\{(Ax, y) : x, y \in V,\ \|x\| = \|x_0\|,\ \|y\| = \|y_0\|,\ (x,y) = (x_0, y_0)\}$$
$$= \{\mathrm{tr}(BU^*AU) : U \text{ unitary}\}$$
$$= W_B(A).$$

By letting x_0, y_0 belong to S in Lemma 6, we see that $W(A:q)$ equals to $W_C(A)$ for some particular C.

THEOREM 2. *Let $C \in \mathrm{Hom}(V,V)$ have rank 1. Then $W_C(A)$ is convex for all $A \in \mathrm{Hom}(V,V)$.*

Proof. If $\operatorname{rank} C = 1$, then $[C]_e^e = ab^*$ for some nonzero column n-tuples a and b. Let $x_0, y_0 \in V$ be such that $[x_0]_e = a$ and $[y_0]_e = b$. Then x_0 and y_0 are nonzero vectors. By Lemma 6

$$W_C(A) = \{(Ax, y) : x, y \in V, \|x\| = \|x_0\|, \|y\| = \|y_0\|, (x, y) = (x_0, y_0)\}$$

$$= \left\{ (A\|x_0\|u, \|y_0\|v) : u, v \in S, (u, v) = \frac{(x_0, y_0)}{\|x_0\|\|y_0\|} \right\}$$

$$= \|x_0\|\|y_0\| W\left(A : \frac{(x_0, y_0)}{\|x_0\|\|y_0\|} \right),$$

the convexity of which is guaranteed by Theorem 1. ■

COROLLARY. *If $n = 2$, then $W_C(A)$ is always convex.*

Proof. Let $n = 2$ and μ be any eigenvalue of C. If $C = \mu I$, then $W_{C-\mu I}(A)$ is a singleton containing 0 as its only element. If $C \neq \mu I$, then $\operatorname{rank}(C - \mu I) = 1$ and hence $W_{C-\mu I}(A)$ is convex by Theorem 2. By the identity

$$W_C(A) = W_{C-\mu I}(A) + \mu \operatorname{tr} A,$$

$W_C(A)$ is always convex. ■

We end our discussion with an example of nonconvex $W_C(A)$ where $\operatorname{rank} C = 2$ and $n = 3$. Let $\dim V = 3$; C be a normal operator with eigenvalues $1, \sqrt{-1}, 0$; and $A = C^*$. Then, since the eigenvalues of C are not collinear, $W_C(A)$ is not convex [2].

REFERENCES

1 Y. H. Au-Yeung and Y. T. Poon, 3×3 orthostochastic matrices and the convexity of generalized numerical ranges, *Linear Algebra Appl.* 27:69–79 (1979).

2 Y. H. Au-Yeung and N. K. Tsing, A conjecture of Marcus on the generalized numerical range, *Linear and Multilinear Algebra*, to appear.

3 K. C. Das and M. Embry-Wardrop, Orthogonality and the numerical range II, *Linear Algebra Appl.* 30:63–67 (1980).

4 M. Goldberg and E. G. Straus, Elementary inclusion relations for generalized numerical ranges, *Linear Algebra Appl.* 18:1–24 (1977).
5 F. Hausdorff, Der Werteworrat einer Bilinearform, *Math. Z.* 3:314–316 (1919).
6 M. Marcus and P. Andresen, Constrained extrema of bilinear functionals, *Monatsh. Math.* 84:219–235 (1977).
7 M. Marcus and M. Sandy, The G-bilinear range, *Linear and Multilinear Algebra* 11:317–332 (1982).
8 F. D. Murnaghan, On the field of values of a square matrix, *Proc. Nat. Acad. Sci. U.S.A.* 18:246–248 (1932).
9 Y. T. Poon, Another proof of a result of Westwick, *Linear and Multilinear Algebra* 9:35–37 (1980).
10 O. Toeplitz, Das algebraische Analogen zu einem Satze von Fejér, *Math. Z.* 2:187–197 (1918).
11 N. K. Tsing, On the shape of the generalized numerical ranges, *Linear and Multilinear Algebra* 10:173–182 (1981).
12 R. Westwick, A theorem on numerical range, *Linear and Multilinear Algebra* 2:311–315 (1975).

Received 21 September 1982