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Analytic ideals and their applications

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Abstract

We study the structure of analytic ideals of subsets of the natural numbers. For example, we prove that for an analytic ideal I, either the ideal $\{X\subset\omega\times\omega\colon \exists n\; X\subset\{0,1,\dots,n\}\times\omega\}$ is Rudin–Keisler below I, or I is very simply induced by a lower semicontinuous submeasure. Also, we show that the class of ideals induced in this manner by lsc submeasures coincides with Polishable ideals as well as analytic P-ideals. We study this class of ideals and characterize, for example, when the ideals in it are F_σ or when they carry a locally compact group topology. We apply these results to Borel partial orders to rederive a theorem of Todorcevic and to Borel equivalence relations to answer a question of Kechris and Louveau. As another application we give a characterization of σ-ideals of μ-zero sets for Maharam submeasures μ on the Cantor set which is to a large extent analogous to a characterization of the meager ideal due to Kechris and the author. © 1999 Elsevier Science B.V. All rights reserved..

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1. Introduction

We study analytic ideals on the set of all natural numbers ω. An ideal is a family of subsets of ω closed under taking finite unions and subsets of its members. We assume throughout the paper that all ideals contain singletons $\{n\}$ for $n\in\omega$. Three natural classes of ideals play particularly important role: ideals induced, in the manner explained below, by lower semicontinuous submeasures, Polishable ideals, and P-ideals. Actually, it turns out that these three classes coincide in the realm of analytic ideals.

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We explain first the relation between submeasures and ideals on ω. We call $\phi : 2^{\omega} \to [0,\infty]$ a *submeasure on* ω if $\phi(\emptyset) = 0$ and $\phi(X) \leqslant \phi(Y)$ whenever $X \subset Y$ and $\phi(X \cup Y) \leqslant \phi(X) + \phi(Y)$ for any $X, Y \in 2^{\omega}$, and $\phi(\{n\}) < \infty$ for any $n \in \omega$. A submeasure on ω is called *lower semicontinuous* if it is lower semicontinuous as a function from 2^{ω} regarded with the product topology, i.e., if $X_n \to X$ then $\liminf_n \phi(X_n) \geqslant \phi(X)$. This condition is equivalent, for submeasures, to $\phi(X_n) \to \phi(X)$ for any non-decreasing sequence (X_n) whose union is $X \in 2^{\omega}$ and also to the condition $\phi(X) = \lim_n \phi(X \cap n)$ for every $X \in 2^{\omega}$. (We sometimes write lsc for lower semicontinuous.) We associate with a lsc submeasure ϕ two ideals on ω. First one called the *exhaustive ideal of* ϕ and the second one the *finite ideal of* ϕ.

$$\mathrm{Exh}(\phi) = \{X \in 2^{\omega}\colon \phi(X \setminus m) \to 0 \text{ as } m \to \infty\},$$

$$\mathrm{Fin}(\phi) = \{X \in 2^{\omega}\colon \phi(X) < \infty\}.$$

It is obvious that $\mathrm{Exh}(\phi) \subset \mathrm{Fin}(\phi)$. A lsc submeasure is called *finite* if $\phi(\omega) < \infty$, i.e., $\mathrm{Fin}(\phi) = 2^{\omega}$. A lsc submeasure on ω is called *exhaustive* if $\phi(X) < \infty$ implies $\phi(X \setminus m) \to 0$, i.e., $\mathrm{Exh}(\phi) = \mathrm{Fin}(\phi)$. It is easy to see that a lsc submeasure ϕ is exhaustive iff for any family X_n, $n \in \omega$, of pairwise disjoint subsets of ω with $\phi(\bigcup_n X_n) < \infty$ we have $\phi(X_n) \to 0$. This shows that our definition agrees with what is usually called an exhaustive submeasure.

An ideal on ω will be sometimes regarded as group with symmetric difference as the group operation; that is, $(X, Y) \to X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. An ideal I on ω is called *Polishable* if there exists a Polish group topology τ on I such that the family of Borel sets with respect to τ is equal to the family of Borel subsets of I with respect to the topology inherited from 2^{ω}. Such a topology is unique if it exists (see [4, Theorem 9.10]). This class of ideals was first studied by Kechris and Louveau in [5].

An ideal I on ω is called a *P-ideal* if for any sequence $X_n \in I$, $n \in \omega$, there exists $X \in I$ such that $X_n \setminus X$ is finite for all n. Analytic P-ideals were studied by Todorcevic [15].

If I and J are two ideals on ω, we write $J \leqslant_f I$ if there exists a finite-to-one function $h : \omega \to \omega$ such that $X \in J$ iff $h^{-1}(X) \in I$. So, $J \leqslant_f I$ means that J is below I with respect to the Rudin–Keisler order and the function witnessing it is finite-to-one. Recall a basic result about $\leqslant_f$, due to Mathias [10], Jalali-Naini, and Talagrand [15], which will be used in the sequel. If I is an ideal on ω which has the Baire property, which is true when, for example, I is analytic, then $[\omega]^{<\omega} \leqslant_f I$ where $[\omega]^{<\omega}$ is the ideal of finite subsets of ω. ($[\omega]^{<\omega}$ is sometimes denoted by Fin.)

Even though all the definitions in this paper are formulated for ideals of subsets of ω, some ideals we will consider do not live on ω but rather are ideals of subsets of another infinite countable set (for example, $\omega \times \omega$, $2^{<\omega}$, etc). Since any such countable set can be identified with ω, this will cause no confusion.

The results in the present paper were announced in [14] which also contains more background information.

2. A dichotomy for analytic ideals

Let I_1 be the ideal on $\omega \times \omega$ consisting of sets included in a finite union of sets of the form $\{n\} \times \omega$, $n \in \omega$.

Theorem 2.1. *Let I be an analytic ideal on ω. Then either $I_1 \leqslant_f I$ or there exists a finite, lower semicontinuous submeasure ϕ on ω such that $I = \mathrm{Exh}(\phi)$. The two possibilities exclude each other.*

Let I be an ideal on ω. A set $A \subset 2^\omega$ is called *small* if there exists $X \in I$ such that $\{Y \cap X\colon Y \in A\}$ is meager in 2^X. Recall that a subset A of 2^X is *hereditary* if subsets of elements of A are themselves in A. Define

$$C(I) = \{K \subset 2^\omega\colon K \text{ hereditary, compact, and}\\ \text{such that } \forall X \in I \exists n \in \omega\; X \setminus n \in K\}.$$

The following lemma relates these two notions to each other.

Lemma 2.2. *Let $K \subset 2^\omega$ be hereditary and compact. Then $K \in C(I)$ if, and only if, K is not small.*

Proof. $\Rightarrow$ is clear. To see $\Leftarrow$, let K be a compact hereditary set which is not small. Let $X \in I$. Then $\{X \cap Y\colon Y \in K\}$ is not meager in 2^X. Since this set is compact, its interior in 2^X is non-empty. Since it is also hereditary, it is not difficult to see that there is m such that $2^{X \setminus m}$ is contained in it. Thus $X \setminus m \in K$. □

The proof of the theorem is based on the key Lemma 2.6. First, however, we will have to prove some auxiliary results.

Lemma 2.3. *Assume that I is an analytic ideal and J is an ideal such that for some infinite $X \subset \omega$, $J \cap 2^X = [X]^{<\omega}$. Then $J \leqslant_f I$ iff there is $Y \in 2^\omega$ and $h : Y \to \omega$ finite-to-1 and such that $J = \{X\colon h^{-1}(X) \in I\}$.*

Proof. The non-obvious direction is $\Leftarrow$. Let $X_0 \in 2^\omega$ be infinite and such that $J \cap 2^{X_0} = [X_0]^{<\omega}$. Let $Y_0 = (\omega \setminus Y) \cup h^{-1}(X_0)$ and let $Y_1 = Y \setminus h^{-1}(X_0)$. Thus, Y_0, Y_1 partition ω. The ideal $I \cap 2^{Y_0}$ is analytic, thus by Mathias–Jalali-Naini–Talagrand's theorem $[\omega]^{<\omega} \leqslant_f I \cap 2^{Y_0}$, hence it follows that there is a finite-to-1 function $f : Y_0 \to X_0$ such that $f^{-1}(X) \in I$ iff $X \in [X_0]^{<\omega}$. But $[X_0]^{<\omega} = J \cap 2^{X_0}$. Thus $h_1 : \omega \to \omega$ defined by $h_1 | Y_0 = f$ and $h_1 | Y_1 = h$ witnesses $J \leqslant_f I$. □

The following lemma contains the crucial technical dichotomy. It will be used several times in the process of establishing Lemma 2.6.

Lemma 2.4. *Let I be an analytic ideal. Then precisely one of the following two possibilities holds*:

(i) $I_1 \leqslant_f I$, *or*

(ii) *countable unions of small sets are small.*

Proof. $\neg$(ii)$\Rightarrow$(i). For $X \in 2^\omega$, let $\pi_X : 2^\omega \to 2^X$ be given by $\pi_X(Y) = Y \cap X$. Let $A_n \subset 2^\omega$, $n \in \omega$, be small and such that $\bigcup_n A_n$ is not small. Let $X_n \in I$ be such that $\pi_{X_n}(A_n)$ is meager in 2^{X_n}. Then there is a partition $\{F_k^n\colon k \in \omega\}$ of X_n into finite sets and sets $B_k^n \subset F_k^n$ such that for any $X \subset X_n$ if $\exists^\infty k\ X \cap F_k^n = B_k^n$, then $X \notin \pi_{X_n}(A_n)$. Using finiteness of the F_k^n's, we can recursively throw away some of them so that, after a re-enumeration, we get

(i) $F_k^n \cap F_l^m = \emptyset$ if $(n,k) \neq (m,l)$;

(ii) $\bigcup_k F_k^n \in I$ for any n;

(iii) $\forall n \forall X \subset X_n$ if $\exists^\infty k\ X \cap F_k^n = B_k^n$, then $X \notin \pi_{X_n}(A_n)$.

Let $\langle\ ,\rangle : \omega \times \omega \to \omega$ be a bijection. Define

$$Z_{p,q} = \bigcup_{n \leqslant p} F^n_{\langle p,q\rangle}.$$

It follows from (i) that

(iv) $Z_{p,q} \cap Z_{p',q'} = \emptyset$ if $(p,q) \neq (p',q')$.

Let $\bar{X} = \bigcup\{Z_{p,q}\colon (p,q) \in S\}$, for some $S \subset \omega \times \omega$, be given. Assume there is p_0 such that $p \leqslant p_0$ for any $(p,q) \in S$. Then by (ii)

$$\bar{X} \subset \bigcup\{Z_{p,q}\colon\ p \leqslant p_0\} \subset \bigcup_{n \leqslant p_0} \bigcup_k F_k^n \in I.$$

Now, assume that no such p_0 exists, i.e., that there are $(p_l, q_l) \in S$, $l \in \omega$, with $p_l < p_{l+1}$. Suppose towards contradiction that $\bar{X} \in I$. Put $Z = \bigcup_l Z_{p_l,q_l}$. Then $Z \subset \bar{X}$, so $Z \in I$. Since $\bigcup_n A_n$ is not small, $\pi_Z(\bigcup_n A_n)$ is not meager in 2^Z. On the other hand, since $Z = \bigcup_l \bigcup_{n \leqslant p_l} F^n_{\langle p_l,q_l\rangle}$, the set

$$\{X \in 2^Z\colon\ \exists^\infty l \forall n \leqslant p_l\ X \cap F^n_{\langle p_l,q_l\rangle} = B^n_{\langle p_l,q_l\rangle}\}$$

is comeager in 2^Z. It follows that some X^0 from this set is also in $\pi_Z(\bigcup_n A_n)$. Thus, to get a contradiction, it is enough to show that $X^0 \notin \pi_Z(A_n)$ for all $n \in \omega$. Fix n. From some l on $n \leqslant p_l$, so, $\exists^\infty l\ X^0 \cap F^n_{\langle p_l,q_l\rangle} = B^n_{\langle p_l,q_l\rangle}$. Let $X^1 \in A_n$ be such that $\pi_Z(X^1) = X^0$. Then, since $F^n_{\langle p_l,q_l\rangle} \subset Z$, we have $\exists^\infty l\ X^1 \cap F^n_{\langle p_l,q_l\rangle} = B^n_{\langle p_l,q_l\rangle}$. But also $F^n_{\langle p_l,q_l\rangle} \subset X_n$, whence $\exists^\infty l\ \pi_{X_n}(X^1) \cap F^n_{\langle p_l,q_l\rangle} = B^n_{\langle p_l,q_l\rangle}$ which contradicts (iii). It follows that

(v) $\bar{X} \in I$ iff $\exists p_0\ S \subset \{(p,q)\colon\ p \leqslant p_0\}$.

Now, (iv) guarantees that $h\colon \bigcup_{(p,q) \in \omega\times\omega} Z_{p,q} \to \omega \times \omega$ given by $h(n) = (p,q)$ iff $n \in Z_{p,q}$ is well-defined and (v) along with Lemma 2.3 shows that h witnesses $I_1 \leqslant_f I$.

(i) $\Rightarrow \neg$(ii). Let $h : \omega \to \omega \times \omega$ witness $I_1 \leqslant_f I$. Define $A_n = \{X \in I\colon\ \forall k \geqslant n \forall m\ h^{-1}((k,m)) \not\subset X\}$. Then $\bigcup_n A_n = I$, so $\bigcup_n A_n$ is not small. But each A_n is small since $\{X \cap h^{-1}(\{n\} \times \omega)\colon\ X \in A_n\}$ is meager in $2^{h^{-1}(\{n\}\times\omega)}$. (Simply note that for any Y in this set, we have $h^{-1}((n,m)) \not\subset Y$ for all m and each $h^{-1}((n,m))$ is finite.)

Lemma 2.5. *Let I be an analytic ideal. If $I_1 \not\leqslant_f I$, then there are $K_n \in C(I)$, $n \in \omega$, such that for any $K \in C(I)$ there is $n \in \omega$ with $K_n \subset K$.*

Proof. First, we show that $C(I)$ is F_σ in the space of all compact subsets of 2^ω with the Vietoris topology. Let $J = \{K \subset 2^\omega\colon\ K$ compact and small$\}$. Since $I_1 \not\leqslant_f I$, by Lemma 2.4, J is a σ-ideal of compact subsets of 2^ω. Note that

$$K \in J \ \text{ iff } \ \exists X \in 2^\omega (X \in I \text{ and } \{Y \cap X\colon\ Y \in K\} \text{ is meager in } 2^X)$$

gives an analytic definition of J. By Kechris–Louveau–Woodin's theorem [6], J is G_δ. By Lemma 2.2, $C(I) = \{K \subset 2^\omega\colon\ K$ compact, hereditary$\} \setminus J$. Thus, $C(I)$ is F_σ.

Now, to get the conclusion of the lemma, it is enough to prove that if $F \subset C(I)$ is compact, then there is a countable family $\mathscr{C} \subset C(I)$ such that for any $K \in F$, $K' \subset K$ for some $K' \in \mathscr{C}$. For $\alpha \in \omega_1$, define F^α, closed subsets of $C(I)$, and $U^\alpha \subset F^\alpha$ relatively open in F^α by letting

(i) $F^0 = F$;
(ii) $F^\lambda = \bigcap_{\alpha<\lambda} F^\alpha$ if λ is limit;
(iii) U^α be a relatively open, nonempty subset of F^α with $\bigcap U^\alpha \in C(I)$ if such a subset exists and $U^\alpha = \emptyset$ otherwise;
(iv) $F^{\alpha+1} = F^\alpha \setminus U^\alpha$.

There is $\alpha_0 < \omega_1$ such that $F^{\alpha_0} = F^{\alpha_0+1}$. Assume α_0 is the smallest such ordinal, and let $F^{\alpha_0} = F'$. If $F' = \emptyset$, the countable family $\mathscr{C} = \{\bigcap U^\alpha\colon\ \alpha < \alpha_0\}$ is as required. Assume towards contradiction that $F' \neq \emptyset$. Clearly, for any $U \subset F'$ nonempty, relatively open in F', $\bigcap U \notin C(I)$; so, since $\bigcap U$ is also compact and hereditary, by Lemma 2.2, $\bigcap U$ is small. Let us fix a countable topological basis $U_n \subset F'$, $n \in \omega$, for F'. We claim that $\bigcup_n \bigcap U_n$ is not small. This will give a contradiction since then by Lemma 2.4, $I_1 \leqslant_f I$. So, suppose, if we can, that $\bigcup_n \bigcap U_n$ is small. Since $\bigcup_n \bigcap U_n$ is hereditary, this means that we can find $X \in I$ such that for all $m, n \in \omega$, $X \setminus m \notin \bigcap U_n$. Let $F'_m = \{K \in F'\colon\ X \setminus m \in K\}$. Then each F'_m is closed and $\bigcup_m F'_m = F'$. Thus, by the Baire Category Theorem, there are $n, m \in \omega$ with $U_n \subset F'_m$, i.e., $X \setminus m \in \bigcap U_n$, a contradiction. $\Box$

Remark. Lemma 2.5 can also be proved without resorting to the Kechris–Louveau–Woodin theorem. Here is an outline of this argument. Fix $f : \omega^\omega \to I$ a continuous surjection. Using an exhaustion argument and Lemma 2.4, we produce a nonempty tree $T \subseteq \omega^{<\omega}$ such that $\overline{f[P_\sigma]}$ is not small for any $\sigma \in T$ where $P_\sigma = \{x \in \omega^\omega\colon\ \sigma \subseteq x$ and $\forall n\ x|n \in T\}$. Now one can check that for $\{K_n\colon\ n \in \omega\}$ we can take an enumeration of the family of sets of the form $\{s \triangle X\colon\ X \in \overline{f[P_\sigma]}\}$ with $s \in [\omega]^{<\omega}$ and $\sigma \in T$.

Lemma 2.6. *Let I be an analytic ideal. If $I_1 \not\leqslant_f I$, then there exist $K_n \in C(I)$, $n \in \omega$, with the following properties*:

(i) $\forall K \in C(I) \exists n\, K_n \subset K$;

(ii) $\forall n\, \{X \cup Y\colon X, Y \in K_{n+1}\} \subset K_n$.

Proof. For $K \in C(I)$, put $K + K = \{X \cup Y\colon X, Y \in K\}$. Let K_n, $n \in \omega$, be as in Lemma 2.5. Since $C(I)$ is closed under finite intersections, we can assume that $K_{n+1} \subset K_n$, or else we let the nth element of the sequence be equal to $\bigcap_{l \leqslant n} K_l$.

Now, the lemma will be proved if we find a sequence $n_k \in \omega$, $k \in \omega$, such that $n_k < n_{k+1}$ and $K_{n_{k+1}} + K_{n_{k+1}} \subset K_{n_k}$. To this end, it is enough to show that for any $K \in C(I)$ there is $K' \in C(I)$ such that $K' + K' \subset K$. Let

$$A_n = \{(X, Y)\colon (X \cup Y) \setminus n \in K\}.$$

Denote by $(A_n)_X$ the vertical section of A_n at X, that is, $(A_n)_X = \{Y\colon (X, Y) \in A_n\}$. It is easy to check that

(i) A_n is compact;

(ii) $A_n \subset A_{n+1}$;

(iii) $(A_n)_X$ is hereditary for any $X \in 2^\omega$;

(iv) if $X \subset Y \in 2^\omega$, then $(A_n)_Y \subset (A_n)_X$.

Let

$$B_{n,m} = \{X \in 2^\omega\colon K_m \subset (A_n)_X\}.$$

Note, that if $X \in I$, then $\bigcup_n (A_n)_X \supset I$. Since $I_1 \not\leqslant_f I$, it follows, by Lemma 2.4, that for some n, $(A_n)_X$ is big. Thus, by (i), (iii), and Lemma 2.2, $(A_n)_X \in C(I)$, so, for some m, $K_m \subset (A_n)_X$. This means that $X \in B_{n,m}$. It follows that $\bigcup_{n,m} B_{n,m} \supset I$. Since $I_1 \not\leqslant_f I$, B_{n_0,m_0} is big for some $n_0, m_0 \in \omega$. Because of (iv), B_{n_0,m_0} is hereditary and by (i) it is compact, so $B_{n_0,m_0} \in C(I)$. Thus, if we let $L = B_{n_0,m_0} \cap K_{m_0}$, then $L \in C(I)$ and, moreover, since $B_{n_0,m_0} \times K_{m_0} \subset A_{n_0}$, $L \times L \subset A_{n_0}$. Put

$$K' = \{X \setminus n_0\colon X \in L\}.$$

Then $K' \in C(I)$, and by the definition of A_{n_0}, $K' + K' \subset K$. □

The next lemma says that if I is reasonable, we can recover I from $C(I)$. It is based on Mathias–Jalali-Naini–Talagrand's theorem.

Lemma 2.7. *Let I be an analytic ideal. Then $X \in I$ if, and only if, for any $K \in C(I)$ there is $n \in \omega$ with $X \setminus n \in K$.*

Proof. The implication $\Leftarrow$ is obvious. To get the other one, let $X \in 2^\omega \setminus I$. We have to show that there exists $K \in C(I)$ such that for no m, $X \setminus m \in K$. Consider $I \cap 2^X$. It is an analytic ideal on X containing all singletons and not containing X. Thus by Mathias–Jalali-Naini–Talagrand's theorem there exists a partition of X into finite sets $\{F_n\colon n \in$

$\omega\}$ such that for any $Y \in 2^\omega$ if $\exists^\infty n\, F_n \subset Y$ then $Y \notin I$. Let $K = \{Y \in 2^\omega\colon \forall n\, F_n \not\subset Y\}$. This K works. □

Proof of Theorem 2.1. First we show that $I = \mathrm{Exh}(\phi)$ for a lsc submeasure ϕ implies that $I_1 \not\leqslant_f I$. If we had $I_1 \leqslant_f I$, where $I = \mathrm{Exh}(\phi)$ for a lsc submeasure ϕ, with a finite-to-1 $h : \omega \to \omega \times \omega$ witnessing it, then $h^{-1}(\{n\} \times \omega) \in I$. Thus, for each n there would exist $k_n \in \omega$ with $\phi(Y_n) < 2^{-n}$ where $Y_n = h^{-1}(\{n\} \times \omega \setminus \{n\} \times k_n)$. Using this and the fact that $Y_n \in I$, we would get $\bigcup_n Y_n \in I$ from which it would follow that $\bigcup_n(\{n\} \times \omega \setminus \{n\} \times k_n) \in I_1$, contradiction.

Now assume I is analytic and $I_1 \not\leqslant_f I$. We will produce a lsc finite submeasure ϕ with $I = \mathrm{Exh}(\phi)$. Let (K_n) be as in Lemma 2.6. We can assume that $K_0 = 2^\omega$ and (by taking every other element of the original sequence (K_n)) that $\{X \cup Y \cup Z\colon X, Y, Z \in K_{n+1}\} \subset K_n$. For $X \in [\omega]^{<\omega}$, put

$$\psi_1(X) = \inf\{2^{-n}\colon X \in K_n\} \quad \text{and}$$

$$\psi_2(X) = \inf\left\{\sum_{i=0}^{m-1} \psi_1(X_i)\colon X \subset \bigcup_{i<m} X_i,\ X_i \in [\omega]^{<\omega},\ m > 1\right\}.$$

These definitions and the proof of the following claim are motivated by the standard proof of the Birkhoff–Kakutani metrization theorem for groups with a countable basis at identity.

Claim.
(i) *For any $X, Y \in [\omega]^{<\omega}$, $\psi_2(X) \leqslant \psi_2(Y)$ if $X \subset Y$, and $\psi_2(X \cup Y) \leqslant \psi_2(X) + \psi_2(Y)$.*
(ii) $\psi_1/2 \leqslant \psi_2 \leqslant \psi_1 \leqslant 1$.

Proof of the claim. Checking (i) and $\psi_2 \leqslant \psi_1 \leqslant 1$ in (ii) is straightforward. So it remains to see that $\psi_1(X) \leqslant 2\psi_2(X)$ for any $X \in [\omega]^{<\omega}$. Since each K_n is hereditary, we have that $\psi_1(X) \leqslant \psi_1(Y)$ whenever $X \subset Y \in [\omega]^{<\omega}$. Thus, it is enough to prove that $\psi_1(\bigcup_{i<m} X_i) \leqslant 2\sum_{i<m} \psi_1(X_i)$ for $X_i \in [\omega]^{<\omega}$. We proceed by induction on m. For $m = 1$ it is obvious. So suppose $m \geqslant 2$. We can assume that $\psi_1(X_0)$ is smallest among $\psi_1(X_i)$, $0 \leqslant i < m$. Let $r \in \omega$ be largest such that $2\sum_{i<r} \psi_1(X_i) \leqslant \sum_{i<m} \psi_1(X_i)$. Clearly $0 < r < m$. By our inductive assumption

$$\psi_1\left(\bigcup_{i<r} X_i\right) \leqslant 2\sum_{i<r} \psi_1(X_i) \leqslant \sum_{i<m} \psi_1(X_i),$$

$$\psi_1\left(\bigcup_{r+1 \leqslant i<m} X_i\right) \leqslant 2\sum_{r+1 \leqslant i<m} \psi_1(X_i) < \sum_{i<m} \psi_1(X_i)$$

and, obviously, $\psi_1(X_r) \leqslant \sum_{i<m} \psi_1(X_i)$. From our choice of the K_n's and the definition of ψ_1, it follows that if $\psi_1(X), \psi_1(Y), \psi_1(Z) \leqslant \varepsilon$, then $\psi_1(X \cup Y \cup Z) \leqslant 2\varepsilon$. Thus, we obtain $\psi_1(\bigcup_{i<m} X_i) \leqslant 2\sum_{i<m} \psi_1(X_i)$. This completes the proof of the claim.

Define for $X \in 2^\omega$

$$\phi(X) = \sup_{l \in \omega} \psi_2(X \cap l).$$

From Claim (i), it follows easily that ϕ is a submeasure on ω. Since for all l the function $X \to \phi_2(X \cap l)$ is continuous, ϕ is lower semicontinuous. By Claim (ii), $\phi \leqslant 1$, so ϕ is finite. Thus, it is enough to see that $I = \mathrm{Exh}(\phi)$. Let $X \in I$ and let $\varepsilon > 0$. Find n such that $2^{-n} < \varepsilon$. For m large enough $X \setminus m \in K_n$. Thus, since K_n is hereditary, $\psi_1(Y) \leqslant 2^{-n}$ for any finite $Y \subset X \setminus m$. So, by Claim (ii), $\psi_2(Y) \leqslant 2^{-n}$ for any such Y. Thus, by the very definition of ϕ, $\phi(X \setminus m) \leqslant 2^{-n} < \varepsilon$. This proves that $X \in \mathrm{Exh}(\phi)$. Now, let $X \in \mathrm{Exh}(\phi)$. To show that $X \in I$, it is enough, by Lemma 2.6(i) and Lemma 2.7, to see that for any n there is an m with $X \setminus m \in K_n$. Fix n. Let m be such that $\phi(X \setminus m) \leqslant 2^{-n-1}$. Then for all l, $\psi_2((X \setminus m) \cap l) \leqslant 2^{-n-1}$, whence by Claim (ii), $\psi_1((X \setminus m) \cap l) \leqslant 2^{-n}$. It follows that for all l, $(X \setminus m) \cap l \in K_n$. But K_n is compact and $(X \setminus m) \cap l \to X \setminus m$ as $l \to \infty$. So $X \setminus m \in K_n$. □

3. Polishable ideals

We prove a theorem which characterizes Polishable ideals and analytic P-ideals as those of the form $\mathrm{Exh}(\phi)$ for a lower semicontinuous submeasure ϕ on ω. Note that, by Theorem 2.1, condition (iii) in the theorem below, and so (i) and (ii) as well, is equivalent to $I_1 \not\leqslant_f I$.

Theorem 3.1. *Let I be an ideal on ω. Then the following conditions are equivalent:*
(i) *I is Polishable.*
(ii) *I is an analytic P-ideal.*
(iii) *There is a finite, lower semicontinuous submeasure ϕ on ω with $I = \mathrm{Exh}(\phi)$.*

Proof. We show that (iii) implies both (i) and (ii) and that (i) and (ii) imply that $I_1 \not\leqslant_f I$ which will complete the proof by Theorem 2.1.

(iii)⇒(i) Let ϕ be a lsc submeasure with $I = \mathrm{Exh}(\phi)$. We can assume that $\phi(\{n\}) > 0$ since if this is not the case, we replace ϕ by $\phi'(X) = \phi(X) + \sum_{n \in X} 2^{-n}$ which satisfies the above condition and fulfills $I = \mathrm{Exh}(\phi')$ as well.

Define

$$d(X, Y) = \phi(X \triangle Y) \quad \text{for } X, Y \in 2^\omega.$$

It is easy to check that d is an invariant metric on 2^ω. (E.g., $d(X, Y) = \phi(X \triangle Y) = \phi\big((X \triangle Z) \triangle (Z \triangle Y)\big) \leqslant \phi\big((X \triangle Z) \cup (Z \triangle Y)\big) \leqslant \phi(X \triangle Z) + \phi(Z \triangle Y) = d(X, Z) + d(Z, Y)$.) To check that d is complete, let (X_n) be d-Cauchy. By taking subsequences, we can assume that $d(X_n, X_{n+1}) < 1/2^n$ and $X_n \to X$ for some $X \in 2^\omega$. We have to show that

$d(X_n, X) \to 0$. But by lower semicontinuity of ϕ, we get

$$d(X_n, X) = \phi(X_n \triangle X) \leqslant \phi\left(\bigcup_{k \geqslant n} X_k \triangle X_{k+1}\right) \leqslant \sum_{k \geqslant n} \phi(X_k \triangle X_{k+1})$$
$$= \sum_{k \geqslant n} d(X_k, X_{k+1}) \to 0.$$

We now show that the restriction of d to $\mathrm{Exh}(\phi)$ is Polish. To see that d is separable on $\mathrm{Exh}(\phi)$, note that $[\omega]^{<\omega}$ is included in $\mathrm{Exh}(\phi)$ and is d-dense in it. Indeed, if $X \in \mathrm{Exh}(\phi)$, then $d(X \cap m, X) = \phi((X \cap m) \triangle X) = \phi(X \setminus m) \to 0$. We will be done if we show that $\mathrm{Exh}(\phi)$ is d-closed in 2^ω. Let $X_n \in \mathrm{Exh}(\phi)$, $n \in \omega$, and $d(X_n, X) \to 0$, i.e., $\phi(X_n \triangle X) \to 0$. Now,

$$\phi(X \setminus m) \leqslant \phi\big((X_n \cup (X_n \triangle X)) \setminus m\big) \leqslant \phi(X_n \setminus m) + \phi(X_n \triangle X).$$

So, $\limsup_m \phi(X \setminus m) \leqslant \phi(X_n \triangle X)$ for all n whence $\phi(X \setminus m) \to 0$.

(iii)⇒(ii) Assume $I = \mathrm{Exh}(\phi)$ for a finite, lower semicontinuous submeasure. First note that $\mathrm{Exh}(\phi)$ is $F_{\sigma\delta}$ so analytic. Indeed,

$$X \in \mathrm{Exh}(\phi) \quad \text{iff } \forall n \exists m\ \phi(X \setminus m) \leqslant 1/(n+1),$$

and since ϕ is lsc, $\{X\colon \phi(X \setminus m) \leqslant 1/(n+1)\}$ is closed for any fixed n and m. Now, let $X_n \in I$, $n \in \omega$. Define recursively $k_n \in \omega$ so that $\sum_{i \leqslant n} \phi(X_i \setminus k_n) < 2^{-n}$. Let $X = \bigcup_n (X_n \setminus k_n)$. Clearly $X_n \setminus X$ is finite for all n. If $m \geqslant k_m$, then by lower semicontinuity of ϕ we get

$$\phi(X \setminus m) \leqslant \phi\left(\bigcup_{i \leqslant n}(X_i \setminus k_n) \cup \bigcup_{i > n}(X_i \setminus k_i)\right)$$
$$\leqslant \sum_{i \leqslant n} \phi(X_i \setminus k_n) + \sum_{i > n} \phi(X_i \setminus k_i) \leqslant 2^{-n} + \sum_{i > n} 2^{-i} = 2^{-n+1}.$$

So, $\phi(X \setminus m) \to 0$, i.e., $X \in \mathrm{Exh}(\phi) = I$. This shows that I is a P-ideal.

(i)⇒ $I_1 \not\leqslant_f I$. Let I be a Polishable ideal. Assume towards a contradiction that $I_1 \leqslant_f I$. Let $h\colon \omega \to \omega \times \omega$ witness it. Note that $\{h^{-1}(X)\colon X \subset \omega \times \omega\}$ is closed in 2^ω. Since, by [4, Theorem 9.10], the Polish group topology τ on I is stronger than the topology inherited from 2^ω, it follows that $I \cap \{h^{-1}(X)\colon X \subset \omega \times \omega\}$ is τ-closed in I. But $I \cap \{h^{-1}(X)\colon X \subset \omega \times \omega\} = \{h^{-1}(X)\colon X \in I_1\}$. Since $(\omega \times \omega \setminus h[\omega]) \in I_1$, we can, and we do, modify h to make it onto. Now, the function $H\colon I_1 \to (I, \tau)$ defined by $H(X) = h^{-1}(X)$ is 1-to-1. Note that H is a group homeomorphism and it is also Borel since the Borel sets generated by τ are the same as the Borel sets in I inherited from 2^ω. Thus, I_1 Borel embeds as a closed subgroup of a Polish group. It follows that I_1 is Polishable. Let τ_1 be a Polish group topology on I_1 witnessing it. Note that for each $n \in \omega$, $n \times \omega$ is a Borel with respect to τ_1 subgroup of I_1. (Actually it is closed.) But $\bigcup_n n \times \omega = I_1$. So, by the Baire Category Theorem, for some $n_0 \in \omega$, $n_0 \times \omega$ is non-meager with respect to τ_1 and therefore it is τ_1-open. Since τ_1 is separable, it follows that $I_1/(n_0 \times \omega)$ is countable which gives a contradiction since as is easy to see $|I_1/(n_0 \times \omega)| = 2^{\aleph_0}$.

(ii)$\Rightarrow I_1 \not\leqslant_f I$. Let I be a P-ideal, and assume towards a contradiction that $I_1 \leqslant_f I$. Let $h : \omega \to \omega \times \omega$ witness it. Then $h^{-1}(\{n\} \times \omega) \in I$ for any n. Let $X \in I$ be such that $h^{-1}(\{n\} \times \omega) \setminus X$ is finite for each n. Since $h^{-1}((n,m))$ is finite for all $(n,m) \in \omega \times \omega$, for each n we can find m_n such that $h^{-1}((n,m_n)) \subset X$. It follows that $h^{-1}(\{(n,m_n) \colon n \in \omega\}) \in I$, so $\{(n,m_n) \colon n \in \omega\} \in I_1$, a contradiction. □

Remark. We would like to present here a direct argument, i.e., not using Theorem 2.1, that if I is Polishable, then it is of the form $I = \mathrm{Exh}(\phi)$ for a finite, lower semicontinuous submeasure on ω. Before starting the proof, we state the following known fact which will be used twice later on. Let G be a group with a Polish group topology on it, and let H be a subgroup of G such that the topology on it inherited from G is Polish. Then H is a closed subgroup of G. (Verification: Note first that the closure of H, $\overline{H}$, is a subgroup of G. Since H is Polish with the inherited topology, it is a G_δ subset of G, so it is a dense G_δ subgroup of $\overline{H}$. Thus each coset of H which lies inside $\overline{H}$ is a dense G_δ subset of $\overline{H}$, so it must intersect H and therefore is equal to H. It follows that $\overline{H} = H$.)

Now assume I is Polishable, and so it carries a Polish group topology which generates the Borel structure on I. This topology is induced by a metric d which is left-invariant (see [2, Theorem 8.3]). Since I is abelian, d is also right-invariant. It is a known fact, discovered by Christensen, that such a d is complete. (Just consider the completion G of I with respect to d. It is easy to check, using the invariance of d, that G is a group with a Polish group topology. By the fact stated at the beginning of this remark, I is closed in G, but since it is dense as well, we have $G = I$.) Since, by [4, Theorem 9.10], the topology induced by d is stronger than the topology inherited by I from the inclusion $I \subset 2^\omega$, we additionally have that $d(X_n, X) \to 0$ implies $X_n \to X$ for $X_n, X \in I$. Let ϕ be defined as follows:

$$\phi(X) = \sup\{d(Y,\emptyset) \colon Y \in I, Y \subset X\}.$$

First, we check that ϕ is a submeasure. The only property that needs justification is $\phi(X \cup Y) \leqslant \phi(X) + \phi(Y)$. Let $\varepsilon > 0$ be given and let $Z \subset X \cup Y$ be such that $\phi(X \cup Y) \leqslant d(Z,\emptyset) + \varepsilon$. Then using the invariance of d, we get

$$\begin{aligned} d(Z,\emptyset) = d\big((Z \cap (X \setminus Y)) \triangle (Z \cap Y), \emptyset\big) &\leqslant d(Z \cap (X \setminus Y), \emptyset) \\ &+ d(Z \cap Y, \emptyset) \leqslant \phi(X) + \phi(Y). \end{aligned}$$

Thus $\phi(X \cup Y) \leqslant \phi(X) + \phi(Y) + \varepsilon$ for any $\varepsilon > 0$.

Claim. *Let $Y \in I$, $X_n \subset Y$, and $X_n \to X$. Then $d(X_n, X) \to 0$.*

Proof of the claim. 2^Y is a compact group with the topology inherited from 2^ω. It is also a d-closed subgroup of I. Thus, by [4, Theorem 9.10], $\mathrm{id} : (2^Y, d) \to 2^Y$ is a homomorphism.

Now, we show that ϕ is lower semicontinuous. Given X, let $\varepsilon > 0$ and let $Y \subset X$ be such that $Y \in I$ and $d(Y, \emptyset) > \phi(X) - \varepsilon$. By Claim, $d(Y \cap m, \emptyset) \to d(Y, \emptyset)$. Thus,

$$\phi(X) \geqslant \limsup_m \phi(X \cap m) \geqslant \liminf_m \phi(X \cap m) \geqslant d(Y, \emptyset) > \phi(X) - \varepsilon,$$

so $\phi(X \cap m) \to \phi(X)$.

We will be done if we show that $I = \mathrm{Exh}(\phi)$. To see $I \subset \mathrm{Exh}(\phi)$, let $X \in I$. Choose $X_m \subset X \setminus m$ so that $X_m \in I$ and $d(X_m, \emptyset) \geqslant \phi(X \setminus m)/2$ or $d(X_m, \emptyset) \geqslant 1$ if $\phi(X \setminus m) = \infty$. (This second possibility cannot really occur.) Then $X_m \to \emptyset$, so by Claim $d(X_m, \emptyset) \to 0$, whence $\phi(X \setminus m) \to 0$, i.e., $X \in \mathrm{Exh}(\phi)$. To show that actually $I = \mathrm{Exh}(\phi)$, define a metric d_1 from ϕ as in the proof of $\Leftarrow$, i.e., $d_1(X, Y) = \phi(X \triangle Y)$. The restriction of d_1 to I, $d_1|I$, is an invariant metric on I which generates the standard Borel structure on I inherited from 2^{ω}. It follows that $\mathrm{id} : (I, d) \to (I, d_1|I)$ is a Borel homomorphism. Since d is Polish, this homomorphism must be continuous by [4, Theorem 9.10]. Note, however, that if $d_1(X_n, \emptyset) \to 0$ for $X_n \in I$, then $\phi(X_n) \to 0$. But $d(X_n, \emptyset) \leqslant \phi(X_n)$ whence $d(X_n, \emptyset) \to 0$. Thus, since both d and $d_1|I$ are invariant, $\mathrm{id} : (I, d_1|I) \to (I, d)$ is also continuous. It follows that d and $d_1|I$ induce the same topology on I. Since d is Polish, so is $d_1|I$. Therefore I is a Polish subgroup of $\mathrm{Exh}(\phi)$ with the topology induced by d_1. Thus, I is d_1-closed in $\mathrm{Exh}(\phi)$ by the fact stated at the beginning of this remark. But $[\omega]^{<\omega} \subset I$, and $[\omega]^{<\omega}$ is d_1-dense in $\mathrm{Exh}(\phi)$. Therefore I is d_1-dense in $\mathrm{Exh}(\phi)$, whence $I = \mathrm{Exh}(\phi)$.

Next we characterize those ideals I which carry a locally compact Polish group topology that generates the same Borel structure as the one inherited from the inclusion I. We call such ideals *locally compact Polishable.*

Lemma 3.2. *Let $I = \mathrm{Exh}(\phi)$ for a lower semicontinuous submeasure ϕ. Given $\varepsilon > 0$, let G be the subgroup of I generated by $\{X\colon \phi(X) < \varepsilon\}$. Then there exists $B \subset \omega$ such that $G = I \cap 2^B$ and $I \cap 2^{\omega \setminus B} = [\omega \setminus B]^{<\omega}$.*

Proof. Assume that $\phi(\{n\}) > 0$ for each $n \in \omega$. (We will show that this assumption is harmless at the end of the proof.) Let $B = \{n \in \omega\colon \phi(\{n\}) < \varepsilon\}$. Since if X is infinite and $\phi(\{n\}) \geqslant \varepsilon$ for all $n \in X$, then $X \notin \mathrm{Exh}(\phi)$, we have $I \cap 2^{\omega \setminus B} = [\omega \setminus B]^{<\omega}$. Also, obviously $G \subset I \cap 2^B$. Consider the Polish group topology on I that agrees with the Borel structure on I and which exists by Theorem 3.1 and is induced by the Polish metric $d(X, Y) = \phi(X \triangle Y)$. (Here we use the fact that ϕ is positive on singletons.) By definition of d, $\{X\colon \phi(X) < \varepsilon\}$ is open in this topology. Now note that G, being generated by an open set, must be open and hence closed in the Polish group topology on I. If $X \in I \cap 2^B$, then, since all finite subsets of B are in G, $X \cap m \in G$ for all m. Moreover, $d(X, X \cap m) = \phi(X \triangle (X \cap m)) = \phi(X \setminus m) \to 0$. Thus, since G is closed in the Polish topology, $X \in G$. This shows that $I \cap 2^B \subset G$.

If ϕ is not positive on all singletons, let $A = \{n \in \omega\colon \phi(\{n\}) = 0\}$. Restrict ϕ to subsets of $\omega \setminus A$ and apply the above argument to this restriction producing some $B \subset \omega \setminus A$. Then $A \cup B$ is the desired set.

Corollary 3.3. *Let I be an ideal on ω. Then I is locally compact Polishable iff there is $A \subset \omega$ such that $I = \{X\colon X \cap A$ is finite$\}$.*

Proof. $\Leftarrow$ is easy, since $I = 2^{\omega \setminus A} \times [A]^{<\omega}$, and we can put the compact product topology on $2^{\omega\setminus A}$ and the discrete topology on $[A]^{<\omega}$.

($\Rightarrow$) Let τ be the locally compact Polish group topology on I. Let $U \subset I$ be a τ-open neighborhood of $\emptyset$ with $\overline{U}^{\tau}$ compact. Then τ and the topology inherited from 2^{ω} coincide on $\overline{U}^{\tau}$, hence $\overline{U}^{\tau}$ is zero-dimensional. Thus (I, τ) is a zero-dimensional locally compact group. From theory of locally compact groups (see [2, Theorem 7.7]), it follows that there exists $H \subset I$ a τ-open, compact subgroup of I. Let ϕ be a lsc submeasure with $I = \mathrm{Exh}(\phi)$. There is $\varepsilon > 0$ such that $V = \{X\colon \phi(X) < \varepsilon\} \subset H$, and let G be the subgroup of I generated by V. Then G is a τ-open, so τ-closed, subgroup of I with $G \subset H$. Thus G is compact. By Lemma 3.2, $G = I \cap 2^{B}$ and $I \cap 2^{\omega\setminus B} = [\omega \setminus B]^{<\omega}$. Since G is compact, $G = 2^{B}$. So, we can put $A = \omega \setminus B$. $\square$

Let Fin^{ω} be the ideal on $\omega \times \omega$ defined by $X \in \mathrm{Fin}^{\omega}$ iff for each n, $\{k\colon (n,k) \in X\}$ is finite.

Theorem 3.4. *Let I be a Polishable ideal on ω. Then precisely one of the following possibilities holds.*

(i) $\mathrm{Fin}^{\omega} \leqslant_{f} I$.

(ii) *There exists a lower semicontinuous, exhaustive submeasure ψ on ω such that* $I = \mathrm{Fin}(\psi) (= \mathrm{Exh}(\psi))$.

Proof. Since I is Polishable, there is a lsc submeasure ϕ with $I = \mathrm{Exh}(\phi)$.

Case 1. $\forall \varepsilon > 0 \exists X$ $\phi(X) < \varepsilon$ and $\phi(X \setminus m) \not\to 0$.

If $\phi(X \setminus m) \not\to 0$, then $\inf_m \phi(X \setminus m) > 0$. This allows us to pick recursively a sequence (X_k) so that $\phi(X_0) < \infty$ and, for any $n \in \omega$, $\inf_m \phi(X_n \setminus m) > \sum_{k>n} \phi(X_k)$. Put $Y_n = X_n \setminus \bigcup_{k>n} X_k$. Note that $Y_n \cap Y_{n'} = \emptyset$ if $n \neq n'$, and

$$\inf_m \phi(Y_n \setminus m) \geqslant \inf_m \left(\phi(X_n \setminus m) - \phi(\bigcup_{k>n} X_k) \right) \geqslant \inf_m \phi(X_n \setminus m) - \sum_{k>n} \phi(X_k) > 0.$$

Fix n. By lower semicontinuity of ϕ, we can find $\delta_n > 0$ and recursively choose an increasing sequence (m_k) so that $\phi(Y_n \cap [m_k, m_{k+1})) \geqslant \delta_n$. Put $F_k^n = Y_n \cap [m_k, m_{k+1})$. The F_k^n's have the following properties.

(i) F_k^n is finite;

(ii) $\sum_n \phi(\bigcup_k F_k^n) < \infty$;

(iii) $\phi(F_k^n) \geqslant \delta_n$.

Let $X = \bigcup_{(n,k) \in S} F_k^n$ for some $S \subset \omega \times \omega$. Assume $\{k\colon (n,k) \in S\}$ is finite for all n. Then, by lower semicontinuity of ϕ, $\phi(X \setminus m) \leqslant \sum_{n \geqslant n_m} \phi(\bigcup_k F_k^n)$ where $n_m = 1 + \max\{n\colon \forall k\ (n,k) \in S \Rightarrow F_k^n \subset m\}$. By (i), $n_m \to \infty$; whence, by (ii), $\phi(X \setminus m) \to 0$, so $X \in \mathrm{Exh}(\phi) = I$. On the other hand, if $\{k\colon (n_0, k) \in S\}$ is infinite for some n_0, then for each m, $F_k^{n_0} \subset X \setminus m$ for some k. Thus for any m, $\phi(X \setminus m) \geqslant \phi(F_k^{n_0}) \geqslant \delta_{n_0}$, so $X \notin I$. It follows that $X \in I$ iff $S \in \mathrm{Fin}^\omega$. Define $h : \bigcup_{k,n} F_k^n \to \omega \times \omega$ by $h(i) = (n,k)$ iff $i \in F_k^n$. Then $Y \in \mathrm{Fin}^\omega$ iff $h^{-1}(Y) \in I$ for $Y \subset \omega \times \omega$. By Lemma 2.3, $\mathrm{Fin}^\omega \leqslant_f I$.

Case 2. $\exists \varepsilon > 0 \forall X \; \phi(X) < \varepsilon \Rightarrow \phi(X \setminus m) \to 0$.

Let G be the subgroup of I generated by $\{X\colon \phi(X) < \varepsilon/2\}$. Then by Lemma 3.2, $G = I \cap 2^B$ and $I \cap 2^{\omega \setminus B} = [\omega \setminus B]^{<\omega}$ for some $B \subset \omega$. Let

$$K = \{X\colon X \subset B \text{ and } \phi(X) \leqslant \varepsilon/2\} \cup \bigcup_{n \notin B} \{X \cup \{n\}\colon X \subset B \text{ and } \phi(X) \leqslant \varepsilon/2\}.$$

By the case assumption $K \subset I$, and by lower semicontinuity of ϕ, K is compact. Moreover, K is hereditary and the subgroup of I generated by K is I itself. Let $K_0 = K$, $K_n = \{X\colon \phi(X) \leqslant 3^{-k}\varepsilon/2\}$, and $K_{-n} = \{\bigcup_{i=1}^{3^n} X_i\colon X_i \in K,\ i = 1,\ldots,3^n\}$, for $n \in \omega$, $n \geqslant 1$.

Now, define ψ_1 and ψ_2 by formulas as in the proof of Theorem 2.1 and let $\psi(X) = \sup\{\psi_2(Y)\colon Y \subset X,\ Y \in [\omega]^{<\omega}\}$. Again as in the proof of Theorem 2.1 it turns out that ψ is a lsc submeasure and that $I \subset \mathrm{Exh}(\psi) \subset \mathrm{Fin}(\psi)$. We want to show that $\mathrm{Fin}(\psi) \subset I$. Assume $\psi(X) < \infty$. Then for some M, $\psi_2(Y) < M$ for any finite $Y \subset X$. Thus $\psi_1(Y) < 2M$ for any such Y. So, we can find an n such that $[X]^{<\omega} \subset K_n$. But K_n is compact and $K_n \subset I$; so the closure of $[X]^{<\omega}$, which is 2^X, is contained in I, that is, $X \in I$. □

4. Applications to equivalence relations and partial orders

Each ideal I induces an equivalence relation E_I on 2^ω defined as follows:

$$XE_IY \Leftrightarrow X \triangle Y \in I.$$

It is a particular case of the following situation. Let a group G act on a set X. Define the orbit equivalence relation E_G by declaring two points $x, y \in X$ to be E_G equivalent if, for some $g \in G$, $gx = y$. In the above particular situation I and 2^ω are regarded as groups with the symmetric difference as the group operation and I acts on 2^ω by translations. The following equivalence relations play an important role in the study of Borel equivalence relations. I list them below along with the symbols used customarily to denote them. By Fin, we denote the ideal of finite subsets of ω and the ideals I_1 and Fin^ω were defined in the remark preceding Theorems 2.1 and 3.4, respectively.

$$E_0 = E_{\mathrm{Fin}}, \qquad E_1 = E_{I_1}, \qquad E_0^\omega = E_{\mathrm{Fin}^\omega}.$$

If E and F are equivalence relations defined on Polish spaces X and Y, then we say the E is *continuously reducible* to F, $E \sqsubseteq_c F$, if there exists a continuous 1-to-1 function $f : X \to Y$ with xEy iff $f(x)Ff(y)$. We write $E \leqslant_B F$ if there exists a Borel function (not necessarily 1-to-1) $f : X \to Y$ such that xEy iff $f(x)Ff(y)$ for $x, y \in X$. It was proved by Kechris and Louveau in [5] that E_1 is not continuously reducible to any orbit equivalence relation induced by a Borel action of a Polish group on a Polish space. They considered the problem if and to what extent the reverse implication might be true. In particular, they asked if for a Borel ideal I the fact that E_1 does not continuously reduce to E_I implies that I is Polishable. Corollary 4.1(i) answers this question in the affirmative. Corollary 4.1(ii) gives a dichotomy for equivalence relations induced by Polishable ideals, and Corollary 4.1(iii) is related to some results of Louveau and Velickovic [9] and Mazur [11] as explained in [14].

Corollary 4.1. *Let I be an analytic ideal:*
(i) *either $E_1 \sqsubseteq_c E_I$ or I is Polishable;*
(ii) *if I is Polishable, then either $E_0^{\omega} \sqsubseteq_c E_I$ or I is F_σ;*
(iii) *$E_I \leqslant_B E$ for a Borel equivalence relation with countable equivalence classes iff $I = \{X \in 2^{\omega}\colon X \cap A$ is finite$\}$ for some $A \in 2^{\omega}$.*

Proof. (i) and (ii) follow immediately from Theorems 2.1 and 3.4 once we notice that $I_1 \leqslant_f J$, for an ideal J, implies $E_1 \sqsubseteq_c E_J$. For if $h : \omega \to \omega \times \omega$ witnesses $I_1 \leqslant_f J$, then $\omega \times \omega \setminus h[\omega \times \omega] \in I_1$, and we can modify h to make it onto. Then $g(X) = h^{-1}(X)$, $g : 2^{\omega\times\omega} \to 2^{\omega}$, witnesses $E_1 \sqsubseteq_c E_J$.

The implication $\Leftarrow$ in (iii) is obvious since the function $X \to X \cap A$ witnesses $E_I \leqslant_B E$ where E is the equivalence relation defined on 2^A by XEY iff $X \triangle Y$ is finite for $X, Y \in 2^A$ and E is a Borel equivalence relation with countable equivalence classes. On the other hand, as pointed out by Kechris if $E_I \leqslant_B E$ for E as in (iii), then by [3, Theorem 1.6 (i)$\Leftrightarrow$(v)] I is locally compact Polishable. Thus, by Corollary 3.3 I is as required. $\square$

Following Todorcevic [16], we consider $(I, \subset)$ as a partial order. Let $\leqslant$ be the partial order on ω^{ω} defined by

$$x \leqslant y \quad \text{iff } \forall n\, x(n) \leqslant y(n).$$

The following result was discovered by Todorcevic [16]; it can also be derived from Theorem 3.1.

Corollary 4.2 (Todorcevic [16]). *Let I be an analytic P-ideal which is not of the form $I = \{X\colon X \cap A$ is finite$\}$ for some $A \in 2^{\omega}$. Then $(I, \subset)$ can be mapped monotonically in a Borel fashion onto a cofinal subset of $(\omega^{\omega}, \leqslant)$.*

Proof. By Theorem 3.4, we have two cases.

Case 1. $\mathrm{Fin}^{\omega} \leqslant_f I$

Let $\phi : \omega \to \omega \times \omega$ be a function witnessing $\mathrm{Fin}^{\omega} \leqslant_f I$. Define $g : I \to \omega^{\omega}$ by letting

$$g(X)(n) = \max\{k \in \omega\colon\ \phi^{-1}((n,k)) \subset X\},$$

where we let $\max \emptyset = 0$. It is easy to check that g is Borel, monotonic as a mapping from $(I, \subset)$ to $(\omega^{\omega}, \leqslant)$, and onto ω^{ω}.

Case 2. $I = \mathrm{Fin}(\phi)$ for some lsc exhaustive submeasure ϕ
Define for $X \in I$,

$$g(X)(n) = \min\{k\colon \phi(X \setminus k) < 2^{-n}\}.$$

Again it is clear that g is Borel and monotonic. So, it remains to show that the image of I under g is cofinal in ω^ω. Let $f \in \omega^\omega$. For any $m, n \in \omega$ there exists $Y \in 2^\omega$ such that $m \leqslant \min Y$ and $2^{-n} \leqslant \phi(Y) < 2^{-n-1}$. Indeed, let $X_0 = \{n \in \omega\colon \phi(\{n\}) < 2^{-n}\}$. Then $\phi(X_0) = \infty$ for otherwise we would have that $I = \{X \in 2^\omega\colon X \cap (\omega \setminus X_0)$ is finite$\}$. Thus also $\phi(X_0 \setminus m) = \infty$. Now let $m' \in \omega$ be smallest such that $\phi((X_0 \setminus m) \cap m') \geqslant 2^{-n}$. Then $Y = (X_0 \setminus m) \cap m'$ works. Thus, we can find Y_n such that $f(n) < \min Y_n$ and $2^{-n} \leqslant \phi(Y_n) < 2^{-n-1}$. Then $\bigcup_n Y_n \in \mathrm{Fin}(\phi) = I$ and $g(\bigcup_n Y_n) \geqslant f$.

5. An application to σ-ideals of Borel sets

In the present section, we address the problem of characterizing ideals of μ-zero sets for Maharam submeasures μ defined on Borel subsets of 2^ω. Kunen [8] formulated certain abstract conditions for σ-ideals of subsets of 2^ω and asked if these conditions characterize the σ-ideals of Lebesgue measure zero sets, of meager sets, and of sets which are meager and have Lebesgue measure zero. Recently, this question was answered in the negative by Roslanowski and Shelah [13]. It turns out however that a slightly different characterization of meager sets is possible and was established by Kechris and the author in [7]. Below we give a characterization of μ-zero sets for Maharam submeasures μ. This characterization is analogous to a large degree to the one from [7].

A family $\mathscr{I}$ of Borel subsets of a Polish space X is called *G_δ-supported* if for any $A \in \mathscr{I}$ there exists a G_δ set $B \in \mathscr{I}$ with $A \subset B$. $\mathscr{I}$ is *analytic on G_δ sets* if, for any Polish space Y and any G_δ set $G \subset X \times Y$, $\{x \in X\colon G_x \in \mathscr{I}\}$ is analytic, where $G_x = \{y\colon (x, y) \in G\}$. A function $\mu\colon \mathrm{Bor}(X) \to [0, \infty)$, X a Polish space, is called a *Maharam submeasure* if

(o) $\mu(\emptyset) = 0$;
(i) $A \subset B$ implies $\mu(A) \leqslant \mu(B)$;
(ii) $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$;
(iii) if $A_n \to A$, then $\mu(A_n) \to \mu(A)$.

In (iii) $A_n \to A$ means that the characteristic functions of the A_n converge pointwise to the characteristic function of A.

Each finite Borel measure on X is a Maharam submeasure. It is an open problem, formulated by Maharam, whether for any Maharam submeasure μ there is a finite Borel measure v such that $\mu(A) = 0$ iff $v(A) = 0$ for any $A \in \mathrm{Bor}(X)$.

For a family $\mathscr{I}$ of subsets of 2^ω let

$$\hat{\mathscr{I}} = \{X \subset 2^{<\omega}\colon [X] \in \mathscr{I}\},$$

where $[X] = \{x \in 2^\omega\colon \exists^\infty k\ x|k \in X\}$. Clearly each $[X]$ is a G_δ but also conversely each G_δ subset of 2^ω is of the form $[X]$ for some $X \subset 2^{<\omega}$. (To see this, represent a G_δ

set $G \subset 2^\omega$ as $G = \bigcap_n U_n$ with $U_n \subset 2^\omega$ open and $U_{n+1} \subset U_n$. For each n, we can find $\sigma_k^n \in 2^{<\omega}$, $k \in \omega$, such that $U_n = \{x \in 2^\omega \colon \exists k\ \sigma_k^n \subset x\}$, σ_k^n and $\sigma_{k'}^n$ are not compatible if $k \neq k'$, and $\sigma_k^n \neq \sigma_l^m$ for $m < n$. Then for $X = \{\sigma_k^n \colon n, k \in \omega\}$ we have $G = [X]$.) Therefore, subsets of $2^{<\omega}$ can be regarded as codes for G_δ subsets of 2^ω; thus, $\hat{\mathscr{I}}$ is the family of codes of G_δ sets in $\mathscr{I}$. If $\mathscr{I}$ is an ideal, then $\hat{\mathscr{I}}$ is an ideal as well. The following lemma shows that in case $\mathscr{I}$ is hereditary and G_δ-supported, then a natural condition on $\hat{\mathscr{I}}$ guarantees that $\mathscr{I}$ is a σ-ideal.

Lemma 5.1. *Let $\mathscr{I}$ be a hereditary G_δ-supported family of Borel sets. If $\hat{\mathscr{I}}$ is a P-ideal, then $\mathscr{I}$ is a σ-ideal.*

Proof. It is enough to show that if $[X_n] \in \mathscr{I}$, $n \in \omega$, for some $X_n \subset 2^{<\omega}$, then there exists $Y \subset 2^{<\omega}$ with $[Y] \in \mathscr{I}$ and $[Y] \supset \bigcup_n [X_n]$. But in this situation $X_n \in \hat{\mathscr{I}}$. By assumption, we can find $Y \in \hat{\mathscr{I}}$ with $X_n \setminus Y$ finite for each n. Then clearly Y is as required. $\square$

Example. One cannot reverse the implication from the above lemma. In [12], a family of ideals was constructed, later called Mycielski ideals, any member of which provides a counterexample to the reverse implication. To define a Mycielski ideal fix a family X_s, $s \in 2^{<\omega}$, of infinite subsets of ω such that $X_s \subset X_t$ if $t \subset s$ and $X_s \cap X_t = \emptyset$ if s and t are incompatible. We let $A \subset 2^\omega$ be in the ideal if for each $s \in 2^{<\omega}$ player II has a winning strategy in the following game: I and II choose $a_n \in \{0, 1\}$; a_n is chosen by I if $n \notin X_s$ and by II otherwise; I wins if the resulting sequence (a_n) is in A. Let us denote by $\mathscr{I}$ an ideal defined as above for some fixed family $\{X_s \colon s \in 2^{<\omega}\}$. It was shown in [11] that $\mathscr{I}$ is a G_δ-supported σ-ideal. We will prove, however, that $\hat{\mathscr{I}}$ is not a P-ideal. Note that $X_{\langle 0 \rangle}$ is coinfinite. Let $X = X_{\langle 0 \rangle}$ and define for $\sigma \in 2^{<\omega}$,

$$\sigma \in Y_n \Leftrightarrow |\{k \in \mathrm{lh}(\sigma) \setminus X \colon \sigma(k) = 0\}| \leqslant n \quad \text{and} \quad \sigma(\max(\mathrm{lh}(\sigma) \setminus X)) = 0.$$

Here $\mathrm{lh}(\sigma)$ denotes the length of σ, that is, the unique $m \in \omega$ with $\sigma \in 2^m$. We also agree that the condition $\sigma(\max(\mathrm{lh}(\sigma) \setminus X)) = 0$ holds vacuously if $\mathrm{lh}(\sigma) \setminus X = \emptyset$. Then for any $x \in 2^\omega$, $\{x|k \colon k \in \omega\} \cap Y_n$ is finite. If it were infinite, we could find natural numbers $0 < k_0 < k_1 < \cdots < k_n$ with $x|k_i \in Y_n$ and with each interval $(0, k_0)$, (k_i, k_{i+1}), $i < n$, containing a member of $\omega \setminus X$. Now if $m_i = \max(k_i \setminus X)$, then $m_i < m_{i+1}$ and $(x|k_n)(m_i) = (x|k_i)(m_i) = 0$ for $i = 0, 1, \ldots, n$, so $|\{k \in k_n \setminus X_s \colon (x|k_n)(k) = 0\}| > n$ hence $x|k_n$ would not belong to Y_n. Thus, $[Y_n] = \emptyset$ which shows that $Y_n \in \hat{\mathscr{I}}$. Now let $F_n \subseteq 2^{<\omega}$ be finite. We claim that $\bigcup_n Y_n \setminus F_n \notin \hat{\mathscr{I}}$ which will witness that $\hat{\mathscr{I}}$ fails to be a P-ideal. It suffices to see that $A = [\bigcup_n (Y_n \setminus F_n)] \notin \mathscr{I}$. In order to do that we show that player I has a winning strategy (for the set A) in the game in which II plays on $X = X_{\langle 0 \rangle}$ and I plays on $\omega \setminus X$. Let $0 = l_0 < l_1 < l_2 < \cdots$ be natural numbers such that $F_n \subset 2^{<l_n}$ and each interval (l_n, l_{n+1}), $n \in \omega$, contains a member of $\omega \setminus X$. Put $m_n = \max(l_n \setminus X)$. Let player I play 1 if $k \in (\omega \setminus X) \setminus \{m_n \colon n \in \omega\}$ and 0 if $k = m_n$ for some n. It is not difficult to see that if $x \in 2^\omega$ is the outcome of a play with I

playing according to the strategy, then for any n, $x|l_n \in Y_n \setminus F_n$, so $x \in [\bigcup_n (Y_n \setminus F_n)]$, that is I wins.

I do not know of any example of a G_δ-supported σ-ideal with $\hat{\mathscr{I}}$ analytic for which $\hat{\mathscr{I}}$ is not a P-ideal. (It follows from results in [1] that for $\mathscr{I}$ a Mycielski ideal, $\hat{\mathscr{I}}$ is complete coanalytic.)

Theorem 5.2. *Let $\mathscr{I}$ be a hereditary family of Borel subsets of 2^ω. Then the following conditions are equivalent*:

(i) *$\hat{\mathscr{I}}$ is a P-ideal, $\mathscr{I}$ is G_δ-supported and analytic on G_δ sets.*

(ii) *$\mathscr{I} = \{A \in \mathrm{Bor}(X)$: $\mu(A) = 0\}$ for a Maharam submeasure μ.*

Proof of (ii)$\Rightarrow$(i). Let μ be a Maharam submeasure. The ideal of μ-zero sets is G_δ supported. To see this, let $A \subset 2^\omega$ is Borel with $\mu(A) = 0$. Let $\mathscr{F}$ be a maximal family of mutually disjoint closed subsets of $2^\omega \setminus A$ with $\mu(F) > 0$ for each $F \in \mathscr{F}$. By condition (iii) in the definition of Maharam measure, it follows that $\mathscr{F}$ is countable. Now note that $\mu^*(X) = \inf\{\mu(B)$: $X \subset B, B$ Borel$\}$ is a capacity. Thus, by Choquet's theorem (see [4, Theorem 30.13]), $\mu(2^\omega \setminus \bigcup \mathscr{F}) = 0$. It follows that $2^\omega \setminus \bigcup \mathscr{F}$ is a μ-zero G_δ containing A.

To see that $\mathscr{I}$ is analytic on G_δ sets, note first that if X is a Polish space and $F \subset X \times 2^\omega$ is closed, then condition (iii) in the definition of Maharam measure implies that $\{x \in X$: $\mu(F_x) \geqslant r\}$ is closed for any real r. Now let $G \subset X \times 2^\omega$ be a G_δ. We can find closed sets $F_{nk} \subset X \times 2^\omega$, $n, k \in \omega$, such that $G = \bigcap_n \bigcup_k F_{nk}$ and, additionally, $\bigcup_k F_{nk} \supset \bigcup_k F_{n+1k}$ and $F_{nk} \subset F_{nk+1}$. Using condition (iii) again, we see that for $x \in X$

$$\mu(G_x) = 0 \quad \text{iff} \quad \forall m \exists n \forall k \; \mu((F_{nk})_x) < 1/(m+1).$$

But the condition on the right-hand side is analytic (actually, Borel) by the remark at the beginning of this paragraph.

It remains to see that $\hat{\mathscr{I}}$ is a P-ideal. Thus it is enough to find a lsc finite submeasure ϕ in $2^{<\omega}$ with $\hat{\mathscr{I}} = \mathrm{Exh}(\phi)$. For $X \subset 2^{<\omega}$, let $[X]_1 = \{x \in 2^\omega$: $\exists k \; x|k \in X\}$. Define, for $X \subset 2^{<\omega}$,

$$\phi(X) = \mu([X]_1).$$

Since $[X \cup Y]_1 = [X]_1 \cup [Y]_1$ and μ is a finite submeasure, ϕ is a finite submeasure as well. To check that ϕ is lsc it is enough to show that $\phi(X) = \sup_m \phi(X \cap 2^{<m})$. But $[X]_1 = \bigcup_m [X \cap 2^{<m}]_1$ and $[X \cap 2^{<m}]_1 \subset [X \cap 2^{<m+1}]_1$; thus since μ is Maharam

$$\phi(X) = \mu([X]_1) = \sup_m \mu([X \cap 2^{<m}]_1) = \sup_m \phi(X \cap 2^{<m}).$$

The last thing that needs checking is that $\mu([X]) = 0$ iff $X \in \mathrm{Exh}(\phi)$. Note that for any X, $[X] = \bigcap_m [X \setminus 2^{<m}]_1$, so since μ is Maharam, we get

$$\mu([X]) = \inf_m \mu([X \setminus 2^{<m}]_1) = \inf_m \phi(X \setminus 2^{<m}).$$

It follows that $\mu([X]) = 0$ iff $\inf_m \phi(X \setminus 2^{<m}) = 0$, that is, $X \in \mathrm{Exh}(\phi)$.

The proof of (i)$\Rightarrow$(ii) will be divided into a sequence of lemmas. In what follows, we let $I = \hat{\mathscr{I}}$.

Since $\mathscr{I}$ is analytic on G_δ sets, I is analytic. Since I is also a P-ideal, by Theorem 3.1 there exists a lsc submeasure $\phi : 2^{(2^{<\omega})} \to [0,\infty)$ such that $I = \mathrm{Exh}(\phi)$. Define for a Borel set $B \subset 2^\omega$

$$\mu(B) = \inf\{\phi(X) \colon X \subset 2^{<\omega} \text{ and } B \subset [X]\}.$$

We will show that μ is a Maharam submeasure and that $\mathscr{I}$ is the σ-ideal of μ-zero sets. This will prove Theorem 5.2. In the following lemma, we gather the facts that are easy to see.

Lemma 5.3. (i) *For $A, B \subset 2^\omega$ Borel, we have $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ and, if $A \subset B$, $\mu(A) \leqslant \mu(B)$.*

(ii) *If $A \in \mathscr{I}$, then $\mu(A) = 0$.*

Proof. (i) is entirely straightforward. To see (ii), let $\varepsilon > 0$ be given. We will show that $\mu(A) < \varepsilon$. Let G be a G_δ such that $A \subset G \in \mathscr{I}$. Let $X \subset 2^{<\omega}$ be such that $G = [X]$. Then $X \in I$, whence there is a finite set $F \subset X$ such that $\phi(X \setminus F) < \varepsilon$. Since clearly $[X \setminus F] = G$, $A \subset [X \setminus F]$ and $\mu(A) < \varepsilon$.

It remains to see that if $\mu(A) = 0$ for a Borel set $A \subset 2^\omega$, then $A \in \mathscr{I}$ and that if $A_n \to A$, $A_n \subset 2^\omega$ Borel, then $\mu(A_n) \to \mu(A)$. The crucial properties of μ are established in Lemmas 5.4–5.8. The rest follows from these properties by fairly standard arguments. $\square$

Lemma 5.4. *Let $K \subset 2^\omega$ be compact. If $\mu(K) = 0$, then $K \in \mathscr{I}$.*

Proof. Let $X_n \subset 2^{<\omega}$ be such that $\phi(X_n) < 1/2^n$ and $K \subset [X_n]$, $n \in \omega$. We can assume that $X_n \cap 2^{<n} = \emptyset$ for otherwise we could replace X_n with $X_n \setminus 2^{<n}$. Note that $K \subset \bigcup\{N_\sigma \colon \sigma \in X_n\}$. Thus, there is a finite set $Y_n \subset X_n$ with $K \subset \bigcup\{N_\sigma \colon \sigma \in Y_n\}$. Let $Y = \bigcup_n Y_n$. Using the facts that $Y_n \cap 2^{<n} = \emptyset$ and that the Y_n's are finite, we easily get $K = [Y]$. Again using finiteness of the Y_n's and

$$\phi\left(\bigcup_{n \geqslant N} Y_n\right) \leqslant \sum_{n \geqslant N} \phi(Y_n) \leqslant \sum_{n \geqslant N} 2^{-n} \to 0,$$

we get that $Y \in I$. Thus, $K \in \mathscr{I}$.

Lemma 5.5. *Let $G \notin \mathscr{I}$ be a G_δ. Then there is $K \subset G$ compact with $K \notin \mathscr{I}$.*

Proof. First we prove two claims. For $\sigma \in 2^{<\omega}$, let $U_\sigma = \{\tau \in 2^{<\omega} \colon \sigma \subseteq \tau \text{ and } \sigma \neq \tau\}$.

Claim 1. *Let $F_i \subseteq 2^{<\omega}$, $i \in \omega$, be finite and $F_{i+1} \subseteq \bigcup_{\sigma \in F_i} U_\sigma$. Then $[\bigcup_i F_i]$ is compact.*

Proof. Since each F_i is finite, we have $x \in [\bigcup_i F_i]$ iff $\exists^\infty i \exists k\, x|k \in F_i$. But if $x|k \in F_i$, then for any $j<i$ there is a k_j with $x|k_j \in F_j$. So,

$$x \in \left[\bigcup_i F_i\right] \quad \text{iff } \forall i \exists k\, x|k \in F_i \quad \text{iff } \forall i \exists k \leqslant \max\{\mathrm{lh}(\sigma)\colon \sigma \in F_i\}\, x|k \in F_i$$

which gives a compact definition of $[\bigcup_i F_i]$. □

Claim 2. *Let $\delta, \varepsilon > 0$, and let $Y \subset 2^{<\omega}$. Assume Y contains infinitely many pairwise disjoint finite sets H with $\phi(H) \geqslant \delta$. Then there is $F \subseteq Y$ finite such that $\bigcup_{\sigma \in F} U_\sigma \cap Y$ contains infinitely many pairwise disjoint sets H with $\phi(H) \geqslant \delta - \varepsilon$.*

Proof. Towards a contradiction assume the conclusion fails. Let $\{\sigma_i\colon i \in \omega\}$ list Y. We can recursively find finite sets $H_k \subseteq Y$, $k \in \omega$, such that $H_{k_1} \cap H_{k_2} = \emptyset$, $\phi(H_k) \geqslant \delta$, and $\phi(H_k \setminus \bigcup_{i<k} U_{\sigma_i}) > \varepsilon$. Let $A_k = H_k \setminus \bigcup_{i<k} U_{\sigma_i}$. Note that if $x \in [\bigcup_k A_k]$, then $x \in [Y]$ since $\bigcup_k A_k \subset Y$. However, this means that for some k_0, $\forall^\infty n\, x|n \in \bigcup_{i<k_0} U_{\sigma_i}$. Thus, $\exists^\infty n\, x|n \in \bigcup_{k<k_0} A_k$ which contradicts the finiteness of the A_k's. Therefore, $[\bigcup_k A_k] = \emptyset \in \mathscr{I}$. It follows that $\bigcup_k A_k \in I$. Since the A_k's are disjoint, $\phi(\bigcup_{k \geqslant n} A_k) \to 0$. But $\phi(A_k) > \varepsilon$ for all k, a contradiction.

Now we are ready to prove the lemma. Let $X \subset 2^{<\omega}$ be such that $[X] = G$. Then $X \notin I = \mathrm{Exh}(\phi)$. By semicontinuity of ϕ, we can find a $\delta > 0$ and infinitely many pairwise disjoint subsets H of X such that $\phi(H) > \delta$. Now we recursively define finite sets $F_k \subset X$, $k \in \omega$, such that

1. $F_k \subset \bigcup_{\sigma \in F_{k-1}} U_\sigma \cap X$ for $k > 0$;
2. $\bigcup_{\sigma \in F_k} U_\sigma \cap X$ contains an infinite disjoint family of finite sets H such that $\phi(H) > \delta$;
3. $\phi(F_k) > \delta$.

If F_{k-1} is defined, we find F'_k fulfilling 1 and 2 using Claim 2. Then we pick an $H \subset \bigcup_{\sigma \in F'_{k-1}} U_\sigma \cap X$ finite with $\phi(H) > \delta$ and let $F_k = F'_k \cup H$.

Now note that since all the F_k's are finite, from 1 and Claim 1, we get that $[\bigcup_k F_k]$ is compact. Since $\bigcup_k F_k \subset X$, $[\bigcup_k F_k] \subset G$. Note also that for any finite $F' \subset 2^\omega$, $\bigcup_k F_k \setminus F'$ contains one of the F_k's (since by 1, $\min\{\mathrm{lh}(\sigma)\colon \sigma \in F_k\} > k$). Thus, $\phi(\bigcup_k F_k \setminus F') > \delta$ by 3. So, $\bigcup_k F_k \notin I$, whence $[\bigcup_k F_k] \notin \mathscr{I}$. □

Lemma 5.6. *Let $G \subset 2^\omega$ be a G_δ, and let $K_n \subset K_{n-1} \subset G$, $n \in \omega$, be compact. If $G \setminus \bigcup_n K_n \in \mathscr{I}$, then $\sup_n \mu(K_n) = \mu(G)$.*

Proof. We will need the following claim.

Claim. *Let $X \subset 2^\omega$, and let $G \subset [X]$ be a G_δ. Then there exists $Y \subset X$ with $G = [Y]$.*

Proof. Let $U_n \subset 2^\omega$ be a sequence of open sets with $G = \bigcap_n U_n$ and $U_{n+1} \subset U_n$. Fix antichains $A_n \subset 2^{<\omega}$ such that $U_n = \bigcup_{\sigma \in A_n} N_\sigma$. Let

$$B_n = \{\sigma \in X\colon \exists \tau \in A_n\, (\sigma \supseteq \tau \text{ and } \forall \sigma' \in X\ \sigma \supseteq \sigma' \supseteq \tau \Rightarrow \sigma = \sigma')\}.$$

Clearly, $G \subset \bigcup_{\sigma \in B_n} N_\sigma \subset U_n$, $B_n \subset X$, and B_n is an antichain. So, since the sequence (U_n) is decreasing, $G = \bigcap_N \bigcup_{n \geqslant N} \bigcup_{\sigma \in B_n} N_\sigma$, and since the B_n's are antichains, $\bigcap_N \bigcup_{n \geqslant N} \bigcup_{\sigma \in B_n} N_\sigma = [\bigcup_n B_n]$. Let $Y = \bigcup_n B_n$.

Let $\varepsilon > 0$. By the claim, we can find $X \subset 2^{<\omega}$ such that $G = [X]$ and $\mu(G) + \varepsilon \geqslant \phi(X)$. Let T_n be the tree corresponding to K_n, and let $Y_n \subset T_n$ be such that $K_n = [Y_n]$ and $\mu(K_n) + (1/n) \geqslant \phi(Y_n)$. Note that for any n, $G = [(X \setminus (2^{<n} \cup T_n)) \cup Y_n]$. It follows that

$$\phi((X \setminus (T_n \cup 2^{<n})) \cup Y_n) + \varepsilon \geqslant \phi(X),$$

whence

$$\phi(Y_n) + \varepsilon \geqslant \phi(X) - \phi(X \setminus (T_n \cup 2^{<n})).$$

It follows that

$$\begin{aligned}\sup_n \mu(K_n) = \sup_n \phi(Y_n) &\geqslant \sup_n \phi(X) - \phi(X \setminus (T_n \cup 2^{<n})) - \varepsilon \\ &= \phi(X) - \varepsilon - \inf_n \phi(X \setminus (T_n \cup 2^{<n})).\end{aligned}$$

So, it is enough to show that $\inf_n \phi(X \setminus (T_n \cup 2^{<n})) = 0$. If this is false, then there is a $\delta > 0$ such that $\phi(X \setminus (T_n \cup 2^{<n})) > \delta$ for all n. Now we can pick finite $F_n \subset X \setminus (T_n \cup 2^{<n})$ with $\phi(F_n) > \delta$. Clearly, $\bigcup_n F_n \notin I$. Thus $[\bigcup_n F_n] \notin \mathscr{I}$. On the other hand, $[\bigcup_n F_n] \subset G$ since $F_n \subset X$ for all n. Also since $T_n \subset T_{n+1}$ and $F_n \cap T_n = \emptyset$, we get that $F_n \cap T_k = \emptyset$ if $n \geqslant k$, so, $[\bigcup_n F_n] \cap T_k = \emptyset$ for any k. Thus $[\bigcup_n F_n] \subset G \setminus \bigcup_k K_k$, whence $[\bigcup_n F_n] \in \mathscr{I}$, a contradiction. □

Lemma 5.7. *Given $\delta > 0$, there is no sequence K_n, $n \in \omega$, such that K_n are compact, $K_n \cap K_m = \emptyset$ if $n \neq m$, and $\mu(K_n) > \delta$.*

Proof. Claim. *Given $\varepsilon > 0$ there does not exist a sequence of compact sets K_n such that $\mu(K_n) > \varepsilon$ and such that there are open sets U_n with $K_n \subset U_n$ and $K_m \cap U_n = \emptyset$ if $m \neq n$.*

Proof of Claim. Let T_n be the tree corresponding to K_n, that is, $[T_n] = K_n$. For each n there is m_n such that if $\sigma \in T_n$ and $\mathrm{lh}(\sigma) \geqslant m_n$, then $N_\sigma \subset U_n$. Since $\mu(K_n) > \varepsilon$, $\phi(T_n \setminus 2^{<m_n}) > \varepsilon$. Thus, there exists a finite set $F_n \subset T_n \setminus 2^{<m_n}$ such that $\phi(F_n) \geqslant \varepsilon$. Note that for any $x \in 2^\omega$ and any n if $x|k \in F_n$ for some k, then $x|k' \notin F_m$ for any k' and any $m \neq n$. It follows that $[\bigcup_n F_n] = \emptyset$, whence $\bigcup_n F_n \in I$. But this is impossible since the F_n's are disjoint and $\phi(F_n) \geqslant \varepsilon$ for any n.

Now note that if K is compact, then for any $\varepsilon > 0$ there exists an open set $U \supset K$ with $\mu(U \setminus K) \leqslant \varepsilon$. For assume otherwise. We recursively construct a sequence of compact sets K_n and open sets U_n as in the claim. Since $\mu(2^\omega \setminus K) > \varepsilon$ and $2^\omega \setminus K$ can be represented as a countable union of compact sets, by Lemma 5.6, we can find a compact set $K_0 \subset 2^\omega \setminus K$ with $\mu(K_0) > \varepsilon$. Let U_0 and V_0 be disjoint open and such that $K_0 \subset U_0$ and $K \subset V_0$. Since $\mu(V_0 \setminus K) > \varepsilon$, again using Lemma 5.6 we can find a compact set $K_1 \subset V_0 \setminus K$ with $\mu(K_1) > \varepsilon$. Let U_1 and V_1 be two disjoint open sets with $K_1 \subset U_1$

and $K \subset V_1$. Continuing this way we obtain sequences K_n and U_n which contradict the claim.

To prove the lemma, let $\delta > 0$ and let K_n, $n \in \omega$, be a sequence of compact disjoint sets with $\mu(K_n) > \delta$. By what was said above, we can find open sets $U_n \supset K_n$ such that $\mu(U_n \setminus K_n) < 2^{-n-2}\delta$. Now let $K'_n = K_n \setminus \bigcup_{i<n} U_i$. Then the sequences K'_n and U_n are as in the claim (with $\varepsilon = \delta/2$) which gives a contradiction. $\square$

Lemma 5.8. *If $A_n \subset A_{n+1} \subset 2^\omega$, $n \in \omega$, are Borel, then there exists a G_δ set $G \supset \bigcup_n A_n$ such that $\mu(G) = \sup_n \mu(A_n)$.*

Proof. The inequality $\geqslant$ is clear. To see $\leqslant$ put $\bigcup_n A_n = A$. Let $\mathscr{K}$ be a maximal disjoint family of compact subsets of $2^\omega \setminus A$ not in $\mathscr{I}$. By Lemmas 5.4 and 5.7, $\mathscr{K}$ is countable. Let $G = 2^\omega \setminus \bigcup \mathscr{K}$. Then G is a G_δ and $A \subset G$, so in particular $\mu(G) \geqslant \mu(A)$. Let $G_n \supset A_n$ be such that $\mu(G_n) \leqslant \mu(A_n) + 1/n$. We can also assume that $G_n \subset G$ and $G_n \subset G_{n+1}$. It will be enough to show that $\sup_n \mu(G_n) \geqslant \mu(G)$.

Let $K^n_m \subset G_n$, $m \in \omega$, be compact and such that $G_n \setminus \bigcup_m K^n_m \in \mathscr{I}$. It is possible to arrange this by Lemmas 5.5 and 5.7. We claim that $G \setminus \bigcup_{n,m} K^n_m \in \mathscr{I}$. If not, being a G_δ it contains a compact set $K \notin \mathscr{I}$. But now $K \cap G_n \in \mathscr{I}$ for all n since $K \cap G_n \subset G_n \setminus \bigcup_m K^n_m$. Thus, $K \cap \bigcup_n G_n \in \mathscr{I}$. So, there is a G_δ set $B \subset K$ with $B \in \mathscr{I}$ and $K \cap \bigcup_n G_n \subset B$. But now $K \setminus B$ is an F_σ not in $\mathscr{I}$ whence, since $\mathscr{I}$ is a σ-ideal, there is a compact set $K_1 \subset K \setminus B$ not in $\mathscr{I}$. But this contradicts maximality of $\mathscr{K}$ as $K_1 \subset G \setminus A$.

Now by Lemma 5.6, we get

$$\mu(G) = \sup_N \mu\left(\bigcup_{n,m \leqslant N} K^n_m\right) \leqslant \sup_N \mu(G_N). \qquad \square$$

Lemma 5.9. *Let $K_n \subset 2^\omega$, $n \in \omega$, be compact and $K_{n+1} \subset K_n$. Then $\mu(\bigcap_n K_n) = \lim_n \mu(K_n)$.*

Proof. $\leqslant$ is clear. To see $\geqslant$, let $K = \bigcap_n K_n$ and assume that for some $\delta > 0$ and all n $\mu(K_n) > \mu(K) + \delta$. Then $\mu(K_n \setminus K) > \delta$. We recursively find compact sets C_n, $n \in \omega$, which are pairwise disjoint and $\mu(C_n) > \delta$ as follows. Since $K_0 \setminus K$ is a G_δ which is a union of compact sets and $\mu(K_0 \setminus K) > \delta$, by Lemma 5.6, we can find $C_0 \subset K_0 \setminus K$ compact with $\mu(C_0) > \delta$. Since $\bigcap_n K_n = K$, there exists n_0 with $K_{n_0} \cap C_0 = \emptyset$. Now, we find $C_1 \subset K_{n_0} \setminus K$ compact with $\mu(C_1) > \delta$ and $n_1 > n_0$ with $C_1 \cap K_{n_1} = \emptyset$. Proceeding this way C_n can be constructed for each n. However by Lemma 5.7 the family $\{C_n : n \in \omega\}$ does not exist, a contradiction. $\square$

Lemma 5.10. *Let $B \subset 2^\omega$ be Borel. Then $\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ compact}\}$.*

Proof. This follows from Choquet's theorem applied to the capacity $\mu^*(X) = \inf\{\mu(B) : X \subset B,\ B \text{ Borel}\}$ (see [4, Theorem 30.13]). That μ^* is a capacity follows from Lemmas 5.8 and 5.9. $\square$

Proof of (i)$\Rightarrow$(ii) **in Theorem 5.2.** First the equality $\mathscr{I} = \{B \in \mathrm{Bor}(2^\omega)\colon \mu(B) = 0\}$. The inclusion $\subset$ is Lemma 5.3(ii). Now, let $\mu(B) = 0$ for a Borel set $B \subset 2^\omega$. By Lemma 5.8, there exists a G_δ set $G \supset B$ with $\mu(G) = 0$. (Simply put $A_n = B$ for each n in Lemma 5.8.) thus by Lemmas 5.5 and 5.4 $G \in \mathscr{I}$, so $B \in \mathscr{I}$.

It remains to check condition (iii) from the definition of Maharam submeasure. First we will show that if B_n, $n \in \omega$, are Borel and $B_{n+1} \subset B_n$, then $\mu(\bigcap_n B_n) = \lim_n \mu(B_n)$. Again $\leqslant$ is clear. To see $\geqslant$, assume towards a contradiction that $\mu(B_n) > \mu(B) + \delta$ for some $\delta >$ where $B = \bigcap_n B_n$. Then $\mu(B_n \setminus B) > \delta$ for any $n \in \omega$. Similarly, as in the proof of Lemma 5.9 we will recursively construct a family of disjoint compact sets $\{C_n\colon n \in \omega\}$ with $\mu(C_n) > \delta$ which will contradict Lemma 5.7. By Lemma 5.8, there exists n_0 with $\mu(B_0 \setminus B_{n_0}) > \delta$. By Lemma 5.10, let $C_0 \subset B_0 \setminus B_{n_0}$ be compact and such that $\mu(C_0) > \delta$. Now find $n_1 > n_0$ with $\mu(B_{n_0} \setminus B_{n_1}) > \delta$ and $C_1 \subset B_{n_0} \setminus B_{n_1}$ compact with $\mu(C_1) > \delta$. We proceed in this fashion until all C_n's are produced. $\square$

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