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A MULTISCALE DERIVATION OF A NEW PARABOLIC EQUATION WHICH INCLUDES DENSITY VARIATIONS

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Abstract—A new parabolic equation is obtained from the acoustic equation by the multiscale method. The new equation incorporates the effects of a variable ocean density. The density can be smooth or piecewise smooth. Thus, the new formulation alleviates the need for interfacial conditions when the density is stratified in a piecewise constant fashion. It also reduces to the standard P.E. when the density is constant. The new equation has the same conservation law as the P.E. A difference equation is presented which has a discrete version of the same law.

1. DERIVATION

The propagation of sound in an ocean with variable density ρ is governed by the elliptic equation

$$\rho\nabla \cdot (1/\rho)\nabla p + k'^2 n^2 p = 0, \tag{1}$$

where p is the acoustic pressure, $k' = \omega/c_0$, ω is the frequency of the time harmonic source, c_0 is a reference sound speed, $n = c_0/c$, and c is the sound speed in the ocean. A time dependence of $e^{-i\omega t}$ is suppressed. Equation (1) is to be solved in a spatial domain D', which contains the water. A simple model is obtained by assuming that both the ocean bottom and the water–air interface are flat. Specifically,

$$D' = \{(x', y', z')|, |x'| < \infty, |y'| < \infty, 0 \le z' \le H'\},$$

where the primed variables denote dimensional quantities. Since Eq. (1) is elliptic, boundary conditions are required to complete the mathematical description of the problem. The conditions used in this report are

$$\partial p/\partial z' = 0, \quad z' = H', \tag{2}$$

and

$$p = 0, \quad z' = 0. \tag{3}$$

Thus, the ocean has a hard bottom and a pressure release (free) surface.

The source deriving Eq. (1) is usually modeled as a point disturbance located at $x' = y' = 0$, $z' = z_0'$. It is omitted from Eq. (1) for simplicity.

In many underwater applications the domain (in polar coordinates) $D' = \{(r', z', \theta)|0 \le r' \le R', 0 \le z' \le H', 0 \le \theta \le 2\pi\}$, where Eq. (1) must be solved is extremely slender. By this we mean that the parameter

$$\epsilon = (H'/R')^2 \tag{4}$$

satisfies $\epsilon \ll 1$, where R' is the maximum range of interest. We now introduce the dimensionless variables r and z used by Tappert[1]—i.e.

$$r = \epsilon k' r' \tag{5}$$

and

$$z = \sqrt{\epsilon} k' z'. \tag{6}$$

Accordingly, D' is transformed into

$$D = \{(r, z, \theta) | 0 \le r \le l, 0 \le z \le l, 0 \le \theta < 2\pi\}, \tag{7}$$

where $l = (k'H')H'/R'$. We assume that this number is fixed and is order one with respect to the parameter ϵ. Introducing this change of variables into Eqs (1)–(3), we find that the acoustic pressure satisfies

$$\epsilon^2\left[p_{rr} + \frac{1}{r}p_r - \frac{1}{\rho}\rho_r p_r\right] + \epsilon\left[p_{zz} - \frac{\rho_z}{\rho}p_z\right] + n^2 p = 0, \tag{8}$$

$$p = 0, \quad z = 0, \tag{9}$$

and

$$\partial p/\partial z = 0, \quad z = l. \tag{10}$$

In addition to the boundary conditions (9) and (10), we demand that p be bounded as $r \to l$.

We now make the assumption that n^2 deviates slightly from a constant and takes the functional form

$$n^2(x', y', z') = 1 + \epsilon f(r, z). \tag{11}$$

The constant 1 in this equation is arrived at by taking c_0 to be the average of c throughout D. The factor ϵ in (11) demonstrates the weak dependence of c on depth and range. (This apparent minor perturbation creates profound effects on acoustic propagation when the range is as short as a few wavelengths!)

We also assume that the density ρ depends upon the variables r and z in a smooth or piecewise smooth fashion.

When (11) is inserted into (8), we observe the presence of the small parameter ϵ in front of nearly every term. To cavilierly set these terms to zero would render a physically meaningless result. Guided by previous experience with such matters, we apply the method of multiple scales to this equation. Specifically, we assume that

$$p(x', y', z') = P(\xi, r, z; \epsilon), \tag{12}$$

where the fast variable ξ is defined by

$$\xi = r/\epsilon. \tag{13}$$

Inserting this variable and (11) into (8), we obtain the equation

$$[P_{\xi\xi} + P] + \epsilon\left[2P_{r\xi} + \frac{1}{r}P - \frac{\rho_r}{\rho}P_\xi + P_{zz} - \frac{\rho_z}{\rho}P_z + fP\right] + \epsilon^2\left[P_{rr} + \frac{1}{r}P_r - \frac{\rho_r}{\rho}P_r\right] = 0. \tag{14}$$

The subscripts denote partial differentiation. Next we make the assumption that P has the asymptotic expansion

$$p \sim \sum_{n=0}^{\infty} \epsilon^n P_n(\xi, r, z, \theta), \quad \epsilon \to 0. \tag{15}$$

When this expression is inserted into (16), we equate to zero the coefficients of the powers of

ϵ. This yields an infinite sequence of equations, of which the first two are

$$LP_0 = P_{0\xi\xi} + P_0 = 0, \tag{16}$$

and

$$LP_1 = 2P_{0r\xi} + \frac{1}{r} P_{0\xi} - \frac{\rho_r}{\rho} P_{0\xi} + P_{0zz} - \frac{\rho_z}{\rho} P_{0z} + fP_0, \tag{17}$$

for $0 < z < l$, $0 < r < l$. Inserting (15) into the boundary conditions (9) and (10) and equating to zero the coefficients of the powers of ϵ, we obtain an infinite sequence of boundary conditions. The companions for (16) and (17) are

$$P_n = 0, \quad n = 0, 1, \quad z = 0, \tag{18}$$

and

$$\partial P_n/\partial z = 0, \quad n = 0, 1, \quad z = l. \tag{19}$$

The solution of Eq. (16) is

$$P_0 = A_0(r, z)\, e^{i\xi} + B_0(r, z)\, e^{-i\xi}, \tag{20}$$

where the amplitudes A_0 and B_0 are functions of the listed variables. Because of the assumed time dependence, $e^{-i\omega t}$, we set

$$B_0(r, z) = 0, \tag{21}$$

Since a failure to do so would yield incoming waves from infinity. Inserting (20) and (21) into (17) gives

$$LP_1 = \left[2iA_{0r} + \frac{i}{r} A_0 - \frac{i\rho_r}{\rho} A_0 + A_{0zz} - \frac{\rho_z}{\rho} A_{0z} + fA_0 \right] e^{i\xi}, \tag{22}$$

which has the general solution

$$P_1 = A_1(r, z)\, e^{i\xi} + \xi(i/2) M(A_0)\, e^{i\xi}, \tag{23}$$

where $M(A_0)$ is the bracketed term on the right side of Eq. (22). We observe that P_1 remains bounded as $\xi = r/\epsilon \to \infty$ only if

$$M(A_0) = 2iA_{0r} + \frac{i}{r} A_0 - \frac{i\rho_r}{\rho} A_0 + A_{0zz} - \frac{\rho_z}{\rho} A_{0z} + fA_0 = 0. \tag{24}$$

If we now substitute

$$A_0(r, z) = \sqrt{\rho(r, z)}\,(u_0/\sqrt{r}) \tag{25}$$

into (24), we find that u_0 must satisfy our new variable density parabolic differential equation (VDPE)

$$-2i \frac{\partial u_0}{\partial r} = \sqrt{\rho} \frac{\partial}{\partial z} \left[\frac{1}{\rho} \frac{\partial}{\partial z} (\sqrt{\rho} u_0) \right] + fu_0. \tag{26}$$

We now make a few interesting observations about this new equation. First, when ρ is a constant, (26) reduces to the standard parabolic equation[1]. Second, the differential operator involving z is symmetric or formally selfadjoint[2]. Third, the quantity

$$E \equiv \int_0^l |u_0|^2 \, dz \tag{27}$$

is independent of range; i.e. $dE/dr \equiv 0$. This follows directly from (26), and the boundary conditions for u_0 are

$$u_0 = 0, \quad z = 0, \tag{28}$$

and

$$\partial(\sqrt{\rho} u_0)/\partial z = 0, \quad z = l. \tag{29}$$

Equations (28) and (29) are direct consequences of (18), (19) and (25). Fourth, we observe that Eq. (26) itself was derived without using the boundary conditions given in (18) and (19). Thus, our new parabolic equation will hold even when more realistic boundary conditions are implemented. Finally, Eq. (26) can be used even when ρ is piecewise smooth. This will allow us to study interfaces that are not planar or straight lines. In this sense our new parabolic equation extends the analysis given by Lee and McDaniel[3, 4].

2. A CONSERVATIVE FINITE-DIFFERENCE SCHEME

In this section we present a finite-difference scheme, which is second-order accurate in depth and first-order accurate in range, for solving Eq. (26). This difference scheme will conserve a discrete analog of $dE/dr = 0$, where E is given by (27). The method of analysis and other examples are given by Kriegsmann and Mahar[5].

We begin by rewriting (26) as

$$-2i \frac{\partial u_0}{\partial r} = a \frac{\partial}{\partial z} \left[b \frac{\partial}{\partial z} (a u_0) \right] + f u_0, \tag{30}$$

where $a \equiv \sqrt{\rho}$ and $b \equiv 1/\rho$. Setting $u_j^n \equiv u_0(r_n, z_j)$, we easily verify by Taylor's theorem that

$$a \frac{\partial}{\partial z} \left[b \frac{\partial}{\partial z} (a u_0) \right]_{(r_n, z_j)} = \hat{L}(u_j^n)(\Delta z)^{-2} + O(\Delta z)^2, \tag{31}$$

where $r_n = n \, \Delta r$, $z_j = j \, \Delta z$, and $\hat{L}$ is defined by

$$\hat{L}(u_j^n) = a_j^n b_{j+1/2}^n [a_{j+1}^n u_{j+1}^n - a_j^n u_j^n] + a_j^n b_{j-1/2}^n [a_{j-1}^n u_{j-1}^n - a_j^n u_j^n]. \tag{32}$$

The Crank–Nicholson scheme for solving (30) is the one we shall use. It is simply

$$-2i[U_j^{n+1} - U_j^n] = \lambda \hat{L}(U_j^{n+1} + U_j^n) + \beta f_j^n (U_j^{n+1} + U_j^n), \quad j = 0, 1, 2, \ldots, N, \tag{33}$$

where $\lambda = \frac{1}{2} \Delta r/(\Delta z)^2$, $\beta = \Delta r/2$, and U_j^n is the numerical approximation of u_j^n. Equation (33) is solved in the usual fashion.

We now define the vector $\mathbf{U}^n$ by

$$\mathbf{U}^n \equiv (U_0^n, U_1^n, \ldots, U_N^n)^T, \tag{34}$$

where the superscript T denotes transpose. The quantity $\hat{E}$ defined as the l_2-norm of $\mathbf{U}^n$, i.e.

$$\hat{E}_n \equiv \|\mathbf{U}^n\|^2 = \sum_{l=0}^{N} |U_l^n|^2, \tag{35}$$

is the discrete analog of E defined by (27). We shall now prove that $\hat{E}$ is range independent; i.e.

$$\hat{E}_{n+1} \equiv \hat{E}_n, \quad \text{for all } n. \tag{36}$$

Defining W_j by

$$W_j \equiv U_j^{n+1} + U_j^n, \tag{37}$$

and multiplying (33) by $\bar{W}_j$, we obtain

$$-2i\{|U_j^{n+1}|^2 - |U_j^n|^2\} + R_j^n = \lambda \bar{W}_j \hat{L} W_j + \beta f_j^n |W_j|^2. \tag{38}$$

The term R_j^n is real and given by

$$R_j^n \equiv -\mathrm{Im}[U_j^n \bar{U}_j^{n+1}]. \tag{39}$$

Summing (38) from $j = 1$ ($z_0 = 0$) to $j = N$ ($z_N = l$), we obtain

$$-2i\{\hat{E}_{n+1} - \hat{E}_n\} = \lambda \sum_{j=0}^{N} \bar{W}_j \hat{L} W_j + \beta \sum_{j=0}^{N} f_j^n |W_j|^2 - \hat{R}_n, \tag{40}$$

where $\hat{R}_n$ is the l_2-norm of the vector $\mathbf{R}^n$, with components R_j^n, defined as in (34). The last two terms on the right side of (40) are *real*. The result given in (36) follows because the term $\lambda \sum_{j=0}^{N} \bar{W}_j \hat{L} W_j$ is real also. To verify this fact, we rewrite this sum as

$$\sum_{j=0}^{N} \bar{W}_j \hat{L} W_j = -g_0 + g_{N+1} - \sum_{j=0}^{N} b_{j-1/2} |a_j W_j - a_{j-1} W_{j-1}|^2, \tag{41}$$

where

$$g_0 \equiv a_0^n b_{1/2} \bar{W}_0 (a_1^n W_1 - a_0^n W_0), \tag{42}$$

and

$$g_{n+1} \equiv a_N^n b_{n+1/2}^n \bar{W}_N (a_{N+1/2}^n W_{N+1} - a_N^n W_N). \tag{43}$$

Now the g_0 term is zero, because $W_0 = U_0^{n+1} + U_0^n$ and both U_0^{n+1} and U_0^n are zero. The third term in (41) is real. The term g_{N+1} vanishes. This is because the parenthetical term in (43) is the discrete implementation of the boundary condition (29).

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