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Distance-Regular Digraphs of Girth 4

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It is shown that all possible parameters for distance-regular digraphs of girth 4 consist of two one-parameter families. The smallest non-trivial example in one family is constructed while in the other family it is shown not to exist. © 1987 Academic Press, Inc.

I. INTRODUCTION

In recent years there has been considerable work done on distance-regular graphs-to cite a few examples cf. [2–4, 9, 10]. The non-symmetric version, distance-regular digraphs, introduced by Lam in [8], was worked on in [1, 7]. But, as is pointed out in [2, 7], examples were not abundant. Here we show that for girth 4, parameters are indeed not common, nevertheless we do build a new example on 64 vertices.

We will use the definitions of Damerell in [7]. Namely, $\mathscr{D} = (V, E)$ is a digraph simply means V is a (finite) set of vertices and E, the edges, is a subset of $V \times V$. We use $\alpha \to \beta$ to denote $(\alpha, \beta) \in E$. The adjacency matrix of $\mathscr{D}$ is the square $(0, 1)$-matrix A with rows and columns indexed by V so that its (α, β)-entry is 1 exactly when $\alpha \to \beta$. By a path from α to β we mean a sequence $\alpha = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_t = \beta$, and t is the length of the path. The distance from α to β, $d(\alpha, \beta)$, is the smallest length of a path from α to β,

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and the diameter of $\mathscr{D}$ is the largest distance between two vertices in $\mathscr{D}$. Throughout we assume $\mathscr{D}$ is connected in the sense that for any α, β, there is a path from α to β. We will let d denote the diameter of $\mathscr{D}$, and for $i=0,\ldots, d$, let $\Gamma_i(\alpha)$ denote the set of points β at distance i from α, $\Gamma_i(\alpha)=\{\beta \mid d(\alpha, \beta)=i\}$. For notation and definitions concerning association schemes, see [2].

With an arbitrary (loopless) digraph $\mathscr{D}=(V, E)$ of diameter d we can associate a collection of digraphs for $i=0,\ldots, d$, where $\alpha \to \beta$ in the ith digraph if $d(\alpha, \beta)=i$ in $\mathscr{D}$. Let $A_0, A_1,\ldots, A_d$ be the adjacency matrices of the respective digraphs, then the following are clear:

(1) A_1 is the adjacency matrix of $\mathscr{D}$,

(2) $A_0=I$,

(3) $A_0+A_1+\cdots+A_d=J$, the matrix of all ones,

(4) $A_i \neq 0$ for $i=0,\ldots, d$,

(5) for any i, $A_1^i+A_1^{i-1}+\cdots+A_1+I$ has the same zero pattern (nonzero entries) as $A_i+A_{i-1}+\cdots+A_1+I$.

If we let $\mathscr{A}=[I, A_1,\ldots, A_d]$ denote the span of $I, A_1,\ldots, A_d$, then also clear is the fact that a matrix X belongs to $\mathscr{A}$ if and only if for any vertices $\alpha, \beta \in V$, the (α, β)-entry of X depends only on $d(\alpha, \beta)$. Now, $\mathscr{A}$ is a vector space closed under Hadamard products, but we also have from Damerell [7],

THEOREM. *$\mathscr{A}$ is a commutative association scheme (i.e., closed under products and transposes) if and only if $A_iA_1^T \in \mathscr{A}$ for any $i=0,\ldots, d$.*

An equivalent (and more customary) condition to the latter half of the theorem is that for any $i=0,\ldots, d$, $|\Gamma_i(\alpha) \cap \Gamma_1(\beta)|$ is the same for all vertices α, β such that $d(\alpha, \beta)=j$.

EXAMPLE. Let $\mathscr{D}$ have adjacency matrix $A_1=\left(\begin{smallmatrix} 0 & X \\ X & 0\end{smallmatrix}\right)$, where

$$X=\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

then $A_2=\left(\begin{smallmatrix} J-I & 0 \\ 0 & J-I\end{smallmatrix}\right)$, $A_3=\left(\begin{smallmatrix} 0 & J-X \\ J-X & 0\end{smallmatrix}\right)$ and $\mathscr{A}$ is closed under products, but not transposes.

We say, following Lam [8], that $\mathscr{D}$ is *distance-regular* if the theorem above is satisfied. If $A_1=A_1^T$, then $\mathscr{D}$ is a *distance-regular graph.*

From now on, we will assume $A_1 \neq A_1^T$ and that $\mathscr{D}$ is distance-regular. It follows then that $A_1 * A_1^T = 0$. We will let g denote the girth of $\mathscr{D}$ (shortest length of a cycle in $\mathscr{D}$), and note $g \geqslant 3$.

EXAMPLE. Let $\mathscr{D}$ be the directed cycle on n vertices, so $A_1 = C_n$, the basic circulant, and for all i, $A_i = C^i$; here $d = n-1$, $g = n$.

The following facts are from [7]:

(1) if $1 \leqslant d(\alpha, \beta) < g$, then $d(\beta, \alpha) = g - d(\alpha, \beta)$;

(2) $d = g$ or $d = g-1$;

(3) if $\mathscr{D}$ satisfies $d = g-1$, and A_1 is its adjacency matrix, then $A_1 \otimes J_m$ for any $m \geqslant 2$ is the adjacency matrix of a distance-regular digraph with $d = g$, and conversely, any such digraph satisfying $d = g$ comes about this way from one with $d = g-1$.

From now on we assume $d = g - 1$.

(4) $A_i^T = A_{d+1-i}$ for $i = 1, \ldots, d$.

If $g = 3$, then it is well-known that

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & I + A_1 - A_1^T & \\ -1 & & & \end{pmatrix}$$

is a skew-Hadamard, and conversely from any skew-Hadamard, an A_1 can so be obtained. Thus, $g = 4$ (and $d = 3$) is the smallest case.

II. PARAMETRIC CONSIDERATIONS

Let $\mathscr{D}$ be a distance-regular digraph with $g = 4$ and $d = 3$. To simplify notation we will write the adjacency matrices as $A = A_1$, $B = B^T = A_2$, $A^T = A_3$, so the Bose–Mesner algebra $\mathscr{A} = [I, A, B, A^T]$. Since $\mathscr{A}$ is four-dimensional, and generated by A, A has four distinct eigenvalues, and since A is normal but not symmetric, two of its eigenvalues are non-real, another one is k, the line sums of A, and hence the fourth one is also real. Writing down the eigenmatrix P for $\mathscr{A}$ we get that P is of the form

$$P = \begin{pmatrix} 1 & k & l & k \\ 1 & \lambda_1 & \lambda_3 & \overline{\lambda_1} \\ 1 & \lambda_2 & \lambda_4 & \lambda_2 \\ 1 & \overline{\lambda_1} & \lambda_3 & \lambda_1 \end{pmatrix},$$

where k and l are the respective line sums of A and B.

Another natural matrix in our discussions will be the (first) intersection matrix $S=(s_{ij})$, where $AA_i=\sum_{j=0}^{3} s_{ij}A_j$ for $i=0, 1, 2, 3$. By girth and other basic considerations we get that

$$S=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & x & y & z \\ k & p & q & p \end{pmatrix},$$

where all entries are nonnegative, b and z are positive.

So $A^2=aA+bB$, hence $\lambda_1^2=a\lambda_1+b\lambda^3$. Let $\lambda_1=\alpha+\beta i$, $\beta\neq 0$. Then $\lambda_1^2=(\alpha^2-\beta^2)+2\alpha\beta i$, so $2\alpha\beta=a\beta$, hence $\alpha=a/2$. Since tr $A=0$, $\lambda_2<0$. Also $\alpha^2-\beta^2=a\alpha+b\lambda_3$, so $\lambda_3=-(1/b)((a^2/4)+\beta^2)$. But $1+\lambda_1+\lambda_3+\overline{\lambda_1}=0$ (since $I+A+B+A^T=J$), thus

$$\lambda_3=-(a+1).$$

Solving for β^2 we get

$$\beta^2=\frac{4(a+1)b-a^2}{4}.$$

We also have $\lambda_2^2=a\lambda_2+b\lambda_4$, and $1+\lambda_2+\lambda_4+\lambda_2=0$, hence eliminating λ_4, we have

$$\lambda_2^2+(2b-a)\lambda_2+b=0.$$

We see then that λ_2 has to be an integer, and that it divides b. Let $\lambda_2=-\lambda$ where λ is a positive integer, and let $b=\lambda\mu$ where μ is also a positive integer. Since $-\lambda$, $-\mu$ are the roots of the quadratic, $\lambda+\mu=2b-a=2\lambda\mu-a$ hence

$$b=\lambda\mu,$$
$$a=2\lambda\mu-\lambda-\mu.$$

From which we get

$$\lambda_1=\lambda\mu-\left(\frac{\lambda+\mu}{2}\right)+\sqrt{\lambda^2\mu^2-\left(\frac{\lambda-\mu}{2}\right)^2}\, i,$$
$$\lambda_2=-\lambda,$$
$$\lambda_3=-(2\lambda\mu-\lambda-\mu+1)=-(a+1),$$
$$\lambda_4=2\lambda-1.$$

By considering the third row of S, one obtains $AB = xA + yB + zA^T$, hence

$$\lambda_1 \lambda_3 = x\lambda_1 + y\lambda_3 + z\overline{\lambda_1}$$

and

$$\lambda_2 \lambda_4 = x\lambda_2 + y\lambda_4 + z\lambda_2.$$

Solving these equations we have

$$\begin{aligned} x &= (\lambda - 1)(2\lambda\mu - \lambda - \mu + 1) = (\lambda - 1)(a + 1), \\ y &= \lambda(2\lambda\mu - \lambda - \mu) = \lambda a, \\ z &= \lambda(2\lambda\mu - \lambda - \mu + 1) = \lambda(a + 1). \end{aligned}$$

Using the first row of P and the second and third rows of S, one has

$$k^2 = ak + bl,$$

and

$$kl = xk + yl + zk.$$

Eliminating l, one obtains a quadratic on k,

$$k^2 - ((2\lambda^2\mu - \lambda^2 + \lambda\mu - \mu) - \lambda)k + (-\lambda)(2\lambda^2\mu - \lambda^2 + \lambda\mu - \mu) = 0,$$

with roots $2\lambda^2\mu - \lambda^2 + \lambda\mu - \mu$, and $-\lambda$. So

$$k = 2\lambda^2\mu - \lambda^2 + \lambda\mu - \mu,$$

and then

$$l = \frac{k(a+1)}{\mu} = 4\lambda^3\mu - 4\lambda^3 + 2\lambda^2 + 2\lambda - 3\lambda\mu + \mu - 1 + \frac{\lambda^2 - \lambda^2}{\mu}.$$

Finally, one computes

$$\begin{aligned} p &= a, \\ q &= \mu(\lambda - 1). \end{aligned}$$

Already we can prove

LEMMA 1. *Let $\lambda = 1$. Then $\mu = 1$ and $A = C_4$.*

Proof. Suppose $\lambda = 1$. Then $A^2 = (\mu - 1)A + \mu B$. Consider $X = \Gamma_1(\alpha)$. Then the subdigraph X is $(\mu - 1)$-regular on $2\mu - 1$ vertices, and the girth of X is at least 4. This means that X is a (regular) tournament. However, a strongly connected tournament has girth 3. Hence $\mu = 1$ and $\mathscr{D} = C_4$.

Observe that the symmetrization of the scheme, $[I, A + A^T, B]$, may exist in some of these cases, e.g., if $\mu = 2$, B is the Petersen graph while if $\mu = 3$, $A + A^T$ is the Clebsch graph.

So far we have not considered the coeigenmatrix Q of the scheme, in particular we have not considered the fact that the multiplicities are integral and that A, B and A^T have trace 0. Let $1, m_1, m_2, m_1$ be the respective multiplicities of $1, \lambda_1, \lambda_2, \overline{\lambda_1}$ in A. Then

$$1 + 2m_1 + m_2 = n = 1 + 2k + l$$

and

$$l + 2m_1\lambda_3 + m_2\lambda_4 = 0$$

since B has trace 0. Eliminating m_2,

$$\delta m_1 = 2k\lambda - k + l\lambda, \tag{2.1}$$

where $\delta = \lambda_4 - \lambda_3 = a + 2\lambda = 2\lambda\mu + \lambda - \mu$. This δ is the standard factor, the difference of the two eigenvalues, that occurs when studying strongly regular graphs. But we also have

$$\delta(\lambda + 1) = k + (2\lambda^2 + \lambda)$$

and

$$\delta(2\lambda^2 + \lambda - 1) = l + \left(6\lambda^3 - \lambda^2 - 3\lambda + 1 - \frac{\lambda^3 - \lambda^2}{\mu}\right).$$

Substituting in (2.1), we obtain

$$m_1 = 2\lambda^3 + 3\lambda^2 - 1 - \frac{1}{\delta}\left(6\lambda^4 + 3\lambda^3 - 3\lambda^2 - \frac{\lambda^4 - \lambda^3}{\mu}\right). \tag{2.2}$$

Thus,

$$6\lambda^4 + 3\lambda^3 - 3\lambda^2 - \frac{\lambda^4 - \lambda^3}{\mu} \equiv 0 \pmod{\delta}. \tag{2.3}$$

Let $\lambda = d\gamma$, $\mu = d\nu$ where γ and ν are relatively prime. Then $\delta = d(2d\gamma\nu + \gamma - \nu)$ and (2.3) becomes

$$6d^4\gamma^4 + 3d^3\gamma^3 - 3d^2\gamma^2 - \frac{d^4\gamma^4 - d^3\gamma^3}{d\nu} \equiv 0 \pmod{d(2d\gamma\nu + \gamma - \nu)}. \tag{2.4}$$

Since we have that $\mu \mid \lambda^3 - \lambda^2$, we can cancel d, and since γ, v and $2d\gamma v + \gamma - v$ are pairwise relatively prime, (2.4) can be reduced to

$$6d^3\gamma^2 v + 3d^2\gamma v - 3dv - d^2\gamma^2 + d\gamma \equiv 0 \pmod{2d\gamma v + \gamma - v},$$

which can be further reduced to

$$2d(2d\gamma + 1) \equiv 0 \pmod{2d\gamma v + \gamma - v}.$$

Multiplying by v and reducing again, we obtain

$$2d(2v - \gamma) \equiv 0 \pmod{2d\gamma v + \gamma - v}.$$

If $2v = \gamma$, then $v = 1$, $\gamma = 2$. If $2v > \gamma$, $4dv - 2d\gamma \geqslant 2d\gamma v + \gamma - v$, so $\gamma \leqslant 2$; and finally $2v < \gamma$ is impossible. These together with the condition $\mu \mid \lambda^3 - \lambda^2$ and the lemma, lead easily to

$$\lambda = \mu \qquad \text{or} \qquad \lambda = 2\mu.$$

III. The Family $\lambda = \mu$

Substituting in the formulas of the previous section, one gets

$$k = 2\mu^3 - \mu = \mu(2\mu^2 - 1),$$
$$l = 4\mu^4 - 4\mu^3 + 2\mu - 1 = (2\mu^2 - 1)(2\mu^2 - 2\mu + 1),$$

so

$$n = 1 + 2k + l = 4\mu^4.$$

Then

$$P = \begin{pmatrix} 1 & \mu(2\mu^2-1) & (2\mu^2-1)(2\mu^2-2\mu+1) & \mu(2\mu^2-1) \\ 1 & \mu^2-\mu+\mu^2 i & -(2\mu^2-2\mu+1) & \mu^2-\mu-\mu^2 i \\ 1 & -\mu & 2\mu-1 & -\mu \\ 1 & \mu^2-\mu-\mu^2 i & -(2\mu^2-2\mu+1) & \mu^2-\mu+\mu^2 i \end{pmatrix}$$

and $Q = \bar{P}$. In particular, $m_1 = k$ and $m_2 = l$. The intersection matrix is given in

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2\mu(\mu-1) & \mu^2 & 0 \\ 0 & (\mu-1)(2\mu^2-2\mu+1) & 2(\mu-1)\mu^2 & \mu(2\mu^2-2\mu+1) \\ \mu(2\mu^2-1) & 2\mu(\mu-1) & \mu(\mu-1) & 2\mu(\mu-1) \end{pmatrix}.$$

Let $E_0, E_1, E_2, E_3 = \overline{E_1} = E_1^T$ be the principal idempotents of $\mathscr{A}$. Then

$$E_1 * E_1 = \frac{\mu-1}{2\mu^3} E_1 + \frac{1}{4\mu^2} E_2,$$

$$E_1 * E_2 = \frac{2\mu^3 - 4\mu^2 + 3\mu - 1}{4\mu^4} E_1 + \frac{\mu-1}{2\mu^2} E_2 + \frac{2\mu^2 - 2\mu + 1}{4\mu^4} E_3,$$

$$E_1 * E_3 = \frac{2\mu^2 - 1}{4\mu^3} E_0 + \frac{\mu-1}{2\mu^3} E_1 + \frac{\mu-1}{4\mu^3} E_2 + \frac{\mu-1}{2\mu^3} E_3.$$

So the scheme would always be Q-polynomial. Moreover, the Krein conditions are satisfied for all μ.

EXAMPLE. If $\mu = 1$, then $\mathscr{D}$ is the 4-cycle.

EXAMPLE. If $\mu = 2$, then $n = 64$, $k = 14$, $l = 35$. Since $Q = \bar{P}$, it is possible to require that the strongly regular graph $A + A^T$ have a regular abelian group of automorphisms. By techniques developed in [5, 6], using $G = Z_4 \times Z_4 \times Z_4$ (where Z_4 is the integers mod 4), one finds that if we let $T \subset G$ be

$$T = \{100, 003, 003, 033, 023, 210, 233, 333, 323, 312, 302, 331, 101, 330\},$$

and let $A = \sum_{g \in T} A_g$, where A_g denotes the 64×64 permutation matrix induced by g in the regular representation of G, then A is the desired digraph. Note that $A + A^T$ is a (64, 28, 12)-design, or equivalently, $I + B - A - A^T$ is a regular Hadamard matrix.

IV. THE FAMILY $\lambda = 2\mu$

Similar substitutions yield

$$k = 8\mu^3 - 2\mu^2 - \mu = \mu(2\mu - 1)(4\mu + 1),$$

$$l = 32\mu^4 - 32\mu^3 + 10\mu^2 + \mu - 1 = (2\mu - 1)(4\mu + 1)(4\mu^2 - 3\mu + 1),$$

$$n = 32\mu^4 - 16\mu^3 + 6\mu^2 - \mu = \mu(4\mu - 1)(8\mu^2 - 2\mu + 1),$$

$$m_1 = (2\mu - 1)(8\mu^2 - 2\mu + 1),$$

$$m_2 = (2\mu - 1)(4\mu - 1)(4\mu^2 - 3\mu + 1),$$

$$\lambda_1 = \tfrac{1}{2}(4\mu^2 - 3\mu + \mu\sqrt{16\mu^2 - 1i}),$$

$$\lambda_2 = -2\mu,$$

$$\lambda_3 = -(4\mu^2 - 3\mu + 1),$$

$$\lambda_4 = 4\mu - 1.$$

The intersection matrix is

$$S=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \mu(4\mu-3) & 2\mu^2 & 0 \\ 0 & (2\mu-1)(4\mu^2-3\mu+1) & 2\mu^2(4\mu-3) & 2\mu(4\mu^2-3\mu+1) \\ k & \mu(4\mu-3) & \mu(2\mu-1) & \mu(4\mu-3) \end{pmatrix}.$$

Although the computations are more tiresome in this case, one can verify that these schemes are never Q-polynomial, and also that the Krein conditions are satisfied for all μ.

The smallest digraph in this family has $n=21$, $k=5$, $l=10$. As a matter of fact A would be the incidence matrix of a plane of order 4, but, unfortunately, one can prove

THEOREM 2. *The digraph in the family $\lambda=2\mu$ with $\mu=1$ does not exist.*

Proof. As noted above, A is 21×21, $A^2=A+2B$ and $AA^T=A^TA=4I+J$. By permutational similarity, we can put A in symmetric block decomposition (of type $1+5+5+5+5$) in the form

$$A=\begin{bmatrix} 0 & 1\cdots 1 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \begin{matrix}0\\ \vdots\\ 0\end{matrix} & A_{11} & A_{12} & A_{13} & A_{14} \\ \begin{matrix}1\\ \vdots\\ 1\end{matrix} & A_{21} & A_{22} & A_{23} & A_{24} \\ \begin{matrix}0\\ \vdots\\ 0\end{matrix} & A_{31} & A_{32} & A_{33} & A_{34} \\ \begin{matrix}0\\ \vdots\\ 0\end{matrix} & A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

One easily gets $A_{12}=0$ since $A^2 * A^T=0$. Similarly, A_{11} is a permutation, and since $A_{11} * I=A_{11} * A_{11}^T=0$, A_{11} is a 5-cycle. Without loss, $A_{11}=C_5$ which we will abbreviate to C. One can also assume $A_{13}=I+C^4$, $A_{14}=I+C^2$. Moreover, A_{21} is also a permutation and since $A_{12}=0$, we can permute so $A_{21}=I$. From the $(2,2)$-position of A^2, we have $(I+C^4)A_{31}+(I+C^2)A_{41}=C+C^2+2C^3$, and A_{31}, A_{41} have row sums one.

These conditions force $A_{31}=C^3$, $A_{41}=C$. Proceeding with the $(3,2)$-positions of A^2 and AA^T we get

$$A_{22}+A_{23}C^3+A_{24}C=C^2+C^3+C^4, \tag{4.1}$$

$$A_{23}+A_{23}C+A_{24}+A_{24}C^3=I+C+C^2+C^3. \tag{4.2}$$

Multiplying (4.1) by C^2 on the right and subtracting from (4.2), we obtain

$$A_{23}C+A_{24}-A_{22}C^2=C^2+C^3-C^4. \tag{4.3}$$

But observing that A_{22}, A_{23} and A_{24} all have columns sums 1, we have

$$A_{22}=C^2,$$

$$A_{23}C+A_{24}=C^2+C^3. \tag{4.4}$$

Substituting in (4.2) we also have

$$A_{23}C+A_{24}C^4=C+C^2, \tag{4.5}$$

and subtracting (4.5) from (4.4),

$$A_{24}-A_{24}C^4=C^3-C.$$

By column sums again, $A_{24}=C^3$ and $A_{24}C^4=C$ are forced, which is nonsense.

On this scanty evidence, one may venture to conjecture that none of the distance-regular digraphs in the family $\lambda=2\mu$ exist.

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