On Boundary Value Problems for Parabolic Equations of Higher Order in Time

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INTRODUCTION

The aim of this paper is to study general initial-boundary value problems for linear parabolic equations of higher order in time. On this subject one can consider more or less two types of results: first of all, optimal regularity results, establishing the existence of linear and topological isomorphisms induced by the problems between certain function spaces. For what concerns higher order parabolic equations, the most general results of this type were obtained by Solonnikov in [12]. He considers a class of problems which is more general than ours (essentially systems instead of equations) and gives results of optimal regularity involving spaces of Sobolev type in the time variable with values in spaces of Sobolev type in the space variables; in the case of equations Grisvard [3] obtains analogous results with a completely different technique. In a recent paper V. Purmonen [11] considered, in the framework of classical $H^s$ spaces, problems with pseudodifferential operators.

The second category of results is directly inspired by the theory of analytic semigroups in Banach spaces. One tries to construct an evolution operator or, simply, to establish the existence and uniqueness of different types of solutions in some weak sense. This is what we do in this paper. We are in particular interested in establishing the existence of “classical solutions,” essentially solutions in a full sense for positive time, but with initial conditions with relatively poor regularity. The existence of classical solutions is a typical phenomenon of parabolic problems and it is not so well put in light by results of optimal regularity. In this framework we just quote the old papers by Lagnese [8] and Obrecht [9] who considered the case of stationary boundary conditions, the coefficients of the equation and of the boundary conditions independent of $t$, and only the initial value of the derivative of order $l - 1$ (if the highest derivative with respect to $t$ is of order $l$) not zero. The case of the coefficients of the equation (but not the boundary conditions) depending on $t$ was treated in [10] (for abstract parabolic problems). By now the most general results are contained in a
paper by Tanabe [14]. He treats the case of boundary conditions depending on time, but not containing derivatives with respect to time and constructs an evolution operator, using the abstract theory of classical [7]. This requires rather strong assumptions of regularity of the coefficients. Moreover, the assumptions on the initial conditions (see Th. 1 in [14]) seem quite restrictive. By a quite different approach we are able to relax the assumptions by Tanabe concerning the regularity of the coefficients and to consider even the case of boundary conditions with derivatives with respect to time. Moreover, we treat also the case of nonhomogeneous boundary conditions and give conditions on the initial values assuring the existence of strict and classical solutions which seem quite natural (see 4.5).

We go now to explain the plan of the paper: the first paragraph treats parabolic problems in \( \mathbb{R}^n \); the results are preliminary to the treatment of general boundary value problems. The method is to reduce the problem in a natural way to a system which is of first order in time. At this point one has the problem of establishing whether certain realizations of elliptic problems with homogeneous boundary conditions generate analytic semigroups in the natural phase space \( Y = W^{(l-1)d,p}(\Omega) \times \cdots \times W^{d,p}(\Omega) \times L^p(\Omega) \). This question is treated in the second paragraph. Essentially it is found that a necessary and sufficient condition is that the order of the boundary operators is sufficiently high. More generally, estimates depending on a parameter are established involving also nonhomogeneous boundary conditions. This is essential to apply the abstract theory developed in [6]. The third paragraph treats the rather restrictive case of boundary conditions of sufficiently high order using the results of the second paragraph and of [6]. Of course, it remains the case of “lower order” boundary conditions. In this situation it is easily seen that for the existence of both strict and classical solutions it is necessary to impose further compatibility conditions on the initial data. By reducing the problem in an appropriate way to a case treated in the second paragraph it is seen that these compatibility conditions are essentially sufficient. A by-product of the foregoing analysis is a result of generation of analytic semigroups in certain closed subspaces of the phase space, generalizing some well known examples in the literature (see 4.6). To conclude, we have omitted to construct a fundamental solution and to give explicit “variation of parameter” formulas. At the light of what we prove and taking into account the results of [6] this seems to be a quite easy task. Moreover, we remark that the method developed here is applicable to problems which are not parabolic in our strict sense (we treat what are also called Petrovskiy parabolic problems), for example to the strongly damped wave equations treated in [16].

Now we introduce the basic notations:

\[ \mathbb{N} := \{1, 2, \ldots\}, \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{R} \text{ is the set of real numbers, } \mathbb{C} \text{ the set of complex numbers, } \mathbb{R}^n \text{ the } n \text{-dimensional euclidean space.} \]
$L^p(\Omega)$ and $W^{k,p}(\Omega)$ are the usual $L^p$ and Sobolev spaces, with $\Omega$ open subset of $\mathbb{R}^n$ or differentiable manifold. $\| \cdot \|_{k,p,\Omega}$ is the norm in $W^{k,p}(\Omega)$. When $\Omega$ is omitted we mean $\Omega = \mathbb{R}^n$.

$BU\mathcal{C}(\mathbb{R}^n)$ is the Banach space of complex valued bounded uniformly continuous functions in $\mathbb{R}^n$.

$S(\mathbb{R}^n)$ is the Schwartz space, $S'(\mathbb{R}^n)$ the space of tempered distributions. $\mathcal{F}$ is the Fourier transform, $\mathcal{F}^{-1}$ the inverse Fourier transform; we shall often write $\hat{f}$ instead of $\mathcal{F}f$.

If $A$ is a subset of a topological space $T$, $\bar{A}$ is the topological closure of $A$.

If $A$ is an open subset of $\mathbb{R}^n$, $Z$ is a Banach space, $k \in \mathbb{N}_0$, $C^k(A, Z)$ is the space of functions of class $C^k$ in $A$ whose derivatives of order less or equal to $k$ are continuously extensible to $\bar{A}$ with values in $Z$.

If $R > 0$, $B_0 := \{ x \in \mathbb{R}^n : |x| < R \}$, $B_+ := \{ x \in \mathbb{R}^n : |x| < R, x_n > 0 \}$.

If $E$, $F$ are Banach spaces, $L(E, F)$ is the space of linear bounded operators from $E$ to $F$.

If $0$ is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial 0$ which is a $C^1$-submanifold of $\mathbb{R}^n$ and $x' \in \partial 0$, $T_x(\partial 0)$ is the linear space of vectors in $\mathbb{R}^n$ which are tangent to $\partial 0$ in $x'$ and pointing outside $\partial 0$, $\gamma$ the trace operator on $\partial 0$.

1. Problems in $\mathbb{R}^n$

1.1. Definition. Let $\mathcal{A}(\tilde{\partial}_t, \tilde{\partial}_x) = \sum_{i=0}^l A_{l,j}(\tilde{\partial}_t, \tilde{\partial}_x) \partial_t^j$ be a linear differential operator with constant coefficients in $\mathbb{R} \times \mathbb{R}^n$, with generic element $(t, x)(t \in \mathbb{R}, x \in \mathbb{R}^n)$ and $l \in \mathbb{N}$. We shall say that it is $d$-parabolic (in the sense of Peytrouliv) with respect to $t$ ($d \in \mathbb{N}$) if:

1. the order of $A_j$ is not larger than $d_j$;
2. set $A_j :=$ the part of order $d_j$ of $A_j$, $\mathcal{A}^0(\lambda, \xi) = \sum_{i=0}^l A^0_{l,i}(\xi) \lambda^i$. Then, $\mathcal{A}^0(\lambda, i\xi) \neq 0$ for any $\lambda \in \mathbb{C}$, with $\text{Re} \lambda \geq 0, \xi \in \mathbb{R}^n, (\lambda, \xi) \neq (0, 0)$.

1.2. Proposition. If $\mathcal{A}(\tilde{\partial}_t, \tilde{\partial}_x)$ is $d$-parabolic with respect to $t$, then $A_0$ is a constant nonvanishing polynomial, $A_j(\partial_x)$ is an elliptic operator of order $dl$ in $\mathbb{R}^n$. Moreover, $d$ is even.

Proof. By definition, ord($A_0$) $\leq 0$. If $A_0 = 0$, $\mathcal{A}^0(\lambda, 0) = 0$ for any $\lambda \in \mathbb{C}$. Moreover, $A^0_j(i\xi) = \mathcal{A}(0, i\xi) \neq 0$ for any $\xi \in \mathbb{R}^n$. 

Finally, assume by contradiction $d$ odd. Let $\xi^* \in \mathbb{R}^n - \{0\}$, $\lambda^* \in \mathbb{C}$ such that $\mathcal{A}\partial(\lambda^*, i\xi^*) = 0$. Then, $\text{Re} \lambda^* < 0$. One has $\mathcal{A}\partial(-\lambda^*, -i\xi^*) = (-1)^d \mathcal{A}\partial(\lambda^*, i\xi^*) = 0$, which is in contradiction with $\text{Re}(-\lambda^*) > 0$.

1.3. \textit{Remark.} Due to 1.2, we shall assume $A_0 = 1$. Also, we put $dl = 2m$.

Now, let $\mathcal{A}(\partial_x, \partial_x)$ be $d$-parabolic with respect to $t$. We start by considering the equation

$$\mathcal{A}(\partial_x, \partial_x) u(t, x) = f(t, x)$$

in $[0, T] \times \mathbb{R}^n$, $(0 < T < +\infty)$.

We set $u_0 := u$, $u_1 := \partial_t u$, ..., $u_{l-1} := \partial^{l-1}_t u$ and obtain the system

\begin{align*}
\partial_t u_0 &= u_1, \\
\partial_t u_{l-2} &= u_{l-1}, \\
\partial_t u_{l-1} &= -\sum_{j=0}^{l-1} A_{l-j}(\partial_x) u_j + f
\end{align*}

So, we put, if $0 \leq i, j \leq l - 1$

$$A_{ij}(\xi) = \begin{cases} 0 & \text{if } 0 \leq i \leq l - 2, j \neq i + 1 \\
1 & \text{if } 0 \leq i \leq l - 2, j = i + 1 \\
-A_{l-j}(\xi) & \text{if } i = l - 1,
\end{cases}$$

\begin{align*}
A_0(\xi) &= (A_{ij}(\xi))_{0 \leq i, j \leq l - 1}.
\end{align*}

One has:

1.4. \textbf{Proposition.} $A_0(\partial_x)$ is a system in $\mathbb{R}^n$ which is elliptic in the sense of Douglis-Nirenberg; this means the following:

\begin{enumerate}
\item if $\pi A_0(\xi)$ is the principal part of $\det A_0(\xi)$, $\pi A_0(\xi) \neq 0$ for any $\xi \in \mathbb{R}^n - \{0\}$;
\item there exist integers $s_0, ..., s_{l-1}, t_0, ..., t_{l-1}$ such that $\text{ord } A_{ij} \leq s_i + t_j$ if $0 \leq i, j \leq l - 1$ and such that, if we indicate with $A_0^0(\xi)$ the part of order $s_i + t_j$ of $A_{ij}$ and with $A_0^0(\xi)$ the matrix $(A_0^0(\xi))_{0 \leq i, j \leq l - 1}$, we have that $\pi A_0^0(\xi)$, the principal part of $\det A_0^0(\xi)$, coincides with the principal part $\pi A_0(\xi)$ of $\det A_0(\xi)$.
\end{enumerate}

In our case, $s_i = -d(l - i - 1)$, $t_j = d(l - j)$.

\textit{Proof.} It is easily seen that $\pi A_0(\xi) = (-1)^l A_0^0(\xi)$. All the following is easy to see.
Fix now $p \in \mathbb{N}^+ \cup \{0\}$. We have that $\mathcal{A}(\partial_\nu)$ is a linear bounded operator from $\prod_{j=0}^{l-1} W^{s_j, p}(\mathbb{R}^n)$ to $\prod_{j=0}^{l-1} W^{s_j-\nu, p}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$. Choosing $s = 0$, we have a linear bounded operator between

$$
\prod_{j=0}^{l-1} W^{s_j-\nu, p}(\mathbb{R}^n) = W^{s_0-\nu, p}(\mathbb{R}^n) \times W^{s_1-\nu, p}(\mathbb{R}^n) \times \cdots \times W^{s_{l-1}-\nu, p}(\mathbb{R}^n)
$$

and

$$
\prod_{j=0}^{l-1} W^{s_j-\nu-1, p}(\mathbb{R}^n) = W^{s_0-\nu-1, p}(\mathbb{R}^n) \times W^{s_1-\nu-1, p}(\mathbb{R}^n) \times \cdots \times L^p(\mathbb{R}^n).
$$

So, if we put $X := \prod_{j=0}^{l-1} W^{s_j, p}(\mathbb{R}^n)$, $Y := \prod_{j=0}^{l-1} W^{s_j-\nu, p}(\mathbb{R}^n)$, we have that $X \subseteq Y$ and we shall think of $\mathcal{A}(\partial_\nu)$ as a linear unbounded operator in $Y$ with domain $X$.

1.6. Proposition. $\mathcal{A}(\partial_\nu)$ is the infinitesimal generator of an analytic semigroup in $Y$.

**Proof.** Let $\lambda \in \mathbb{C}$, $\Re \lambda \geq 0$. Assume $\lambda = r^d e^{i\theta}$, with $r \geq 0$, $-\pi/2 \leq \theta \leq \pi/2$, $F := (f_0, \ldots, f_{l-1}) \in Y$ and consider the problem

$$
\lambda U - \mathcal{A}^0(\partial_\nu) U = F. \quad (4)
$$

By Fourier transform, we obtain

$$
\hat{r^d e^{i\theta}} \hat{U} - \mathcal{A}^0(\partial_\nu) \hat{U} = \hat{F}.
$$

It is easily seen that $\det(\hat{r^d e^{i\theta}} I - \mathcal{A}^0(\partial_\nu)) = \det(\hat{r^d e^{i\theta}} I \xi)$, so that, owing to the parabolicity, if $r > 0$, (4) has a unique solution $u \in \mathcal{S}(\mathbb{R}^n)$,

$$
U = \mathcal{F}^{-1}((\lambda I - \mathcal{A}^0(\partial_\nu))^{-1} \mathcal{F} F).
$$

Next, we show that, as $F \in Y$, $U \in X$ and, if $r \geq 1$,

$$
\|U\|_X \leq C \|F\|_Y,
$$

with $C$ independent of $r$, $\theta$ and $F$. To this aim, we want to analyze the matrix $(\hat{r^d e^{i\theta}} I - \mathcal{A}^0(\partial_\nu))^{-1}$ ($r \geq 1$, $-\pi/2 \leq \pi/2$, $\xi \in \mathbb{R}^n$). First of all, remark that, if we set

$$
S_{ik}(\rho) = \delta_{ik} \rho^k (0 \leq i, k \leq l-1, \rho \neq 0),
$$

$$
T_{ij}(\rho) = \delta_{ij} \rho^j (0 \leq i, j \leq l-1, \rho \neq 0),
$$

$$
S(\rho) := (S_{ik}(\rho))_{0 \leq i, k \leq l-1},
$$

$$
T(\rho) := (T_{ij}(\rho))_{0 \leq i, j \leq l-1},
$$

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we have, for any $\zeta \in C^n$, $\rho \in C - \{0\}$

$$\mathcal{A}_0^0(\rho\zeta) = S(\rho) \mathcal{A}_0^0(\zeta) T(\rho)$$

so that, as for $j = 0, \ldots, l-1$ we have $s_j + t_j = d$,

$$(\text{rp})^d e^{i\theta} I - \mathcal{A}_0^0(i\rho \zeta) = S(\rho) \left[ r^d e^{i\theta} I - \mathcal{A}_0^0(i\zeta) \right] T(\rho).$$

So, if $r > 0$, $\rho > 0$, $-\pi/2 < \theta < \pi/2$, $\zeta \in \mathbb{R}^n$,

$$[(\text{rp})^d e^{i\theta} I - \mathcal{A}_0^0(i\rho \zeta)]^{-1} = T(\rho^{-1}) \left[ r^d e^{i\theta} I - \mathcal{A}_0^0(i\zeta) \right]^{-1} S(\rho^{-1}).$$

From this formula one gets the fact that the $(j, i)$-term $P_{ji}(r, \zeta)$ of the matrix $[r^d e^{i\theta} I - \mathcal{A}_0^0(i\zeta)]^{-1}$ is positively homogeneous of degree $-t_j = s_i$ in $(r, \zeta)$. On the other hand, the well known algorithm to compute the inverse of a square matrix implies that $P_{ji}(r, \zeta)$ is a linear combination of summands of the form

$$r^d A_j^0(i\zeta) / \mathcal{A}_0^0(r^d e^{i\theta}, i\zeta),$$

with $\alpha \leq 2m - d$. As $A^0_\alpha$ is positively homogeneous of degree $2m$ in $(r, \zeta)$, necessarily $\alpha + dh - 2m = -t_j - s_i$, or $s_i + t_j = (l - h) d - \alpha$.

One has, for $j = 0, \ldots, l-1$

$$U_j(\zeta) = \sum_{i=0}^{l-1} P_{ji}(r, \zeta) \hat{F}_i(\zeta),$$

so that, for $j = 0, \ldots, l-1$ and Mikhlin’s multiplier theorem,

$$\|U_j\|_{2m - jd, p} \leq \sum_{i=0}^{l-1} \|\mathcal{F}^{-1}(P_{ji}(r, \zeta) \hat{F}_i(\zeta))\|_{2m - jd, p}$$

$$\leq C \sum_{i=0}^{l-1} \|\mathcal{F}^{-1}[(1 + |\zeta|^2)^{2m - jd}/2 P_{ji}(r, \zeta)$$

$$\times (1 + |\zeta|^2)^{(i+1)d - 2m}/2]$$

$$\times \mathcal{F}^{-1}(1 + |\zeta|^2)^{(2m - (i+1)d)/2} \hat{F}_i(\zeta))\|_{0, p}.$$
with \( x + dh = 2m - d(i + j + 1) \), so that, if \( \beta \in \mathbb{N}_0^n \),
\[
|\partial^\beta_\mu(x, r)| \leq C(\beta)(1 + |x|)^{-|\beta|},
\]
with \( C(\beta) \) independent of \( r \) if \( r \geq 1 \).

So, again by Mikhlin’s multiplier theorem,
\[
\|U\|_{2m-\mu,d,p} \leq C \sum_{i=0}^{l-1} \|\mathcal{F}^{-1}[1 + |\xi|^2]^{2m-(i+1)d/2} \hat{F}(\xi)\|_{0,p} \leq C \|F\|_y,
\]
that is, \( \|U\|_x \leq C \|F\|_y \), with \( C \) independent of \( r \geq 1 \) and \( \theta \in [-\pi/2, \pi/2] \).

From (4) it follows
\[
\|U\|_y = \|\lambda^{-1}(A_0(\partial_x) U + F)\|_y \leq C |\lambda|^{-1} \|F\|_y.
\]

Consider now the equation
\[
\lambda U - A_0(\partial_x) U = F, \tag{5}
\]
with \( F \in Y \), \( \text{Re} \lambda \geq 0 \). This is equivalent to
\[
\lambda U - A_0(\partial_x) U = [A(\partial_x) - A_0(\partial_x)] U + F. \tag{6}
\]

Set \( G := \lambda U - A_0(\partial_x) U \). Then,
\[
G = [A(\partial_x) - A_0(\partial_x)](\lambda - A_0(\partial_x))^{-1} G + F.
\]

Consider the operator in \( Y \)
\[
T_\lambda := [A(\partial_x) - A_0(\partial_x)](\lambda - A_0(\partial_x))^{-1} \in Y.
\]
If \( H \in Y \),
\[
\|T_\lambda H\|_y \leq C \sum_{i=0}^{l-1} \|(\lambda - A_0(\partial_x))^{-1} H\|_{2m-id-1,p} \leq (\text{by interpolation}) C |\lambda|^{-1/d} \|H\|_y,
\]
that is,
\[
\|G\|_y \leq C \|F\|_y
\]
with \( C \) independent of \( \lambda \) and \( F \). Setting \( U := (\lambda - A_0(\partial_x))^{-1} G \), it is easily seen that \( U \in X \), it is the unique solution of (5) and satisfies estimates like \( \|U\|_x \leq C \|F\|_y \), \( \|U\|_y \leq C |\lambda|^{-1} \|F\|_y \). As \( X \) is surely dense in \( Y \), the result is completely proved.
1.7. The next step is to extend the previous result to the case of operators with not necessarily constant coefficients; we shall consider a differential operator in \([0, T] \times \mathbb{R}^n\) \((0 < T < +\infty)\):

\[
\mathcal{A}(t, x, \partial_t, \partial_x) = \sum_{k=0}^l A_{l-k}(t, x, \partial_x) \partial_t^k
\]

and we shall assume that the following assumptions are satisfied:

(i) for any \(t \in [0, T]\), \(x \in \mathbb{R}^n, \mathcal{A}(t, x, \partial_t, \partial_x)\) is \(d\)-parabolic in the sense of 1.1, with \(d\) independent of \(t\) and \(x\);

(ii) the coefficients of \(A_{l-k}(t, x, \partial_x)\) belong to \(C([0, T]; BUC(\mathbb{R}^n))\) \((k = 0, \ldots, l)\), for some \(\epsilon > 0\);

(iii) there exists \(C > 0\) such that

\[
|\mathcal{A}^0(t, x, r\xi^0, i\xi)| \geq C(r + |\xi|)^{2m}
\]

for any \(r > 0\), \(\theta \in [-\pi/2, \pi/2], \xi \in \mathbb{R}^n, t \in [0, T], x \in \mathbb{R}^n\).

We put

\[
A_i(t, x, \xi) = \begin{cases} 0 & \text{if } 0 \leq i \leq l - 2, j \neq i + 1, \\ 1 & \text{if } 0 \leq i \leq l - 2, j = i + 1, \\ -A_{l-1}(t, x, \xi) & \text{if } i = l - 1, 
\end{cases}
\]

\(A(t, x, \xi) = (A_i(t, x, \xi))_{0 \leq i, j \leq l - 1}\).

We have the following "a priori" estimate:

1.8. LEMMA. There exists \(R > 0\) such that for any \(\lambda \in \mathbb{C}\), with \(\Re \lambda \geq 0\), \(|\lambda| \geq R\), for any \(U \in X\), for any \(t \in [0, T]\)

\[
|\lambda| \|U\|_Y + \|U\|_X \leq C (|\lambda - \lambda(t, x, \xi)| U)_{-Y}
\]

The proof of 1.8 can be obtained in a rather standard way from 1.6, by a "localization" of the estimate, using a partition of unity (for the same type of argument see [4] Lemma 2.4).

1.9. PROPOSITION. Put \(D(\mathcal{A}(t)) = X, \mathcal{A}(t) U := \mathcal{A}(t, x, \partial) U\). Then \(\mathcal{A}(t)\) is the infinitesimal generator of an analytic semigroup in \(Y\).

Proof. Owing to 1.8, we have only to show that the problem \(\dot{U} - \mathcal{A}(t) U = F\) has a unique solution \(U \in X\) for any \(F \in Y\), if \(\lambda \in \mathbb{C}, \Re \lambda \geq 0\) and \(|\lambda|\) is sufficiently large. Set for \(R > 0\)
\[ A_R(t, x, \xi) = \begin{cases} A(t, x, \xi) & \text{if } |x| \geq R, \\ A(t, R|x|, \xi) & \text{if } |x| > R, \end{cases} \]

\[ A_0(t, x, \xi) = \hat{A}(t, 0, \xi) \]

By 1.8, owing to the uniform estimates with respect to \( R \), there exist \( C > 0, \lambda > 0 \) such that for any \( \delta \in \mathbb{C} \), with \( \Re \delta \geq 0 \) and \( |\delta| \geq \lambda \), for any \( R \geq 0 \), for any \( U \in X \),

\[ |\delta| \| U \|_Y + \| U \|_X \leq C \| (\lambda - A_R(t, x, \delta)) U \|_Y. \] (10)

In force of the case \( R = 0 \) (constant coefficients) and the continuity method, if \( |\delta| \geq \lambda \), for any \( F \in Y \), for any \( R \geq 0 \) there exists a unique \( U_R \in X \) such that

\[ (\lambda - A_R(t, x, \delta)) U_R = F. \]

By (10), \( \{ U_R | R \geq 0 \} \) is bounded in \( X \). As \( X \) is reflexive, there exists a sequence \( (R_k)_{k \in \mathbb{N}} \) tending to \( +\infty \) such that \( U_{R_k} \) converges to \( U \) in \( X_w \) (\( = X \) with the weak topology), so that \( A(t, x, \delta) U_{R_k} \) converges to \( A(t, x, \delta) U \) in \( Y_w \). This implies that for any \( F \in \mathcal{D}(\mathbb{R}^n)^{m} \), if \( k \) is large enough,

\[ \int_{\mathbb{R}^n} F^T \Phi \, dx = \int_{\mathbb{R}^n} ((\lambda - A_{R_k}(t, x, \delta) U_{R_k})^T \Phi \, dx \]

\[ \to \int_{\mathbb{R}^n} ((\lambda - A(t, x, \delta)) U)^T \phi \, dx(k \to +\infty), \]

so that \( \lambda U - A(t) U = F \).

1.10. Consider the problem

\[ \mathcal{A}(t, x, \partial_t, \partial_x) u(t, x) = f(t, x) \quad \text{in } [0, T] \times \mathbb{R}^n, \]

\[ u(0, x) = u_0(x) \]

\[ \ldots \]

\[ \partial_t^{l-1} u(0, x) = u_{l-1}(x) \]

under the assumptions (h1)–(h3). A \textit{strict solution} of (11) is by definition a function \( u \in \bigcap_{k=0}^{r-1} C^{|l|-k}([0, T]; W^{d, p}(\mathbb{R}^n)) \) solving (11). A \textit{classical solution} is a solution of (11) belonging to \( \bigcap_{k=0}^{r-1} C^{|l|-k}([0, T]; W^{d, p}(\mathbb{R}^n)) \cap \bigcap_{k=0}^{r-1} C^{|l|-k}([0, T]; W^{d, p}(\mathbb{R}^n)) \).

It is immediately seen that, if we put \( U = (u, \partial_t u, \ldots, \partial_t^{l-1} u) \), \( F = (0, \ldots, 0, f) \), \( u \) is a strict solution of (11) if and only if \( U \) is a strict solution...
of \( \partial_t U - A(t) U = F \) (that is, \( U \in C^1([0, T]; Y) \cap C([0, T]; X) \)), \( u \) is a classical solution of (11) if and only if \( U \) is a classical solution of \( \partial_t U - A(t) U = F \) (that is, \( U \in C^1([0, T]; Y) \cap C([0, T]; X) \cap C([0, T]; Y) \)). An immediate consequence of 1.9 and Tanabe’s well known theory (see [13] Ch. 5) is the following

1.11. Proposition. Let \( 1 < p < + \infty \). If \( f \in C^\infty([0, T]; L^p(R^n)) \) \((\varepsilon > 0)\), \( u_0 \in W^{2m, p}(R^n) \), \( u_j \in W^{2m-j, p}(R^n) \), \( u_{j-1} \in W^{d, p}(R^n) \), (11) has a unique strict solution.

If \( f \in C^\infty([0, T]; L^p(R^n)) \) \((\varepsilon > 0)\),

\[
    u_0 \in W^{2m-d, p}(R^n), \quad u_j \in W^{2m-(j+1)d, p}(R^n), \quad u_{j-1} \in L^p(R^n),
\]

(11) has a unique classical solution.

2. Estimates for Certain Systems in Open Bounded Subsets of \( \mathbb{R}^n \)

The starting point of our discussion are the following assumptions:

\( (k1) \) \( m \in \mathbb{N}, \Omega \) is a bounded open subset of \( \mathbb{R}^n \) lying on one side of its boundary \( \partial \Omega \), a submanifold of \( \mathbb{R}^n \) of class \( C^m \);

\( (k2) \) \( \mathcal{A}(x, \partial_x, \partial_t) = \sum_{k=0}^d A_{-k}(x, \partial_x) \partial_t^k \) is a differential operator with coefficients in \( C(\overline{\Omega}) \). For any \( x \in \Omega \) \( \mathcal{A}(x, \partial_x, \partial_t) \) is \( d \)-parabolic (\( dl = 2m \));

\( (k3) \) for \( \mu = 1, \ldots, m, \mathcal{B}_{\mu}(x, \xi, \lambda) = \sum_{k=0}^{d-1} B_{\mu,k}(x, \xi) \lambda^k \) \((x \in \overline{\Omega}, \xi \in \mathbb{C}^n, \lambda \in \mathbb{C})\), with \( B_{\mu,k}(x, \cdot) \) polynomial of degree less or equal to \( \mu - dk(\sigma_{\mu} \in \mathbb{N}_0, \sigma_\mu \leq 2m-1) \) and coefficients of class \( C^{2m-\sigma_{\mu}(\overline{\Omega})} \) in \( x \) (of course, \( B_{\mu,k}(x, \cdot) = 0 \) if \( \mu - dk < 0 \));

\( (k4) \) (Complementing condition) Consider the O.D.E. problem

\[
    \mathcal{A}(x', \lambda', \xi') + \nu(x') \partial_t \) w(\tau) = 0 \quad \text{in } \mathbb{R},
\]

\[
    \mathcal{B}(x', \lambda', \xi' + \nu(x') \partial_t \) w(0) = g_\mu \in \mathbb{C}, \quad \mu = 1, \ldots, m,
\]

\( w \) bounded in \( \mathbb{R}^+ \), with \( \xi' \in T_x(\partial \Omega), \Re \lambda' \geq 0, (\lambda', \xi') \neq (0, 0), (g_1, \ldots, g_m) \in \mathbb{C}^m \).

Then the problem has a unique solution;
(k5) \( \min_{\mu} \sigma_{\mu} \geq 2m - d. \)

Here we indicate with \( \mathcal{B}_{\mu,k}^{0}(x, \partial) \) the part of order \( \sigma_{\mu} - dk \) of \( \mathcal{B}_{\mu,k} \) and put

\[
\mathcal{B}_{\mu,k}^{0}(x, \lambda, \zeta) = \sum_{k=0}^{l-1} \mathcal{B}_{\mu,k}^{0}(x, \zeta) \lambda^{k}.
\]

We construct the elliptic system \( \mathcal{A}(x, \partial \zeta) \) as in (3) and set

\[
X := \prod_{j=0}^{l-1} W^{d(l-j), p}(\Omega), \quad Y := \prod_{i=0}^{l-1} W^{d(l-i-1), p}(\Omega);
\]

we want to study the problem

\[
(\lambda - \mathcal{A}(x, \partial)) U = F(x) \quad \text{in } \Omega,
\]

\[
\gamma(\mathcal{B}_{\mu}(x, \partial \zeta)) U - g_{\mu} := \gamma\left( \sum_{k=0}^{l-1} \mathcal{B}_{\mu,k}(x, \partial \zeta) U_{k} - g_{\mu} \right) = 0, \quad \text{in the cylinder } V := \mathbb{R}^{n} \times \Omega,
\]

\[
U = (U_{0}, \ldots, U_{l-1}), \quad \mu = 1, \ldots, m
\]

with \( F \in Y, g_{\mu} \in W^{2m - \sigma_{\mu}, p}(\Omega) \), for some \( p \in ]1, + \infty[. \)

We are interested in solutions \( U \) in \( X \) and in estimates of the solutions depending on the parameter \( \lambda \). Precisely, our goal in this section is to prove Propositions 2.4 and 2.5.

We start with a couple of preliminary estimates:

2.1. LEMMA. Let \( U \in X, \ Re \lambda \geq \lambda, \lambda U - \mathcal{A}(x, \partial) U = 0 \) in \( \Omega \). For \( \mu = 1, \ldots, m \) let \( g_{\mu} \in W^{2m - \sigma_{\mu}, p}(\Omega) \) such that \( \gamma(\mathcal{B}_{\mu}(x, \partial \zeta)) U - g_{\mu} = 0 \). Then, there exist \( A \geq 0, C > 0 \) such that, if \( |\lambda| \geq A \),

\[
|\lambda| \| U \|_{Y} + \| U \|_{X} \leq C \left( \sum_{\mu=1}^{m} \| g_{\mu} \|_{2m - \sigma_{\mu}, p, \Omega} + \sum_{\mu=1}^{m} |\lambda|^{(2m - \sigma_{\mu})/d} \| g_{\mu} \|_{p, \Omega} \right).
\]

Proof. The proof can be obtained in a rather standard way through a well known method due to Agmon (see [13] 3.8). Here one must use the well known a priori estimates of [1] applied to the elliptic system \( L(x, \partial \zeta) := e^{\mu(\partial_{\zeta} - d\partial)} - \mathcal{A}(x, \partial \zeta) \) in the cylinder \( V := \mathbb{R}^{n} \times \Omega \), with the boundary conditions \( \mathcal{B}_{\mu,k}^{0}(x, \partial \zeta) \) on \( \partial V \).

2.2. LEMMA. Let \( s \in ]0, p^{-1}[. \ Let \( U \in X, \ Re \lambda \geq \lambda, \lambda U - \mathcal{A}(x, \partial) U = 0 \) in \( \Omega \). For \( \mu = 1, \ldots, m \) let \( g_{\mu} \in W^{2m - \sigma_{\mu}, p}(\Omega) \) such that \( \gamma(\mathcal{B}_{\mu}(x, \partial \zeta)) U - g_{\mu} = 0 \). Then, there exist \( A \geq 0, C > 0 \) such that, if \( |\lambda| \geq A \),

\[
|\lambda| \| U \|_{Y} + \| U \|_{X} \leq C \left( \sum_{\mu=1}^{m} \| g_{\mu} \|_{2m - \sigma_{\mu}, p, \Omega} + \sum_{\mu=1}^{m} |\lambda|^{(2m - \sigma_{\mu} - s)/d} \| g_{\mu} \|_{p, \Omega} \right).
\]
Proof. A proof can be obtained with the same method of [4] Lemma 2.14 (the idea comes from [15]).

2.3. Remark. Lemma 2.2 is still valid if \( \Omega = \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x_n > 0 \} \), just assuming the coefficients of \( B(x, \partial) \) uniformly continuous and bounded on the closure of \( \mathbb{R}^n_+ \) and the coefficients of \( B_\mu \) with all the derivatives of order less or equal to \( 2m - \sigma \_p \) uniformly continuous and bounded.

Now we can prove the main estimate:

2.4. Proposition. Fix \( 0 < \sigma < p^{-1} \). Under the assumptions (k1)–(k5) there exist \( A > 0, C > 0, \) such that for any \( U \in X, \) for any \( \lambda \in \mathbb{C} \), with \( \Re \lambda \geq 0, |\lambda| \geq A, \) if \( \lambda U - \lambda_0(x, \partial) U = F \) in \( \Omega \) and \( \gamma(B_\mu U - g_\mu) = 0 \) \((\mu = 1, \ldots, m, \ g_\mu \in W^{2m-\sigma_\_p}(\Omega))\), one has

\[
|\lambda| \| U \|_\gamma + \| U \|_\chi \leq C \left( \| F \|_\gamma + \sum_{\mu = 1}^m \| g_\mu \|_{2m-\sigma_\_p, \Omega} \right.
+ \sum_{\mu = 1}^m |\lambda|^{(\sigma_\_p - \sigma)/d} \| g_\mu \|_{\sigma_\_p, \Omega} \).
\]

Proof. Let \( x^0 \in \overline{\Omega} \). We distinguish the cases: \( x^0 \in \Omega \) and \( x^0 \in \partial \Omega \). If \( x^0 \in \Omega \), there is a ball \( U(x^0) \) contained in \( \Omega \) and with centre in \( x^0 \). If \( x^0 \in \partial \Omega \), there exists a neighbourhood \( U(x^0) \) of \( x^0 \) in \( \mathbb{R}^n \), \( R > 0 \) and a diffeomorphism \( \Phi : U(x^0) \to B_R \) of class \( C^{2m} \) such that \( \Phi(U(x^0) \cap \Omega) = B_R^+ \) and \( \Phi(U(x^0) \cap \partial \Omega) = \{ y \in B_R \mid y_n = 0 \} \).

We set \( \lambda_0(v) := \lambda_0(v, \partial) (v \cdot \Phi) \cdot \Phi^{-1}, \ B_\mu \phi(v) := B_\mu(v \cdot \Phi) \cdot \Phi^{-1} \). It is well known that \( \Phi \) can be chosen in such a way that \( \Phi \) and \( B_\mu \) satisfy (k1)–(k5) (of course, (k4) must be satisfied only in \( \{ y \in B_R \mid y_n = 0 \} \)). Let \( \overline{\Omega} \subset \bigcup_{1 \leq s \leq S} U_s \) with \( U_s = U(x^0) \) for some \( x^0 \in \overline{\Omega} \). Let \( \{ \phi_s \mid 1 \leq s \leq S \} \) be a partition of unity subordinated to this covering of \( \overline{\Omega} \). Assume first that \( U_s \subseteq \Omega \). Therefore, \( U_s \) is a ball and it is easily seen that it is possible to extend the coefficients of \( \sigma(x, \partial, \partial_\_s) \) to \( \mathbb{R}^n \) in such a way that (h1)–(h3) are satisfied. One has in \( U_s \):

\[
\lambda \phi_s, U - \lambda_0(x, \partial, \partial_\_s)(\phi_s, U) = \phi_s F + \sum_{j = 0}^{l-1} c_{l-j}(x, \partial, \partial_\_s) U_j,
\]

with order of \( c_j(x, \partial, \partial_\_s) \) less or equal to \( d(l - j) - 1 \). This and 1.8 give

\[
|\lambda| \| \phi_s, U \|_\gamma + \| \phi_s, U \|_\chi \leq C \left( \| \phi_s F \|_\gamma + \sum_{j = 0}^{l-1} \| U_j \|_{2m-\sigma_\_p - 1, \Omega} \right.
+ \sum_{\mu = 1}^m |\lambda|^{(\sigma_\_p - \sigma)/d} \| g_\mu \|_{\sigma_\_p, \Omega})
\]

(if \( \Re \lambda \geq 0, |\lambda| \geq A \)).
One has in $R(h_1)(h_3)$ in to 1.9, if $Re^*$ with order of $A$ $V_s$ satisfies 

$$\lambda V_s - \alpha_\phi V_s = (\phi, F) \Phi_s^{-1} + (A^* u) \Phi_s^{-1},$$

$$\gamma'(\beta_{\mu, \phi}, V_s - (\phi, g_s) \Phi_s - B^*_s u \Phi_s^{-1}) = 0,$$

with $\gamma'$ trace operator on $\gamma^*_u = 0$, $A^* = (A^*_u)^{0 \leq i, j \leq l - 1}$, $A^*_u = 0$ if $0 \leq i \leq l - 2$, order of $A^*_{l-1,j}$ less or equal to $2m - jd - 1$, $B^*_s U = \sum_{k=0}^{l-1} B^*_s U_k$ and order of $B^*_s$ less or equal to $\sigma_m - dk - 1$. Let $F_s := (\phi, F) \Phi_s^{-1}$, $g_s := (\phi, g_s) \Phi_s^{-1} + B^*_s u \Phi_s^{-1}$. One can extend $\alpha_\phi$ and $\beta_{\mu, \phi}$ to $R^n$ in such a way that (k1)-(k5) are satisfied for $Q = R^n$, and $\alpha_\phi$ satisfies (h1)-(h3) in $R^n$ (see for this [5] Lemma 3.1). Now extend with 0 $F_s$ and $g_{\mu, s}$ to $R^n$ (remark that in such a way $F_s \in Y(R^n) = \prod_{j=0}^{l-1} W^{p^*+j^*} (R^n)_{p^*}$ and $g_{\mu, s} \in W_1^{2m-s_\alpha, p^*} (R^n)_{p^*}$). Next, let $F'_s \in Y(R^n) = \prod_{j=0}^{l-1} W^{p^*+j^*} (R^n)_{p^*}$ such that $F_s$ is the restriction of $F'_s$ to $R^n$ and $\|F'_s\|_{Y(R^n)} \leq 2 \|F_s\|_{Y(R^n)}$. Owing to 1.9, if $Re^* \geq 0$ and $|\lambda|$ is sufficiently large, the problem

$$\lambda V - \alpha_\phi(x, \theta) V = F'_s$$

has a unique solution $V'_s \in X(R^n) = \prod_{j=0}^{l-1} W^{p^*+j^*} (R^n)$ and

$$|\lambda| \|V'_s\|_{Y(R^n)} + \|V'_s\|_{X(R^n)} \leq C \|F'_s\|_{Y(R^n)} \leq C \|F_s\|_{Y(R^n)}.$$

One has in $R^n$:

$$\lambda(v_s - V'_s) - \alpha_\phi (V_s - V'_s) = 0$$

$$\gamma'(\beta_{\mu, \phi}, (V_s - V'_s) - g_{\mu, s} + \beta_{\mu, \phi}, V'_s) = 0$$

So, from 2.3 we have for any $\sigma \in [0, p^* - 1]$:

$$|\lambda| \|V_s - V'_s\|_{Y(R^n)} + \|V_s - V'_s\|_{X(R^n)} \leq C \left( \sum_{\mu=1}^{m} \|g_{\mu, s}\|_{2m-s_\alpha, p^*_{\mu}} + \sum_{\mu=1}^{m} \|\beta_{\mu, \phi}, V'_s\|_{2m-s_\alpha, p^*_{\mu}} \right. \left. + \sum_{\mu=1}^{m} |\lambda|^{2m - s_\alpha} \|g_{\mu, s}\|_{s_\alpha, p^*_{\mu}} \right. \left. + \sum_{\mu=1}^{m} |\lambda|^{2m - s_\alpha} \|\beta_{\mu, \phi}, V'_s\|_{s_\alpha, p^*_{\mu}} \right).$$

(Of course we have identified $V'_s$ with its restriction to $R^n$).
One has
\[ \| B^\mu \phi^s V'_s \|_{-m} \leq C \| V'_s \|_{X(\|v\|)} \leq C \| F_s \|_{Y(\|v\|)}, \]
\[ \| B^\mu \phi^s V'_s \|_{s, \sigma} \leq C \sum_{k=0}^{l-1} \| V'_s, k \|_{s, \sigma + \sigma - dk, p, \|v\|} \quad \text{(} V'_s = (V'_s, 0, \ldots, V'_s, l-1) \text{)} \]

If \( \sigma > 0, \quad \sigma + \sigma - dk > 2m - d(k+1) + \sigma > 2m - d(k+1) \). From 1.9, by interpolation, one has
\[ \| V'_s, k \|_{s, \sigma, p, \|v\|} \leq C (\| F_s \|_{Y(\|v\|)} + \sum_{\mu=1}^{m} \| g^\mu \|_{2m-\sigma, p, \|v\|} + \sum_{\mu=1}^{m} (\| \phi^s \|_{s, \sigma+\sigma - \sigma})^d \| g^\mu \|_{s, p, \|v\|}). \]

So, we have
\[ |\lambda| \| V'_s \|_{Y(\|v\|)} + \| V'_s \|_{X(\|v\|)} \leq C \left( \| F_s \|_{Y(\|v\|)} + \sum_{\mu=1}^{m} \| g^\mu \|_{2m-\sigma, p, \|v\|} + \sum_{\mu=1}^{m} (\| \phi^s \|_{s, \sigma+\sigma - \sigma})^d \| g^\mu \|_{s, p, \|v\|} \right). \]

if \( 0 \leq \sigma < p^{-1} \).

Coming back to \( \Omega \), we obtain
\[ |\lambda| \| \phi^s U \|_{Y} + \| \phi^s U \|_{X} \leq C \left( \| F \|_{Y} + \sum_{\mu=1}^{m} \| \phi^s \|_{s, \sigma} \| g^\mu \|_{2m-\sigma, \|v\|} \right) \]

Summing up in \( s \), we obtain
\[ |\lambda| \| U \|_{Y} + \| U \|_{X} \leq C \left( \| F \|_{Y} + \sum_{\mu=1}^{m} \| g^\mu \|_{2m-\sigma, \|v\|} \right) \]

if \( 0 \leq \sigma < p^{-1} \).
From this formula, by interpolation one has

$$
|\hat{a}|^{1/d} \sum_{j=0}^{l-1} \|U_j\|_{2m-jd-1,p,\Omega} + \sum_{\mu=1}^{m} \sum_{j=0}^{l-1} \|\hat{a}_{2m-\sigma_j+1/d}^{j} U_j\|_{2m-jd-1+\sigma,\Omega} \\
\leq C \left( \|F\|_Y + \sum_{\mu=1}^{m} \|g_{\mu}\|_{2m-\sigma_{\mu},p,\Omega} + \sum_{\mu=1}^{m} \|\hat{a}_{2m-\sigma_j+1/d}^{j} U_j\|_{2m-jd-1+\sigma,\Omega} \\
+ \sum_{\mu=1}^{m} \sum_{j=0}^{l-1} \|\hat{a}_{2m-\sigma_j+1/d}^{j} U_j\|_{2m-jd-1+\sigma,\Omega} \right),
$$

which implies our estimate if $|\hat{a}|$ is suitably large.

2.5. Proposition. Assume (k1)-(k5) are satisfied. Then, there exists $A > 0$, such that, if $\Re \hat{a} \geq 0$, $|\hat{a}| \geq A$, problem (12) has a unique solution $u \in X$ for any $f \in Y$, for any $g_1, \ldots, g_m \in W^{2m-\sigma_l,p}(\Omega) \times \cdots \times W^{2m-\sigma_m,p}(\Omega)$.

Define $D(A) := \{ u \in \mathcal{X} \mid \gamma(\Theta(\cdot, \partial) U) = 0, \mu = 1, \ldots, m \}$, $A \mathcal{U} := A(x, \partial) U$. Then, $A$ is the infinitesimal generator of an analytic semigroup in $Y$.

Proof. The uniqueness of the solution of (12) is a consequence of 2.4. We are going to prove the existence.

Putting in (12) $U_1 = \lambda U_0 - F_0, \ldots, U_{l-1} = \lambda^{l-1} U_0 - \lambda^{l-2} F_0 \cdots - F_{l-2}$ it is easily seen that, if $U$ is the solution of (12), $U = (U_0, \ldots, U_{l-1})$, $U_0$ is a solution of the system

$$
\mathcal{A}(x, \lambda, \partial_x) U_0 = f, \\
\gamma(\Theta(\cdot, \partial) U_0 - g_\mu) = 0, \quad \mu = 1, \ldots, m,
$$

with $f \in L^p(\Omega)$, depending only on $\lambda$ and $F$, $g_\mu \in W^{2m-\sigma,\Omega}$, with $g_\mu$ depending only on $\lambda, g_1, \ldots, g_m$.

Systems of type (13) depending on the parameter $\lambda$ were studied by Agranovich and Visik. Following their method (see [2] Theorem 5.1) (that is, constructing a parametrix), using estimate 2.4, one can prove the existence of a solution $U_0 \in W^{2m,p}(\Omega)$ if $\Re \lambda > 0$ and $|\hat{a}|$ is sufficiently large. Putting $U_{l-1} = \lambda U_0 - F_0, \ldots, U_{l-2} = \lambda U_{l-2} - F_{l-2}$, one obtains a solution $U = (U_0, \ldots, U_{l-1})$ of (12). Therefore, $\lambda - A$ is onto $Y$ and, from 2.4, one has the estimate

$$
|\hat{a}| \|U\|_Y \leq C \|\lambda - A\|_Y U_Y,
$$

for any $u \in D(A)$. The density of $D(A)$ in $Y$ follows from this estimate and from the fact that $Y$ is reflexive.
2.6. Remark. The result of generation of a semigroup given in 2.5 is impossible if (k5) is not satisfied, just because $D(A)$ is not dense. We note also that, as $Y$ is reflexive, even estimates of maximal decay of the resolvent are impossible.

3. PARABOLIC PROBLEMS I

In this section we want to study the problem

$$\mathcal{A}(t, x, \partial_t, \partial_x) u(t, x) = f(t, x) \quad \text{in } [0, T] \times \Omega$$

$$(\mathcal{B}_\mu(t, \cdot, \partial_t, \partial_x) u(t, \cdot) - g_\mu(t, \cdot)) = 0, \quad \mu = 1, \ldots, m, \quad t \in [0, T],$$

$$u(0, x) = u_0(x), \quad x \in \Omega$$

$$\partial_t^{l-1} u(0, x) = u_{l-1}(x), \quad x \in \Omega$$

under the assumptions that for any $t \in [0, T]$ (k1)-(k5) are satisfied by $\mathcal{A}(t, x, \partial_t, \partial_x)$ and $\mathcal{B}_\mu(t, x, \partial_t, \partial_x)$ (with $\sigma_1, \ldots, \sigma_m$ independent of $t$). In the next section we shall see how the situation changes dropping (k5).

Before considering our specific problem, we shall recall some abstract results of [6] which will be used in the sequel.

In the mentioned paper we have considered the following abstract situation:

(i1) we have two Banach spaces $E_0$ and $E_1$, with $E_1 \subseteq E_0$ (continuous imbedding) and norms $\| \cdot \|_0$ and $\| \cdot \|_1$.

(i2) For $j = 1, \ldots, r$ ($r \in \mathbb{N}$) $\mu_1, \ldots, \mu_j$ are real numbers with $0 < \mu_1 \leq \ldots \leq \mu_j < 1$, $E_{1-\mu_1}, \ldots, E_{1-\mu_j}$, $F_{\theta_0}, F_{\theta_0+\mu_1}, \ldots, F_{\theta_0+\mu_j}$ are Banach spaces such that, if $A, B, C \in \{ E, F \}$, $\zeta, \eta, \rho \in \{ 0, 1-\mu_1, \ldots, 1-\mu_j, \theta_0, \theta_0+\mu_1, \ldots, \theta_0+\mu_j \}$, $\theta, \zeta \in ]0,1[$ and $(1-\theta) \zeta + \theta \eta = \rho$, $C_\rho$ is of type $\theta$ between $A_\zeta$ and $B_\eta$. Further, if $\zeta < \eta$, $B_\eta \subseteq A_\zeta$. We indicate with $| \cdot |_\zeta$ the norm in $F_{\zeta}$.

(i3) $\theta_0 + \mu_j < 1$.

(i4) $0 < T < +\infty$. $\mathcal{A} : [0, T] \rightarrow \mathscr{L}(E_1, E_0)$, for $j = 1, \ldots, r$ $\mathcal{B}_j : [0, T] \rightarrow \mathscr{L}(E_1, E_{1-\mu_j}) \cap \mathscr{L}(F_{\theta_0+\mu_j}, F_{\theta_0})$ and there exists $\beta \in ]0,1[$ such that, for $0 \leq s \leq t \leq T$,

$$\| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathscr{L}(E_1, E_0)} + \sum_{j=1}^r \| \mathcal{B}_j(t) - \mathcal{B}_j(s) \|_{\mathscr{L}(E_1, E_{1-\mu_j}) \cap \mathscr{L}(F_{\theta_0+\mu_j}, F_{\theta_0})}$$

$$\leq C(t-s)^\beta.$$
(i5) \( \beta + \theta_0 + \mu_1 > 1 \).

(i6) \( Z \) is a Banach space and \( \tau \in \mathcal{L}(E_1 - \mu, Z) \).

(i7) There exists \( A > 0 \) such that for any \( \lambda \in \mathbb{C} \), \( |\lambda| \geq A \), the problem
\[
(\lambda - \mathcal{A}(t)) u = f \\
\tau(\mathcal{B}(t) u - g_j) = 0, \quad j = 1, ..., r
\]
has a unique solution \( u \in E_1 \) for any \( f \in E_0 \), \( (g_1, ..., g_r) \in E_1 - \mu_1 \times \cdots \times E_1 - \mu_r \), and the following estimate is available:
\[
|\lambda| \| u \|_0 + \| u \|_1 \leq C \left( \| f \|_0 + \sum_{j=1}^r \| g_j \|_1 - \mu_j + \sum_{j=1}^r |\lambda|^{1 - \mu_j - \theta_0} \| g_j \|_{\theta_0} \right)
\]
(\( C > 0 \) independent of \( f, g_1, ..., g_r, \lambda \)).

A careful inspection of the proof shows (of course, in order to obtain the desired results) that it is not necessary to assume that \( E_1 - \mu_1, ..., E_1 - \mu_r, F_{\theta_0} \) are intermediate between \( E_0 \) and \( E_1 \) and it suffices to require that \( E_1 - \mu_j \subseteq F_{\theta_j} \), for \( j = 1, ..., r \). It is also convenient to take as new parameters \( \mu_j, ..., \mu_r \) instead of \( \mu_1, ..., \mu_r \). So, in the new formulation of (i1)-(i7) we are going to give we set \( v_j := \theta_0 + \mu_j \) and write \( E_j \) instead of \( E_1 - \mu_j \) and also \( F \) instead of \( F_{\theta_0} \). We replace (i1)-(i7) with the following (a)-(e):

Let \( E_0, E_1 \) be a couple of Banach spaces, with \( E_j \subseteq E_0 \) (continuous inclusion) and norms \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \). For \( j = 1, ..., r \) \((r \in \mathbb{N})\), \( v_1, ..., v_r \) are real numbers and \( E_{\mu_1}, ..., E_{\mu_2}, E, F_{v_1}, ..., F_{v_r}, Z \) are Banach spaces such that

(a) \( 0 < v_j, ..., v_r < 1 \) and \( F_{v_j} \) is a space of type \( v_j \) between \( E_0 \) and \( E_1 \), \( E_{\mu_j} \subseteq F \) for any \( j \).

(b) \( \tau \in [0, +\infty[ \), \( \mathcal{A}: [0, T] \to \mathcal{L}(E_1, E_0) \).

(c) For \( j = 1, ..., r \), \( \mathcal{B}_j: [0, T] \to \mathcal{L}(E_1, E_{\mu_j}) \cap \mathcal{L}(F_{v_j}, F) \) and there exist \( C, \beta > 0 \) such that
\[
\| \mathcal{A}(t) - \mathcal{A}(s) \|_{\mathcal{L}(E_1, E_0)} + \sum_{j=1}^r \| \mathcal{B}_j(t) \mathcal{B}_j(t) - \mathcal{B}_j(s) \|_{\mathcal{L}(E_1, E_{\mu_j}) \cap \mathcal{L}(F_{v_j}, F)} \leq C(t-s)^{\beta}.
\]

(d) \( \tau \in \mathcal{L}(\bigoplus_{j=1}^r F_{v_j}, Z) \), \( \beta + v_j > 1 \) for any \( j = 1, ..., r \).

(e) There exists \( A > 0 \) such that for any \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \geq 0 \), \( |\lambda| \geq A \) the problem
\[
(\lambda - \mathcal{A}(t)) u = f \\
\tau(\mathcal{B}_j(t) u - g_j) = 0, \quad j = 1, ..., r
\]
has a unique solution \( u \in E_1 \) for any \( f \in E_0 \), \((g_1, \ldots, g_r) \in E_{\mu_1} \times \cdots \times E_{\mu_r}\), and the following estimate is available:

\[
|\lambda| \|u\|_0 + \|u\|_1 \leq C \left( \|f\|_0 + \sum_{j=1}^r \|g_j\|_{E_{\mu_j}} + \sum_{j=1}^r |\lambda|^{-\gamma_j} \|g_j\|_r \right).
\]

Consider the problem

\[
\frac{du}{dt}(t) = A(t)u(t) + f(t)
\]

\[
\tau(\beta_j(t) u(t) - g_j(t)) = 0, \quad j = 1, \ldots, r
\]

\[
u(0) = u_0,
\]

with \( u_0 \in E_0, f \in C([0, T]; E_0), g_j \in C([0, T]; E_j) \).

By definition, a strict solution of (15) is a function \( u \in C^1([0, T]; E_1) \cap C([0, T]; E_0) \) solving (15) pointwise for any \( t \in [0, T] \) (of course, this implies \( u_0 \in E_1 \)).

A classical solution of (15) is a function \( u \in C^1([0, T]; E_0) \cap C([0, T]; E_j) \cap C([0, T]; E_0) \), satisfying the two first conditions in (15) only for \( t \in [0, T] \).

One can prove the following result, analogous to Theorem 4.14 in [6] (the proof can be obtained just changing the notations):

3.1. Theorem. Under the assumptions (a)–(e) for any \( f \in C^\nu([0, T]; E_0) \) \((\nu > 0)\), for any \( g_j \in C^\nu([0, T]; E_j) \cap C^{1-\gamma_j}(\nu, \nu), \) for any \( u_0 \) in the closure of \( E_1 \) in \( E_0 \) (15) has a unique classical solution \( u \). Moreover, if \( u_0 \in E_1, \beta(0) u_0 + f(0) \) belongs to the closure of \( E_1 \) in \( E_0 \) and for \( j = 1, \ldots, r \)

\[
\tau(\beta_j(0) u_0 - g_j(0)) = 0
\]

the classical solution is strict. Finally,

\[
\|u\|_{C_1(0, r}; E_0) \leq \text{const} \left( \|u_0\|_0 + \|f\|_{C_1(0, r}; E_0) + \sum_{j=1}^r \|g_j\|_{C_1(0, r}; E_j) \right)
\]

Now we come back to problem (14) and, as usual, we set: \( u_0 := u, \ldots, u_j := \partial^j u, \ldots, u_{l-1} := \partial^{l-1} u \) to obtain, putting \( U := (u_0, \ldots, u_{l-1}) \), the system

\[
\partial_t U(t, x) = \beta(t, x, \partial_x) U(t, x) + F(t, x), \quad t \in [0, T], \quad x \in \Omega,
\]

\[
\gamma(\beta(t, \cdot, \partial_x) U(t, \cdot) - g_\mu(t, \cdot)) = 0, \quad \mu = 1, \ldots, m, \quad t \in [0, T], \quad U(0, x) = U_0(x),
\]

with \( F(t, x) = (0, \ldots, 0, f(t, x)), U_0(x) = (u_0(x), \ldots, u_{l-1}(x)) \).
Next, we put $E_0 := Y = W^{2m-d}p(\Omega) \times \cdots \times W^{2m-(i+1)d}p(\Omega) \times \cdots \times L^p(\Omega)$, $E_i := X = W^{2m-i}p(\Omega) \times \cdots \times W^{2m-j}p(\Omega) \times \cdots \times W^{2d}p(\Omega)$, $E_{i+1} := W^{2m-\sigma_2-i}p(\Omega)$, ..., $E_{2m} := W^{2m-\sigma_2}p(\Omega)$, $F_0 := W^{m}p(\Omega)$ (for some $\sigma \in ]0, p^{-1}[)$, $F_{i+1} := W^{m+\sigma-i}p(\Omega) \times \cdots \times W^{m+\sigma-i}p(\Omega) \times \cdots \times W^{m+\sigma-(i-1)d}p(\Omega)$.

In this case we have $v_i = (\sigma_1 + \sigma - (2m-d))/p$, ..., $v_m = (\sigma_m + \sigma - (2m-d))/d$.

Finally, we set $\mathcal{A}(t) := \mathcal{A}(t, x, \partial_x)$, $\mathcal{B}_\mu(t) := \mathcal{B}_\mu(t, x, \partial_x)$, $\tau = \gamma$, $Z := L^p(\partial\Omega)$.

Now we are going to precise what we mean for classical and strict solution of (14):

A classical solution of (14) is a function $u \in C^0([0, T]; L^p(\Omega)) \cap \cdots \cap C^{1-j}(0, T]; W^{jd}p(\Omega)) \cap \cdots \cap C^{1-j}(0, T]; W^{jd}p(\Omega)) \cap \cdots \cap C([0, T]; W^{2m-d}p(\Omega))$ satisfying the two first conditions for $t \in [0, T]$.

A strict solution of (14) is a function $u \in C^0([0, T]; L^p(\Omega)) \cap \cdots \cap C^{1-j}(0, T]; W^{jd}p(\Omega)) \cap \cdots \cap C([0, T]; W^{2m-d}p(\Omega))$, satisfying the two first conditions for $t \in [0, T]$.

A straightforward application of 3.1 (consequence of 2.4 and 2.5) is the following

3.2. Theorem. Assume that (k1)-(k5) are satisfied in problem (14) for any $t \in [0, T]$, with $d, \sigma_1, ..., \sigma_m$ independent of $t$. Let $\beta > 0$, $\beta > (2m - \sigma_j - p^{-1})/d$ for any $j \in \{1, ..., m\}$. Assume that the coefficients of $\mathcal{A}(t, x, \partial_x)$ are of class $C^0([0, T]; C(\overline{\Omega}))$, for $\mu = 1, ..., m$ the coefficients of $\mathcal{B}_\mu(t, x, \partial_x)$ are of class $C^0([0, T]; C^{2m-n}(\overline{\Omega}))$. Let $\sigma \in ]0, p^{-1}[$ be such that $\beta > (2m - \sigma_j - \sigma)/d$ for any $j \in \{1, ..., m\}$.

Then, for any $f \in C^0([0, T]; L^p(\Omega)) (c > 0), g \in C^0([0, T], W^{2m-\sigma_j}p(\Omega)) \cap C^{0}([0, T]; W^{2m-d}p(\Omega)) (1 \leq j \leq m)$, $u_0 \in W^{2m-d}p(\Omega)$, ..., $u_{m+1} \in W^{2m-(j+1)d}p(\Omega)$, $u_{m+1} \in L^p(\Omega)$ (14) has a unique classical solution, which is strict if $u_{m+1} \in W^{2m-d}p(\Omega)$, ..., $u_{m} \in W^{2m-\sigma_j}p(\Omega)$, $u_{m-1} \in W^{2m-d}p(\Omega)$ and for $\mu = 1, ..., m$, $\gamma(\sum_{k=1}^{m-1} B_{\mu,k}(0, \cdot, \partial_x) u_k - g_\mu(0, \cdot)) = 0$.

4. PARABOLIC PROBLEMS

Now we consider (14) in case (k1)-(k4) but not necessarily (k5) are satisfied. To this aim we shall treat certain parabolic problems on $\partial \Omega$, which we recall again, is a compact submanifold of class $C^{2m}$ of $\mathbb{R}^n$. 
Let $r \in \mathbb{N}$, $r < m$.

A strongly elliptic operator on $\partial \Omega$ of order $2m$ is an operator (defined for example on functions of class $C^{2r}$ on $\partial \Omega$) which in local coordinates is of the form

$$
\sum_{|\alpha| \leq 2r} a_\alpha(y) \partial^\alpha_y
$$

(17)

with $a_\alpha$ of class $C^{1|\alpha}$ and $\text{Re}\{(-1)^r \sum_{|\alpha| - 2r} a_\alpha(y) \xi^\alpha\} < 0$ if $\xi \in \mathbb{R}^n - \{0\}$.

It is immediately seen that such an operator is extensible to a linear bounded operator from $W^{2r,p}(\partial \Omega)$ to $L^p(\partial \Omega)$. For any $r \in \{1, \ldots, m-1\}$ an operator of this type can be easily constructed: let $\{(\Phi_j, U_j) \mid j = 1, \ldots, q\}$ be a $C^{2m}$-atlas of $\partial \Omega$ and $\{\phi_j \mid j = 1, \ldots, q\}$ a $C^{2m}$-partition of unity such that $\text{supp}(\phi_j) \subseteq U_j$. Define (for $u \in C^2(\partial \Omega)$)

$$
A, u := (-1)^{r+1} \sum_{j=1}^q \phi_j \left[ A'(u \cdot \Phi_j^{-1}) \cdot \Phi_j \right]
$$

($A$ is the usual Laplace operator).

We fix an operator $H$ of this type and order $2r$ and prove:

4.1. Proposition. Let $1 < p < +\infty$. Set $Y := L^p(\partial \Omega)$, $D(A) := W^{2r,p}(\partial \Omega)$, $Au = Hu$. Then $A$ is the infinitesimal generator of an analytic semigroup in $Y$.

Proof. We consider the problem

$$
\lambda u - Au = f,
$$

(18)

with $\text{Re}\, \lambda \geq 0$, $u \in D(A)$, $f \in Y$. By local charts, using well known results on elliptic problems in $\mathbb{R}^n$, on obtains the following a priori estimate: if $|\lambda|$ is sufficiently large, $u \in D(A)$ and (18) is satisfied,

$$
|\lambda| \|u\|_{0,p,\partial \Omega} + \|u\|_{2r,p,\partial \Omega} \leq C \|f\|_{0,p,\partial \Omega}
$$

(19)

with $C > 0$, independent of $\lambda$, $u$, $f$. It remains to establish the existence of a solution of (18) if $|\lambda|$ is sufficiently large; to this aim it is sufficient to recall that the dual operator $A'$ of $A$ is of the same form in the space $L^p(\partial \Omega)$ and satisfies therefore an estimate like (19). From (19) it follows that the range of $\lambda - A$ is closed in $L^p(\partial \Omega)$ and from the corresponding estimate for $A'$ that $\lambda - A'$ is injective; a simple duality argument gives therefore the desired result.

4.2. Remark. Consider, for example, the case $H = A_r$, with $4r < 2m - 1$.

It is easily seen that there exists a differential operator $K(x, \partial_x)$ of order
2r whose coefficients are defined and of class $C^{2r}$ on the closure of $\Omega$ such that for any $u \in W^{2r+1,2}(\Omega) \cap (K(\gamma, \partial_{x} u)) = H^{r}u$.

If $x' \in \partial\Omega$,

$$K^{0}(x', \xi) = (-1)^{r+1} \sum_{j=1}^{q} \phi_{j}(x') \left( \sum_{k=1}^{n} \xi_{k} \left( \frac{\partial \Phi_{j}}{\partial y_{j}} \cdot \Phi_{j} \right)(x') \right)^{2}.$$ 

4.3. We come back to problem (14) without assuming (k5). Therefore we admit boundary conditions such that $2m - 2d < \sigma_{p} < 2m - d$ and let $u$ be a strict solution of (14). For example, assume $2m - 2d < \sigma_{p} \leq 2m - d - 1$. Necessarily, $B_{p, l} = 0$.

Assume the coefficients of $B_{\mu, k}$ of class $C([0, T]; C^{2m-d}(\Omega)) \cap C^{1}([0, T]; C^{2m-d}(\Omega))$, $g_{\mu} \in C([0, T]; W^{2m-d}(\Omega) \cap C^{1}([0, T]; W^{2m-d}(\Omega))$, $\mu \in \mathbb{N}$.

Then, $\sum_{k=0}^{l-2} B_{\mu, k}(\gamma, \partial_{x}) \partial_{x}^{l} u - g_{\mu}$ has a null trace on $\partial\Omega$. Precisely,

$$\gamma \left( \sum_{k=0}^{l-2} B_{\mu, k}(0, \partial_{x}) u_{k} + \sum_{k=0}^{l-2} B_{\mu, k}(0, \partial_{x}) u_{k+1} - g_{\mu}^{(0)}(0) \right) = 0$$

is necessary to assure that (14) has a strict solution. In general, if $2m - (r + 1) d \leq \sigma_{p} < 2m - rd$ $(0 \leq r \leq l - 1)$, if the coefficients of $B_{\mu, k}$ are of class $C([0, T]; C^{2m-d}(\Omega)) \cap C^{1}([0, T]; C^{2m-d}(\Omega)) \cap \cdots \cap C^{1}([0, T]; C^{2m-d}(\Omega)) \cap \cdots \cap C^{1}([0, T]; W^{2m-d}(\Omega))$, $B_{\mu, k} = 0$ if $k \geq l - r$ and we obtain the following necessary compatibility conditions:

for $j = 0, \ldots, r$

$$\gamma \left( \sum_{k=0}^{l-r-1} \sum_{\rho=0}^{j} \frac{j!}{\rho!} B_{\mu, k}(\gamma, \partial_{x}) u_{k+\rho} - g_{\mu}^{(j)}(0) \right) = 0$$

(Of course we indicate with $B_{\mu, k}(t, x, \partial_{x})$ the operator obtained differentiating the coefficients of $B_{\mu, k}(t, x, \partial_{x})$ with respect to $t$).

We turn now to classical solutions; under the same assumptions of regularity on the coefficients of the system and on the functions $g_{\mu}$, one can
verify that, if $2m - (r + 1) d \leq \sigma_{\mu} < 2m - rd$ ($0 \leq r \leq l - 1$) a necessary condition for the existence of a classical solution is that

$$
\gamma \left( \sum_{k=0}^{l-r-1} \sum_{\rho=0}^{j} \left( \frac{j}{\rho} \right) B_{\mu,k}^{r} \cdot (0, \cdot, \partial_\nu) u_{k+\rho} - g^j(0) \right) = 0
$$

(no conditions of $r = 0$).

Now we want to show that these necessary conditions are essentially sufficient to guarantee the existence of strict and classical solutions. We start with the following

4.4. Lemma. Assume (k1)–(k4) are satisfied. Moreover, for a certain $\mu \in \{1, \ldots, m\}$ assume that $\sigma_{\mu} < 2m - d$. Put $H := \Lambda_{d,2}$ and define

$$
\mathcal{B}_{\mu}^n(x, \partial_\nu, \partial_\nu) u := (\partial_\nu - K_{d,2}(x, \partial_\nu))(\mathcal{B}(\cdot, \partial_\nu, \partial_\nu) u).
$$

Then, substituting to $\mathcal{B}_{\mu}^n u$, one obtains a system again satisfying (k1)–(k4).

Proof. We verify only (k4). We have to consider the O.D.E. problem

\begin{align*}
&\mathcal{A}^0(x', \lambda, \lambda, i\zeta' + v(x') \partial_\nu) w(\tau) = 0 \quad \text{in } \mathbb{R}, \\
&\mathcal{B}_{\mu}^n(x', \lambda, \lambda, i\zeta' + v(x') \partial_\nu) w(0) = g_j, \quad j \in \{1, \ldots, m\} \setminus \{\mu\}, \\
&\mathcal{B}_{\mu}^n(x', \lambda, \lambda, i\zeta' + v(x') \partial_\nu) w(0) = g_{\mu}
\end{align*}

with $w$ bounded on $\mathbb{R}^+$

with $(g_1, \ldots, g_\mu, \ldots, g_m) \in \mathbb{C}^m$, $x' \in \partial \Omega$, $\Re \lambda > 0$, $\zeta' \in T_{x'}(\partial \Omega)$, $(\lambda, \zeta') \neq (0, 0)$. One has from 4.2

$$
\mathcal{B}_{\mu}^n(x', \lambda, \lambda, i\zeta' + v(x') \partial_\nu) = (\lambda - K_{d,2}(x', i\zeta')) B_{\mu}^n(x', \lambda, \lambda, i\zeta' + v(x') \partial_\nu)
$$

with $K_{d,2}(x', i\zeta') \leq -\delta |\zeta'|^d$ and $\delta > 0$.

From this the proof follows easily.

Now we are able to state and prove our main result

4.5. Theorem. Consider problem (14) with a fixed $p \in ]1, +\infty[$ under the following assumptions:

\begin{enumerate}
  \item[(L1)] (k1)–(k4) are satisfied for any $t \in [0, T]$ with $m, d, l, \sigma_1, \ldots, \sigma_m$ independent of $t$;
  \item[(L2)] the coefficients of $A_k$ ($0 \leq k \leq l$) are of class $C^p([0, T]; \mathbb{C}(\Omega))$ ($\beta > 0$ whose value is specified in the following);
\end{enumerate}
(L3) if \( 2m - rd \leq \sigma_\mu < 2m - (r - 1) d \) \((1 \leq r \leq l)\) the coefficients of \( B_{\mu, k} \) \((0 \leq k \leq l - 1)\) are of class \( C^0([0, T]; \mathbb{C}^{2m-\sigma_\mu}([\Omega])) \cap C^1([0, T]; \mathbb{C}^{2m-\sigma_\mu} - (r - 1) d ([\Omega])); \)

(L4) for any \( \mu \in \{1, \ldots, m\} \), if \( 2m - rd \leq \sigma_\mu < 2m - (r - 1) d \) \((1 \leq r \leq l)\),

\[
\beta > [2m - (r - 1) d - \sigma_\mu - p - 1]/d;
\]

(L5) there exists \( \sigma \in [0, p^{-1} [ \) such that, for \( \mu = 1, \ldots, m \), if \( 2m - rd \leq \sigma_\mu < 2m - (r - 1) d \), \( \beta > [2m - (r - 1) d - \sigma_\mu]/d \), \( g_\mu \in C^4([0, T]; \mathbb{C}^{2m-\sigma_\mu}([\Omega])) \cap C^{1+}(0, T]; \mathbb{C}^{2m-\sigma_\mu} - (r - 1) d ([\Omega])); \) then, for any \( (u_1, \ldots, u_{l-1}) \in \mathbb{W}^{2m, p}(\Omega) \times \cdots \times \mathbb{W}^{d, p}(\Omega) \) such that, for any \( \mu \in \{1, \ldots, m\} \), if \( 2m - rd \leq \sigma_\mu < 2m - (r - 1) d \) \((1 \leq r \leq l)\)

for \( j = 0, \ldots, r - 1 \)

\[
\gamma \left( \sum_{k=0}^{l-r} \sum_{\rho=0}^{j} \left( \frac{j}{\rho} \right) B_{\mu, k}^{(j-\rho)}(0, \cdot, \partial_t) u_{k+\rho} - g_\mu^{(j)}(0) \right) = 0
\]

(20)

(14) has a unique strict solution; for any \( (u_0, \ldots, u_{l-1}) \in \mathbb{W}^{2m-d, p}(\Omega) \times \cdots \times \mathbb{W}^{d, p}(\Omega) \times L^p(\Omega) \) such that, for any \( \mu \in \{1, \ldots, m\} \), if \( 2m - rd \leq \sigma_\mu < 2m - (r - 1) d \) \((1 \leq r \leq l)\)

for \( j = 0, \ldots, r - 2 \)

\[
\gamma \left( \sum_{k=0}^{l-r} \sum_{\rho=0}^{j} \left( \frac{j}{\rho} \right) B_{\mu, k}^{(j-\rho)}(0, \cdot, \partial_t) u_{k+\rho} - g_\mu^{(j)}(0) \right) = 0
\]

(21)

(no conditions if \( r = 1 \)).

(14) has a unique classical solution.

Proof. Assume \( u \) is a strict solution. Then, necessarily, for any \( \mu \in \{1, \ldots, m\} \), if \( 2m - rd \leq \sigma_\mu < 2m - (r - 1) d \) \((1 \leq r \leq l)\),

\[
(\partial_t - A_{d/2})^{-1} \gamma \left( \sum_{k=0}^{l-r} B_{\mu, k}(t, \cdot, \partial_t) \partial_t u - g_\mu(t, \cdot) \right) = 0,
\]

which implies

\[
\gamma \left( (\partial_t - K_{d/2})^{-1} \sum_{k=0}^{l-r} B_{\mu, k}(t, \cdot, \partial_t) \partial_t u - (\partial_t - K_{d/2})^{-1} g_\mu(t, \cdot) \right) = 0.
\]

Replace now \( A_{d/2} \) with the operator

\[
A_{d/2}(t, x, \partial_t, \partial_\xi) := (\partial_t - K_{d/2})^{-1} \sum_{k=0}^{l-r} B_{\mu, k}(t, \cdot, \partial_t) \partial_\xi,
\]
to show that $u$ is the infinitesimal generator of a semigroup in a Banach space $X$.

Remark now that $v$ solves (14). One has to verify that, for $\mu = 1, \ldots, m$,

$$\gamma(\mathcal{B}_\mu(t, \cdot, \partial_x, \partial_x) u(t, \cdot) - g_\mu(t, \cdot)) = 0,$$

Assume that $0 < r < l$. It remains to show that $u$ solves (14). One has to verify that, for $\mu = 1, \ldots, m$,

$$\gamma(\mathcal{B}_\mu(t, \cdot, \partial_x, \partial_x) u(t, \cdot) - g_\mu(t, \cdot)) = 0.$$

Assume $2m - rd \leq \sigma \mu < 2m - (r - 1)d$ with $2 \leq r \leq l$.

We know that $\gamma(\mathcal{B}_\mu(t, \cdot, \partial_x, \partial_x) u(t, \cdot) - g_\mu(t, \cdot)) = 0$, which means

$$(\partial_t - A_d) \gamma((\partial_x - K_d)^{r-2} (\mathcal{B}_\mu(t, \cdot, \partial_x, \partial_x) u(t, \cdot) - g_\mu(t, \cdot))) = 0$$

Remark now that $v(t) := \gamma((\partial_x - K_d)^{r-2} (\mathcal{B}_\mu(t, \cdot, \partial_x, \partial_x) u(t, \cdot) - g_\mu(t, \cdot))) \in C^1([0, T]; W^{2m - rd - (r - 2r - 1)p^{-1}, p}(Q))$ and, from (L5), $v(0) = 0$. It follows from 4.1 that $v \equiv 0$, that is, $\gamma((\partial_x - K_d)^{r-2} (\mathcal{B}_\mu(t, \cdot, \partial_x, \partial_x) u(t, \cdot) - g_\mu(t, \cdot))) \equiv 0$. Iterating this procedure one obtains $\gamma((\mathcal{B}_\mu(t, \cdot, \partial_x, \partial_x) u(t, \cdot) - g_\mu(t, \cdot))) \equiv 0$.

The proof in the case of classical solutions is similar (recalling that if $A$ is the infinitesimal generator of a semigroup in a Banach space $X$, $T > 0$, $u \in C^1([0, T]; X) \cap C([0, T]; X) \cap C([0, T]; D(A))$, $u(0) = 0$ and $u'(t) - Au(t) = 0$ for any $t \in [0, T]$, $u(t) = 0$ for any $t \in [0, T]$).

An easy consequence of 4.5 is the following

4.6. Corollary. Assume that $\Omega$, $\mathcal{A}(x, \partial_x, \partial_x)$ and, for $\mu = 1, \ldots, m$, $\mathcal{B}_\mu(x, \partial_x, \partial_x)$ satisfy the assumptions (k1)–(k4). Set

$$H := \{ (u_0, \ldots, u_{j-1}) \in \prod_{i=0}^{l-1} W^{\alpha(l-i-1), i}(\Omega) \mid \text{for any } \mu \in \{ 1, \ldots, m \},$$

if $2m - rd \leq \sigma \mu < 2m - (r - 1)d$ with $r \in \mathbb{N}$, $r \geq 2$, $\gamma \left( \sum_{k=0}^{l-r} B_\mu, k(x, \partial_x) u_{k+j} \right) = 0$ for $j = 0, \ldots, r - 2$.
Let
\[ D(A) := \{ (u_0, \ldots, u_{j-1}) \in \prod_{j=0}^{l-1} W_h^{l-j,j}(\Omega) \mid \text{for any } yu \in \{1, \ldots, m\}, \]
if \(2m - rd \leq \sigma_n < 2m - (r - 1)d\) with \(r \in \mathbb{N}\),
\[ \gamma \left( \sum_{k=0}^{l-r} \beta_{n,k}(x, \partial_x) u_{k+j} \right) = 0 \text{ for } j = 0, \ldots, r - 1, \]
\[ Au := (u_1, \ldots, u_{j-1}, - \sum_{j=0}^{l-1} A_{j-}(x, \partial_x) u_j) \quad (u = (u_0, \ldots, u_{j-1})) \]
Then, \(A\) is the infinitesimal generator of an analytic semigroup in \(H\).

**Proof.** From 4.5 one has that, for any \((u_0, \ldots, u_{j-1}) \in H\) the problem
\[ U'(t) = AU(t), \]
\[ U(0) = (u_0, \ldots, u_{j-1}), \quad (23) \]
has a classical solution.

**REFERENCES**