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Convexity in Directed Graphs*

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In this paper the concept of convexity in directed graphs is described. It is shown that the set of convex subgraphs of a directed graph G partially ordered by inclusion forms a complete, semimodular, A-regular lattice, denoted $\mathcal{S}_G$. The lattice theoretic properties of the convex subgraph lattice lead to inferences about the path structure of the original graph G. In particular, a graph factorization theorem is developed. In Section 4, several graph homomorphism concepts are investigated in relation to the preservation of convexity properties. Finally we characterize an interesting class of locally convex directed graphs.

1. PRELIMINARY CONCEPTS

A directed graph, or more briefly, *digraph*, $G = (P, E)$ is a relation E on a point set P. Ordered pairs $(p, q) \in E$ are called *edges* of G. G will be said to contain a *path* from p to q, denoted $\rho(p, q)$, if there exists a sequence of points

$$p = p_0, p_1, \ldots, p_n = q, \qquad n \geqslant 0,$$

such that (1) $p_{i-1} \neq p_i$ and (2) $(p_{i-1}, p_i) \in E$ for $1 \leqslant i \leqslant n$. In this case the path is said to have length n, denoted $|\rho(p, q)| = n$. A path of length $\geqslant 2$ from p to itself is called a *cycle*; in view of condition (1) it is impossible to have cycles of length 1, and in particular loops are not considered cycles. The symbolism $\rho(p, q)$ will also be used to denote "there exists a path from p to q in G."

Given a digraph $G = (P, E)$ we can define a new relation Π on the point set P as follows: $(p, q) \in \Pi$ if and only if $\rho(p, q)$ in G. This derived digraph (P, Π) is simply the *transitive closure* of G, usually denoted G^t. The relation Π is clearly transitive and we admit paths of length zero in

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our definition so that Π is also reflexive. Further, it is not hard to show that if $G = (P, E)$ is acyclic, then Π is a partial ordering of P.

A digraph $H = (P_H, E_H)$[1] is said to be a *subgraph*[2] of $G = (P_G, E_G)$, denoted $H \leqslant G$, if (1) $P_H \subseteq P_G$ and (2) $(p, q) \in E_H$ if and only if $p, q \in P_H$, $(p, q) \in E_G$. Thus any set of points $P \subseteq P_G$ defines a unique subgraph of G which we denote by $[P]$.

A subgraph H of G is said to be *convex* in G if for any pair of points $p, q \in P_H$, every path $\rho_G(p, q)$ is completely contained in P_H. If $H_1 \leqslant H_2 \leqslant G$ it follows immediately that (1) H_1 convex in G implies H_1 is convex in H_2, and (2) H_1 convex in H_2 and H_2 convex in G implies H_1 is convex in G. Also any digraph G is convex in itself, while the empty digraph $\varnothing = [\varnothing]$ is convex in any graph.

If the intersection of two subgraphs $H_1 = (P_1, E_1)$ and $H_2 = (P_2, E_2)$ is defined as $H_1 \cap H_2 = (P_1 \cap P_2, E_1 \cap E_2)$, it is easy to show that an arbitrary intersection of convex subgraphs is convex. As is the case with normal subgroups, the union of convex subgraphs need not be convex; indeed the union need not be a subgraph. Instead we define the *convex hull* of a subset $P \subseteq P_G$, denoted $\mathrm{ch}(P)$, as the intersection of all convex subgraphs of G which contain P.

Since convexity is preserved under arbitrary intersections, the set of all convex subgraphs of G, partially ordered by $\leqslant$, is a complete lattice [1]. We denote this *convex subgraph lattice* of G by $\mathscr{S}_G$. Evidently if $\{H_i = (P_i, E_i)\}$ are elements of $\mathscr{S}_G$ (i.e., convex subgraphs of G) then the sup and inf operators in $\mathscr{S}_G$ are defined by (1) $\bigwedge H_i = \bigcap H_i$ and (2) $\bigvee H_i = \mathrm{ch}(\bigcup P_i)$. Figure 1 illustrates three typical digraphs and their associated convex subgraph lattices.

Since convexity is a condition imposed upon paths between points of a subgraph, it is easy to show that, if $G = (P, E)$ and $G^t = (P, \Pi)$ is its transitive closure, then $\mathscr{S}_G = \mathscr{S}_{G^t}$. Further any two digraphs on a common point set P which have the same transitive closure, must have the same convex subgraph lattice.

The next theorem provides a useful characterization of the convex hull operator.

THEOREM 1. *If P is any subset of P_G, then* $\mathrm{ch}(P) = [\{q \mid \rho(p_1, q)$ *and* $\rho(q, p_2)$ *for some* $p_1, p_2 \in P\}]$ *where* p_1, p_2 *need not be distinct.*

Proof. Clearly this subgraph (call it Q) must be contained in any convex subgraph which contains P, and hence $Q \subseteq \mathrm{ch}(P)$. On the other

[1] We use subscripts, where necessary, to distinguish point sets and relations. Similarly $\rho_H(p, q)$ denotes a path from p to q contained entirely in H.

[2] Several definitions of "subgraph" are commonly found in the literature.

hand Q itself is convex. Indeed let r be any point of G such that $\rho(q_1, r)$ and $\rho(r, q_2)$ where $q_1, q_2 \in Q$. Then by transitivity $\rho(p_1, r)$ and $\rho(r, p_2)$ for some $p_1, p_2 \in P$, implying $r \in Q$. Since $P \subseteq Q$, $\text{ch}(P) \subseteq Q$. ||

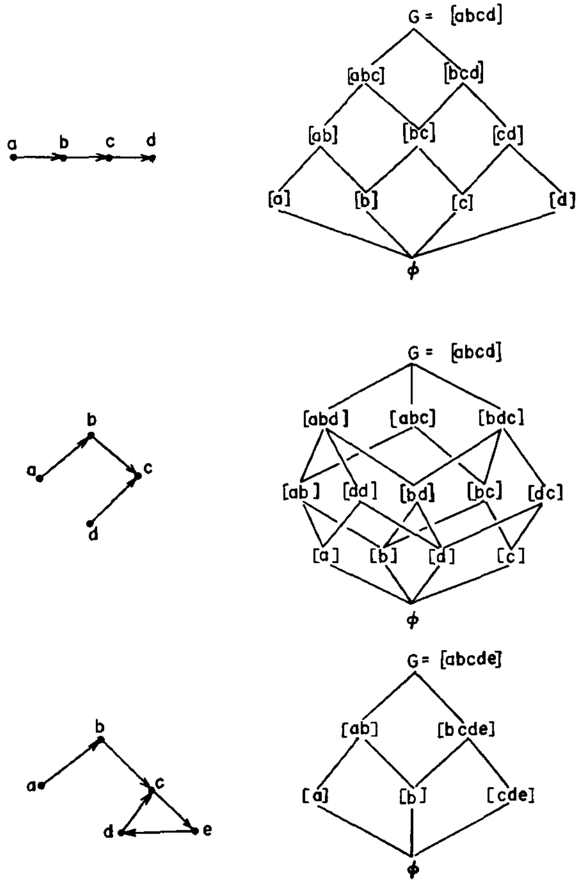


FIG. 1

As a simple corollary of this theorem, it follows that if p is any point of G, then $\text{ch}(\{p\}) = [\{q \mid \rho(p, q) \text{ and } \rho(q, p)\}]$. Thus $p \in \text{ch}\{q\}$ if and only if $q \in \text{ch}\{p\}$, or equivalently the convex hull of a single point is the subgraph on all cycles which include the point.

If one further defines the *border* of a subgraph to be its set of extremal points, then clearly if G is acyclic every subgraph has a border. One can then state an immediate analog of the Krein-Milman theorem, if G is acyclic then every convex subgraph is the convex hull of its border.

2. The Convex Subgraph Lattice $\mathscr{S}_G$

Just as the lattice of normal subgroups may be used to characterize aspects of the internal structure of a group, the convex subgraph lattice may be used to investigate properties of a digraph. It is of particular value in studying the effects of graph homomorphisms which are considered in Section 4. In this section we first investigate properties of the lattices themselves.

THEOREM 2. *For any digraph G, its convex subgraph lattice $\mathscr{S}_G$ is lower semimodular.*

Proof. Recall that a lattice $\mathscr{L}$ is lower semimodular (LSM) if $x \vee y$ covers both x and y implies that both x and y cover $x \wedge y$. We suppose H_1 and H_2 are convex subgraphs of G, hence elements of $\mathscr{S}_G$; that $\tilde{H} = H_1 \vee H_2$ and $\hat{H} = H_1 \wedge H_2$; and that $\tilde{H}$ covers both H_1 and H_2. By symmetry it will suffice to show that H_1 covers $\hat{H}$.

Suppose there existed a convex subgraph H' such that $\hat{H} < H' < H_1$. Let $p_0 \in H' \sim \hat{H}$, so that p_0 is in H_1 but not H_2; and let $p_1 \in H_1 \sim H'$ so that $p_1 \neq p_0$ and $p_1 \notin H_2$. Now $H_2 < [P_{H_2} \cup \{p_0\}] < \tilde{H}$, and since $\tilde{H}$ covers H_2 this implies that the subgraph $[P_{H_2} \cup \{p_0\}]$ is not convex. Hence there exists a path between p_0 and some point $q_0 \in H_2$ which does not lie completely in $[P_{H_2} \cup \{p_0\}]$. We may assume the path is of the form $\rho(q_0, p_0)$; in the case $\rho(p_0, q_0)$ we simply give a symmetric argument below.

Let $X = \{p \in H' \sim \hat{H} \mid \rho(q, p)$ for some $q \in H_2\}$. Since $p_0 \in X$, X is non-empty and we have $H_2 < [P_{H_2} \cup X]$. Further, since $p_1 \notin X$ (in fact, $p_1 \notin H'$), we have $[P_{H_2} \cup X] < \tilde{H}$, so that this subgraph $[P_{H_2} \cup X]$ too cannot be convex. There thus exists a path $\rho(r, s)$ between two points r and s of $P_{H_2} \cup X$ which contains a point t in neither H_2 nor X. It is safe to assume that $t \in \tilde{H}$ since $[P_{H_2} \cup X]$ cannot be convex in $\tilde{H}$ either. We now consider four possible cases:

Case I. r and s are both in H_2. This yields an immediate contradiction to the convexity of H_2.

Case II. $r \in X$, $s \in H_2$. Since $r \in X$, we have $r \notin H_2$, and $\rho(q, r)$ for some $q \in H_2$. This together with $\rho(r, s)$ contradicts convexity of H_2.

Case III. r and s are both in X. Since $r \in X$, there is a path $\rho(q, r)$ for some $q \in H_2$, which together with $\rho(r, t)$ implies $\rho(q, t)$. But $t \notin X$ so we must have $t \notin H' \sim \hat{H}$. Since $t \notin H_2$, it cannot be in $\hat{H}$, so $t \notin H'$. But $r, s \in X \subseteq H'$, thus contradicting assumed convexity of H'.

Case IV. $r \in H_2$, $s \in X$. Here again since $\rho(r, t)$ and $t \notin X$, we must have $t \notin H' \sim \hat{H}$; and since $t \notin H_2$ this implies $t \notin H'$. Now $\tilde{H}$ covers H_2 and $t \notin H_2$, so we must have $\text{ch}(P_{H_2} \cup \{t\}) = \tilde{H}$; thus every point of $\tilde{H}$ is in t itself, is in H_2, or else lies on some path between t and a point of H_2 (in some direction). In particular, consider $p_0 \in X$. We know $\rho(q_0, p_0)$ and that $\rho(t, s)$. If $p_0 \in \rho(q, t)$ where $q \in H_2$, then we have $t \in \rho(p_0, s)$ contradicting the convexity of H'. On the other hand, if $p_0 \in \rho(t, q)$, then $p_0 \in \rho(q_0, q)$ contradicting convexity of H_2. ||

It is known [1] that the Jordan-Dedekind chain condition, which asserts that all unrefinable chains between two elements have the same length, holds in any LSM lattice. In the case of $\mathscr{S}_G$ it is easy to prove a stronger result which is analogous to the Jordan-Hölder theorem in that it relates the increments in cardinality between successive convex subgraphs in two such unrefinable chains, under an appropriate permutation.

We also recall that in lattices which satisfy the Jordan-Dedekind condition the length of any unrefinable chain from A to B is equal to the dimension of B over A (denoted $\dim(B : A)$, we shall abbreviate $\dim(B : 0)$ by $\dim B$). Further dimension is additive, justifying later proofs using induction on dimension in $\mathscr{S}_G$.

The atoms of a lattice $\mathscr{L}$, are those elements which cover the least element 0. We call a convex subgraph H an *atom* of G if it is an atom in $\mathscr{S}_G$; that is, H contains no proper convex subgraphs. From the observations following Proposition 1, it is evident that a convex subgraph of G is an atom if and only if it is the convex hull of a single point; that all points on a cycle belong to the same atom; and that the atoms of a digraph are simply its strongly connected components. We next prove an important property which relates the dimension of H in $\mathscr{S}_G$ to the number of atoms contained in H.

LEMMA. *Let H cover H' in $\mathscr{S}_G$, and let p, q be points of H but not of H'; then* $\text{ch}(\{p\}) = \text{ch}(\{q\})$.

Proof. Since H covers H', we have $\text{ch}(P_{H'} \cup \{p\}) = \text{ch}(P_{H'} \cup \{q\}) = H'$. Hence p lies in a path between q and a point of H', and vice versa. If $\rho(p, q)$ and $\rho(q, p)$, we are done; but if both paths have the same direction (say) from p to q, we have $\rho(p, r)$ and $\rho(s, p)$ with r, s points of H', contradicting convexity of H'. ||

In any finite dimensional lattice $\mathscr{L}$, we can let A_x denote the set of atoms $a \in \mathscr{L}$ such that $a \leqslant x$. Evidently if $\mathscr{L} = \mathscr{S}_G$ we have $A_H = \{\text{ch}(p) \mid p \in H\}$ for all $H \in \mathscr{S}_G$. Also $A_\varnothing = \varnothing$, and $A_{\text{ch}(p)} = \{\text{ch}(p)\}$ for all p.

THEOREM 3. *dim* $H = |A_H|$ *for all* $H \in \mathcal{S}_G$.

Proof. This is clear if dim $H = 1$ since H is then an atom in $\mathcal{S}_G$. Suppose it is true for all convex subgraphs of dimension $k > 1$ and let dim $H = k + 1$. Let H cover the subgraph H'; then dim $H' = k$ so that by induction hypothesis $|A_{H'}| = k$. Moreover by the preceding lemma there is exactly one atom below H which is not below H', so that $|A_H| = k + 1$. ||

If x is an arbitrary element of any finite dimensional lattice, we will call $|A_x|$ the *A-rank* (short for "atom-rank") of x. A finite dimensional lattice in which $|A_x| = \dim x$ for all $x \in \mathcal{L}$ will be called *A-regular.*

One may now consider a class of abstract lattices, called *G-lattices*, which are both lower semimodular and A-regular. Clearly every lattice of convex subgraphs $\mathcal{S}_G$ is a G-lattice; although the converse is not true. G-lattices have many interesting lattice theoretic properties [6] which we will simply assert without proofs.

PROPOSITION A. *In any finite dimensional lattice* $\mathcal{L}$,

(1) $x \leqslant y$ *implies* $A_x \subseteq A_y$,

(2) $A_{x \vee y} \supseteq A_x \cup A_y$,

(3) $A_{x \wedge y} = A_x \cap A_y$.

If further $\mathcal{L}$ *is A-regular,*

(4) $A_x = A_y$ *if and only if* $x = y$,

(5) $A_x \subset A_y$ *if and only if* $x < y$.

A set $S = \{a_1, \ldots, a_n\}$ of atoms in a lattice is called *full* if $A_{\sup S} = S$.

PROPOSITION B. *In an A-regular lattice* $\mathcal{L}$,

(1) S *is full if and only if* $S = A_x$ *for some* $x \in \mathcal{L}$,

(2) *The relation* $\{(x, A_x) \mid x \in \mathcal{L}\}$ *is an order isomorphism of* $\mathcal{L}$ *with the full sets of atoms of* $\mathcal{L}$ *under inclusion.*

PROPOSITION C. *If* $\mathcal{L}$ *is A-regular and satisfies the Jordan-Dedekind condition, then*

(1) $\mathcal{L}$ *is lower semimodular.*

(2) *If* $x \vee y$ *covers both* x *and* y, *then* $A_{x \vee y} = A_x \cup A_y$.

PROPOSITION D. *In a G-lattice,*

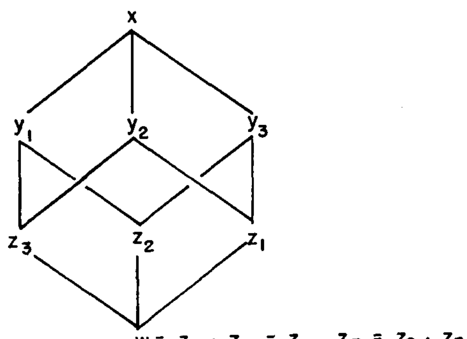
(1) $\dim x \geqslant 2$ *implies the element x covers at least two elements.*

(2) *x covers exactly two elements if and only if x is the* sup *of a unique pair of distinct atoms,* a_1 *and* a_2.

(3) *if x covers three (or more) elements, say* y_1, y_2, y_3, *and we let* $z_3 = y_1 \wedge y_2$, $z_2 = y_1 \wedge y_3$, $z_1 = y_2 \wedge y_3$, *then* $z_1 \wedge z_2 = z_1 \wedge z_3 = z_2 \wedge z_3$.

Proposition D asserts that G-lattices, and in particular convex subgraph lattices $\mathscr{S}_G$, show a surprising regularity in their internal structure, even though they are not in general modular. It is this regularity which makes possible the derivation of significant algebraic results concerning convex subgraph lattices, and permits easy transition from $\mathscr{S}_G$ to the digraph G itself. For example, Proposition D(2) asserts that, if $H \in \mathscr{S}_G$ covers exactly two elements, then $H = \text{ch}(\{a_1\} \cup \{a_2\})$ and hence H is precisely convex subgraph consisting of all paths between the atoms (points if G is acyclic) a_1 and a_2 in G. Further all such subgraphs which consist of all paths between two atoms are characterized in $\mathscr{S}_G$ by this covering property.

Proposition D(3) accounts for the very striking "cubic" structure beneath all other elements in $\mathscr{S}_G$:



By way of illustration consider the digraph G in Figure 2 and its convex subgraph lattice $\mathscr{S}_G$. The paths a, b, c and a, b, d stand out in $\mathscr{S}_G$ since they are of dimension $\geqslant 3$ but still cover only two elements.

3. GRAPHS WITH A GIVEN CONVEX SUBGRAPH LATTICE

In this section we assume $\mathscr{S}_G$ is given and we are concerned with those properties of the digraph G that are reflected in $\mathscr{S}_G$. It is easily verified that non-isomorphic digraphs may have isomorphic subgraph lattices,

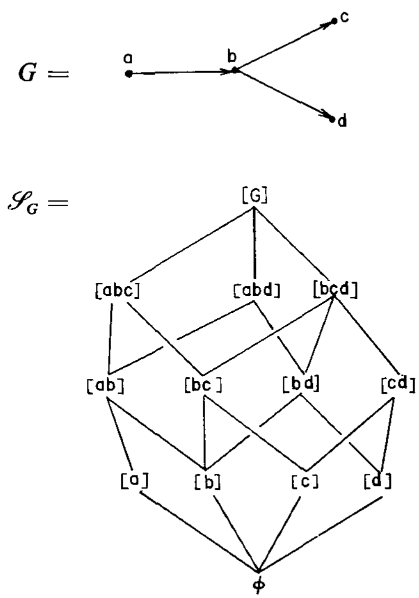


FIG. 2

so that, in particular, $\mathscr{S}_G$ does not completely characterize G. Our first result is a formal statement of the last observation of the preceding section.

THEOREM 4. *Let $H = A_1 \vee A_2$, where A_1 and A_2 are atoms of $\mathscr{S}_G$ and let* $\dim H \geqslant 3$. *If p and q are any points of A_1 and A_2, respectively, then every point $r \in H$ lies on some path* (with unspecified direction) *between p and q.*

Proof. Since $\dim H \geqslant 3$, $| A_H | \geqslant 3$, implying there exists at least one other atom $< H$. So there exists a point $r_0 \in H$ which is not a point of either A_1 or A_2. Since $r_0 \in \text{ch}(A_1, A_2)$ it is a member of some path between $s \in A_1$ and $t \in A_2$. For concreteness we may assume $r_0 \in \rho(s, t)$. But A_1 and A_2 are atoms, thus strongly connected which implies $\rho(p, s)$ and $\rho(t, q)$. Thus by transitivity of paths, $r_0 \in \rho(p, q)$. We are done if $r_0 = r$. To complete the proof suppose $r \in A_1$ (or A_2). Since the existence of at least one path $\rho(p, q)$ has been established, and A_1 is strongly connected, the desired result follows immediately. ||

If G is acyclic, so that the atoms of $\mathscr{S}_G$ are just the points of G themselves, then the preceding theorem may be restated as: (1) there exists a path between p and q through r if and only if $r < p \vee q$ in $\mathscr{S}_G$; or, equivalently, (2) there exists a path of length $\geqslant 2$ between p and q if and only if $\dim p \vee q \geqslant 3$ in $\mathscr{S}_G$.

Let p be any point in a digraph G, and inductively define

$$C^0(p) = \{p\}$$
$$C^1(p) = \{q \mid \rho(p, q) \text{ or } \rho(q, p)\}$$
$$\vdots$$
$$C^k(p) = \bigcup_q \{C^1(q) \mid q \in C^{k-1}(p)\}.$$

Thus $C^1(p)$ is the set of all points which are elements of paths to or from p. (Note that $C^0(p) \subseteq C^1(p)$ since by definition there exists a path of length zero from p to itself.) Similarly, $C^k(p)$ is the set of all points that are joined to p by a sequence of k or fewer paths. Readily the subgraph $C(p) = [\bigcup_k C^k(p)]$, called the *component* of G containing p, is simply the maximal "connected" subgraph containing p, and thus corresponds to the standard definition [2, 7]. It is well known that the components of G form a disjoint partition of G and clearly every component is convex in G.

THEOREM 5. *Let G be acyclic and let $(p, q) \in E_G$, then the orientation of every path in $C(p) = C(q)$ is uniquely determined by $\mathscr{S}_G$.*

Proof. Let $r \in C^1(p)$ be different from p, so that either $\rho(p, r)$ or $\rho(r, p)$, but not both, since G is acyclic. Now, in $\mathscr{S}_G$, either $p < q \vee r$ or $p \nless q \vee r$. If $p \nless q \vee r$, then we must have $\rho(p, r)$, since the alternative $\rho(r, p)$ together with $\rho(p, q)$ imply $p \in \rho(r, q)$ or equivalently $p < q \vee r$, a contradiction. On the other hand, $p < q \vee r$ implies p lies on some path between q and r by the preceding theorem. The case $p \in \rho(q, r)$ implies, in particular, $\rho(q, p)$, contradicting acyclicity. Thus we must have $\rho(r, q)$ and, in particular, $\rho(r, p)$. Thus $\mathscr{S}_G$ determines the orientation of every path between p and any point $r \in C^1(p)$.

Let s, t be any points in $C(p)$ that are joined by a path. $s \in C^j(p)$ for some j, hence $t \in C^{j+1}(p)$. It is thus sufficient to show, for any j, that the orientation of all paths between points of $C^j(p)$ and $C^{j+1}(p)$ is determined by ζ_G. This has already been shown for $j = 0$; we proceed by induction. Let $t \in C^{j+1}(p)$ and there exists $r \in C^{j-1}(p)$ such that either $\rho(r, s)$ or $\rho(s, r)$ where, by the induction hypothesis, the orientation is determined by ζ_G. We can now apply the argument used for $C^1(p)$, with r and s assuming the roles of p and q if $\rho(r, s)$, or of q and p if $\rho(s, r)$. ||

Consequently we see that a convex subgraph lattice $\mathscr{S}_G$ determines the existence of a path of length $\geqslant 2$ (although not necessarily its constituent edges) between any two points of G. Further, if G is acyclic and the orientation of a single edge is known, then $\mathscr{S}_G$ also determines the orientation of all of these paths (including all other shorter paths and edges) in that component.

As a corollary to Theorem 5, one sees that, if G' is obtained from G by reversing the orientation of some set S of the edges in E_G, then $\mathscr{S}_G = \mathscr{S}_{G'}$ if and only if S consists of all the edges in one or more components of G.

It is customary to partition the point-set P of a digraph G in terms of either strongly connected or just connected components. Such partitions lead toward the idea of simplifying a digraph by factoring it into constituent components. We next develop a natural factorization concept which yields a partition of S which is finer than, yet at the same time compatible with, that induced by its connected components.

The *cardinal product* of the lattices $\mathscr{L}_1, ..., \mathscr{L}_n$ (denoted $\mathscr{L}_1 \times \cdots \times \mathscr{L}_n$) is the set of all n-tuples $(x_1, ..., x_n)$, where $x_i \in \mathscr{L}_i$, under the order relation $\leqslant$ defined by $(x_1, ..., x_n) \leqslant (y_1, ..., y_n)$ if and only if $x_i \leqslant y_i$ in $\mathscr{L}_i$ for all i. It is not difficult to show

PROPOSITION E. *$\mathscr{L}_1 \times \cdots \times \mathscr{L}_n$ is a G-lattice if and only if each factor $\mathscr{L}_i$ is a G-lattice.*

Suppose that $G_1, ..., G_n$ are subgraphs of G and that $\mathscr{S}_G \cong \mathscr{S}_{G_1} \times \cdots \times \mathscr{S}_{G_n}$ under the natural correspondence in which any atom A_i in G_i corresponds to the atom $(0, ..., A_i, ..., 0)$ in $\mathscr{S}_{G_1} \times \cdots \times \mathscr{S}_{G_n}$; then we call $G_1, ..., G_n$ *factors* of G and write $G = G_1 \times \cdots \times G_n$.

THEOREM 6. *If G is acyclic, then $G = G_1 \times G_2$ if and only if P_G can be partitioned into two disjoint sets P_1 and P_2 such that*

(i) *$G_1 = [P_1]$ and $G_2 = [P_2]$ are convex;*

(ii) *any path of length $\geqslant 2$ lies entirely in G_1 or entirely in G_2.*

Proof. If $G = G_1 \times G_2$, we have $\mathscr{S}_G \cong \mathscr{S}_{G_1} \times \mathscr{S}_{G_2}$ under the natural correspondence. Hence $P_{G_1} \cap P_{G_2} = \varnothing$, since otherwise $p \in P_{G_1} \cap P_{G_2}$ would imply that p corresponds to both (p, o) and (o, p). Similarly $P_{G_1} \cup P_{G_2} = P_G$, since otherwise the correspondence would not be onto. Moreover $G_1 = [P_{G_1}]$ is convex in G, since $(G_1, 0) \leftrightarrow H \in \mathscr{S}_G$. Under the correspondence $p \leqslant H$ in $\mathscr{S}_G$ if and only if $(p, o) \leqslant (G_1, 0)$ in $\mathscr{S}_{G_1} \times \mathscr{S}_{G_2}$, so that $[P_{G_1}] = H$ which is known to be convex in G. Similarly $G_2 = [P_{G_2}]$ is convex in G. Finally, we consider a path $\rho(p, q)$ where $|\rho(p, q)| \geqslant 2$. By a corollary remark to Theorem 4, $\dim p \vee q \geqslant 3$. If we had $p \in G_1$, $q \in G_2$, then $p \leftrightarrow (p, o)$ and $q \leftrightarrow (o, q)$ under the natural correspondence, so that $p \vee q \leftrightarrow (p, o) \vee (o, q) = (p, q)$; but $\dim(p, q) = 2$ in $\mathscr{S}_{G_1} \times \mathscr{S}_{G_2}$, a contradiction.

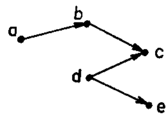
Conversely suppose that there exists a partition of P_G satisfying the hypotheses. We may evidently establish a 1-1 correspondence between the atoms of $\mathcal{S}_G$ and $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$ by letting $p \leftrightarrow (p, o)$ if $p \in P_1$ and $q \leftrightarrow (o, q)$ if $q \in P_2$. We shall now show that a set of atoms is full in $\mathcal{S}_G$ if and only if the corresponding set is full in $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$. Let $S = \{(p_1, o),..., (p_m, o), (o, q_1),..., (o, q_n)\}$ be full in $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$, so that $S = A_{(H_1, H_2)}$ for some element $(H_1, H_2) \in \mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$, where $H_1 \leqslant G_1$ and $H_2 \leqslant G_2$. By properties (i) and (ii) the subgraph $H_1 \cup H_2 = [P_{H_1} \cup P_{H_2}]$ is readily convex in G, so that $\{p_1,..., p_m, q_1,..., q_n\} = A_{H_1 \cup H_2}$ is full in $\mathcal{S}_G$. Conversely, if S is not full, there exists another atom (p, o) (or (o, q)) such that $(p, o) \leqslant \sup S = (p_1 \vee \cdots \vee p_m, q_1 \vee \cdots \vee q_n)$, implying $p < p_1 \vee \cdots \vee p_m$ so that $\{p_1,..., p_m, q_1,..., q_n\}$ is also not full. We thus have a $1 - 1$ correspondence between the full sets of atoms of $\mathcal{S}_G$ and $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$. Consequently we may extend the original correspondence, which was defined only on the atoms, as follows:

$$(H_1, H_2) \leftrightarrow H_1 \cup H_2 \quad \text{if and only if} \quad A_{(H_1, H_2)} \leftrightarrow A_{H_1 \cup H_2}.$$

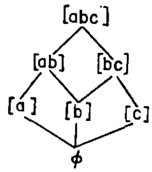
Since both $\mathcal{S}_G$ and $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$ are G-lattices, by Proposition B there is a $1 - 1$ correspondence between their elements and full sets of atoms so that the correspondence defined above is a $1 - 1$ correspondence between the elements of $\mathcal{S}_G$ and $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$. Moreover the correspondence is obviously order preserving in both directions and hence is the natural isomorphism. ||

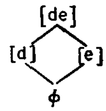
The conditions of this theorem are analogous to those required in algebra, for a group G to be isomorphic (in the natural way) to the direct product of its normal subgroups: namely, $G_1 \cap G_2 = \{e\}$ and $G_1 \cdot G_2 = G$. Figure 3 illustrates the ideas of Theorem 6. The reader can verify that $G = G_1 \times G_2$, where $G_1 = [abcd]$ and $G_2 = [e]$, is also a factorization. Here $\mathcal{S}_{G_1}$ is the second example of Figure 1, while $\mathcal{S}_{G_2}$ is the trivial lattice on two elements. The unique factorization of G into indecomposible factors is $G = G_1 \times G_2 \times G_3$, where $G_1 = [abc]$, $G_2 = [d]$, $G_3 = [e]$.

It is known that any lattice has a unique factorization as a cardinal product of lattices which are themselves indecomposable (as cardinal products) [1]. Now if $G = G_1 \times \cdots \times G_n$ where $\mathcal{S}_{G_i}$ is decomposable, say $\mathcal{S}_{G_i} = \mathcal{L}_1 \times \mathcal{L}_2$, there are principal ideals of $\mathcal{S}_{G_i}$ order isomorphic to the $\mathcal{L}_i$'s, and these ideals must be convex subgraph lattices of two convex subgraphs of H which are as in the theorem. Thus repeated application of the decomposition procedure of Theorem 6 must eventually yield the unique factorization of both G and $\mathcal{S}_G$. Further it is evident that, if G_i is a component of G, then G_i must be a factor of G, although not necessarily an indecomposable factor.

$G = G_1 \times G_2 =$ 

where $G_1 = [abc]$ and $G_2 = [de]$ are factors.

$\mathscr{S}_{G_1} =$ 

$\mathscr{S}_{G_2} =$ 

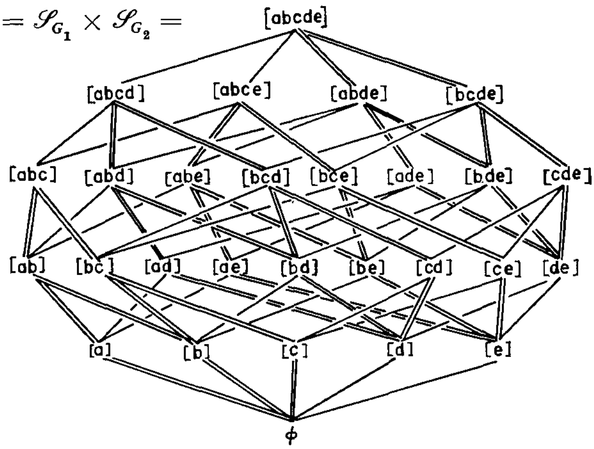
$\mathscr{S}_G = \mathscr{S}_{G_1} \times \mathscr{S}_{G_2} =$ 

FIG. 3. The sublattice (or ideal) under the element $(G_1, 0)$ has been indicated by double lines, as have three other replicas of it generated by the cardinal product. The sublattice under $(0, G_2)$ and its copies are indicated by single lines.

Let $\mathscr{S}$ be any convex subgraph lattice. A digraph G is said to be a *realization*[3] of $\mathscr{S}$ if $\mathscr{S}_G \cong \mathscr{S}$. It is said to be *minimal realization* of $\mathscr{S}$ (or briefly minimal) if no graph on fewer points or with fewer edges has the same convex subgraph lattice.

THEOREM 7. *Any convex subgraph lattice $\mathscr{S}$ has a minimal realization.*

Proof. Let $\mathscr{G}$ be the set of all realizations of $\mathscr{S}$, and let $\mathscr{G}_1$ be the subset of $\mathscr{G}$ consisting of all digraphs with fewest points. From $\mathscr{G}_1$ choose any

[3] It can be shown that there exist G-lattices $\mathscr{L}$ which cannot be realized, that is there exist no graphs G such that $\mathscr{S}_G \cong \mathscr{L}$.

digraph G with fewest edges; we will show that G is minimal. If not, let $G' \in \mathscr{G}$ have fewest edges; since G is not minimal, we have $| E_{G'} | < | E_G |$ implying that $G' \notin \mathscr{G}_1$. If G' were acyclic, G could not have fewer points than G', since G' would already have as few points as there are atoms in $\mathscr{S}$. Hence at least two points of G' belong to the same atom, so that the graph G'' obtained from G' by condensing this atom to a single point has the same convex subgraph lattice, but fewer edges, contradiction. ||

Finally it can be shown that, if G is minimal, then G is loop-free, acyclic and every edge belongs to a path of length $\geqslant 2$. (The converse of this statement, however, is not true.) Also if $G_1, ..., G_n$ are connected components of a minimal graph G, then $G_1 \times \cdots \times G_n$ is the unique factorization of G into indecomposable factors and each G_i is minimal.

4. Graph Homomorphisms

Several different graph homomorphism concepts have appeared in the literature, depending on what properties of the graph are to be preserved under the mapping. Of particular interest will be convexity concepts under homomorphic maps. A mapping $\varphi : G \to G'$ is called a *graph homomorphism* if φ maps P_G onto $P_{G'}$ and

(i) $(p, q) \in E_G$ implies $(\varphi(p), \varphi(q)) \in E_{G'}$,

(ii) $(p', q') \in E_{G'}$ implies there exist $p \in \varphi^{-1}(p')$ and $q \in \varphi^{-1}(q')$ such that $(p, q) \in E_G$.

If the mapping satisfies only condition (i), it will be called a *weak graph homomorphism*. If instead we require that $(p, q) \in E_G$ if and only if $(\varphi(p), \varphi(q)) \in E_{G'}$, then we call φ an *E-homomorphism*. The standard definition of graph isomorphism is a 1-1 E-homomorphism.

We note that, if $G = (P, E)$ is a digraph, and φ is any mapping of P onto a point set P', then there exist digraphs on P' that are homomorphic, or weakly homomorphic images of G under φ, while in general there may be no digraph on P' which is an E-homomorphic image of G. Further one can show that, given $\varphi : P \to P'$, the digraph on P' which is a graph homomorphic image of G under φ is unique. Also every 1-1 graph homomorphism is an isomorphism (i.e., an E-homomorphism).

Let $\varphi : G \to G'$ be any homomorphism, and let $H = (P_H, E_H)$ be a subgraph of G. By $\varphi(H)$ we shall mean $[\varphi(P_H)]$. Note that $\varphi(H)$ need not be the same as the graph obtained by restricting φ to H. In particular $(p', q') \in E_{\varphi(H)}$ need not imply that there exist $p, q \in P_H$ such that

$(p, q) \in E_H$. If H' is a subgraph of G', then by $\varphi^{-1}(H')$ we shall mean $[\varphi^{-1}(P_H)]$, the subgraph of G on the set of points which map onto $P_{H'}$ under φ.

THEOREM 8. *Let $\varphi : G \to G'$ be a homomorphism* (of any kind). *If H' is convex in G', then $\varphi^{-1}(H')$ is convex in G.*

Proof. Readily $\rho_G(p, q)$ implies $\rho_{G'}(\varphi(p), \varphi(q))$. If there were a path between two points p, q of $\varphi^{-1}(H')$ containing a point r not in $\varphi^{-1}(H')$, then $\varphi(r) \notin H'$ would lie on a path between $\varphi(p)$ and $\varphi(q)$ in H', contradicting convexity of H'. ||

Thus homomorphisms of digraphs may be likened to continuous functions,[4] in that convexity is preserved under inverse mappings. In general, however, $\varphi(H)$ need not be convex even though H is. Analogously to the definition of open mappings, a homomorphism will be called *convex* if it takes convex subgraphs onto convex subgraphs.

THEOREM 9. *Let $\varphi : G \to G'$ be a convex homomorphism. If A is an atom of G, then $\varphi(A)$ is an atom of G'. Conversely if A' is an atom of G', then there exists an atom A of G such that $\varphi(A) = A'$.*

Proof. If $\varphi(A)$ were not an atom, there would exist an atom $\varnothing < A' < \varphi(A)$. But then $\varnothing < \varphi^{-1}(A') \cap A < A$, where $\varphi^{-1}(A') \cap A$ is convex, contradicting the fact that A is an atom. Conversely let A' be an atom and let A be any atom contained in $\varphi^{-1}(A')$. Then $\varnothing < \varphi(A) \leqslant A'$, and since A' is an atom this implies $\varphi(A) = A'$. ||

Thus it follows that convex homomorphisms take acyclic digraphs onto acyclic digraphs; since G acyclic implies every point p is an atom and consequently its image, a point p' in G' must be an atom. In the remainder of this section we consider convex homomorphisms whose images are acyclic.

If φ is a homomorphism, then $\rho_G(p, q)$ implies $\rho_{G'}(p', q')$ while the converse is not true in general. Under suitable conditions, however, we can prove a partial converse.

THEOREM 10. *Let $\varphi : G \to G'$ be a convex homomorphism with G' acyclic. If $\rho_{G'}(p', q')$ is a path of length $\geqslant 2$ in G', then for any $p \in \varphi^{-1}(p')$, $q \in \varphi^{-1}(q')$ there exists a path $\rho_G(p, q)$.*

[4] If one defines topologies on G and G' by means of the natural Galois-connection closure operator [4], then every homomorphism is a continuous function with respect to these topologies, and indeed convexity is preserved under the inverse of all continuous functions, $f: G \to G'$.

Proof. We cannot have $\rho_G(q, p)$, since the existence of such a path would imply $\rho_{G'}(q', p')$, contradicting acyclicity of G'. Suppose there were no path $\rho_G(p, q)$. We would then have $H = \text{ch}\{p\} \cup \text{ch}\{q\}$ convex in G. But $\varphi(H) = [\varphi(\text{ch}\{p\}) \cup \varphi(\text{ch}\{q\})] = [p' \cup q']$ cannot be convex since $|\rho_{G'}(p', q')| \geqslant 2$ implies this path contains a point other than p' and q'. ||

In the same spirit one can construct a proof by induction on path length to show that, if ρ' is any path $p' = p_0', \ldots, p_n' = q'$ of length $\geqslant 2$ and p, q are *any* preimages of p' and q', then there exists a path from p to q in G which contains, in sequence, preimages of $p_1', \ldots, p_{n-1}'$.

Since convex homomorphisms map convex subgraphs onto convex subgraphs, and convexity is preserved under inverse images, every convex homomorphism $\varphi : G \to G'$ induces an onto correspondence between $\mathscr{S}_G$ and $\mathscr{S}_{G'}$. It is thus natural to ask whether this correspondence homomorphically preserves the algebraic sup and inf properties of the lattices involved.

It appears that the "correct" question to ask is whether $\mathscr{S}_{G'}$ is a lower semihomomorphic image of $\mathscr{S}_G$; where *lower semihomomorphisms* are defined as onto mappings $\sigma : \mathscr{L} \to \mathscr{L}'$ such that

(i) $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in \mathscr{L}$;

(ii) if $x \vee y$ covers x and y in $\mathscr{L}$, then $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$.

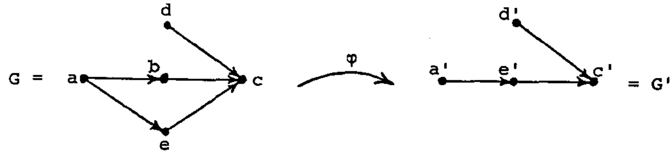
Lower semihomomorphisms (LSH's) of lower semimodular lattices have many interesting properties [5, 6]. We need only the facts that:

(1) every LSH on a G-lattice is completely characterized by its kernel, $k(\sigma)$ (the set of elements of $\mathscr{L}$ which map onto $0'$ under σ);

(2) σ is a lattice isomorphism if and only if $k(\sigma) = \{0\}$; and

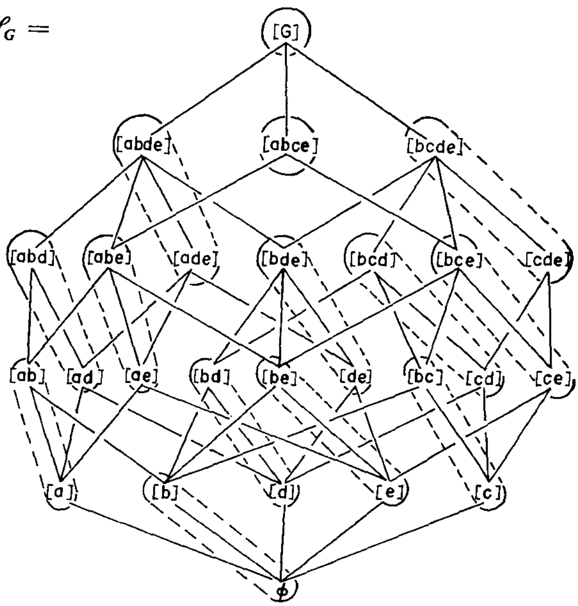
(3) σ is dimension reducing, so that it maps atoms of $\mathscr{L}$ either onto atoms of $\mathscr{L}'$ or $0'$.

Now let $\varphi : G \to G'$ be a convex homomorphism and let $\sigma : \mathscr{S}_G \to \mathscr{S}_{G'}$ be an LSH. We say that σ is *compatible* with φ if $A_{\varphi(H)} = \sigma(A_H)$ (deleting 0's) for all $H \in \mathscr{S}_G$. For example, let φ be defined by



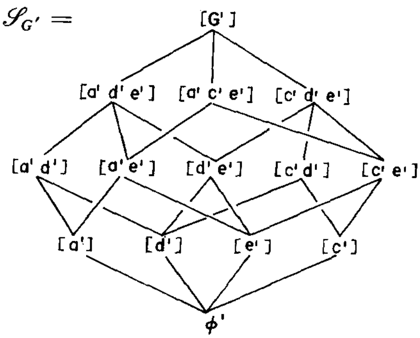
i.e., $\varphi(a) = a'$, $\varphi(b) = \varphi(e) = e'$, $\varphi(c) = c'$, and $\varphi(d) = d'$.

$\mathscr{S}_G =$



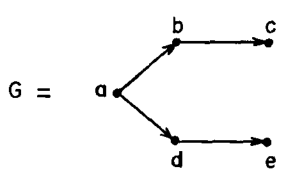
Then the subgraph lattice $\mathscr{S}_{G'}$ shown below is the image of $\mathscr{S}_G$ under the compatible LSH σ whose kernel is $\{0, [b]\}$:

$\mathscr{S}_{G'} =$

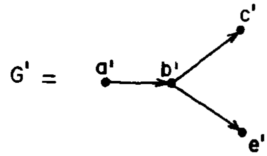


Equivalence classes on $\mathscr{S}_G$ induced by σ have been circled. Note that this σ is not the only LSH compatible with φ; for example, σ^* with kernel $\{\varnothing, [e]\}$ is also compatible.

Not every convex homomorphism has a compatible LSH. In particular let



and let



be its image under the convex homomorphism which sets $\varphi(b) = \varphi(d) = b'$. It can be shown that $\mathscr{S}_{G'}$ cannot be an image of $\mathscr{S}_G$ under any LSH.

A convex homomorphism $\varphi : G \to G'$ for which there exists a compatible LSH $\sigma : \mathscr{S}_G \to \mathscr{S}_{G'}$ we will call *strong*:

THEOREM 11. *Let $\varphi : G \to G'$ be a convex homomorphism, where G' is acyclic, and suppose that whenever $\varphi(p) = \varphi(q)$ we have $C^1(p) \subseteq C^1(q)$ or $C^1(q) \subseteq C^1(p)$; then φ is strong.*

Proof. For each atom $p' \in G'$, the set $\{C^1(p) \mid \varphi(p) = p'\}$ is totally ordered by inclusion; let $C^1(p^*)$ be its greatest element. Define σ by

(i) $\sigma(\varnothing) = \varnothing'$,

(ii) $\sigma(\text{ch}\{p^*\}) = p'$, and $\sigma(p) = \varnothing'$ for all other $p \in \varphi^{-1}(p')$,

(iii) $\sigma(H) = \sup(\sigma(A_H))$ for all H.

Clearly σ is order preserving. Let $H' \in \mathscr{S}_{G'}$ and let $H = \varphi^{-1}(H')$; by Theorem 8, $H \in \mathscr{S}_G$. Readily $\sigma(A_H) = A_H$; hence $\sigma(H) = \sup(\sigma(A_H)) = \sup(A_{H'}) = H'$, proving σ onto. Note that, by (ii), the preimage of any atom consists of precisely one atom.

We next prove that σ takes full sets of atoms into full sets of atoms. Let A be full, and suppose that $q' \leqslant \sup(\sigma(A))$, i.e., $q' \in \text{ch}(\sigma(A))$. Thus either $q' = \sigma(q)$ for some $q \in A$, in which case we are done, or there exist r^*, s^* in A, with $\sigma(r^*) = r' \neq \varnothing'$ and $\sigma(s^*) = s' \neq \varnothing'$, such that $q' \in \rho(r', s')$. By the remark following Theorem 10, there exists a path $\rho_1(r^*, s^*)$ containing a point q such that $\varphi(q) = q'$. Thus r^*, s^* are in $C^1(q) \subseteq C^1(q^*)$. In fact, q^* must lie on a path $\rho_2(r^*, s^*)$ since, by the convexity of $\varphi^{-1}(q')$, $\rho_3(q^*, r^*)$ and $\rho_1(r^*, q)$ implies $r^* \in \varphi^{-1}(q')$, contradiction, and similarly $\rho_4(s^*, q^*)$ yields a contradiction. Since A is full, this implies $q^* \in A$, so that $\sigma(q^*) = q' \in \sigma(A)$.

It follows that $p' \leqslant \sigma(H)$ implies $p^* \leqslant H$. Indeed, since $\sigma(A_H)$ is full, $p' \leqslant \sup(\sigma A_H))$ implies $p' \in \sigma(A_H)$; but, since the only atom mapped onto p' by σ is p^*, we thus have $p^* \in A_H$.

We can now show that σ is a meet homomorphism. Let

$$q' \in A_{\sigma(H_1) \wedge \sigma(H_2)} = A_{\sigma(H_1)} \cap A_{\sigma(H_2)} ;$$

then, as just shown, $q^* \in A_{H_1} \cap A_{H_2} = A_{H_1 \wedge H_2}$, so that $q' \in A_{\sigma(H_1 \wedge H_2)}$. Thus $A_{\sigma(H_1) \wedge \sigma(H_2)} \subseteq A_{\sigma(H_1 \wedge H_2)}$, implying $\sigma(H_1) \wedge \sigma(H_2) \leqslant \sigma(H_1 \wedge H_2)$, while $\geqslant$ is immediate since σ is order preserving.

Finally, let $H_1 \vee H_2$ cover H_1 and H_2, say $A_{H_1 \vee H_2} = A_{H_1} \cup \{r\} = A_{H_2} \cup \{s\}$. If $\sigma(r) = \sigma(s) = \varnothing'$ we have $\sigma(H_1 \vee H_2) = \sigma(H_1) = \sigma(H_2) = \sigma(H_1) \vee \sigma(H_2)$. If $\sigma(r) = r' \neq \varnothing'$ and $\sigma(s) = s' \neq \varnothing'$ we have $A_{\sigma(H_1 \vee H_2)} = A_{\sigma(H_1)} \cup \{r'\} = A_{\sigma(H_2)} \cup \{s'\}$, so that $\sigma(H_1 \vee H_2)$ covers $\sigma(H_1)$ and $\sigma(H_2)$, and since r and s are distinct so are r' and s', proving that $\sigma(H_1 \vee H_2) = \sigma(H_1) \vee \sigma(H_2)$. On the other hand, suppose that $\sigma(r) = r' \neq \varnothing$ and $\sigma(s) = \varnothing'$ (or vice versa). By LSM, H_1 covers $H_1 \wedge H_2$, so that we must have $A_{H_1} = A_{H_1 \wedge H_2} \cup \{s\}$. Hence $\sigma(A_{H_1}) = \sigma(A_{H_1 \wedge H_2})$, whence $\sigma(H_1) = \sigma(H_1 \wedge H_2) = \sigma(H_1) \wedge \sigma(H_2)$, i.e., $\sigma(H_1) \leqslant \sigma(H_2)$, so that $\sigma(H_1) \vee \sigma(H_2) = \sigma(H_2)$. Since $\sigma(A_{H_1 \vee H_2}) = \sigma(A_{H_2})$, we thus have $\sigma(H_1 \vee H_2) = \sigma(H_2) = \sigma(H_1) \vee \sigma(H_2)$. ||

The hypothesis of Theorem 11 holds in the case of the following important example. Let $G = (P, E)$ be any digraph and define the relation S on P by $(p, q) \in S$ if and only if p, q belong to the same atom of $\mathscr{S}_G$. Clearly S is an equivalence relation on P. Let $G^* = (P^*, E^*)$ the digraph in which $P^* = P/S$ and $(S_p, S_q) \in E^*$ whenever $(p, q) \in E$; this graph is called the *condensation graph* of G.[5] The cannonical function φ of S is evidently a homomorphism of G onto G^*. Readily we have:

(1) G^* is acyclic,

(2) $G^* = G$ if and only if G is acyclic,

(3) φ is a convex homomorphism, and

(4) $\varphi(p) = \varphi(q)$ implies $C^1(p) = C^1(q)$.

Hence we can apply Theorem 11 to obtain

COROLLARY 12. *If G^* is the condensation graph of G, then $\mathscr{S}_G \cong \mathscr{S}_{G^*}$.*

Proof. Let σ be as in Theorem 11. By definition G^* has the same number of atoms as G, hence $k(\sigma) = \{0\}$ making σ an isomorphism. ||

This result yields considerable insight into the significance of the convex subgraph lattice. From several earlier results we see that $\mathscr{S}_G$ describes what might be called the path structure of a digraph G, in particular the longer paths involving more than a single edge of G. Further, in view of Corollary 12, $\mathscr{S}_G$ describes only the acyclic "core" of the path structure, that is, the condensation graph of G.

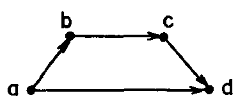
[5] As remarked earlier, the atoms of a graph are precisely its strongly connected components, so that this definition of G agrees with that in [3].

Homomorphic mappings of graphs are designed to preserve the presence of local edge structure, and by transitivity the presence of paths. As the conditions for homomorphism become stronger, they also preserve to some degree the absence of edges and paths. Convex homomorphisms preserve convexity properties that are derived from the presence and absence of paths in G, while strong homomorphisms preserve those aspects of the path structure which reflect a (possibly incomplete) partial ordering of the points of G. Finally E-homomorphisms, which form a proper subset of all the other homomorphism classes, totally preserve both the presence and absence of edges.

5. Locally Convex Diagraphs

In this final section, we describe an interesting class of digraphs which can be defined by means of convexity concepts.

Let p be a point of the digraph G. By the *left neighborhood* of p, denoted $L(p)$, we mean the subgraph on the set $\{q \mid |\rho(q, p)| = 1\}$. Similarly we define the *right neighborhood* of p by $R(p) = [\{q \mid |\rho(p, q)| = 1\}]$. Note that a point p cannot be in its own left or right neighborhood, even if there is a loop at p. G will be called *locally convex at p* if both $L(p)$ and $R(p)$ are convex subgraphs of G. It will be called *locally convex* if it is locally convex at each of its points. For example:

G = 

is not locally convex since $R(a) = [\{b\} \cup \{d\}]$ is not convex. Similarily $L(d)$ is not convex.

Theorem 13. *Every locally convex digraph is acyclic.*

Proof. Suppose there existed points $p, q \in G$ such that $\rho(p, q)$ and $\rho(q, p)$. Let $p = p_0, \ldots, p_m = q$ and $q = q_0, \ldots, q_n = p$ be such paths. Then $p_1 \in R(p)$ but the path $p_1, \ldots, p_m = q_0, \ldots, q_n = p, p_1$ is not contained in $R(p)$, since $p \notin R(p)$. ||

A digraph G is called *transitive* if $G = G^t$, that is, if $r \in \rho(p, q)$ implies (r, q) and $(p, r) \in E$. The following weaker property turns out to be closely related to local convexity. We call a digraph *semitransitive* if $\rho(p, q)$ *and* $(p, q) \in E$ together imply that for all points $r \in \rho(p, q)$, (p, r), $(r, q) \in E$.

THEOREM 14. *A digraph is locally convex if and only if it is acyclic and semitransitive.*

Proof. Let G be locally convex, let there be an edge from p to q and let $p, r_1, ..., r_{n-1}, q$ be a path $\rho(p, q)$. Since r_1 and q are both in $R(p)$, by local convexity every point of the path is contained in $R(p)$, so there is an edge from p to every point r_i of the path. Similarly, since r_{n-1} and p are in $L(q)$, there is an edge from every point r_i to q.

Conversely let G be semitransitive and acyclic, and let $\rho(r, s)$ be a path between the points r, s of $L(p)$. Thus $(s, p) \in E$ implies $\rho(r, p)$ and, since $(r, p) \in E$, there is an edge from every point of ρ to p. Moreover, since G is acyclic, p itself cannot lie on ρ, so all of these edges are paths of length 1, implying $\rho(r, s) \subseteq L(p)$. Similarily for $R(p)$. ||

THEOREM 15. *Convex graph homomorphisms take locally convex digraphs onto locally convex digraphs.*

Proof. By Theorem 13 and the remark following Theorem 9, G' is acyclic, hence we need only show that G' is semitransitive. Let $(p', q') \in E_{G'}$ and let $r' \in (p', q')$. Since φ is a graph homomorphism, there exist preimages p, q of p', q' such that $(p, q) \in E_G$. Moreover, by the remark following Theorem 10, there exists a path $\rho(p, q)$ containing a preimage r of r'. Since G is locally convex, it is semitransitive, so there exist edges (p, r) and (r, q) in E; hence there exist edges (p', r') and (r', q') in E'. ||

REFERENCES

1. G. BIRKHOFF, "Lattice Theory," American Mathematical Society (Colloquium Pub. 25) (revised), Providence, R. I., 1948.
2. R. G. BUSACKER AND T. L. SAATY, "Finite Graphs and Networks," McGraw-Hill, New York, 1965.
3. F. HARARY, R. Z. NORMAN, AND D. CARTWRIGHT, "Structural Models: An Introduction to the Theory of Directed Graphs," Wiley, New York, 1965.
4. O. ORE, "Theory of Graphs," American Mathematical Society, (Colloquium Pub. 38), Providence, T.I., 1962.
5. J. L. PFALTZ, Semi-homomorphisms of semi-modular lattices, *Proc. Am. Math. Soc.* **22**, 2 (1969) 418–425.
6. J. L. PFALTZ, "Convexity in Graphs," Univ. of Maryland, Computer Science Center Technical Report 68-74, 1968.
7. W. T. TUTTE, "Connectivity in Graphs," University of Toronto Press, Toronto, 1966.