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Spectra of algebras of holomorphic functions of bounded type

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ABSTRACT

We prove that if U is a balanced $\mathcal{H}_b(U)$-domain of holomorphy in Tsirelson's space then the spectrum of $\mathcal{H}_b(U)$ is identified with U. We derive theorems of Banach–Stone type for algebras of holomorphic functions and algebras of holomorphic germs.

INTRODUCTION

Let E be a Banach space and let U be an open subset of E. In [2], it is proved that if E is Tsirelson's space, then the spectrum of $\mathcal{H}_b(U)$ is identified with U, when $U = E$. In [11], J. Mujica generalizes this result for absolutely convex open subsets of Tsirelson's space, and asks if the result can be improved for a more general class of open subsets of E, for instance, polynomially convex open subsets. In this paper we give a partial answer to this question, i.e., we show that the result remains true for balanced $\mathcal{H}_b(U)$-domains of holomorphy on Tsirelson's space. In Section 1 we define $\mathcal{H}_b(U)$-convex open subsets and present properties and examples of such sets. We also give some auxiliary results before proving the main result. Most of them are generalizations to U-bounded sets of known results for compact sets. In Section 2 we present the main result and a corollary on finitely generated ideals of the algebra $\mathcal{H}_b(U)$. In Section 3 we present theorems of Banach–Stone type for the

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algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$, and also for algebras of holomorphic germs $\mathcal{H}(K)$ and $\mathcal{H}(L)$, improving results from [14].

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1. PRELIMINARIES

We refer to [7] and [10] for background information on infinite-dimensional complex analysis. E and F will always denote Banach spaces. Let $\mathcal{P}(E; F)$ denote the Banach space of all continuous polynomials from E into F. $\mathcal{P}(^mE; F)$ denotes the Banach space of all continuous m-homogeneous polynomials from E into F. $\mathcal{P}_f(^mE; F)$ denotes the subspace of $\mathcal{P}(^mE; F)$ generated by all polynomials of the form $P(x) = \varphi(x)^m b$, for all $x \in E$, where $\varphi \in E'$ and $b \in F$. Such polynomials are called *of finite type*. When $F = \mathbb{C}$, we write $\mathcal{P}(E)$, $\mathcal{P}(^mE)$ and $\mathcal{P}_f(^mE)$ instead of $\mathcal{P}(E; \mathbb{C})$, $\mathcal{P}(^mE; \mathbb{C})$ and $\mathcal{P}_f(^mE; \mathbb{C})$, respectively.

Let U be an open subset of E. We say that a subset $A \subset U$ is *U-bounded* if A is bounded and there exists $\varepsilon > 0$ such that $A + B(0, \varepsilon) \subset U$.

We will denote by $\mathcal{H}_b(U; F)$ the vector space of all holomorphic mappings $f: U \longrightarrow F$ which are bounded on every U-bounded subset. Such mappings are called *holomorphic mappings of bounded type*. If $F = \mathbb{C}$, we write $\mathcal{H}_b(U)$ instead of $\mathcal{H}_b(U; \mathbb{C})$. We denote by τ_b the topology on $\mathcal{H}_b(U; F)$ of the uniform convergence on all U-bounded subsets. $\mathcal{H}_b(U; F)$ is a Fréchet space for this topology, and likewise $\mathcal{H}_b(U)$ is a Fréchet algebra. If U is balanced, it follows from the Cauchy inequalities that the Taylor series of each $f \in \mathcal{H}_b(U; F)$ at the origin converges uniformly on each U-bounded subset. In particular, if ρ_U denotes the restriction of mappings to U, then $\rho_U(\mathcal{P}(E; F))$ is τ_b-dense in $\mathcal{H}_b(U; F)$.

We denote by $\mathcal{S}_b(U)$ the spectrum of the algebra $\mathcal{H}_b(U)$, i.e., the set of all continuous complex homomorphisms (and by that we mean linear and multiplicative) of $\mathcal{H}_b(U)$. Every point of U can be associated with an element of $\mathcal{S}_b(U)$ as follows: for each $z \in U$ fixed, let $\delta_z : \mathcal{H}_b(U) \longrightarrow \mathbb{C}$ be defined by $\delta_z(f) = f(z)$, for all $f \in \mathcal{H}_b(U)$. Each δ_z is called *evaluation at z*. It is clear that $\delta_z \in \mathcal{S}_b(U)$, for all $z \in U$, and the mapping $\delta : U \longrightarrow \mathcal{S}_b(U)$ is used in order to identify U with the subset $\delta(U)$ of $\mathcal{S}_b(U)$. Note that δ is injective because the continuous linear forms already separate the points of E.

In this paper we will show that under certain hypotheses on E and U, all the elements of $\mathcal{S}_b(U)$ are evaluations at some point of U, and in this sense we say that *$\mathcal{S}_b(U)$ is identified with $\delta(U)$*.

In the following we give some definitions, examples and auxiliary results to the main result.

Let X be a subset of E, A be a subset of X, and $\mathcal{F} \subset \mathcal{C}(X)$. Then the *$\mathcal{F}$-hull of A* is the following set:

$$\widehat{A}_{\mathcal{F}} = \left\{ x \in X \colon \left| f(x) \right| \leqslant \sup_A |f|, \text{ for all } f \in \mathcal{F} \right\}.$$

Definitions 1.1. Let E be a Banach space and let U be an open subset of E. We say that U is:

(1) $\mathcal{P}_b(E)$-*convex* if $\widehat{A}_{\mathcal{P}(E)} \cap U$ is U-bounded, for every U-bounded subset A;
(2) *strongly* $\mathcal{P}_b(E)$-*convex* if $\widehat{A}_{\mathcal{P}(E)} \subset U$ and is U-bounded, for every U-bounded subset A;
(3) $\mathcal{H}_b(E)$-*convex* if $\widehat{A}_{\mathcal{H}_b(E)} \cap U$ is U-bounded, for every U-bounded subset A;
(4) *strongly* $\mathcal{H}_b(E)$-*convex* if $\widehat{A}_{\mathcal{H}_b(E)} \subset U$ and is U-bounded, for every U-bounded subset A;
(5) $\mathcal{H}_b(U)$-*convex* if $\widehat{A}_{\mathcal{H}_b(U)}$ is U-bounded, for every U-bounded subset A.

The following lemma shows that the notions of (strongly) $\mathcal{P}_b(E)$-convex and (strongly) $\mathcal{H}_b(E)$-convex set coincide.

Lemma 1.2. *Let A be a bounded subset of E. Then $\widehat{A}_{\mathcal{H}_b(E)} = \widehat{A}_{\mathcal{P}(E)}$.*

Proof. Since $\mathcal{P}(E) \subset \mathcal{H}_b(E)$, we have that $\widehat{A}_{\mathcal{H}_b(E)} \subseteq \widehat{A}_{\mathcal{P}(E)}$. Now let us suppose that there exists $a \in \widehat{A}_{\mathcal{P}(E)}$ such that $a \notin \widehat{A}_{\mathcal{H}_b(E)}$. Let $f \in \mathcal{H}_b(E)$ be such that $|f(a)| > \sup_A |f|$. Since $\widehat{A}_{\mathcal{P}(E)}$ is bounded and $\mathcal{P}(E)$ is dense in $\mathcal{H}_b(E)$ for the τ_b topology, given $\varepsilon > 0$ there exists $P \in \mathcal{P}(E)$ such that $\sup_{\widehat{A}_{\mathcal{P}(E)}} |f - P| < \frac{\varepsilon}{2}$. In particular we have that $\sup_A |P| \leqslant \sup_A |P - f| + \sup_A |f| < \frac{\varepsilon}{2} + \sup_A |f|$. Finally we get that $|f(a)| \leqslant |f(a) - P(a)| + |P(a)| < \frac{\varepsilon}{2} + \sup_A |P| < \varepsilon + \sup_A |f|$, for all $\varepsilon > 0$, which is a contradiction. $\square$

The next lemma shows that the last condition on Definition 1.1.2 (and 1.1.4) is superfluous.

Lemma 1.3. *If $\widehat{A}_{\mathcal{H}_b(E)} \subset U$, for every U-bounded subset A, then U is strongly $\mathcal{H}_b(E)$-convex.*

Proof. We follow ideas of [10, Lemma 54.8]. Let A be a U-bounded subset. We must show that $\widehat{A}_{\mathcal{H}_b(E)}$ is U-bounded. Since it is clear that $\widehat{A}_{\mathcal{H}_b(E)}$ is bounded, it remains to show that there exists $\varepsilon > 0$ such that $\widehat{A}_{\mathcal{H}_b(E)} + B(0, \varepsilon) \subset U$. Let $\varepsilon > 0$ be such that $A + B(0, \varepsilon)$ is U-bounded. Then $(A + B(0, \varepsilon))\widehat{\ }_{\mathcal{H}_b(E)} \subset U$. Let $y \in \widehat{A}_{\mathcal{H}_b(E)}$, $t \in B(0, \varepsilon)$ and $0 < \theta < 1$. Then for each $f \in \mathcal{H}_b(E)$ we have that

$$\left|f(y + \theta t)\right| \leqslant \sum_{m=0}^{\infty} \theta^m \left|P_t^m(f)(y)\right| \leqslant \sum_{m=0}^{\infty} \theta^m \sup_A \left|P_t^m(f)\right| \leqslant (1 - \theta)^{-1} \sup_{A + B(0,\varepsilon)} |f|,$$

where the second inequality follows because $P_t^m(f) \in \mathcal{H}_b(E)$ and $y \in \widehat{A}_{\mathcal{H}_b(E)}$. The third inequality follows by applying [10, Corollary 7.3], with $t \in B(0, \varepsilon)$ and $r = 1$.

Next we apply the above inequality to f^n, take n-th roots and let $n \to \infty$ to get that $|f(y + \theta t)| \leqslant \sup_{A + B(0,\varepsilon)} |f|$, that is, $y + \theta t \in (A + B(0, \varepsilon))\widehat{\ }_{\mathcal{H}_b(E)} \subset U$. By letting $\theta \to 1$ we have that $y + t \in U$, and the conclusion follows. $\square$

Lemma 1.4. *Let $\mathcal{F} \subset \mathcal{H}_b(E)$ be a family with the property that the function $x \mapsto f(\lambda x)$ is an element of $\mathcal{F}$, for every $f \in \mathcal{F}$ and $|\lambda| \leqslant 1$. Let $A \subseteq E$ be a balanced subset. Then $\widehat{A}_{\mathcal{F}}$ is balanced.*

Proof. Let $f \in \mathcal{F}$. For each $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant 1$, let $f_\lambda \in \mathcal{F}$ be such that $f_\lambda(x) = f(\lambda x)$, for all $x \in E$. Let $y \in \widehat{A}_{\mathcal{F}}$. Then $|f(\lambda y)| = |f_\lambda(y)| \leqslant \sup_A |f_\lambda| \leqslant \sup_A |f|$, proving that $\lambda y \in \widehat{A}_{\mathcal{F}}$, and hence $\widehat{A}_{\mathcal{F}}$ is balanced. □

It is clear that strongly $\mathcal{H}_b(E)$-convex open subsets are always $\mathcal{H}_b(E)$-convex. Also, it is easy to see that $\widehat{A}_{\mathcal{H}_b(U)} \subset \widehat{A}_{\mathcal{H}_b(E)} \cap U$, and hence we have that $\mathcal{H}_b(E)$-convex open subsets are always $\mathcal{H}_b(U)$-convex. The next proposition shows that if U is balanced, then all these notions coincide. Moreover, this proposition is the heart of the proof of the main result.

Proposition 1.5. *Let $U \subset E$ be a balanced open subset. Then the following conditions are equivalent.*

(1) *U is strongly $\mathcal{P}_b(E)$-convex.*
(2) *U is strongly $\mathcal{H}_b(E)$-convex.*
(3) *U is $\mathcal{P}_b(E)$-convex.*
(4) *U is $\mathcal{H}_b(E)$-convex.*
(5) *U is $\mathcal{H}_b(U)$-convex.*

Proof. The implications (1) ⇔ (2) and (3) ⇔ (4) were proved in Lemma 1.2. The implications (2) ⇒ (4) ⇒ (5) were commented above.

(5) ⇒ (4) We show that $\widehat{A}_{\mathcal{H}_b(U)} = \widehat{A}_{\mathcal{P}(E)} \cap U = \widehat{A}_{\mathcal{H}_b(E)} \cap U$, for every U-bounded subset A, and then conclude that U is $\mathcal{H}_b(E)$-convex. Let $y \in \widehat{A}_{\mathcal{P}(E)} \cap U$ and we show that $y \in \widehat{A}_{\mathcal{H}_b(U)}$. Let $f \in \mathcal{H}_b(U)$ fixed. Since U is $\mathcal{H}_b(U)$-convex, the set $B = \widehat{A}_{\mathcal{H}_b(U)} \cup \{y\}$ is U-bounded, and since U is balanced, given $\varepsilon > 0$, there is $P \in \mathcal{P}(E)$ such that $\sup_B |f - P| < \frac{\varepsilon}{2}$. Then $|f(y)| \leqslant |f(y) - P(y)| + |P(y)| < \frac{\varepsilon}{2} + \sup_A |P|$. On the other hand:

$$\sup_A |P| \leqslant \sup_A |f - P| + \sup_A |f| < \frac{\varepsilon}{2} + \sup_A |f|.$$

And finally we get that $|f(y)| < \varepsilon + \sup_A |f|$, for all $\varepsilon > 0$, which implies that $y \in \widehat{A}_{\mathcal{H}_b(U)}$.

(4) ⇒ (2) Let A be an U-bounded subset. By Lemma 1.3, it suffices to prove that $\widehat{A}_{\mathcal{H}_b(E)} \subset U$. First we assume that A is balanced. Let $x \in \widehat{A}_{\mathcal{P}(E)}$. Define $D_1 = \{\lambda \in \mathbb{C}\colon\ |\lambda| \leqslant 1\}$ and $D = \{\lambda \in D_1\colon\ \lambda x \in U\}$. Then D is a disk centered at the origin because U is balanced, and D is an open subset of D_1 because U is open. Let $\varepsilon > 0$ be such that $\widehat{A}_{\mathcal{P}(E)} \cap U + B(0, \varepsilon) \subset U$. Let $\lambda \in D_1$, $\lambda x \in U$, and let $\mu \in D_1$ be such that $|\mu - \lambda|\|x\| < \varepsilon$. Then $\lambda x \in \widehat{A}_{\mathcal{P}(E)} \cap U$ because $x \in \widehat{A}_{\mathcal{P}(E)}$ and $\widehat{A}_{\mathcal{P}(E)}$ is balanced by Lemma 1.4. Furthermore $\|\mu x - \lambda x\| < \varepsilon$, hence $\mu x \in \widehat{A}_{\mathcal{P}(E)} \cap U + B(0, \varepsilon)$, and therefore $\mu x \in U$. This implies that any point on the boundary of D belongs to D,

and D is an open and closed subset of D_1, and therefore $D = D_1$. It follows that $x = 1x \in U$. Since this holds for any $x \in \widehat{A}_{\mathcal{P}(E)}$, we have proved that $\widehat{A}_{\mathcal{P}(E)} \subset U$.

If A is not balanced, we consider $B = ba(A)$, the balanced hull of A. If follows by [5, Lemma 1.3(b)] that B is a balanced U-bounded subset. Then we apply the arguments above and get that $\widehat{A}_{\mathcal{H}_b(E)} \subset \widehat{B}_{\mathcal{H}_b(E)} \subset U$. □

Next we give some examples of balanced $\mathcal{H}_b(E)$-convex open subsets.

Example 1.6. *Let $P \in \mathcal{P}(^m E; F)$ and let $U = \{x \in E \colon \|P(x)\| < 1\}$. Then U is a balanced $\mathcal{H}_b(U)$-convex open set.*

Proof. Clearly U is a balanced open set. Let A be an U-bounded subset of U. Let $\varepsilon > 0$ denote the distance from A to the boundary of U, and let $r = \sup_{x \in A} \|x\|$. If $x \in A$ and $1 \leqslant \lambda < 1 + \frac{\varepsilon}{r}$, then $\|\lambda x - x\| = |\lambda - 1|\|x\| < \varepsilon$, hence $\lambda x \in U$, and therefore $\|P(x)\| = \|P((\frac{1}{\lambda})\lambda x)\| = \lambda^{-m}\|P(\lambda x)\| < \lambda^{-m}$. Taking in the right-hand side the infimum over all λ such that $1 \leqslant \lambda < 1 + \frac{\varepsilon}{r}$, we conclude that $\|P(x)\| \leqslant c := (1 + \frac{\varepsilon}{r})^{-m} < 1$ for every $x \in A$.

Let us show that $\widehat{A}_{\mathcal{H}_b(E)} \subset U$. Let $y \in \widehat{A}_{\mathcal{H}_b(E)}$ and $\varphi \in F'$. Then $\varphi \circ P \in \mathcal{H}_b(E)$ and hence $|\varphi \circ P(y)| \leqslant \sup_A |\varphi \circ P|$. Now

$$\left\|P(y)\right\| = \sup_{\varphi \in B_{F'}} \left|\varphi\big(P(y)\big)\right| \leqslant \sup_{\varphi \in B_{F'}} \sup_{x \in A} \left|\varphi\big(P(x)\big)\right| \leqslant \sup_{x \in A} \left\|P(x)\right\| = c < 1,$$

and hence $y \in U$.

This shows that $\widehat{A}_{\mathcal{H}_b(E)}$ is U-bounded, because if $0 < c < 1$, then every bounded subset of $\{x \in E \colon \|P(x)\| \leqslant c\}$ is U-bounded. Hence U is strongly $\mathcal{H}_b(E)$-convex by Lemma 1.3. Finally U is $\mathcal{H}_b(U)$-convex by Proposition 1.5. □

Corollary 1.7. *Let $P \in \mathcal{P}(^m E)$ and let $U = \{x \in E \colon |P(x)| < 1\}$. Then U is a balanced $\mathcal{H}_b(U)$-convex open set.*

Corollary 1.8. *Let $A \in \mathcal{L}(E_1, \ldots, E_m; F)$ and $E = E_1 \times \cdots \times E_m$. Then*

$$U = \left\{(x_1, \ldots, x_m) \in E \colon \left\|A(x_1, \ldots, x_m)\right\| < 1\right\}$$

is a balanced $\mathcal{H}_b(U)$-convex open set.

Proof. By [10, Theorem 3.6] it follows that A, viewed as a mapping from E to F, is a homogeneous polynomial of degree m. Then the result follows by Example 1.6. □

Corollary 1.9. *Let $U = \{(x, \lambda) \in E \times \mathbb{C} \colon \|\lambda x\| < 1\}$. Then U is a balanced $\mathcal{H}_b(U)$-convex open set.*

In [13], B. Tsirelson constructed a reflexive Banach space X, with an unconditional Schauder basis, that does not contain any subspace which is isomorphic to c_0 or to any ℓ_p. R. Alencar, R. Aron and S. Dineen proved in [1] that $\mathcal{P}_f({}^m X)$ is norm-dense in $\mathcal{P}({}^m X)$, for all $m \in \mathbb{N}$. Inspired by this result, we will say that a Banach space E is a *Tsirelson-like space* if E is reflexive and $\mathcal{P}_f({}^m E)$ is norm-dense in $\mathcal{P}({}^m E)$, for all $m \in \mathbb{N}$.

The following theorem is the main result of this paper.

Theorem 2.1. *Let E be a Tsirelson-like space, and let U be a balanced $\mathcal{H}_b(U)$-convex open subset of E. Then the spectrum of $\mathcal{H}_b(U)$ is identified with U.*

Proof. Since U is balanced and $\mathcal{H}_b(U)$-convex, it follows by Proposition 1.5 that U is strongly $\mathcal{H}_b(E)$-convex. Now we follow the ideas of [11, Theorem 1.1]. Let $T : \mathcal{H}_b(U) \longrightarrow \mathbb{C}$ be a continuous homomorphism. Then there exists $C > 0$ and an U-bounded subset $A \subset U$ such that

$$\left|T(f)\right| \leqslant C \sup_A |f|, \quad \text{for all } f \in \mathcal{H}_b(U).$$

Since T is multiplicative, we have that $|T(f)|^n = |T(f^n)| \leqslant C \sup_A |f|^n$ for every $n \in \mathbb{N}$. Taking n-th roots and making $n \to \infty$ we conclude that actually $C = 1$. Let $r > 0$ such that $A \subset B(0,r)$. In particular, we have that $|T(f)| \leqslant \sup_A |f| \leqslant \sup_{B(0,r)} |f|$, for all $f \in E'$. Hence we have that $T|_{E'} \in E'' = E$, so there exists a unique $a \in E$ such that $T(f) = f(a)$, for all $f \in E'$, and hence $T(P) = P(a)$, for all $P \in \mathcal{P}_f({}^m E)$, for all $m \in \mathbb{N}$. Since $\mathcal{P}_f({}^m E)$ is norm-dense in $\mathcal{P}({}^m E)$, for all $m \in \mathbb{N}$, it follows that $T(P) = P(a)$, for all $P \in \mathcal{P}(E)$. Then we have that $|P(a)| = |P(f)| \leqslant \sup_A |P|$, for all $P \in \mathcal{P}(E)$, which implies that $a \in \widehat{A}_{\mathcal{P}(E)} = \widehat{A}_{\mathcal{H}_b(E)} \subset U$. Since U is balanced, we have that $\mathcal{P}(E)$ is τ_b-dense in $\mathcal{H}_b(U)$, and then we conclude that $T(f) = f(a)$, for all $f \in \mathcal{H}_b(U)$, proving the theorem. □

Definition 2.2. Let E be a Banach space and let U be an open subset of E. We say that U is a *$\mathcal{H}_b(U)$-domain of holomorphy* if there are no open sets V and W in E with the following properties:

(1) V is connected and not contained in U;
(2) $\emptyset \neq W \subset U \cap V$;
(3) for each $f \in \mathcal{H}_b(U)$ there exists $\tilde{f} \in \mathcal{H}(V)$ such that $\tilde{f} = f$ on W.

The following corollary is the announced result for balanced $\mathcal{H}_b(U)$-domains of holomorphy.

Corollary 2.3. *Let E be a Tsirelson-like space, and let U be a balanced $\mathcal{H}_b(U)$-domain of holomorphy in E. Then the spectrum of $\mathcal{H}_b(U)$ is identified with U.*

Proof. By [6, Theorem 1] or [8, Theorem 1], we have that U is $\mathcal{H}_b(U)$-convex. Then apply Theorem 2.1. □

The following result is a consequence of Corollary 2.3. It says that, under the hypotheses of Corollary 2.3, every proper finitely generated ideal of $\mathcal{H}_b(U)$ has a common zero.

Theorem 2.4. *Let E be a Tsirelson-like space. Let $U \subset E$ be a balanced $\mathcal{H}_b(U)$-domain of holomorphy. Then given $f_1, \ldots, f_n \in \mathcal{H}_b(U)$ without common zeros, we can find $g_1, \ldots, g_n \in \mathcal{H}_b(U)$ such that $\sum_{i=1}^{n} f_i g_i = 1$.*

Proof. The proof of [11, Theorem 1.5] applies. □

3. THEOREMS OF BANACH–STONE TYPE

In [3], S. Banach proved that two compact metric spaces X and Y are homeomorphic if and only if the Banach algebras $C(X)$ and $C(Y)$ are isometrically isomorphic. M.H. Stone, in [12], generalized this result to arbitrary compact Hausdorff topological spaces, the well-known Banach–Stone theorem.

In [14], we present similar results for algebras of holomorphic functions of bounded type, using results on the spectrum of such algebras. More specifically, let E and F be reflexive spaces, one of them a Tsirelson-like space, and let $U \subset E$ and $V \subset F$ be absolutely convex opens subsets. Then it is shown that the algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ are topologically isomorphic, if and only if there is a special type of biholomorphic mapping between U and V. To show these results we use the characterization of the spectra of $\mathcal{H}_b(U)$ with U due to J. Mujica, [11, Theorem 1.1].

In this section we generalize this result to balanced $\mathcal{H}_b(U)$-domains of holomorphy, using the characterization of the spectrum of $\mathcal{H}_b(U)$, Corollary 2.3 of this paper.

Let E and F be Banach spaces, and $U \subset E$ and $V \subset F$ be open subsets of E and F, respectively. We denote by $\mathcal{H}_b(V, U)$ the set of all holomorphic mappings $\varphi : V \longrightarrow E$, with $\varphi(V) \subset U$, such that φ maps V-bounded subsets into U-bounded subsets.

Next theorem is the result announced, and improves [14, Corollary 14].

Theorem 3.1. *Let E and F be reflexive Banach spaces, one of them a Tsirelson-like space. Let $U \subset E$ and $V \subset F$ be balanced $\mathcal{H}_b$-domains of holomorphy. Then the following conditions are equivalent.*

(1) *There exists a bijective mapping $\varphi : V \longrightarrow U$ such that $\varphi \in \mathcal{H}_b(V, U)$ and $\varphi^{-1} \in \mathcal{H}_b(U, V)$.*
(2) *The algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ are topologically isomorphic.*

Proof. The proof of [14, Corollary 14] applies. □

In [14, Theorem 16] it is shown that if $K \subset E$ and $L \subset F$ are absolutely convex compact subsets of Tsirelson-like spaces, then the algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic if and only if K and L are biholomorphically equivalent. The key to the proof of such result is a theorem of Banach–Stone type between algebras of holomorphic functions of bounded type [14, Corollary 14]. We are going to present a generalization of this result to greater class of compact sets, using Theorem 3.1. But before we need some preparation.

Let E be a Banach space, and let $K \subset E$ be a compact subset. We define $\mathcal{H}(K)$ to be the algebra of all functions that are holomorphic on some open neighborhood of K. The elements of $\mathcal{H}(K)$ are called *germs of holomorphic functions*. We endow $\mathcal{H}(K)$ with the locally convex inductive limit of the locally convex algebras $(\mathcal{H}(U), \tau_\omega)$, where U varies among the open subsets of E such that $K \subset U$. If $U_n = K + B(0, \frac{1}{n})$, for all $n \in \mathbb{N}$, then it is easy to see that

$$\big(\mathcal{H}(K), \tau_\omega\big) = \varinjlim_{n \in \mathbb{N}} \mathcal{H}_b(U_n).$$

We refer to [4,7] or [9] for background information on algebras of germs of holomorphic functions.

Definition 3.2. Let E be a Banach space, let K be a compact subset of E and let $m \in \mathbb{N}$. We say that K is *$\mathcal{P}(^m E)$-convex* if $K = \widehat{K}_{\mathcal{P}(^m E)}$.

Before we present examples of balanced $\mathcal{P}(^m E)$-convex compact sets, we shall need the next lemma, which is inspired in [10, Proposition 11.1]. If A is a subset of a Banach space, we denote by $\overline{\Gamma}(A)$ the closed, absolutely convex hull of A.

Lemma 3.3. *Let E be a Banach space and let A be a bounded subset of E. Then $\widehat{A}_{\mathcal{P}_f(^m E)} \subset \overline{\Gamma}(A)$, for all $m \in \mathbb{N}$.*

Proof. Let $y \notin \overline{\Gamma}(A)$. By the Hahn–Banach theorem, there exists $\varphi \in E'$ such that $|\varphi(y)| > \sup_{x \in \overline{\Gamma}(A)} |\varphi(x)|$. Hence $|\varphi^m(y)| > \sup_{x \in \overline{\Gamma}(A)} |\varphi^m(x)| \geqslant \sup_A |\varphi^m|$, which shows that $y \notin \widehat{A}_{\mathcal{P}_f(^m E)}$. □

Example 3.4. Every absolutely convex compact subset of a Banach space E is $\mathcal{P}(^m E)$-convex, for all $m \in \mathbb{N}$.

Proof. Let $K \subset E$ be an absolutely convex compact set. Since $\mathcal{P}_f(^m E) \subset \mathcal{P}(^m E)$, we have that $\widehat{K}_{\mathcal{P}(^m E)} \subset \widehat{K}_{\mathcal{P}_f(^m E)} \subset \overline{\Gamma}(K) = K$, where the last inclusion follows by Lemma 3.3. □

Example 3.5. Let E be a Banach space, and $L \subset E$ be a compact, balanced and $\mathcal{P}(^m E)$-convex set. Let $P \in \mathcal{P}(^m E)$. Then $K = \{x \in L\colon |P(x)| \leqslant 1\}$ is compact, balanced and $\mathcal{P}(^m E)$-convex.

Remark 3.6. If K is a $\mathcal{P}(^m E)$-convex compact set, then it is clear that K is polynomially convex. But the converse is not true. Indeed, it is easy to see that if $K = \widehat{K}_{\mathcal{P}(^m E)}$, then K is balanced. Now let K be a convex compact set, which is not balanced. Then K is polynomially convex by [10, Examples 24.2(a)], but is not $\mathcal{P}(^m E)$-convex, for any $m \in \mathbb{N}$.

The next theorem will be useful to prove the main result of this section.

Theorem 3.7. *Let E be a Banach space and let K be a compact, balanced and $\mathcal{P}(^m E)$-convex subset of E, for some $m \in \mathbb{N}$. Let U be an open subset of E such that $K \subset U$. Then there exists an open set $V \subset E$ which is balanced and $\mathcal{H}_b(V)$-convex, such that $K \subset V \subset U$.*

Proof. We begin with a slight modification of [10, Lemma 24.7]. If $\overline{\Gamma}(K) \subset U$, then we take $V = \overline{\Gamma}(K) + B(0, \varepsilon)$, where ε is such that $\overline{\Gamma}(K) + B(0, \varepsilon) \subset U$. If $\overline{\Gamma}(K)$ is not contained in U, then for each $a \in \overline{\Gamma}(K) \setminus U$ there is $P \in \mathcal{P}(^m E)$ such that $\sup_K |P| < 1 < |P(a)|$. Since $\overline{\Gamma}(K) \setminus U$ is compact, we can find polynomials $P_1, \ldots, P_k \in \mathcal{P}(^m E)$ such that

$$\overline{\Gamma}(K) \setminus U \subset \bigcup_{j=1}^{k} \{x \in E\colon \left|P_j(x)\right| > 1\}.$$

Now it is easy to see that $\{x \in \overline{\Gamma}(K)\colon |P_j(x)| \leqslant 1, \text{ for } j = 1, \ldots, k\} \subset U$. Next we follow the arguments of [10, Theorem 28.2], finding a positive number $\delta > 0$ such that $V = (\overline{\Gamma}(K) + B(0, \delta)) \cap \{x \in E\colon |P_j(x)| < 1, \text{ for } j = 1, \ldots, k\} \subset U$. Now V is balanced and $\mathcal{H}_b(V)$-convex, by Corollary 1.7. □

Let E and F be Banach spaces, and let $K \subset E$ and $L \subset F$ be compact subsets. We say that K and L are *biholomorphically equivalent* if there exist open subsets $U \subset E$ and $V \subset F$ with $K \subset U$ and $L \subset V$ and a biholomorphic mapping $\varphi\colon V \longrightarrow U$ such that $\varphi(L) = K$. The next theorem is the announced result for algebras of holomorphic germs, and generalizes [14, Theorem 16].

Theorem 3.8. *Let E and F be Tsirelson-like spaces. Let $K \subset E$ and $L \subset F$ be balanced compact subsets, such that K is $\mathcal{P}(^m E)$-convex, and L is $\mathcal{P}(^k F)$-convex, for some $m, k \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) *K and L are biholomorphically equivalent.*
(2) *The algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic.*

Proof. (1) $\Rightarrow$ (2) The proof of [14, Theorem 16] applies.

(2) $\Rightarrow$ (1) We claim that $\mathcal{H}(K)$ is the inductive limit of a sequence of Fréchet spaces $\mathcal{H}_b(V_n)$, where each V_n is balanced and $\mathcal{P}_b(E)$-convex (and the same for $\mathcal{H}(L)$). Indeed, let $U_n = K + B(0; \frac{1}{n})$, for every $n \in \mathbb{N}$. Applying Theorem 3.7, for each $n \in \mathbb{N}$ there exists a balanced $\mathcal{H}_b$-convex open subset V_n

such that $K \subset V_n \subset U_n$. Since $\mathcal{H}(K) = \varinjlim_{n\in\mathbb{N}} \mathcal{H}_b(U_n)$ and the inclusion $\mathcal{H}_b(U_n) \hookrightarrow \mathcal{H}_b(V_n)$ is continuous, we have that $\mathcal{H}(K) = \varinjlim_{n\in\mathbb{N}} \mathcal{H}_b(V_n)$, and our claim is proved. Next we apply the same arguments of (2) $\Rightarrow$ (1) of [14, Theorem 16], replacing [14, Corollary 14] there by Theorem 3.1 here. □

CONCLUDING REMARKS

We go back to the introduction of this paper, where we say that we give a partial answer to Mujica's question. He asks if, in Tsirelson-like spaces, $\mathcal{S}_b(U)$ can be identified with U, when U is a polynomially convex open subset. On the one hand, we think that maybe this question could be reformulated to $\mathcal{P}_b(E)$-convex open subsets. Then, as showed in Lemma 1.2, $\mathcal{P}_b(E)$-convex open sets are always $\mathcal{H}_b(E)$-convex, and hence $\mathcal{H}_b(U)$-convex open subsets, so in this sense Theorem 2.1 gives a partial answer to Mujica's question. On the other hand, it is clear that $\mathcal{P}_b(E)$-convex open subsets are always polynomially convex, but we don't know if the converse holds in general.

We still don't know whether it is possible to remove the hypothesis that U is balanced on our results. It is known that if E is separable and has the bounded approximation property, then the spectrum of $(\mathcal{H}(U), \tau_0)$ is identified with U if and only if U is a domain of holomorphy (see [10, Theorem 58.11]). We also ask whether it is possible to state an analogous result for the algebra $\mathcal{H}_b(U)$.

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