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Journal of Functional Analysis 257 (2009) 2476–2496

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Unitarizable representations and fixed points of groups of biholomorphic transformations of operator balls

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Received 11 November 2008; accepted 22 June 2009

Available online 8 July 2009

Communicated by D. Voiculescu

Abstract

We show that the open unit ball of the space of operators from a finite-dimensional Hilbert space into a separable Hilbert space (we call it "operator ball") has a restricted form of normal structure if we endow it with a hyperbolic metric (which is an analogue of the standard hyperbolic metric on the unit disc in the complex plane). We use this result to get a fixed point theorem for groups of biholomorphic automorphisms of the operator ball. The fixed point theorem is used to show that a bounded representation in a separable Hilbert space which has an invariant indefinite quadratic form with finitely many negative squares is unitarizable (equivalent to a unitary representation). We apply this result to find dual pairs of invariant subspaces in Pontryagin spaces. In Appendix A we present results of Itai Shafrir about hyperbolic metrics on the operator ball.

Keywords: Hilbert space; Bounded representation; Unitary representation; Hyperbolic space; Fixed point; Normal structure; Biholomorphic transformation; Indefinite quadratic form

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.06.021

1. Introduction

Let K, H be Hilbert spaces; by $L(K, H)$ we denote the Banach space of all linear bounded operators from K to H. We will denote the open unit ball of $L(K, H)$ by $\mathcal{B}$ and call it *operator ball*. We say that a subset M of $\mathcal{B}$ is *separated from the boundary* if it is contained in a ball $r\mathcal{B}$, for some $r \in [0, 1)$.

A group G of transformations of $\mathcal{B}$ will be called *elliptic* if all its orbits are separated from the boundary (this terminology goes back to [9]).

We call G *equicontinuous* if, for each $\varepsilon > 0$ there is $\delta > 0$ such that if $A, B \in \mathcal{B}$ and $\|A - B\| < \delta$, then $\|g(A) - g(B)\| < \varepsilon$ for all $g \in G$. This condition can be also called *global equicontinuity* because it is possible also to consider equicontinuity in a point.

Since $\mathcal{B}$ is a bounded open set of a Banach space, one may consider holomorphic maps from $\mathcal{B}$ to Banach spaces. We will deal with invertible holomorphic maps from $\mathcal{B}$ onto $\mathcal{B}$; such maps are called *biholomorphic automorphisms* of $\mathcal{B}$. Our aim is to prove that if one of the spaces K, H is finite-dimensional and the other is separable, then any elliptic group of biholomorphic automorphisms of $\mathcal{B}$ has a common fixed point. More precisely we will prove the following result.

Theorem 1.1. *Let* $\dim K < \infty$ *and* H *be separable. For a group* G *of biholomorphic automorphisms of* $\mathcal{B}$*, the following statements are equivalent*:

(i) *G is elliptic on $\mathcal{B}$*;
(ii) *at least one orbit of G is separated from the boundary*;
(iii) *G is equicontinuous*;
(iv) *G has a common fixed point in $\mathcal{B}$.*

Remark 1.2. The assumption $\dim K < \infty$ is essential, some of the results of this paper are known to fail without it, see, for example, the last paragraph of Section 8. As for separability of H, it is just a technical convenience, our approach works for non-separable H also, with a bit more complicated proofs.

The result will be applied to the orthogonalization (or similarity) problem for bounded group representations on Hilbert spaces. This problem can be formulated as follows. Let π be a representation of a group G on a Hilbert space $\mathcal{H}$. Under which conditions there is an invertible operator V such that the representation σ of G, defined by the formula $\sigma(g) = V\pi(g)V^{-1}$, is unitary?

Clearly a necessary condition is the boundedness of π: $\sup_{g\in G} \|\pi(g)\| < \infty$. In general it is not sufficient. Some sufficient conditions (on G or π) are known, see the book [12]. We will show that a bounded representation π of a group G on a Hilbert space $\mathcal{H}$ is similar to a unitary representation if it preserves a quadratic form η with finite number of negative squares. The last condition means that $\eta(x) = \|Px\|^2 - \|Qx\|^2$ and P, Q are orthogonal projections in $\mathcal{H}$ with $P + Q = 1$ and $\dim(Q\mathcal{H}) < \infty$.

As a consequence we obtain that each bounded group of J-unitary operators on a Pontryagin space Π_k has an invariant dual pair of subspaces. In other words the space can be decomposed into J-orthogonal direct sum $H_+ + H_-$ of positive and negative subspaces which are invariant for all operators in the group.

The proof of Theorem 1.1 is based on the analysis of the structure of the operator ball as a metric space with respect to the Carathéodory distance (see Chapters 4 and 5 of [6]). It was proved by Shafrir [15] that $\mathcal{B}$ is a hyperbolic space with respect to this distance. Since [15] is not easily accessible, we present a proof of this result in Appendix A, with the kind permission of the author. We will show that $\mathcal{B}$ has a restricted form of a *normal structure* if $\dim(K) < \infty$.

In the case where K is one-dimensional Theorem 1.1 was obtained in [17]; a transparent proof can be found in [10, Section 23].

2. Hyperbolic spaces

In our definition of hyperbolic spaces we follow fixed point theory literature (see e.g. [13,14]). In geometric literature (see e.g. [3]) hyperbolic spaces are defined differently.

By a *line* in a metric space $(\mathcal{X}, \rho)$ we mean a subset of $\mathcal{X}$ which is isometric to the real line $\mathbb{R}$ with its usual metric (in the literature lines are also called *metric lines* or *geodesic lines*).

Let $(\mathcal{X}, \rho)$ be a metric space with a distinguished set $\mathcal{M}$ of lines. We say that $\mathcal{X}$ is a *hyperbolic space* if the following conditions are satisfied:

(1) (Uniqueness of a distinguished line through a given pair of points) For each $x, y \in \mathcal{X}$, there is exactly one line $\ell \in \mathcal{M}$ containing both x and y.
(2) (Convexity of the metric) To state the condition (see (2.3)) we need to introduce some more definitions and notation. The *segment* $[x, y]$ is defined as the part of the line $\ell \in \mathcal{M}$ containing both x and y, which consists of all $z \in \ell$ satisfying

$$\rho(x, y) = \rho(x, z) + \rho(z, y). \tag{2.1}$$

We write

$$z = (1-t)x \oplus ty \tag{2.2}$$

if $z \in [x, y]$, $\rho(z, x) = t\rho(x, y)$, and $\rho(z, y) = (1-t)\rho(x, y)$ (where $t \in [0, 1]$).

The convexity condition is:

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \leqslant \frac{1}{2}\rho(y, z). \tag{2.3}$$

Hyperbolic spaces satisfy also the following stronger form of the condition (2.3):

$$\rho\big((1-t)x \oplus ty, (1-t)w \oplus tz\big) \leqslant (1-t)\rho(x, w) + t\rho(y, z). \tag{2.4}$$

(To get (2.4) from (2.3) we observe that, if for some value of t we have the inequalities $\rho((1-t)x \oplus ty, (1-t)x \oplus tz) \leqslant t\rho(y, z)$ and $\rho((1-t)x \oplus tz, (1-t)w \oplus tz) \leqslant (1-t)\rho(x, w)$, then, by the triangle inequality, we have (2.4) for that value of t. Using this observation repeatedly we prove the inequalities for t of the form $\frac{k}{2^n}$ ($k \in \mathbb{N}$, $1 \leqslant k \leqslant 2^n$). Then we use continuity.)

A subset $C \subset \mathcal{X}$ is called *convex* if $x, y \in C$ implies $[x, y] \subset C$. Sometimes we say *ρ-convex* instead of convex, to avoid confusion with other natural notions of convexity for the same set.

We use the notation $E_{a,r}$ for $\{x \in \mathcal{X}\colon\ \rho(a,x) \leqslant r\}$ and call such sets *closed balls*. The condition (2.4) implies that in a hyperbolic space all closed balls are convex.

3. Normal structure

Let M be a subset in a metric space $(\mathcal{X}, \rho)$. The *diameter* of M is defined by

$$\operatorname{diam} M = \sup\bigl\{\rho(x, y)\colon\ x, y \in M\bigr\}. \tag{3.1}$$

A point $a \in M$ is called *diametral* if

$$\sup\bigl\{\rho(a, x)\colon\ x \in M\bigr\} = \operatorname{diam} M.$$

A hyperbolic space $\mathcal{X}$ is said to have *normal structure* if every convex bounded subset of $\mathcal{X}$ with more than one element has a non-diametral point.

This notion goes back to Brodskii and Milman [4] who proved that uniformly convex Banach spaces (they are hyperbolic spaces) have normal structure. Takahashi [18] introduced and studied normal structure in somewhat more general context. See [2, Chapter 3] for a nice account on those aspects of fixed point theory which are related to the geometry of Banach spaces.

Lemma 3.1. *Let M be a separable bounded convex subset of a hyperbolic space $\mathcal{X}$ and α be the diameter of M. If all points of M are diametral, then M contains a sequence $\{a_n\}$ with the property*: $\lim_{n\to\infty} \rho(a_n, x) = \alpha$ *for each $x \in M$.*

Proof. Let $\{c_n\}$ be a dense sequence in M. We define a sequence $\{b_n\}$ of "centers of mass" by the following rule: $b_1 = c_1$, $b_{n+1} = \frac{n}{n+1} b_n \oplus \frac{1}{n+1} c_{n+1}$. By convexity of ρ we have

$$\rho(x, b_n) \leqslant \frac{1}{n}\sum_{k=1}^{n} \rho(x, c_k) \tag{3.2}$$

for all $n \in \mathbb{N}$. Indeed for $n = 1$ this is obvious. If it is true for some n, then $\rho(x, b_{n+1}) \leqslant \frac{1}{n+1}\rho(x, c_{n+1}) + \frac{n}{n+1}\rho(x, b_n) \leqslant \frac{1}{n+1}\rho(x, c_{n+1}) + \frac{n}{n+1}\frac{1}{n}\sum_{k=1}^{n} \rho(x, c_k) = \frac{1}{n+1}\sum_{k=1}^{n+1} \rho(x, c_k)$.

By convexity of M we have $b_n \in M$ for each $n \in \mathbb{N}$. Our assumption implies that b_n is diametral, hence there is a point $a_n \in M$ with $\rho(b_n, a_n) \geqslant (1 - \frac{1}{n^2})\alpha$. It follows that $(1 - \frac{1}{n^2})\alpha \leqslant \frac{1}{n}\sum_{k=1}^{n} \rho(a_n, c_k)$. If $\rho(a_n, c_j) < (1 - \frac{1}{n})\alpha$, for some $j \leqslant n$, then $\frac{1}{n}\sum_{k=1}^{n} \rho(a_n, c_k) < \frac{1}{n}(1 - \frac{1}{n})\alpha + \frac{n-1}{n}\alpha = (1 - \frac{1}{n^2})\alpha$, a contradiction. Hence $\rho(a_n, c_j) \geqslant (1 - \frac{1}{n})\alpha$ for $j \leqslant n$. This shows that $\lim_{n\to\infty} \rho(a_n, c_j) = \alpha$ for each fixed j. Since the sequence $\{c_j\}$ is dense in M, the lemma is proved. □

4. The invariant distance in the operator ball

Recall that K, H denote Hilbert spaces and $\mathcal{B}$ is the open unit ball of $L(K, H)$. For $A, X \in \mathcal{B}$ set

$$M_A(X) = (1 - AA^*)^{-1/2}(A + X)(1 + A^*X)^{-1}(1 - A^*A)^{1/2}. \tag{4.1}$$

Clearly all M_A are holomorphic on $\mathcal{B}$. They are called *the Möbius transformations*. It can be proved that $M_A^{-1} = M_{-A}$ (see [8, Theorem 2]). Hence each Möbius transformation is a biholomorphic automorphism of $\mathcal{B}$. Since $M_A(0) = A$ the group of all biholomorphic automorphisms is transitive on $\mathcal{B}$.

We set

$$\rho(A, B) = \tanh^{-1}\left(\left\|M_{-A}(B)\right\|\right). \tag{4.2}$$

It is easy to see that ρ coincides with the Carathéodory distance $c_{\mathcal{B}}$ in $\mathcal{B}$. Indeed, by [6, Theorem 4.1.8], $c_{\mathcal{B}}(0, B) = \tanh^{-1}(\|B\|)$ (this holds for the unit ball of every Banach space). Since $c_{\mathcal{B}}$ is invariant and M_{-A} sends A to 0 we get:

$$c_{\mathcal{B}}(A, B) = \tanh^{-1}\left\|M_{-A}(B)\right\| = \rho(A, B). \tag{4.3}$$

Hence ρ is invariant with respect to biholomorphic automorphisms. I. Shafrir [15] proved that the space $(\mathcal{B}, \rho)$ is hyperbolic. We present a proof of this result in Appendix A.

A set in $\mathcal{B}$ is called *bounded* if it is contained in some ρ-ball, or equivalently in a multiple $r\mathcal{B}$ of the operator ball with $r < 1$. So a set is bounded if and only if it is separated from the boundary of $\mathcal{B}$ in the sense of Section 1.

The following lemma is a special case of a more general result proved in [6, Theorem IV.2.2].

Lemma 4.1. *On any bounded set the hyperbolic metrics is equivalent to the operator norm.*

5. WOT-topology

As before, let $\mathcal{B}$ be the unit ball of the space of operators from K to H. We suppose that K is finite-dimensional, $\dim K = n$, and that H is separable. We consider biholomorphic maps on $\mathcal{B}$. By WOT we denote the weak operator topology (see [5, p. 476]). Because of the separability, the restriction of this topology to $\mathcal{B}$ is metrizable, so in our arguments we may consider only sequences, not nets.

Lemma 5.1. *If K is finite-dimensional and H is separable, then all biholomorphic maps of $\mathcal{B}$ are* WOT-*continuous.*

Proof. Let us firstly show that all Möbius transforms M_B are WOT-continuous (this was noticed and used already in the paper of Krein [11]). Indeed let $B \in \mathcal{B}$ be fixed, then the map $\varphi : X \mapsto 1 + B^*X$ from $(\mathcal{B}, \text{WOT})$ to $(L(K, K), \text{WOT})$ is continuous. Moreover, since K is finite-dimensional, φ remains continuous if instead of WOT we endow $L(K, K)$ with its norm topology. The map $T \to T^{-1}$ is norm continuous on the group of invertible operators on K. Hence the map $\psi : X \mapsto (1 + B^*X)^{-1}$ is continuous from $(\mathcal{B}, \text{WOT})$ to $L(K, K)$ with its norm topology.

It follows that the map $\omega : X \to (X + B)(1 + B^*X)^{-1}$ is continuous from $(\mathcal{B}, \text{WOT})$ to $(\mathcal{B}, \text{WOT})$. Indeed, if $X_n \to X$, then $\omega(X_n) - \omega(X) = (X_n + B)(\psi(X_n) - \psi(X)) + (X_n - X)\psi(X)$, where ψ was defined above. The first summand tends to zero in norm while the second one tends to zero in WOT.

By a result of Harris [7], if a biholomorphic map of $\mathcal{B}$ preserves the point 0, then it coincides with the restriction to $\mathcal{B}$ of an isometric linear map $h : L(K, H) \to L(K, H)$. Since K is finite-dimensional, the WOT-topology on $L(K, H)$ coincides with the weak topology (indeed $L(K, H)$

is linearly homeomorphic to the direct sum of n copies of H); since any bounded linear map is weakly continuous, h is WOT-continuous. On the other hand, if φ is a biholomorphic map of $\mathcal{B}$ and $A = \varphi(0)$, then $\psi = M_{-A} \circ \varphi$ is a biholomorphic map preserving 0. Hence ψ is an isometric linear map and $\varphi = M_{-A}^{-1} \circ \psi = M_A \circ \psi$ is a composition of two WOT-continuous maps. Thus φ is WOT-continuous. $\square$

Corollary 5.2. *If* $\dim K < \infty$ *and* H *is separable, then each ball* $E_{A,r}$ *is* WOT*-compact.*

Proof. Since there is a Möbius transform that maps $E_{A,r}$ onto $E_{0,r}$, and since all Möbius transforms are WOT-continuous, it suffices to consider the case $A = 0$. But $E_{0,r}$ is a usual closed operator ball; its WOT-compactness follows from the Banach–Alaoglu theorem. $\square$

6. Restricted normal structure of $\mathcal{B}$

The purpose of this section is to show that in the case when $\dim K < \infty$ and H is separable, the (open) operator ball $\mathcal{B}$ with the metric (4.2) has a restricted form of normal structure in the sense that WOT-compact ρ-convex subsets in it have non-diametral points. As we already mentioned $\mathcal{B}$ with the metric (4.2) is a hyperbolic space (see Appendix A). Our assumptions on K and H imply that $\mathcal{B}$ is separable in the norm-topology and hence, by Lemma 4.1, with respect to ρ.

Theorem 6.1. *Let* K *be finite-dimensional and* H *be separable. Let* M *be a weakly compact,* ρ*-convex subset of* $\mathcal{B}$ *endowed with its hyperbolic metric. If* M *is not a singleton, then* M *contains a non-diametral point.*

Proof. Let $\alpha = \operatorname{diam} M > 0$. Assume the contrary, that is, all points in M are diametral. By Lemma 3.1, there is a sequence $\{A_n\}$ in M such that $\lim_{n\to\infty} \rho(A_n, X) = \alpha$ for each $X \in M$.

Since M is weakly compact, the sequence $\{A_n\}_{n=1}^{\infty}$ contains a weakly convergent subsequence. Let W be its limit, we have $W \in M$ (since M is weakly compact).

Throughout this proof we will not change our notation after passing to a subsequence.

Since $W \in M$ we get

$$\lim_{n\to\infty} \rho(W, A_n) = \alpha. \tag{6.1}$$

We will get a contradiction by proving

$$\sup_{n,m} \rho(A_n, A_m) > \alpha. \tag{6.2}$$

We may assume without loss of generality that $W = 0$ (since a Möbius transformation which maps W to 0 is a ρ-isometry and weak homeomorphism).

Let $\beta = \tanh \alpha$. Then (6.1) leads to $\lim_{n\to\infty} \|A_n\| = \beta$ and it suffices to show that

$$\sup_{n,m} \bigl\| M_{A_m}(-A_n) \bigr\| > \beta.$$

Since K is finite-dimensional and $A_n \in L(K, H)$, we can select a strongly convergent subsequence in the sequence $\{A_n^* A_n\}$. Assume that $A_n^* A_n \to P$, where $P \in L(K, K)$. It is clear that $P \geqslant 0$ and $\|P\| = \beta^2$.

Choose $\varepsilon > 0$ and fix a number m with $\|A_m^* A_m - P\| < \varepsilon$. For brevity, denote $A_m^* A_m$ by Q. We prove that $\lim_{n\to\infty} \|M_{A_m}(-A_n)\| > \beta$ if $\varepsilon > 0$ is small enough. By the definition,

$$M_{A_m}(-A_n) = \left(1 - A_m A_m^*\right)^{-1/2}(A_m - A_n)\left(1 - A_m^* A_n\right)^{-1}\left(1 - A_m^* A_m\right)^{1/2}. \tag{6.3}$$

Since A_m^* is of finite rank, $A_m^* A_n \to 0$ in the norm topology. Hence $\lim_{n\to\infty} \|M_{A_m}(-A_n)\| = \lim_{n\to\infty} \|T_n\|$ where

$$\begin{aligned} T_n &= \left(1 - A_m A_m^*\right)^{-1/2}(A_m - A_n)\left(1 - A_m^* A_m\right)^{1/2} \\ &= A_m - \left(1 - A_m A_m^*\right)^{-1/2} A_n \left(1 - A_m^* A_m\right)^{1/2}. \end{aligned}$$

It follows from the identity

$$(1-t)^{-1/2} - 1 = \frac{t}{(1-t)(1 + (1-t)^{-1/2})}$$

that the operator $(1 - A_m A_m^*)^{-1/2}$ is a finite rank perturbation of the identity operator. Since $A_n \to 0$ in WOT, we obtain that $\|T_n - S_n\| \to 0$, where $S_n = A_m - A_n(1 - A_m^* A_m)^{1/2}$.

Denote $A_n(1 - A_m^* A_m)^{1/2}$ by B_n. Since $B_n \to 0$ in WOT, the sequence

$$(A_m - B_n)^*(A_m - B_n) - A_m^* A_m - B_n^* B_n = -A_m^* B_n - B_n^* A_m$$

tends to zero in norm topology. Furthermore,

$$B_n^* B_n = (1 - Q)^{1/2} A_n^* A_n (1 - Q)^{1/2}$$

tends in norm topology to $(1 - Q)^{1/2} P (1 - Q)^{1/2}$. Therefore

$$(A_m - B_n)^*(A_m - B_n) \to Q + (1 - Q)^{1/2} P (1 - Q)^{1/2}.$$

Since $\|P - Q\| < \varepsilon$, we have that

$$\left\| Q + (1 - Q)^{1/2} P (1 - Q)^{1/2} - \left(Q + (1 - Q) Q\right) \right\| < \varepsilon.$$

The inequalities

$$\beta^2 - \varepsilon \leqslant \|Q\| \leqslant \beta^2$$

imply

$$\left\| Q + (1 - Q) Q \right\| \geqslant 2\beta^2 - \beta^4 - 2\varepsilon,$$

whence

$$\lim_{n\to\infty} \left\| S_n^* S_n \right\| = \lim_{n\to\infty} \left\| (A_m - B_n)^*(A_m - B_n) \right\| \geqslant 2\beta^2 - \beta^4 - 3\varepsilon > \beta^2,$$

if ε is sufficiently small. □

7. Fixed points

The main purpose of this section is to establish the existence of a common fixed point for an elliptic group G of biholomorphic transformations of the operator ball $\mathcal{B}$. As it is shown in Appendix A, a biholomorphic transformation of $\mathcal{B}$ is a bijective isometric transformation of the metric space $(\mathcal{B}, \rho)$ which maps the set $\mathcal{M}$ onto itself (and hence segments onto segments).

Lemma 7.1. *If G is an elliptic group of biholomorphic transformations of $\mathcal{B}$, then there is a non-empty* WOT*-compact ρ-convex G-invariant subset of $\mathcal{B}$.*

Proof. Let $A \in \mathcal{B}$ be such that the orbit $G(A) := \{g(A)\colon\ g \in G\}$ is bounded. Therefore $G(A)$ is contained in some closed ball $E_{a,r}$. Let M be the intersection of all closed balls containing $G(A)$. It is clear that this intersection is non-empty (it contains $G(A)$), WOT-compact and ρ-convex (as an intersection of WOT-compact ρ-convex sets). It remains to check that it is G-invariant. To see this it suffices to observe that each element $g \in G$ maps the set of balls containing $G(A)$ bijectively onto itself. □

Lemma 7.2. *Let G be an elliptic group of biholomorphic transformations of $\mathcal{B}$. Let M be a minimal* WOT*-compact ρ-convex G-invariant subset in $(\mathcal{B}, \rho)$. Then M is a singleton.*

Proof. We use the approach suggested in [4]. Assume the contrary, let $\operatorname{diam} M = \alpha > 0$. By Theorem 6.1 M contains a non-diametral point N, so that $M \subset \{A\colon\ \rho(A, N) \leqslant \delta\}$ for some $\delta < \alpha$. Consider the set

$$O = \bigcap_{B \in M} E_{B,\delta}.$$

The set O is non-empty because $N \in O$. The set O is weakly compact and ρ-convex since each of the balls $E_{B,\delta}$ is weakly compact and ρ-convex. The set O is a proper subset of M since M has diameter $\alpha > \delta$.

Since G is a group of isometric transformations and M is invariant under each element of G, the action of G on M is by isometric bijections. Therefore O is G-invariant. We get a contradiction with the minimality of M. □

Proof of Theorem 1.1. The implication (i) ⇒ (ii) is obvious. On the other hand if $G(X_0)$ is separated from the boundary, for some $X_0 \in \mathcal{B}$, then $\sup_{g \in G} \rho(0, g(X_0)) < \infty$ whence, for each $X \in \mathcal{B}$, $\sup_{g \in G} \rho(0, g(X)) \leqslant \sup_{g \in G}(\rho(0, g(X_0)) + \rho(g(X_0), g(X))) = \sup_{g \in G}(\rho(0, g(X_0)) + \rho(X_0, X)) < \infty$. This means that the orbit $G(X)$ is separated from the boundary. We proved that (i) ⇔ (ii).

The implication (i) ⇒ (iv) can be derived from Lemmas 7.1 and 7.2 as follows. It is clear that families of WOT-compact ρ-convex G-invariant sets with the finite intersection property have non-empty intersections which are also WOT-compact ρ-convex and G-invariant. Therefore, by the Zorn Lemma, there is a minimal non-empty WOT-compact ρ-convex G-invariant set M_0. By Lemma 7.2, M_0 is a singleton and (iv) is proved.

If (iv) is true and A is a fixed point of G, then $G_1 = M_{-A} G M_A$ is a group of biholomorphic maps of $\mathcal{B}$ preserving 0. Hence it consists of restrictions to $\mathcal{B}$ of isometric linear maps (see the beginning of Section 4 in this connection). Thus G_1 is equicontinuous.

Note that each Möbius transform is a Lipschitz map: $\|M_A(X) - M_A(Y)\| \leqslant C\|X - Y\|$ for each $X, Y \in \mathcal{B}$, where the constant $C > 0$ depends on A. Indeed setting $F(X) = (A + X) \times (1 + A^*X)^{-1}$ and $D = (1 - \|A\|)^{-1}$ we have

$$\begin{aligned}\big\|F(X) - F(Y)\big\| &= \big\|(A + X)\big((1 + A^*X)^{-1} - (1 + A^*Y)^{-1}\big) + (X - Y)(1 + A^*Y)^{-1}\big\| \\ &= \big\|(A + X)(1 + A^*X)^{-1}A^*(Y - X)(1 + A^*Y)^{-1} + (X - Y)(1 + A^*Y)^{-1}\big\| \\ &\leqslant 2D^2\|X - Y\| + D\|X - Y\| \leqslant 3D^2\|X - Y\|.\end{aligned}$$

Hence

$$\begin{aligned}\big\|M_A(X) - M_A(Y)\big\| &= \big\|(1 - AA^*)^{-1/2}\big(F(X) - F(Y)\big)(1 - A^*A)^{1/2}\big\| \\ &\leqslant D^{\frac{1}{2}}\big\|F(X) - F(Y)\big\| \leqslant 3D^{\frac{5}{2}}\|X - Y\|.\end{aligned}$$

Since $G = M_A G_1 M_{-A}$ and the maps M_A, M_{-A} are Lipschitz, G is also equicontinuous. We proved that (iv) $\Rightarrow$ (iii).

Let now (iii) hold, we have to prove (ii). We will show that the orbit of 0 is separated from the boundary. Assuming the contrary we get that for any $\delta > 0$ there is $g \in G$ with $\|g(0)\| > 1 - \delta$. Let $A = g(0)$; we may assume that $\delta < 1/2$ so $\|A\| > 1/2$.

By the already mentioned result of [7], $g = M_A \circ h$ where h is a linear isometry. Let P be the spectral projection of $T = A^*A$ corresponding to the eigenvalue $\|T\| = \|A\|^2$ (recall that T is an operator in a finite-dimensional space). Then

$$\big\|(1 - T)P\big\| = 1 - \|T\| \leqslant 2\big(1 - \|A\|\big) < 2\delta.$$

Set $X_1 = 0$, $X_2 = h^{-1}(\frac{1}{2}AP)$. Then $\|X_2 - X_1\| = \frac{1}{2}\|AP\| = \|A\|/2 > 1/4$.

On the other hand

$$\begin{aligned}\big\|g(X_2) - g(X_1)\big\| &= \left\|M_A\left(\frac{1}{2}AP\right) - M_A(0)\right\| \\ &= \left\|(1 - AA^*)^{-1/2}\left(\frac{1}{2}AP + A\right)\left(1 + \frac{1}{2}A^*AP\right)^{-1}(1 - A^*A)^{1/2} - A\right\| \\ &= \left\|A(1 - T)^{-1/2}\left(\frac{1}{2}P + 1\right)\left(1 + \frac{1}{2}TP\right)^{-1}(1 - T)^{1/2} - A\right\| \\ &= \left\|A\left(\frac{1}{2}P + 1\right)\left(1 + \frac{1}{2}TP\right)^{-1} - A\right\| = \left\|\frac{1}{2}A(1 - T)P\left(1 + \frac{1}{2}TP\right)^{-1}\right\| \\ &\leqslant \frac{1}{2}\|A\|\big\|(1 - T)P\big\| < \frac{1}{2}2\delta = \delta.\end{aligned}$$

This contradicts to the assumption of equicontinuity. Indeed for each δ we get points $Y_i = g(X_i)$ with $\|Y_1 - Y_2\| < \delta$ and $\|g^{-1}(Y_1) - g^{-1}(Y_2)\| > 1/4$. Thus (ii) holds. □

8. Orthogonalization

Theorem 8.1. *If a bounded representation π of a group G on a Hilbert space $\mathcal{H}$ preserves a quadratic form η with finite number of negative squares, then it is similar to a unitary representation.*

Proof. By our assumptions, $\mathcal{H} = H_1 \oplus H_2$, $\dim(H_2) < \infty$, and $\eta(x) = \|Px\|^2 - \|Qx\|^2$ where P, Q are the projections onto H_1 and H_2 respectively. We write $H_1 = H$ and $H_2 = K$, for brevity.

We will relate to each invertible operator T on $\mathcal{H}$ preserving the form η a biholomorphic map w_T of $\mathcal{B}$ in such a way that

$$w_{T_1 T_2} = w_{T_1} \circ w_{T_2}. \tag{8.1}$$

Let us call a subspace L of $\mathcal{H}$ *positive* (*negative*) if $\eta(y) > 0$ (respectively $\eta(y) < 0$) for all non-zero $y \in L$. Since each negative subspace L is finite-dimensional, there is $\varepsilon > 0$ such that

$$\eta(y) \leqslant -\varepsilon\|y\|^2 \quad \text{for all non-zero } y \in L.$$

The supremum of all such ε is called the *degree of negativeness* of L and is denoted by $\varepsilon(L)$.

For each operator $A \in \mathcal{B}$, the set

$$L(A) = \{Ax \oplus x\colon\ x \in K\}$$

is a negative subspace of $\mathcal{H}$. Furthermore the condition

$$\eta(y) \leqslant -\varepsilon\|y\|^2 \quad \text{for all } y \in L(A)$$

means that

$$-\|x\|^2 + \|Ax\|^2 \leqslant -\varepsilon\big(\|x\|^2 + \|Ax\|^2\big)$$

for all $x \in K$. That is

$$\varepsilon \leqslant \frac{1 - \|A\|^2}{1 + \|A\|^2}.$$

It follows that the degree of negativeness of $L(A)$ is related to $\|A\|$ by the equality

$$\varepsilon\big(L(A)\big) = \frac{1 - \|A\|^2}{1 + \|A\|^2}. \tag{8.2}$$

Since $\dim(L(A)) = \dim(K)$, $L(A)$ is a maximal negative subspace in $\mathcal{H}$. Indeed if some subspace M of $\mathcal{H}$ strictly contains $L(A)$, then its dimension is greater than codimension of H, whence $M \cap H \neq \{0\}$. But all non-zero vectors in H are positive.

Conversely, each maximal negative subspace Q of $\mathcal{H}$ coincides with $L(A)$, for some $A \in \mathcal{B}$. Indeed, since $Q \cap H = \{0\}$, there is an operator $A : K \to H$ such that each vector of Q is of the form $Ax \oplus x$. Since Q is negative, we have $\eta(Ax \oplus x) = \|Ax\|^2 - \|x\|^2 < 0$, and therefore $\|A\| < 1$, so $A \in \mathcal{B}$. Thus $Q \subset L(A)$; and, by maximality, $Q = L(A)$.

It is easy to see that the map $A \to L(A)$ from $\mathcal{B}$ to the set $\mathcal{E}$ of all maximal negative subspaces is injective and therefore bijective.

Now we can define w_T. Indeed, if a subspace L of $\mathcal{H}$ is maximal negative, then its image TL under T is also maximal negative (because T is invertible and preserves η). Hence, for each $A \in \mathcal{B}$, there is $B \in \mathcal{B}$ such that $L(B) = TL(A)$. We let $w_T(A) = B$.

The equality (8.1) follows easily because $L(w_{T_1}(w_{T_2}(A))) = T_1 L(w_{T_2}(A)) = T_1 T_2 L(A) = L(w_{T_1 T_2}(A))$ and the map $A \to L(A)$ is injective.

Our next goal is to check that w_T is biholomorphic. Since $w_T^{-1} = w_{T^{-1}}$ it suffices to show that w_T is holomorphic.

Let $T = (T_{ij})_{i,j=1}^2$ be the matrix of T with respect to the decomposition $\mathcal{H} = H_1 \oplus H_2$. Then $T(Ax \oplus x) = (T_{11}Ax + T_{12}x) \oplus (T_{21}Ax + T_{22}x)$. Since $T(Ax \oplus x) \in L(w_T(A))$, we conclude that

$$w_T(A)(T_{21}Ax + T_{22}x) = T_{11}Ax + T_{12}x.$$

Thus

$$w_T(A) = (T_{11}A + T_{12})(T_{21}A + T_{22})^{-1}. \tag{8.3}$$

This shows that w_T is a holomorphic map on $\mathcal{B}$.

Suppose now that π is a bounded representation of a group G on $\mathcal{H}$ preserving η. Then $W = \{w_{\pi(g)}\colon g \in G\}$ is a group of biholomorphic maps of $\mathcal{B}$. Moreover since π is bounded, the group W is elliptic. To see this, note that for each negative subspace L, one has

$$\eta(y) \leqslant -\varepsilon(L)\|y\|^2 \quad \text{for all } y \in L.$$

If T is an invertible operator preserving η, then $T^{-1}x \in L$, for each $x \in TL$, whence

$$\eta(x) = \eta\left(T^{-1}x\right) \leqslant -\varepsilon(L)\left\|T^{-1}x\right\|^2 \leqslant -\varepsilon(L)\|T\|^{-2}\|x\|^2.$$

Thus

$$\varepsilon(TL) \geqslant \varepsilon(L)\|T\|^{-2}.$$

For $L = L(A)$, $TL = L(w_T(A))$. This gives

$$\frac{1 - \|w_T(A)\|^2}{1 + \|w_T(A)\|^2} \geqslant \|T\|^{-2}\frac{1 - \|A\|^2}{1 + \|A\|^2}$$

if one takes into account (8.2). Thus, if $\|\pi(g)\| \leqslant C$ for all $g \in G$, then

$$\frac{1 - \|w_{\pi(g)}(A)\|^2}{1 + \|w_{\pi(g)}(A)\|^2} \geqslant C^{-2}\frac{1 - \|A\|^2}{1 + \|A\|^2}.$$

Therefore

$$1 - \left\|w_{\pi(g)}(A)\right\|^2 \geqslant C^{-2}\frac{1 - \|A\|^2}{1 + \|A\|^2}$$

and

$$\sup_{g\in G}\left\| w_{\pi(g)}(A)\right\| < 1$$

for each $A \in \mathcal{B}$.

By Theorem 1.1, there is $D \in \mathcal{B}$ with $w_{\pi(g)}(D) = D$ for all $g \in G$. Hence $\pi(g)L(D) = L(D)$ for all $g \in G$.

Let U be an operator on $\mathcal{H}$ with the matrix (U_{ij}) where $U_{11} = (1_H - DD^*)^{-1/2}$, $U_{12} = -D(1_K - D^*D)^{-1/2}$, $U_{21} = -D^*(1_H - DD^*)^{-1/2}$, $U_{22} = (1_K - D^*D)^{-1/2}$. It can be checked that U preserves η and maps $L(D)$ onto K. Then all operators $\tau(g) = U\pi(g)U^{-1}$ preserve η, and the subspace K is invariant for them. It follows that H is also invariant for operators $\tau(g)$. Hence these operators preserve the scalar product on $\mathcal{H}$. Thus $g \mapsto \tau(g)$ is a unitary representation similar to π. □

It should be noted that Theorem 8.1 does not extend to the case when η has infinite number of negative (and positive) squares, that is, to the case that both H_1 and H_2 are infinite-dimensional [16].

9. *J*-unitary operators on Pontryagin spaces

The Pontryagin space is a linear space $\mathcal{E}$ supplied with an indefinite scalar product $x, y \to [x, y]$ which has a finite number of negative squares. More precisely this means that one can choose a usual scalar product $x, y \to (x, y)$ with respect to which $\mathcal{E}$ is a Hilbert space and $[x, y] = (Jx, y)$, where J is a selfadjoint involutive operator on this Hilbert space with $\operatorname{rank}(1 - J) < \infty$. An invertible operator T on $\mathcal{E}$ is called *J-unitary* if $[Tx, Ty] = [x, y]$ for all $x, y \in \mathcal{E}$.

It should be noted that the terminology does not seem to be successful because the choice of the operator J and the corresponding scalar product is not unique while the set of J-unitary operator is completely determined by the original indefinite scalar product $[\cdot,\cdot]$. However, this terminology is widely used (see, for example, [1,10] and references therein). It is important that all scalar products defining $[\cdot,\cdot]$ via J-operators are equivalent, so one can speak, for example, about boundedness of a set of operators, without indicating which scalar product is chosen.

A subspace $X \subset \mathcal{E}$ is called *positive* (*negative*) if $[x, x] > 0$ (respectively $[x, x] < 0$) for all $x \in X$. A *dual pair of subspaces* in $\mathcal{E}$ is a pair Y, Z, where Y is a positive subspace, Z is a negative subspace and $Y + Z = \mathcal{E}$. The study of dual pairs invariant for a given set of J-unitary operators was started by Sobolev and intensively developed by Pontryagin, Krein, Phillips, Naimark and other prominent mathematicians.

The previous theorem on the orthogonalization of representations implies the following result.

Corollary 9.1. *A group of J-unitary operators on a Pontryagin space has an invariant dual pair if and only if it is bounded.*

Proof. Choose a scalar product $(\cdot,\cdot)$ and the corresponding operator J. Denote by $\mathcal{H}$ the Hilbert space $(\mathcal{E}, (\cdot,\cdot))$. Since J is an Hermitian involutive operator, there are orthogonal subspaces H, K of $\mathcal{H}$ such that $J = P_H - P_K$. By our assumption on J, the subspace K is finite-dimensional.

Let G be a group of J-unitary operators. If it is bounded, then the identity map can be regarded as a bounded representation of G on $\mathcal{H}$. Moreover it preserves the form $\eta(x) = [x, x]$. Since it has a finite number of negative squares, Theorem 8.1 implies that there is an invertible operator V such that the representation $\tau(g) = T^{-1}gT$ is unitary. It follows from [10, Theorem 5.8] that G has an invariant dual pair of subspaces.

For completeness we include the proof of this fact. Passing to adjoints in the equality $T\tau(g) = gT$ and taking into account that $g^* = Jg^{-1}J$, $\tau(g)^* = \tau(g^{-1})$ we obtain that $\tau(g^{-1})T^* = T^*Jg^{-1}J$. Using this identity for g instead of g^{-1} and multiplying both sides by JT we get:

$$\tau(g)T^*JT = T^*JgJJT = T^*JgT = T^*JT\tau(g).$$

Thus the invertible selfadjoint operator $R = T^*JT$ commutes with the group $\tau(G)$ of unitary operators. It follows that its spectral subspaces H_1 and K_1 corresponding to positive and negative parts of spectrum are invariant for $\tau(G)$. Note that $(Rx, x) > 0$ for $x \in T^{-1}H\backslash\{0\}$ and $(Rx, x) < 0$ for $x \in T^{-1}K\backslash\{0\}$. It follows that $\dim K_1 = \dim K$. Now the subspaces $H_2 = TH_1$ and $K_2 = TK_1$ form an invariant dual pair for G.

The converse implication is simple. If G has an invariant dual pair H, K, then the scalar product $(h_1 + k_1, h_2 + k_2) = [h_1, h_2] - [k_1, k_2]$ is invariant for G. Thus G is a group of unitary operators on $\mathcal{H} = (\mathcal{E}, (\cdot,\cdot))$, hence it is bounded. □

As a consequence we obtain the following result proved in [16]:

Corollary 9.2. *A J-symmetric representation of a unital C^*-algebra on a Pontryagin space is similar to a *-representation.*

For a proof it suffices to notice that restricting the representation to the unitary group of the C^*-algebra we obtain a bounded group of J-unitary operators.

Acknowledgments

We wish to express our gratitude to Professor Itai Shafrir for informing us about results of his dissertation [15] and to Ekaterina Shulman for providing us with a copy of [15] and for helping us with its translation. The second author also would like to thank Alexei Loginov and Natal'ya Yaskevich for helpful discussions on the subject of this paper many years ago.

Appendix A. Hyperbolicity of $\mathcal{B}$ (after Itai Shafrir)

For any bounded domains D_1, D_2 of complex Banach spaces we denote by $Hol(D_1, D_2)$ the set of all holomorphic maps from D_1 to D_2. If $D_1 = D_2 = D$, then $Hol(D_1, D_2)$ is a semigroup with respect to the composition, and by $Aut(D)$ we denote the set of all its invertible elements (biholomorphic automorphisms of D). The group $Aut(\mathcal{B})$ acts transitively on $\mathcal{B}$. Indeed, for each $A \in \mathcal{B}$ the Möbius transform M_A is biholomorphic and sends 0 to A.

As usually the Carathéodory metric on $\mathcal{B}$ is defined by the equality:

$$c_{\mathcal{B}}(A, B) = \sup\bigl\{\omega\bigl(f(A), f(B)\bigr):\ f \in Hol(\mathcal{B}, \Delta)\bigr\}$$

where Δ is the unit disk and ω is the Poincaré distance:

$$\omega(z_1, z_2) = \tanh^{-1}\left|\frac{z_1 - z_2}{1 - \overline{z_1}z_2}\right|.$$

As it was mentioned in Section 4, $c_{\mathcal{B}}$ coincides with the metric ρ defined by the formula (4.2). Clearly $c_{\mathcal{B}}$ is invariant under biholomorphic maps of $\mathcal{B}$.

We shall prove that $\mathcal{B}$ is a hyperbolic space with respect to this metric.

Furthermore the differential Carathéodory metrics on $\mathcal{B}$ is defined by

$$\alpha(A, V) = \sup_{f \in Hol(\mathcal{B},\Delta)} \frac{|\mathcal{D}f(A)V|}{1-|f(A)|^2} \tag{A.1}$$

for all $A \in \mathcal{B}$, $V \in L(K, H)$, where $\mathcal{D}f(A)$ is the differential of f in A (see [6], where α is denoted by $\gamma_{\mathcal{B}}$).

Lemma A.1. *For each $A \in \mathcal{B}$, $V \in L(K, H)$*

$$\mathcal{D}M_B(A)V = (1 - BB^*)^{1/2}(1 + AB^*)^{-1}V(1 + B^*A)^{-1}(1 - B^*B)^{1/2}. \tag{A.2}$$

In particular,

$$\mathcal{D}M_B(0)V = (1 - BB^*)^{1/2}V(1 - B^*B)^{1/2}.$$

Proof. By definition, $M_B(X) = (1 - BB^*)^{-1/2}(B + X)(1 + B^*X)^{-1}(1 - B^*B)^{1/2}$. We have to calculate the coefficient c of t in the Taylor decomposition of the function $t \to M_B(A + tV)$. For this, note that if P is an invertible operator then $(P + tQ)^{-1} = P^{-1} - tP^{-1}QP^{-1} + o(t)$. It follows immediately that

$$\begin{aligned}
c &= (1 - BB^*)^{-1/2}\big(V(1 + B^*A)^{-1} - (B + A)(1 + B^*A)^{-1}B^*V(1 + B^*A)^{-1}\big)(1 - B^*B)^{1/2} \\
&= (1 - BB^*)^{-1/2}\big(1 - (B + A)(1 + B^*A)^{-1}B^*\big)V(1 + B^*A)^{-1}(1 - B^*B)^{1/2} \\
&= (1 - BB^*)^{-1/2}\big(1 - (B + A)B^*(1 + AB^*)^{-1}\big)V(1 + B^*A)^{-1}(1 - B^*B)^{1/2} \\
&= (1 - BB^*)^{-1/2}\big(\big(1 + AB^* - (B + A)B^*\big)(1 + AB^*)^{-1}\big)V(1 + B^*A)^{-1}(1 - B^*B)^{1/2} \\
&= (1 - BB^*)^{1/2}(1 + AB^*)^{-1}V(1 + B^*A)^{-1}(1 - B^*B)^{1/2}. \qquad \square
\end{aligned}$$

Lemma A.2. $\alpha(A, V) = \|(1 - AA^*)^{-1/2}V(1 - A^*A)^{-1/2}\|$ *for all $A \in \mathcal{B}$ and $V \in L(K, H)$.*

Proof. By [6, Lemma V.1.5]

$$\alpha(0, V) = \|V\|.$$

Let now A be arbitrary. Then by [6, Proposition V.1.2]

$$\alpha\big(M_A(0), \mathcal{D}M_A(0)X\big) = \alpha(0, X) = \|X\|.$$

On the other hand, by Lemma A.1,

$$\alpha\big(M_A(0), \mathcal{D}M_A(0)X\big) = \alpha\big(A, (1 - AA^*)^{1/2}X(1 - A^*A)^{1/2}\big).$$

Setting now $V = (1 - AA^*)^{1/2}X(1 - A^*A)^{1/2}$, we obtain $X = (1 - AA^*)^{-1/2}V(1 - A^*A)^{-1/2}$ and hence $\alpha(A, V) = \|(1 - AA^*)^{-1/2}V(1 - A^*A)^{-1/2})\|$. $\square$

For any bounded operator D, set $D^{(1)} = D$, $D^{(3)} = DD^*D$, $D^{(5)} = DD^*DD^*D, \dots,$ $D^{(2k+1)} = (DD^*)^k D$.

Let

$$\operatorname{Th} D = \sum_{n=0}^{\infty} a_{2n+1} D^{(2n+1)} \tag{A.3}$$

where a_j are the Taylor coefficients of $\tanh t$, i.e., $\tanh t = \sum_{n=0}^{\infty} a_{2n+1} t^{2n+1}$.

It follows from the definition that $\operatorname{Th} D = \tanh(D)$ if D is selfadjoint.

If $D = J|D|$ is the polar decomposition of D (that is, $|D| = (D^*D)^{1/2}$ and J is a partial isometry such that $(J^*J)|D| = |D|(J^*J) = |D|$), then

$$D^{(2n+1)} = J|D|^{2n+1},$$

and hence

$$\operatorname{Th} D = J \tanh |D|.$$

On the other hand, we can write $D = |D^*|J$, where $|D^*| = J|D|J^* = (DD^*)^{1/2}$, therefore

$$\operatorname{Th} D = \bigl(\tanh |D^*|\bigr) J.$$

For the space $(\mathcal{B}, \rho)$ we define the set $\mathcal{M}$ of lines as follows: for $A \in \mathcal{B}$, $D \in \partial\mathcal{B}$ (i.e., $\|D\| = 1$) we let

$$\gamma_{A,D} = \bigl\{\gamma_{A,D}(t) := M_A\bigl(\operatorname{Th}(tD)\bigr)\colon\ t \in \mathbb{R}\bigr\} \tag{A.4}$$

and set

$$\mathcal{M} = \{\gamma_{A,D}\colon\ A \in \mathcal{B},\ D \in \partial\mathcal{B}\}.$$

Proposition A.3. *$\gamma_{A,D}$ is a metric line.*

Proof. It suffices to show that $\rho(\gamma_{A,D}(s), \gamma_{A,D}(t)) = |s - t|$. Since ρ is invariant with respect to M_A we can assume that $A = 0$. We have $\rho(\gamma_{0,D}(s), \gamma_{0,D}(t)) = \tanh^{-1} \|M_B(\operatorname{Th}(tD))\|$, where $B = -\operatorname{Th}(sD)$. Using polar decomposition $D = J|D|$ we have that $\operatorname{Th}(tD) = J\tanh(t|D|)$, $\operatorname{Th}(tD)^* \operatorname{Th}(sD) = \tanh(t|D|)\tanh(s|D|)$, whence

$$\begin{aligned} M_B\bigl(\operatorname{Th}(tD)\bigr) &= \bigl(1 - \operatorname{Th}(sD)\operatorname{Th}(sD)^*\bigr)^{-1/2}\bigl(\operatorname{Th}(tD) - \operatorname{Th}(sD)\bigr) \\ &\quad \times \bigl(1 - \operatorname{Th}(sD)^* \operatorname{Th}(sD)\bigr)^{-1}\bigl(1 - \operatorname{Th}(sD)^* \operatorname{Th}(sD)\bigr)^{1/2} \\ &= J\bigl(1 - \tanh^2\bigl(s|D|\bigr)\bigr)^{-1/2} J^*J\bigl(\tanh\bigl(t|D|\bigr) - \tanh\bigl(s|D|\bigr)\bigr) \\ &\quad \times \bigl(1 - \tanh\bigl(s|D|\bigr)\tanh\bigl(t|D|\bigr)\bigr)^{-1}\bigl(1 - \tanh^2\bigl(s|D|\bigr)\bigr)^{1/2} \\ &= J\tanh\bigl((t-s)|D|\bigr) = \operatorname{Th}\bigl((t-s)D\bigr) \end{aligned}$$

giving the statement. □

We have to prove that $\gamma_{A,D}(t)$ is a *metric curve*, in the sense that the metric of its derivation equals 1, i.e., $\alpha(\gamma(t),\gamma'(t))=1$.

Lemma A.4. *Let* $\gamma(t)=\mathrm{Th}(tD)$, $D\in\partial\mathcal{B}$. *Then*

$$\gamma'(t)=D-\gamma(t)D^*\gamma(t). \tag{A.5}$$

Proof. We have $\gamma(t)=J\tanh(t|D|)$ and

$$\gamma'(t)=J|D|\cosh\bigl(t|D|\bigr)^{-2}=D\bigl(\cosh\bigl(t|D|\bigr)\bigr)^{-2}.$$

On the other hand

$$\begin{aligned}D-\gamma(t)D^*\gamma(t)&=D-J\tanh\bigl(t|D|\bigr)|D|J^*J\tanh\bigl(t|D|\bigr)=D-D\tanh^2\bigl(t|D|\bigr)\\&=D\bigl(\cosh\bigl(t|D|\bigr)\bigr)^{-2}\end{aligned}$$

giving (A.5). □

Lemma A.5. *Let* $\gamma(t)=\mathrm{Th}(tD)$, $D\in\partial\mathcal{B}$. *Then*

$$(1-\gamma\gamma^*)^{-1/2}(D-\gamma D^*\gamma)(1-\gamma^*\gamma)^{-1/2}=D. \tag{A.6}$$

Proof. Setting $D=J|D|$, we have

$$\gamma^*\gamma=\mathrm{Th}(tD)^*\,\mathrm{Th}(tD)=\tanh\bigl(t|D|\bigr)J^*J\tanh\bigl(t|D|\bigr)=\tanh^2\bigl(t|D|\bigr).$$

Furthermore

$$\gamma\gamma^*=\tanh\bigl(t|D^*|\bigr)JJ^*\tanh\bigl(t|D^*|\bigr)=\tanh^2\bigl(t|D^*|\bigr).$$

Since

$$D|D|=|D^*|D,$$

we have that

$$Df\bigl(|D|\bigr)=f\bigl(|D^*|\bigr)D$$

for any bounded Borel function f. Taking $f(x)=1-\tanh^2(tx)$, we get

$$D(1-\gamma^*\gamma)=(1-\gamma\gamma^*)D.$$

Next

$$D^*\gamma=\gamma^*D$$

because $D^*\,\mathrm{Th}(tD)=D^*J\tanh(t|D|)=|D|\tanh(t|D|)$ is selfadjoint.

Hence

$$(1-\gamma\gamma^*)^{-1/2}(D-\gamma D^*\gamma)(1-\gamma^*\gamma)^{-1/2} = (1-\gamma\gamma^*)^{-1/2}(1-\gamma\gamma^*)D(1-\gamma^*\gamma)^{-1/2}$$
$$= (1-\gamma^*\gamma)^{1/2}(1-\gamma^*\gamma)^{-1/2}D = D. \qquad \square$$

Proposition A.6. *Let $\gamma(t) = \gamma_{A,D}(t)$, $A \in \mathcal{B}$, $D \in \partial\mathcal{B}$. Then*

$$\alpha\big(\gamma(t), \gamma'(t)\big) = 1.$$

Proof. It suffices to prove this for $A = 0$, since $\alpha(F(X), \mathcal{D}F(X)V) = \alpha(X, V)$ for any $F \in Aut(\mathcal{B})$, $X \in \mathcal{B}$, $V \in L(H, K)$ (see [6, Proposition V.1.2]), and hence

$$\alpha\big(M_A\big(\mathrm{Th}(tD)\big), \big(M_A\big(\mathrm{Th}(tD)\big)\big)'\big) = \alpha\big(M_A\big(\mathrm{Th}(tD)\big), \mathcal{D}M_A\big(\mathrm{Th}(tD)\big)\big(\mathrm{Th}(tD)\big)'\big)$$
$$= \alpha\big(\mathrm{Th}(tD), \big(\mathrm{Th}(tD)\big)'\big).$$

Assume therefore that $\gamma(t) = \mathrm{Th}(tD)$. By Lemmas A.2 and A.5 we have

$$\alpha\big(\gamma(t), \gamma'(t)\big) = \big\|(1-\gamma\gamma^*)^{-1/2}(D-\gamma D^*\gamma)(1-\gamma^*\gamma)^{-1/2}\big\| = \|D\| = 1. \qquad \square$$

The next step is to prove that the family $\mathcal{M}$ of all lines is invariant with respect to the biholomorphic maps of $\mathcal{B}$.

Lemma A.7. *Let $\eta(t) = M_A(\gamma(t))$ where $\gamma(t) = \mathrm{Th}(tD)$. Then, for each biholomorphic map $h : \mathcal{B} \to \mathcal{B}$, the curve $h(\eta(t))$ belongs to the family $\mathcal{M}$.*

Proof. By [8, Theorems 3 and 4], there is a linear isometry L of the space $L(K, H)$ to itself satisfying the condition

$$L(AB^*A) = L(A)L(B)^*L(A) \quad \text{for all } A, B \in L(K, H) \tag{A.7}$$

and such that

$$h = M_{h(0)} \circ L = L \circ M_{-h(0)}.$$

It follows from (A.7) (see a remark after [8, Corollary 5]) that

$$L \circ M_A = M_{L(A)} \circ L$$

for all $A \in \mathcal{B}$.

So it suffices to consider the cases $h = L$ and $h = M_B$. Let us firstly prove that $L(\eta(t)) \in \mathcal{M}$. Indeed,

$$L\big(\eta(t)\big) = L\big(M_A\big(\gamma(t)\big)\big) = M_{L(A)}\big(L\big(\gamma(t)\big)\big)$$
$$= M_{L(A)}\big(L\big(\mathrm{Th}(tD)\big)\big) = M_{L(A)}\big(\mathrm{Th}\big(tL(D)\big)\big) \in \mathcal{M}.$$

Now we have to prove that $M_B(\eta(t)) \in \mathcal{M}$. Applying [8, Theorems 3 and 4] to $h(x) = M_B(M_A(x))$ we get a linear isometry L satisfying (A.7) and such that

$$M_B \circ M_A = M_C \circ L$$

where $C = h(0) = M_B(A)$. Thus

$$M_B\bigl(\eta(t)\bigr) = M_B\bigl(M_A\bigl(\gamma(t)\bigr)\bigr) = M_C\bigl(L\bigl(\gamma(t)\bigr)\bigr) = M_C\bigl(\mathrm{Th}\bigl(tL(D)\bigr)\bigr) \in \mathcal{M}. \qquad \square$$

Our next goal is to show that for each $A, B \in \mathcal{B}$ there is a unique line in $\mathcal{M}$ which passes through A, B.

Lemma A.8. *The set of all lines in $\mathcal{M}$ that go through A is $\{\gamma_{A,D}\colon D \in \partial(\mathcal{B})\}$.*

Proof. It suffices to assume that $A = 0$. Suppose that a line $\gamma(t) = M_B(\mathrm{Th}(tD))$ goes through 0, i.e., $\gamma(s) = 0$ for some $s \in \mathbb{R}$. Then clearly $B = -\mathrm{Th}(sD)$. Using the arguments from the proof of Proposition A.3 we obtain $\gamma(t) = \mathrm{Th}((t-s)D)$. Thus $\gamma = \gamma_{0,D}$. $\square$

Corollary A.9. *For each $A, B \in \mathcal{B}$, there is a unique line in $\mathcal{M}$ that passes through them.*

Proof. We may assume that $A = 0$. Let $B = J|B|$ be the polar decomposition of B and let $C = \tanh^{-1}|B|/t_0$ for $t_0 > 0$ be such that $\|C\| = 1$. Then for $D = JC$ the line $\gamma_{0,D}$ passes through 0 and B.

If there are two lines, γ_{0,D_1} and γ_{0,D_2}, going through B then by the above lemma, $B = \mathrm{Th}(tD_1) = \mathrm{Th}(sD_2)$ for some $t, s \in \mathbb{R}$. We may suppose that $t, s > 0$. Taking polar decompositions of $D_1 = J_1|D_1|$ and $D_2 = J_2|D_2|$ we see that $J_1 = J_2$ and $\tanh(t|D_1|) = \tanh(s|D_2|)$, which imply that $t|D_1| = s|D_2|$. But this clearly shows that the lines coincide. $\square$

Lemma A.10.

$$\|A\| \leqslant \bigl\|(1 - BB^*)^{-1/2}(A - BA^*B)(1 - B^*B)^{-1/2}\bigr\| \tag{A.8}$$

for each A, $B \in \mathcal{B}$.

Proof. Consider the polar decomposition $B = J|B|$. Then $|B^*| = (BB^*)^{1/2} = J|B|J^*$. Let $P = \tanh^{-1}(|B^*|)$, and $Q = \tanh^{-1}(|B|)$. Then

$$(1 - BB^*)^{-1/2}(A - BA^*B)(1 - B^*B)^{-1/2} = (\cosh P)A(\cosh Q) - (\sinh P)JAJ^*(\sinh Q).$$

For any $\varepsilon > 0$, there are unit vectors x, y such that

$$\bigl((\cosh P)A(\cosh Q)x, y\bigr) \geqslant \bigl\|(\cosh P)y\bigr\|\,\|A\|\,\bigl\|(\cosh Q)x\bigr\| - \varepsilon.$$

Since $\|(\cosh P)y\|^2 - \|(\sinh P)y\|^2 = \|y\|^2$, and $\|(\cosh Q)x\|^2 - \|(\sinh Q)x\|^2 = \|x\|^2$ one can find numbers a, b such that

$$\bigl\|(\sinh P)y\bigr\| = \sinh b, \qquad \bigl\|(\cosh P)y\bigr\| = \cosh b,$$
$$\bigl\|(\sinh Q)x\bigr\| = \sinh a, \qquad \bigl\|(\cosh Q)x\bigr\| = \cosh a.$$

Hence

$$\begin{aligned}&\big\|(\cosh P)A(\cosh Q)-(\sinh P)JAJ^*(\sinh Q)\big\|\\&\quad\geqslant\big(\big((\cosh P)A(\cosh Q)-(\sinh P)JAJ^*(\sinh Q)\big)x,y\big)\\&\quad\geqslant(\cosh b)(\cosh a)\|A\|-\varepsilon-(\sinh b)(\sinh a)\|A\|\\&\quad\geqslant\cosh(b-a)\|A\|-\varepsilon\geqslant\|A\|-\varepsilon,\end{aligned}$$

giving the statement. □

Lemma A.11. *Let us consider two lines*: $\gamma(t)=M_A(\mathrm{Th}(tC))$, $\eta(t)=M_A(\mathrm{Th}(tD))$. *Then*

$$2\rho\big(\gamma(s),\eta(s)\big)\leqslant\rho\big(\gamma(2s),\eta(2s)\big) \tag{A.9}$$

for each $s>0$.

Proof. Since ρ is invariant with respect to the transformations M_A we may assume $A=0$.

Let $C(t)$ be a curve $\gamma_{B,E}(t)$ which joins $\gamma(2s)$ with $\eta(2s)$, we assume that $C(0)=\gamma(2s)$, $C(t_0)=\eta(2s)$ for some $t_0>0$ (such curve exists by Corollary A.9). Define now a new curve C_1 by

$$C_1=\mathrm{Th}\left(\frac{1}{2}\mathrm{Th}^{-1}C\right).$$

Then $C_1(0)=\gamma(s)$, $C_1(t_0)=\eta(s)$ and

$$C(t)=2C_1(t)\big(1+C_1(t)^*C_1(t)\big)^{-1}. \tag{A.10}$$

As usually we denote by $L(C_1)$ the length of the curve C_1: $L(C_1)=\int_0^{t_0}\alpha(C_1(t),C_1'(t))\,dt$.

If we could show that

$$L(C)\geqslant 2L(C_1) \tag{A.11}$$

for all curves C, C_1 satisfying (A.10) then we would obtain that

$$\rho\big(\gamma(2s),\eta(2s)\big)=L(C)\geqslant 2L(C_1)\geqslant 2\rho\big(\gamma(s),\eta(s)\big)$$

(the first equality follows from Propositions A.3 and A.6, the last inequality holds because the length of any curve is not smaller then the distance between its ends).

Thus our goal is the inequality (A.11). It suffices to show that

$$2\alpha\big(C_1(t),C_1'(t)\big)\leqslant\alpha\big(C(t),C'(t)\big). \tag{A.12}$$

Since

$$2C_1=C\big(1+C_1^*C_1\big),$$

we have

$$C'\big(1+C_1^*C_1\big)+C\big(C_1'^*C_1+C_1^*C_1'\big)=2C_1',$$

whence

$$C'=\big(\big(2-CC_1^*\big)C_1'-CC_1'^*C_1\big)\big(1+C_1^*C_1\big)^{-1}. \tag{A.13}$$

Since

$$2-CC_1^*=2-2C_1\big(1+C_1^*C_1\big)^{-1}C_1^*=2\big(1-C_1C_1^*\big(1+C_1C_1^*\big)^{-1}\big)=2\big(1+C_1C_1^*\big)^{-1},$$

substituting this into (A.13) we obtain

$$\begin{aligned}C'&=2\big(\big(1+C_1C_1^*\big)^{-1}C_1'-C_1\big(1+C_1^*C_1\big)^{-1}C_1'^*C_1\big)\big(1+C_1^*C_1\big)^{-1}\\&=2\big(1+C_1C_1^*\big)^{-1}\big(C_1'-C_1C_1'^*C_1\big)\big(1+C_1^*C_1\big)^{-1}.\end{aligned}$$

Now it follows from Lemma A.2 that the inequality (A.11) is equivalent to the following

$$\begin{aligned}&\big\|\big(1-C_1C_1^*\big)^{-1/2}C_1'\big(1-C_1^*C_1\big)^{-1/2}\big\|\\&\quad\leqslant\big\|(1-CC^*)^{-1/2}\big(1+C_1C_1^*\big)^{-1}\big(C_1'-C_1C_1'^*C_1\big)\big(1+C_1^*C_1\big)^{-1}(1-C^*C)^{-1/2}\big\|.\end{aligned} \tag{A.14}$$

But

$$\begin{aligned}1-CC^*&=1-4C_1\big(1+C_1^*C_1\big)^{-2}C_1^*=1-4C_1C_1^*\big(1+C_1C_1^*\big)^{-2}\\&=\big(\big(1+C_1C_1^*\big)^2-4C_1C_1^*\big)\big(1+C_1C_1^*\big)^{-2}=\big(1-C_1C_1^*\big)^2\big(1+C_1C_1^*\big)^{-2}.\end{aligned}$$

Similarly

$$(1-C^*C)^{-1/2}=\big(1+C_1^*C_1\big)\big(1-C_1^*C_1\big)^{-1}.$$

It follows now that (A.14) is equivalent to the inequality

$$\begin{aligned}&\big\|\big(1-C_1C_1^*\big)^{-1/2}C_1'\big(1-C_1^*C_1\big)^{-1/2}\big\|\\&\quad\leqslant\big\|\big(1-C_1C_1^*\big)^{-1}\big(C_1'-C_1C_1'^*C_1\big)\big(1-C_1^*C_1\big)^{-1}\big\|.\end{aligned} \tag{A.15}$$

But (A.15) follows from Lemma A.10 by substituting $B=C_1$ and $A=(1-C_1C_1^*)^{-1/2}C_1'\times(1-C_1^*C_1)^{-1/2}$ into inequality (A.8). □

The above results establish

Theorem A.12. *$\mathcal{B}$ is a hyperbolic space.*

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