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Ciarlet–Raviart mixed finite element approximation for an optimal control problem governed by the first bi-harmonic equation ☆

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A B S T R A C T

The Ciarlet–Raviart mixed finite element approximation is constructed to solve the constrained optimal control problem governed by the first bi-harmonic equation. The optimality conditions consisting of the state and the co-state equations is derived. Also, the a priori error estimates are analyzed. In the analysis of the a priori error estimates, the improved convergent rate of the higher order than existed results is proved. Some numerical experiments are performed to confirm the theoretical analysis for the a priori error estimate.

1. Introduction

There have been many works on finite element methods for the fourth order partial differential equations (PDEs, for short), of course, containing the bi-harmonic equation as well, such as in [1–12,25,26] and so on. The problems described by bi-harmonic equations arise from fluid mechanics and solid mechanics, such as bending of elastic plates. For the fourth order PDEs, the mixed finite elements scheme is naturally introduced, which will reduce the order of PDEs in the mixed system so as to be solved easily. There has been much research about mixed finite element methods for the 4th order PDEs, for example, Ciarlet–Raviart elements, Herrmann–Miyoshi elements, Hellan–Herrmann–Johnson elements. More details can be found in [13,4–6,9–12] and the references cited therein. Among the mixed finite element methods, Ciarlet–Raviart mixed finite element method of the piecewise linear elements is the special case, for which the weaker convergent rate was proved by Scholz in [12]. Optimal control problems governed by the fourth order PDEs also are encountered in many engineering applications. In [14], Li and Liu introduced a mixed finite element method for the optimal boundary control problem governed by the bi-harmonic equation.

The purpose of this article is to research the C–R mixed finite element method for the optimal distributed control problem governed by the bi-harmonic equations. We investigate the a priori error estimate of the mixed finite element approximation. In the analysis of the a priori error estimates, we improve Scholz's results in [12] for the piecewise linear C–R mixed elements and other C–R mixed elements of polynomial of higher degree in [6,11].

The paper is organized as follows. The model description and the notations used throughout the article are introduced, and also the optimality conditions are given in Section 2. In Section 3, the mixed finite element approximation of the optimal control problem is presented. The a priori error estimates are given and are proved in Section 4. At last, in Section 5, some numerical experiments are performed to confirm the a priori error estimate given in Section 4.

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2. Optimal control problem

Let Ω be a convex domain in R^2 with the Lipschitz boundary $\partial\Omega$. In this paper, we adopt the standard notations $W^{m,q}(\Omega)$ for the Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$ and seminorm $|\cdot|_{m,q,\Omega}$. Let $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$ for $m \geq 1$ and denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$) with the norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. Set $V = H_0^1(\Omega)$, $W = H^1(\Omega)$ and $U = L^2(\Omega)$. Define the functional:

$$J(z, v) = \frac{1}{2}\left\{\int_\Omega (z - y_d)^2 + \int_\Omega (\Delta z)^2 + \alpha \int_\Omega (v - u_0)^2\right\}.$$

Consider the following constrained optimal-control problem:

$$\min_{v \in K} J(z, v) \tag{2.1}$$

governed by the first bi-harmonic equation:

$$\begin{cases} \Delta^2 z = f + v, & \text{in } \Omega, \\ z = \dfrac{\partial z}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $K = \{v \in L^2(\Omega),\ v \geq 0\}$ is a closed convex set in $L^2(\Omega)$, $y_d \in L^2(\Omega)$ is the observation, $f \in L^2(\Omega)$ and u_0 in $L^2(\Omega)$ are given functions, $\alpha > 0$ is a constant.

We will use the mixed form to approach the state equation. Introduce the auxiliary variable:

$$q = -\Delta z$$

and define the functional:

$$\mathcal{J}(z, q, v) = \frac{1}{2}\left\{\int_\Omega (z - y_d)^2 + \int_\Omega q^2 + \alpha \int_\Omega (v - u_0)^2\right\}.$$

Then the problem (2.1)–(2.2) can be rewritten as

$$\min_{v \in K} \mathcal{J}(z, q, v) \tag{2.3}$$

subject to

$$\begin{cases} -\Delta z = q, & \text{in } \Omega, \\ -\Delta q = f + v, & \text{in } \Omega, \\ z = \dfrac{\partial z}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

To construct the mixed finite element approximation of the above optimal control problem, we first give its weak form:

$$\min_{v \in K} \mathcal{J}(z, q, v) \tag{2.5}$$

subject to

$$\begin{cases} (q, w) - (\nabla z, \nabla w) = 0, & \forall\, w \in W, \\ (\nabla q, \nabla r) = (f + v, r), & \forall\, r \in V, \end{cases} \tag{2.6}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

It has been proved, (for example, see the Chapter 2 of [15]), that the convex optimal control problem (2.5)–(2.6) has the unique solution (y, p, u) and that a triplet (y, p, u) is the solution of (2.5)–(2.6) if and only if there is a co-state $(y^*, p^*) \in V \times W$ such that (y, p, u, y^*, p^*) satisfies the following optimality conditions :

$$\begin{aligned} &\text{(a) } (p, w) - (\nabla y, \nabla w) = 0, \quad \forall\, w \in W, \\ &\text{(b) } (\nabla p, \nabla v) = (f + u, v), \quad \forall\, v \in V, \\ &\text{(c) } (p^*, w) - (\nabla y^*, \nabla w) = (-p, w), \quad \forall\, w \in W, \\ &\text{(d) } (\nabla p^*, \nabla v) = (y - y_d, v), \quad \forall\, v \in V, \\ &\text{(e) } (\alpha(u - u_0) + y^*, z - u) \geq 0, \quad \forall\, z \in K. \end{aligned} \tag{2.7}$$

It follows from (2.7)(e) that

$$u = \max\left\{0, -\frac{1}{\alpha} y^* + u_0\right\}. \tag{2.8}$$

The nonlinear system (2.7) gives another approach to solve the optimal control problem (2.5)–(2.6), which suits to be solved by mixed finite element methods. The following lemma gives the regularity of the solution of the first bi-harmonic equation.

Lemma 2.1 *([1,3]). Let Ω be a convex polygon and $F \in H^{-1}(\Omega)$. The solution y_F of the boundary value problem*

$$-\Delta^2 y_F = F, \quad \text{in } \Omega, \qquad y_F = \frac{\partial y_F}{\partial n} = 0, \quad \text{on } \partial\Omega$$

is in $H^3(\Omega)$, and (y_F, p_F), ($p_F = -\Delta y_F$), is the unique solution of the mixed system:

$$\begin{cases} (p_F, w) - (\nabla y_F, \nabla w) = 0, & \forall\, w \in W, \\ (\nabla p_F, \nabla v) = \langle F, w\rangle, & \forall\, v \in V, \end{cases} \tag{2.9}$$

where $\langle\cdot,\cdot\rangle$ stands for the dual product on $H^{-1}(\Omega) \times H^1(\Omega)$. Moreover, there exist $C > 0$, such that for all F,

$$\|y_F\|_{3,\Omega} + \|p_F\|_{1,\Omega} \le C\|F\|_{-1,\Omega}. \tag{2.10}$$

As the consequence, we have the conclusion on the regularity of the solution of the optimal control system.

Lemma 2.2. *Let Ω be a convex polygon. There exists the constant C such that*

$$\|y\|_{3,\Omega} + \|y^*\|_{3,\Omega} + \|p\|_{1,\Omega} + \|p^*\|_{1,\Omega} \le C\Big\{\|u_0\|_{0,\Omega} + \|f\|_{0,\Omega} + \|y_d\|_{0,\Omega}\Big\}, \tag{2.11}$$

where C is independent of $\|f\|_{0,\Omega}$, $\|y_d\|_{0,\Omega}$ and $\|u_0\|_{0,\Omega}$.

Proof. Let y_1 satisfy

$$\Delta^2 y_1 = f, \quad \text{in } \Omega; \qquad y_1 = \frac{\partial y_1}{\partial n} = 0, \quad \text{on } \partial\Omega.$$

It follows from Lemma 2.1 that

$$\|y_1\|_{3,\Omega} \le C\|f\|_{0,\Omega}.$$

Since (y, u) is the solution of the optimal control problem, hence

$$J(y,u) \le J(y_1, 0) = \frac{1}{2}\Big\{\int_\Omega (y_1 - y_d)^2 + \int_\Omega (\Delta y_1)^2 + \alpha\int_\Omega u_0^2\Big\}$$

such that

$$\alpha\|u\|_{0,\Omega}^2 \le 2\big(\alpha\|u_0\|_{0,\Omega}^2 + J(y,u)\big) \le 3\alpha\|u_0\|_{0,\Omega}^2 + 2\|y_1\|_{0,\Omega}^2 + \|\Delta y_1\|_{0,\Omega}^2 + 2\|y_d\|_{0,\Omega}^2,$$

which leads to

$$\|u\|_{0,\Omega} \le C\Big\{\|u_0\|_{0,\Omega} + \|f\|_{0,\Omega} + \|y_d\|_{0,\Omega}\Big\}. \tag{2.12}$$

Again using Lemma 2.1, we know that

$$\|p\|_{1,\Omega} \le \|y\|_{3,\Omega} \le C\Big\{\|f\|_{0,\Omega} + \|u\|_{0,\Omega}\Big\} \le C\Big\{\|u_0\|_{0,\Omega} + \|f\|_{0,\Omega} + \|y_d\|_{0,\Omega}\Big\}. \tag{2.13}$$

On the other hand, the system (2.7) (c)–(d) lead to

$$p^* = -\Delta y^* - p, \qquad -\Delta p^* = y - y_d$$

such that

$$\Delta^2 y^* = f + u + y - y_d, \quad \text{in } \Omega; \qquad y^* = \frac{\partial y^*}{\partial n} = 0, \quad \text{on } \partial\Omega.$$

As the results of Lemma 2.1, we know that

$$\|y^*\|_{3,\Omega} \le C_2\Big\{\|f\|_{0,\Omega} + \|u\|_{L^2(\Omega)} + \|y - y_d\|_{0,\Omega}\Big\} \le C\Big\{\|u_0\|_{0,\Omega} + \|f\|_{0,\Omega} + \|y_d\|_{0,\Omega}\Big\} \tag{2.14}$$

such that

$$\|p^*\|_{1,\Omega} \le \|y^*\|_{3,\Omega} + \|p\|_{1,\Omega} \le C\Big\{\|u_0\|_{0,\Omega} + \|f\|_{0,\Omega} + \|y_d\|_{0,\Omega}\Big\}. \tag{2.15}$$

The proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let Ω be a convex polygon. For $1 \le q \le \infty$, if $u_0 \in W^{1,q}(\Omega)$, then $u \in W^{1,q}(\Omega)$ and*

$$\|u\|_{1,q,\Omega} \le C\Big\{\|u_0\|_{1,q,\Omega} + \|f\|_{0,\Omega} + \|y_d\|_{0,\Omega}\Big\}. \tag{2.16}$$

The a priori estimate (2.16) is the direct result of (2.8) and (2.11).

3. Mixed finite element approximation

Let us consider the approximation of the control problem (2.5). Here we consider only n-simplex elements, as they are widely used in engineering applications. For simplicity, we assume that Ω is a polygonal domain.

Let T^h be a partitioning of Ω into disjoint regular n-simplices τ such that $\bar{\Omega} = \bigcup_{\tau\in T^h} \bar{\tau}$, in which each element has at most one face on $\partial\Omega$, and $\bar{\tau}$ and $\bar{\tau}'$ have either only one common vertex or a whole edge or face if τ and $\tau' \in T^h$. Associated with T^h is a finite dimensional subspace W^h of $C(\bar{\Omega})$, such that $\chi|_\tau$ are polynomial function of the degree lesser than and equal to k, $(k \geq 1)$, for each $\chi \in W^h$ and $\tau \in T^h$. Let $V^h = W^h \cap H_0^1(\Omega)$. It is easy to see that $V^h \subset V$, $W^h \subset W$ and $V^h \subset W^h$.

Let T_U^h be another partitioning of Ω into disjoint regular n-simplices τ_U such that $\bar{\Omega} = \bigcup_{\tau_U\in T_U^h} \bar{\tau}_U$. Assume that $\bar{\tau}_U$ and $\bar{\tau}'_U$ have at most either only one common vertex or a whole edge or face if τ_U and $\tau'_U \in T_U^h$. Associated with T_U^h is another finite dimensional subspace U^h of $L^2(\Omega)$, such that $z_h|_{\tau_U}$ is a constant for each $z_h \in U^h$ and $\tau_U \in T_U^h$. Define $K^h = \{z_h \in U^h;\ z_h \geq 0\}$ as the approximation of K.

Set $h(h_U)$ denote the maximum diameter of the element $\tau(\tau_U)$ in $T^h(T_U^h)$. In addition, c and C denote some general positive constants and ε and δ some arbitrary small positive constants, which are independent of h and h_U.

The mixed finite element approximation of (2.5)–(2.6) is as follows:

$$\min_{v_h\in K^h} \mathcal{J}(z_h, q_h, v_h) \tag{3.1}$$

subject to

$$\begin{aligned} &(q_h, w_h) - (\nabla z_h, \nabla w_h) = 0, \quad \forall\, w_h \in W^h,\\ &(\nabla q_h, \nabla r_h) = (f + v_h, r_h), \quad \forall\, r_h \in V^h. \end{aligned} \tag{3.2}$$

Again, similar to the exact case, it can be proved that the control problem (3.1)–(3.2) has the unique solution (y_h, p_h, u_h), and that a triplet $(y_h, p_h, u_h) \in V^h \times W^h \times U^h$ is the solution of (3.1)–(3.2) if and only if there is a co-state $(y_h^*, p_h^*) \in V^h \times W^h$ such that $(y_h, p_h, u_h, y_h^*, p_h^*)$ satisfies the following discretized optimality conditions:

$$\begin{aligned} &\text{(a) } (p_h, w_h) - (\nabla y_h, \nabla w_h) = 0, \quad \forall\, w_h \in W^h,\\ &\text{(b) } (\nabla p_h, \nabla v_h) = (f + u_h, v_h), \quad \forall\, v_h \in V^h,\\ &\text{(c) } (p_h^*, w_h) - (\nabla y_h^*, \nabla w_h) = (-p_h, w_h), \quad \forall\, w_h \in W^h,\\ &\text{(d) } (\nabla p_h^*, \nabla v_h) = (y_h - y_d, v_h), \quad \forall\, v_h \in V^h,\\ &\text{(e) } (\alpha(u_h - u_0) + y_h^*, z_h - u_h) \geq 0, \quad \forall\, z_h \in K^h. \end{aligned} \tag{3.3}$$

It follows from (3.3)(e) that

$$u_h = \max\left\{0, -\frac{1}{\alpha}\mathcal{P}_h y_h^* + \mathcal{P}_h u_0\right\}, \tag{3.4}$$

where $\mathcal{P}_h$ is the L^2-projection from $L^2(\Omega)$ on to U^h such that

$$(\mathcal{P}_h z, q_h) = (z, q_h), \quad \forall\, q_h \in U^h.$$

It is obvious that

$$\mathcal{P} z|_{\tau_U} = \frac{1}{|\tau_U|}\int_{\tau_U} z, \quad \forall\, \tau_U \in T_{h_U}.$$

4. A priori error estimate

In this section, we analyze the a priori error estimates for the mixed finite element approximation (3.3). We need some regularity assumptions. One is that there exists the constant $\hat{C}_1$ such that

$$\text{(H1)} \begin{cases} \|u_0\|_{1,\Omega} + \|y\|_{3,\Omega} + \|y^*\|_{3,\Omega} \leq \hat{C}_1, & \text{for } k = 1,\\ \|u_0\|_{1,\Omega} + \|y\|_{k+1,\Omega} + \|y^*\|_{k+1,\Omega} \leq \hat{C}_1 & \text{for } k \geq 2. \end{cases}$$

In the cases of $k = 1$ and $k = 2$, it follows from Lemma 2.2 that the condition (H1) holds. Another is that there exists the constant $\hat{C}_2$ such that

$$\text{(H2)} \begin{cases} \|u_0\|_{1,\Omega} + \|y\|_{3,\Omega} + \|y^*\|_{3,\Omega} + \|y\|_{2,\infty,\Omega} + \|y^*\|_{2,\infty,\Omega} \leq \hat{C}_2, & \text{for } k = 1,\\ \|u_0\|_{1,\Omega} + \|y\|_{k+\frac{3}{2},\Omega} + \|y^*\|_{k+\frac{3}{2},\Omega} + \|y\|_{k+1,\infty,\Omega} + \|y^*\|_{k+1,\infty,\Omega} \leq \hat{C}_2 & \text{for } k \geq 2. \end{cases}$$

We will see that there exits better convergent rate for the co-state under the stronger condition (H2). The main conclusion is the following theorem.

Theorem 4.1. *Let (y, p, u, y^*, p^*) and $(y_h, p_h, u_h, y_h^*, p_h^*)$ be the solutions of (2.7) and (3.3), respectively. Assume that the condition (H1) holds.*

In the case of $k = 1$, there hold the a priori error estimates:

$$\|u - u_h\|_{0,\Omega} + \|y - y_h\|_{1,\Omega} + \|y^* - y_h^*\|_{1,\Omega} \le C\left\{h_U + h^{1-\varepsilon}\right\} \tag{4.1}$$

and

$$\|p - p_h\|_{0,\Omega} + \|p^* - p_h^*\|_{0,\Omega} \le C\left\{h_U + h^{\frac{1}{2}-\varepsilon}\right\}, \tag{4.2}$$

where $0 < \varepsilon \ll 1$ and C is the constant dependent upon ε and $\hat{C}_1$ but not h and h_U.

In the case of $k \ge 2$, there hold the a priori error estimates:

$$\|u - u_h\|_{0,\Omega} + \|y - y_h\|_{1,\Omega} + \|y^* - y_h^*\|_{1,\Omega} \le C\left\{h_U + h^{k}\right\} \tag{4.3}$$

and

$$\|p - p_h\|_{0,\Omega} + \|p^* - p_h^*\|_{0,\Omega} \le C\left\{h_U + h^{k-1}\right\}, \tag{4.4}$$

where C is the constant dependent upon $\hat{C}_1$ but not h and h_U.

Furthermore, if the condition (H2) holds, then

$$\|p - p_h\|_{0,\Omega} + \|p^* - p_h^*\|_{0,\Omega} \le C\left\{h_U + h^{k-\frac{1}{2}}\right\}, \tag{4.5}$$

where C is the constant dependent upon $\hat{C}_2$ but not h and h_U.

Before the proof of Theorem 4.1, we remark that the results in Theorem 4.1 improve Scholz's results in [12]. For the piecewise linear C–R mixed element system

$$\begin{aligned} &\text{(a) } (p_h, w_h) - (\nabla y_h, \nabla w_h) = 0, \quad \forall\, w_h \in W^h,\\ &\text{(b) } (\nabla p_h, \nabla v_h) = (f + u_h, v_h), \quad \forall\, v_h \in V^h. \end{aligned} \tag{4.6}$$

Scholz proved the following a priori error estimate:

$$\|p - p_h\|_{0,\Omega} \le Ch^{\frac{1}{2}}|\ln h|\|y\|_{4,\Omega}, \qquad \|y - y_h\|_{1,\Omega} \le Ch^{3/4}|\ln h|\|y\|_{4,\Omega}.$$

From Theorem 4.1 in the case of $k = 1$, we get the a priori error estimate

$$\|y - y_h\|_{1,\Omega} + h^{\frac{1}{2}}\|p - p_h\|_{0,\Omega} \le Ch^{1-\varepsilon}\|y\|_{3,\Omega}$$

and

$$\|p - p_h\|_{0,\Omega} \le Ch^{\frac{1}{2}}\left\{\|y\|_{3,\Omega} + \|y\|_{2,\infty,\Omega}\right\}.$$

So (4.1)–(4.5) are better results. In the case $k \ge 2$, the following result is given in [1,6,16,11]:

$$\|p - p_h\|_{0,\Omega} \le Ch^{k-1}\|y\|_{k+1,\Omega}.$$

It follows from Theorem 4.1 that

$$\|p - p_h\|_{0,\Omega} \le Ch^{k-\frac{1}{2}}.$$

This also is the better result.

To derive the a priori error estimates, we need the following useful lemmas.

Lemma 4.1 *([17,12]). Define the Riesz operators $R_h^0 : H_0^1(\Omega) \mapsto V^h$ by*

$$(\nabla(v - R_h^0 v), \nabla v_h) = 0, \quad \forall\, v_h \in V^h$$

and $R_h : H^1(\Omega) \mapsto W^h$ by

$$(\nabla(w - R_h w), \nabla w_h) = 0, \quad \forall\, w_h \in W^h, \qquad \int_\Omega (w - R_h w) = 0,$$

then for $1 \le s \le \infty$ the following error estimates hold:

$$\|\nabla(v - R_h^0 v)\|_{0,s,\Omega} \le Ch^k\|v\|_{k+1,s,\Omega}, \quad \forall\, v \in W_0^{k+1,s}(\Omega), \tag{4.7}$$

and

$$\|\nabla(w - R_h w)\|_{0,s,\Omega} \le Ch^k\|w\|_{k+1,s,\Omega}, \quad \forall\, w \in W^{k+1,s}(\Omega). \tag{4.8}$$

Lemma 4.2. *Let R_h^0 be the Riesz operators defined in Lemma 4.1, then there exists the constant C such that*

$$(\nabla(v - R_h^0 v), \nabla w_h) \le Ch^{\frac{1}{2}-\varepsilon}\|v\|_{3,\Omega}\|w_h\|_{0,\Omega}, \quad \forall\, w_h \in W^h,\ v \in V \cap H^3(\Omega) \tag{4.9}$$

for $k = 1$, *where* $0 < \varepsilon \ll 1$ *and C is the constant dependent upon* ε *but not h, and*

$$(\nabla(v - R_h^0 v), \nabla w_h) \le Ch^{k-\frac{1}{2}} \|v\|_{k+1,\infty,\Omega} \|w_h\|_{0,\Omega}, \quad \forall\, w_h \in W^h,\ v \in V \cap H^{k+1}(\Omega) \tag{4.10}$$

and

$$(\nabla(v - R_h^0 v), \nabla w_h) \le Ch^{k-1} \|v\|_{k+1,\Omega} \|w_h\|_{0,\Omega}, \quad \forall\, w_h \in W^h,\ v \in V \cap H^{k+1}(\Omega), \tag{4.11}$$

for $k \ge 2$, *where C is the constant independent of h.*

Proof. Divide $\bar{\Omega} = \bigcup_{\tau \in T^h} \bar{\tau}$ into two parts $\bar{\Omega}_0$ and $\bar{\Omega}_1$, where $\bar{\Omega}_1$ is defined by the boundary triangles corresponding to the partition T^h. Then the measure of $\bar{\Omega}_1$ is of size Ch. Relevantly, w_h may be divided into two parts $w_{h1} \in V^h$ such that $w_{h1} = w_h$ at the nodes in $\bar{\Omega}_0$ and $w_{h1} = 0$ at the nodes on $\partial\Omega$, and $w_{h2} \in W^h$ such that $w_{h2} = 0$ at the nodes in $\bar{\Omega}_0$ and $w_{h2} = w_h$ at the nodes on $\partial\Omega$. It follows from Lemma 4.2 and the inverse inequality of the finite element spaces that

$$\begin{aligned}
(\nabla(v - R_h^0 v), \nabla w_h) &= (\nabla(v - R_h^0 v), \nabla(w_{h1} + w_{h2})) \\
&= (\nabla(v - R_h^0 v), \nabla w_{h2}) \\
&= \int_{\Omega_1} \nabla w_{h2} \cdot \nabla(v - R_h^0 v) \\
&\le C \|\nabla w_{h2}\|_{0,2,\Omega_1} \|\nabla(v - R_h^0 v)\|_{0,q_0,\Omega_1} \|1\|_{0,q_1,\Omega_1} \\
&\le Ch^{-\frac{1}{2}} \|w_h\|_{0,2,\partial\Omega} h \|v\|_{2,q_0,\Omega_1} h^{\frac{1}{q_1}} \\
&\le Ch^{\frac{1}{q_1}} \|w_h\|_{0,\Omega} \|v\|_{2,q_0,\Omega_1}
\end{aligned}$$

where

$$\frac{1}{2} + \frac{1}{q_0} + \frac{1}{q_1} = 1, \quad 2 \le q_0 \le \infty,$$

such that

$$\frac{1}{q_1} = \frac{1}{2} - \frac{1}{q_0}.$$

On the other hand, $H^1(\Omega)$ embeds into $L^{q_0}(\Omega)$ for each $1 \le q_0 < \infty$ in 2-dimensional case such that $\|v\|_{2,q_0,\Omega_1} \le C\|v\|_{3,\Omega}$. So we have

$$|(\nabla(v - R_h^0 v), \nabla w_h)| \le Ch^{\frac{1}{2}-\frac{1}{q_0}} \|w_h\|_{0,\Omega} \|v\|_{3,\Omega}.$$

This is (4.9) for $q_0 = 1/\varepsilon$.

Next, we see that

$$\begin{aligned}
(\nabla(v - R_h^0 v), \nabla w_h) &= \int_{\Omega_1} \nabla w_{h2} \cdot \nabla(v - R_h^0 v) \\
&\le Ch^{\frac{1}{2}} \|\nabla w_{h2}\|_{0,\Omega_1} \|\nabla(v - R_h^0 v)\|_{\infty,\Omega_1} \\
&\le Ch^{k} \|w_h\|_{0,\partial\Omega} \|v\|_{k+1,\infty,\Omega} \\
&\le Ch^{k-\frac{1}{2}} \|w_h\|_{0,\Omega} \|v\|_{k+1,\infty,\Omega}.
\end{aligned}$$

This is (4.10).

Finally, we have

$$(\nabla(v - R_h^0 v), \nabla w_h) \le Ch^{k} \|\nabla w_h\|_{0,\Omega} \|v\|_{k+1,\Omega} \le Ch^{k-1} \|w_h\|_{0,\Omega} \|v\|_{k+1,\Omega}.$$

This is (4.11). □

We need to introduce the following auxiliary equations: $(y_h(u), p_h(u)) \in V^h \times W^h$ such that

$$\begin{aligned}
&\text{(a)}\ (p_h(u), w_h) - (\nabla y_h(u), \nabla w_h) = 0, \quad \forall\, w_h \in W^h, \\
&\text{(b)}\ (\nabla p_h(u), \nabla v_h) = (f + u, v_h), \quad \forall\, v_h \in V^h.
\end{aligned} \tag{4.12}$$

The following lemma gives the error estimate between (y, p) and $(y_h(u), p_h(u))$.

Lemma 4.3. *Let* $(y_h(u), p_h(u))$ *be the solutions of* (4.12). *Assume that the condition* (H1) *holds.*

In the case of $k = 1$, *there hold the a priori error estimates:*

$$\|p - p_h(u)\|_{0,\Omega} \le Ch^{\frac{1}{2}-\varepsilon}, \tag{4.13}$$

where $0 < \varepsilon \ll 1$ *and C is the constant dependent upon* ε *and* $\hat{C}_1$ *but not h.*

In the case of $k \geq 2$, *there hold the a priori error estimates:*

$$\|p - p_h(u)\|_{0,\Omega} \leq Ch^{k-1}, \tag{4.14}$$

where C *is the constant dependent upon* $\hat{C}_1$ *but not* h.
Furthermore, if the condition $(H2)$ *holds, then*

$$\|p - p_h(u)\|_{0,\Omega} \leq Ch^{k-\frac{1}{2}}, \quad k \geq 1, \tag{4.15}$$

where C *is the constant dependent upon* $\hat{C}_2$ *but not* h.

Proof. It is clear that

(a) $(p - p_h(u), w_h) - (\nabla(y - y_h(u)), \nabla w_h) = 0, \quad \forall\, w_h \in W^h$,
(b) $(\nabla(p - p_h(u)), \nabla v_h) = 0, \quad \forall\, v_h \in V^h$

such that

$$\begin{aligned}(R_hp - p_h(u), R_hp - p_h(u)) &= (R_hp - p, R_hp - p_h(u)) + (\nabla(y - y_h(u)), \nabla(R_hp - p_h(u))) \\ &= (R_hp - p, R_hp - p_h(u)) + (\nabla(y - R_h^0 y), \nabla(R_hp - p_h(u))).\end{aligned} \tag{4.16}$$

In the case of $k = 1$, it follows from (4.16) and (4.9) that

$$(R_hp - p_h(u), R_hp - p_h(u)) \leq C\left\{h\|p\|_{1,\Omega} + h^{\frac{1}{2}-\varepsilon}\|y\|_{3,\Omega}\right\}\|R_hp - p_h(u)\|_{0,\Omega}$$

such that

$$\|R_hp - p_h(u)\|_{0,\Omega} \leq Ch^{\frac{1}{2}-\varepsilon}\|y\|_{3,\Omega},$$

which implies

$$\|p - p_h(u)\|_{0,\Omega} \leq Ch^{\frac{1}{2}-\varepsilon}\|y\|_{3,\Omega}.$$

This is (4.13).
In the case of $k \geq 2$, it follows from (4.16) and (4.10) that

$$(R_hp - p_h(u), R_hp - p_h(u)) \leq Ch^{k-1}\left\{\|p\|_{k-1,\Omega} + \|y\|_{k+1,\Omega}\right\}\|R_hp - p_h(u)\|_{0,\Omega}$$

such that

$$\|R_hp - p_h(u)\|_{0,\Omega} \leq Ch^{k-1}\|y\|_{k+1,\Omega},$$

which implies

$$\|p - p_h(u)\|_{0,\Omega} \leq Ch^{k-1}\|y\|_{k+1,\Omega}.$$

This is (4.14).
On the other hand, it follows from (4.16) and (4.11) that

$$(R_hp - p_h(u), R_hp - p_h(u)) \leq C\left\{h^{k-\frac{1}{2}}\|p\|_{k-\frac{1}{2},\Omega} + h^{k-\frac{1}{2}}\|y\|_{k+1,\infty,\Omega}\right\}\|R_hp - p_h(u)\|_{0,\Omega}$$

such that

$$\|R_hp - p_h(u)\|_{0,\Omega} \leq Ch^{k-\frac{1}{2}}\left\{\|y\|_{k+\frac{3}{2},\Omega} + \|y\|_{k+1,\infty,\Omega}\right\},$$

which implies

$$\|p - p_h(u)\|_{0,\Omega} \leq Ch^{k-\frac{1}{2}}\left\{\|y\|_{k+\frac{3}{2},\Omega} + \|y\|_{k+1,\infty,\Omega}\right\}.$$

This is (4.15). The proof of Lemma 4.3 is completed. □

Lemma 4.4. *Let* $(y_h(u), p_h(u))$ *be the solutions of* (4.12). *Then under the condition* (H1) , *there hold*

$$\|y - y_h(u)\|_{1,\Omega} \leq Ch^{1-\varepsilon} \tag{4.17}$$

for $k = 1$, *where* $0 < \varepsilon \ll 1$ *and* C *is the constant dependent upon* ε *and* $\hat{C}_1$ *but not* h, *and*

$$\|y - y_h(u)\|_{1,\Omega} \leq Ch^k \tag{4.18}$$

for $k \geq 2$, *where* C *depends upon* $\hat{C}_1$ *but not* h.

Proof. By taking $w = y - y_h(u)$ in (2.9), we obtain

$$(\nabla p_F, \nabla(y - y_h(u))) = \langle F, y - y_h(u) \rangle.$$

We need to estimate the term $(\nabla p_F, \nabla(y - y_h(u)))$, i.e.,

$$(\nabla p_F, \nabla(y - y_h(u))) = (p - p_h(u), p_F) + [(\nabla p_F, \nabla(y - y_h(u))) - (p - p_h(u), p_F)]. \tag{4.19}$$

Bound the two terms of the right hand side of (4.19) one by one.

In the case of $k = 1$, we have

$$\begin{aligned}(\nabla p_F, \nabla(y - y_h(u))) - (p - p_h(u), p_F) &= (\nabla(p_F - R_h p_F), \nabla(y - y_h(u))) - (p - p_h(u), p_F - R_h p_F)\\ &= (\nabla(p_F - R_h p_F), \nabla(y - R_h^0 y)) - (p - p_h(u), p_F - R_h p_F)\\ &\le Ch\Big\{\|y\|_{2,\Omega} + \|p - p_h(u)\|_{0,\Omega}\Big\}\|\nabla p_F\|_{0,\Omega}.\end{aligned}$$

Then, it follows from Lemmas 4.2 and 2.2 that

$$\begin{aligned}(p - p_h(u), p_F) &= (\nabla y_F, \nabla(p - p_h(u)))\\ &= (\nabla(y_F - R_h^0 y_F), \nabla(p - R_h p)) + (\nabla(y_F - R_h^0 y_F), \nabla(R_h p - p_h(u)))\\ &\le C\Big\{h\|p\|_{1,\Omega}\|y_F\|_{2,\Omega} + h^{\frac{1}{2}-\varepsilon}\|R_h p - p_h(u)\|_{0,\Omega}\|y_F\|_{3,\Omega}\Big\}\\ &\le C\Big\{h\|p\|_{1,\Omega} + h^{1-\varepsilon}\|y\|_{3,\Omega}\Big\}\|y_F\|_{3,\Omega}\end{aligned}$$

such that

$$|(\nabla p_F, \nabla(y - y_h(u)))| \le Ch^{1-\varepsilon}\|y\|_{3,\Omega}\Big\{\|p_F\|_{1,\Omega} + \|y_F\|_{3,\Omega}\Big\} \le Ch^{1-\varepsilon}\|y\|_{3,\Omega}\|F\|_{-1,\Omega}. \tag{4.20}$$

Thus we obtain the estimate for $\|y(u_h) - y_h\|_{1,\Omega}$

$$\begin{aligned}\|y - y_h(u)\|_{1,\Omega} &= \sup_{F \in H^{-1}(\Omega)} \frac{\langle F, y - y_h(u)\rangle}{\|F\|_{-1,\Omega}}\\ &= \sup_{F \in H^{-1}(\Omega)} \frac{(\nabla p_F, \nabla(y - y_h(u)))}{\|F\|_{-1,\Omega}}\\ &\le Ch^{1-\varepsilon}\|y\|_{3,\Omega}.\end{aligned}$$

This is (4.17).

In the case of $k \ge 2$, we have

$$\begin{aligned}(\nabla p_F, \nabla(y - y_h(u))) - (p - p_h(u), p_F) &= (\nabla(p_F - R_h p_F), \nabla(y - R_h^0 y)) - (p - p_h(u), p_F - R_h p_F)\\ &\le C\Big\{h^k\|y\|_{k+1,\Omega} + h\|p - p_h(u)\|_{0,\Omega}\Big\}\|\nabla p_F\|_{0,\Omega}\\ &\le Ch^k\|\nabla p_F\|_{0,\Omega}.\end{aligned}$$

On the other hand, it follows from Lemmas 4.2 and 2.2 that

$$\begin{aligned}(p - p_h(u), p_F) &= (\nabla(y_F - R_h^0 y_F), \nabla(p - R_h p)) + (\nabla(y_F - R_h^0 y_F), \nabla(R_h p - p_h(u)))\\ &\le C\Big\{h^k\|p\|_{k-1,\Omega}\|y_F\|_{2,\Omega} + h^2\|\nabla(R_h p - p_h(u))\|_{0,\Omega}\|y_F\|_{3,\Omega}\Big\}\\ &\le C\Big\{h^k\|y\|_{k+1,\Omega}\|y_F\|_{2,\Omega} + h\|R_h p - p_h(u)\|_{0,\Omega}\|y_F\|_{3,\Omega}\Big\}\\ &\le Ch^k\|y_F\|_{3,\Omega}\end{aligned}$$

such that

$$|(\nabla p_F, \nabla(y - y_h(u)))| \le Ch^k\|F\|_{-1,\Omega}. \tag{4.21}$$

We obtain the estimate for $\|y(u_h) - y_h\|_{1,\Omega}$

$$\|y - y_h(u)\|_{1,\Omega} = \sup_{F \in H^{-1}(\Omega)} \frac{\langle F, y - y_h(u)\rangle}{\|F\|_{-1,\Omega}} \le Ch^k.$$

The proof of Lemma 4.4 is completed. □

As the consequence, we have the following conclusion.

Lemma 4.5. *Let* (y, p, u, y^*, p^*) *and* $(y_h, p_h, u_h, y_h^*, p_h^*)$ *be the solutions of* (2.7) *and* (3.3), *respectively. Assume that the condition* (H1) *holds. Then*

$$\|y - y_h\|_{1,\Omega} \leq C\left\{\|u - u_h\|_{0,\Omega} + h^{1-\varepsilon}\right\} \tag{4.22}$$

and

$$\|p - p_h\|_{0,\Omega} \leq C\left\{\|u - u_h\|_{0,\Omega} + h^{\frac{1}{2}-\varepsilon}\right\} \tag{4.23}$$

for $k = 1$, *where* $0 < \varepsilon \ll 1$ *and C depends upon* ε *and* $\hat{C}_1$ *but not h, and*

$$\|y - y_h\|_{1,\Omega} \leq C\left\{\|u - u_h\|_{0,\Omega} + h^{k}\right\} \tag{4.24}$$

and

$$\|p - p_h\|_{0,\Omega} \leq C\left\{\|u - u_h\|_{0,\Omega} + h^{k-1}\right\} \tag{4.25}$$

for $k \geq 2$, *where C depends upon* $\hat{C}_1$ *but not h. Furthermore if the condition* (H2), *then*

$$\|p - p_h\|_{0,\Omega} \leq C\left\{\|u - u_h\|_{0,\Omega} + h^{k-\frac{1}{2}}\right\}, \quad k \geq 1, \tag{4.26}$$

where C only depends upon $\hat{C}_2$ *but not h.*

Proof. It follows from (3.3) and (4.12) that

$$\begin{aligned}
&\text{(a)}\ (p_h - p_h(u), w_h) - (\nabla(y_h - y_h(u)), \nabla w_h) = 0, \quad \forall\, w_h \in W^h,\\
&\text{(b)}\ (\nabla(p_h - p_h(u)), \nabla v_h) = (u - u_h, w_h), \quad \forall\, v_h \in V^h.
\end{aligned} \tag{4.27}$$

Taking $y_h - y_h(u) \in V^h$ in (4.27)(a), we get

$$\begin{aligned}
&\|\nabla(y_h - y_h(u))\|_{0,\Omega}^2 = (\nabla(y_h - y_h(u)), \nabla(y_h - y_h(u)))\\
&\quad = (p_h - p_h(u), y_h - y_h(u)) \leq \|p_h - p_h(u)\|_{0,\Omega}\|y_h - y_h(u)\|_{0,\Omega},
\end{aligned}$$

which leads to

$$\|\nabla(y_h - y_h(u))\|_{1,\Omega} \leq \|p_h - p_h(u)\|_{0,\Omega}.$$

Taking $w_h = p_h - p_h(u)$ in (4.27)(a), we have

$$\begin{aligned}
&\|p_h - p_h(u)\|_{0,\Omega}^2 = (\nabla(y_h - y_h(u)), \nabla(p_h - p_h(u)))\\
&\quad = (u - u_h, y_h - y_h(u)) \leq \|u - u_h\|_{0,\Omega}\|y_h - y_h(u)\|_{0,\Omega}.
\end{aligned}$$

Thus we obtained

$$\|p_h - p_h(u)\|_{0,\Omega} + \|y_h - y_h(u)\|_{1,\Omega} \leq C\|u - u_h\|_{0,\Omega}. \tag{4.28}$$

Noting that

$$\begin{aligned}
&\|p - p_h\|_{0,\Omega} \leq \|p - p_h(u)\|_{0,\Omega} + \|p_h(u) - p_h\|_{0,\Omega},\\
&\|y - y_h\|_{1,\Omega} \leq \|y - y_h(u)\|_{1,\Omega} + \|y_h(u) - y_h\|_{1,\Omega}
\end{aligned}$$

and using (4.28), Lemmas 4.3 and 4.4, we derive (4.22)–(4.26). The proof of Lemma 4.5 ends. □

Next, to estimate the term $\|u - u_h\|_{0,\Omega}$, we need to introduce another auxiliary equations: $(y_h^*(u), p_h^*(u)) \in V^h \times W^h$ such that

$$\begin{aligned}
&\text{(a)}\ (p_h^*(u), w_h) - (\nabla y_h^*(u), \nabla w_h) = (-p_h(u), w_h), \quad \forall\, w_h \in W^h,\\
&\text{(b)}\ (\nabla p_h^*(u), \nabla v_h) = (y_h(u) - y_d, v_h), \quad \forall\, v_h \in V^h.
\end{aligned} \tag{4.29}$$

Now we are in the position of deducing the estimates for $y^* - y_h^*(u)$ and $p^* - p_h^*(u)$.

Lemma 4.6. *Let (y^*, p^*, u) and $(y_h^*(u), p_h^*(u))$ be the solutions of (2.7) and (4.29), respectively. Then under the condition* (H1)*, there holds*

$$\|p^* - p_h^*(u)\|_{0,\Omega} \le Ch^{\frac{1}{2}-\varepsilon} + \delta\|y^* - y_h^*(u)\|_{0,\Omega} \tag{4.30}$$

for $k = 1$, where $0 < \varepsilon, \delta \ll 1$ and C only depends upon ε, δ and $\hat{C}_1$ but not h, and

$$\|p^* - p_h^*(u)\|_{0,\Omega} \le Ch^{k-1} + \delta\|y^* - y_h^*(u)\|_{0,\Omega} \tag{4.31}$$

for $k \ge 2$, where C only depends upon $\hat{C}_1$ but not h. Furthermore, under the condition $(H2)$, there holds

$$\|p^* - p_h^*(u)\|_{0,\Omega} \le Ch^{k-\frac{1}{2}} + \delta\|y^* - y_h^*(u)\|_{0,\Omega}, \quad k \ge 1, \tag{4.32}$$

where C only depends upon $\hat{C}_2$ but not h.

Proof. The proof of the estimate for $\|p^*(y_h) - p_h^*\|_{0,\Omega}$ is similar to that in Lemma 4.3. It is clear that

$$\text{(a) } (p^* - p_h^*(u), w_h) - (\nabla(y^* - y_h^*(u)), \nabla w_h) = (p_h(u) - p, w_h), \quad \forall\, w_h \in W^h,$$
$$\text{(b) } (\nabla(p^* - p_h^*(u)), \nabla v_h) = (y - y_h(u), v_h), \quad \forall\, v_h \in V^h$$

such that

$$(\nabla(R_hp^* - p_h^*(u)), \nabla v_h) = (\nabla(p^* - p_h^*(u)), \nabla v_h) = (y - y_h(u), v_h), \quad \forall\, v_h \in V^h,$$

which implies

$$\begin{aligned}(R_hp^* - p_h^*(u), R_hp^* - p_h^*(u)) &= (R_hp^* - p^*, R_hp^* - p_h^*(u)) + (\nabla(y^* - y_h^*(u)),\\ &\quad \nabla(R_hp^* - p_h^*(u))) + (p_h(u) - p, R_hp^* - p_h^*(u))\\ &= (R_hp^* - p^*, R_hp^* - p_h^*(u)) + (\nabla(y^* - R_h^0y^*),\\ &\quad \nabla(R_hp^* - p_h^*(u))) - (y - y_h(u), y^* - R_h^0y^*) + (y - y_h(u), y^* - y_h^*(u))\\ &\quad + (p_h(u) - p, R_hp^* - p_h^*(u)).\end{aligned} \tag{4.33}$$

In the case of $k = 1$, it follows from (4.33) and (4.9) that

$$\begin{aligned}(R_hp^* - p_h^*(u), R_hp^* - p_h^*(u)) &\le C\Big\{\big(h\|p^*\|_{1,\Omega} + \|p - p_h(u)\|_{0,\Omega} + h^{\frac{1}{2}-\varepsilon}\|y^*\|_{3,\Omega}\big)\|R_hp^* - p_h^*(u)\|_{0,\Omega}\\ &\quad + h^{1-\varepsilon}\|y\|_{3,\Omega}\|y^* - y_h^*(u)\|_{0,\Omega} + h^{2-\varepsilon}\|y^*\|_{2,\Omega}\|y\|_{3,\Omega}\Big\}\end{aligned}$$

such that

$$\|p^* - p_h^*(u)\|_{0,\Omega} \le Ch^{\frac{1}{2}-\varepsilon} + \delta\|y^* - y_h^*(u)\|_{0,\Omega}.$$

This (4.30).

In the case of $k \ge 2$, it follows from (4.33) and (4.10) that

$$\begin{aligned}(R_hp^* - p_h^*(u), R_hp^* - p_h^*(u)) &\le C\Big\{\big[h^{k-1}\big(\|p^*\|_{k-1,\Omega} + \|y^*\|_{k+1,\Omega}\big) + \|p - p_h(u)\|_{0,\Omega}\big]\|R_hp^* - p_h^*(u)\|_{0,\Omega}\\ &\quad + h^{k-1}\|y\|_{k-1,\Omega}\|y^* - y_h^*(u)\|_{0,\Omega} + h^{2k}\|y^*\|_{k+1,\Omega}\|y\|_{k+1,\Omega}\Big\}\end{aligned}$$

such that

$$\|p^* - p_h^*(u)\|_{0,\Omega} \le Ch^{k-1} + \delta\|y^* - y_h^*(u)\|_{0,\Omega}.$$

This (4.31).

Furthermore, it follows from (4.33) and (4.11) that

$$\begin{aligned}(R_hp^* - p_h^*(u), R_hp^* - p_h^*(u)) &\le C\Big\{\big[h^{k-\frac{1}{2}}\big(\|p^*\|_{k-\frac{1}{2},\Omega}\|y^*\|_{k+1,\infty,\Omega}\big) + \|p - p_h(u)\|_{0,\Omega}\big]\|R_hp^* - p_h^*(u)\|_{0,\Omega}\\ &\quad + h^{k-\frac{1}{2}}\big(\|y\|_{k+\frac{3}{2},\Omega} + \|y\|_{k+1,\infty,\Omega}\big)\|y^* - y_h^*(u)\|_{0,\Omega}\\ &\quad + h^{2k+\frac{1}{2}}\|y^*\|_{k+1,\Omega}\big(\|y\|_{k+\frac{3}{2},\Omega} + \|y\|_{k+1,\infty,\Omega}\big)\Big\}\end{aligned}$$

such that

$$\|p^* - p_h^*(u)\|_{0,\Omega} \le Ch^{k-\frac{1}{2}} + \delta\|y^* - y_h^*(u)\|_{0,\Omega}.$$

This (4.32). Then the proof of Lemma 4.6 is completed. □

Lemma 4.7. *Let* (y^*, p^*, u) *and* $(y_h^*(u), p_h^*(u))$ *be the solutions of* (2.7) *and* (4.29)*, respectively. Then under the condition* (H1) *, there holds the a priori error estimate*

$$\|y^* - y_h^*(u)\|_{1,\Omega} \le Ch^{1-\varepsilon} \tag{4.34}$$

for $k = 1$*, where* $0 < \varepsilon \ll 1$ *and C only depends upon* ε *and* $\hat{C}_1$ *but not h, and*

$$\|y^* - y_h^*(u)\|_{1,\Omega} \le Ch^k \tag{4.35}$$

for $k \ge 2$*, where C only depends upon* $\hat{C}_1$ *but not h*

Proof. For any $F \in H^{-1}(\Omega)$, by taking $w = y_h^*(u) - y^*$ in (2.11), we have

$$(\nabla p_F, \nabla(y_h^*(u) - y^*)) = \langle F, y_h^*(u) - y^*\rangle. \tag{4.36}$$

We now rewrite the term $(\nabla p_F, \nabla(y^*(u_h) - y_h^*))$ in the following form:

$$(p_h^*(u) - p^* + p_h(u) - p, p_F) + (\nabla p_F, \nabla(y_h^*(u) - y^*)) - (p_h^*(u) - p^* + p_h(u) - p, p_F). \tag{4.37}$$

We bound the terms on the right-hand side of (4.37). In the case of $k = 1$, it follows Lemmas 4.2 and 4.3 that

$$\begin{aligned}
(p_h^*(u) - p^* + p_h(u) - p, p_F) &= (\nabla(p_h^*(u) - p^*), \nabla y_F) + (\nabla(p_h(u) - p), \nabla y_F)\\
&= (\nabla(p_h^*(u) - p^*), \nabla(y_F - R_h^0 y_F)) + (\nabla(p_h(u) - p),\\
&\quad \nabla(y_F - R_h^0 y_F)) + (y_h(u) - y, R_h^0 y_F)\\
&= (\nabla(R_h p^* - p^*), \nabla(y_F - R_h^0 y_F)) + (\nabla(p_h^*(u) - R_h p^*),\\
&\quad \nabla(y_F - R_h^0 y_F)) + (\nabla(R_h p - p), \nabla(y_F - R_h^0 y_F)) + (\nabla(p_h(u) - R_h p),\\
&\quad \nabla(y_F - R_h^0 y_F)) + (y_h(u) - y, R_h^0 y_F)\\
&\le C\Big\{h\big(\|p\|_{1,\Omega} + \|p^*\|_{1,\Omega}\big)\|y_F\|_{2,\Omega} + h^{\frac{1}{2}-\varepsilon}(\|p_h^*(u) - R_h p^*\|_{0,\Omega}\\
&\quad + \|p_h(u) - R_h p\|_{0,\Omega})\|y_F\|_{3,\Omega} + \|y_h(u) - y\|_{0,\Omega}\|R_h^0 y_F\|_{0,\Omega}\Big\}\\
&\le C\Big\{h^{1-\varepsilon}\Big(\|y\|_{3,\Omega} + \|y^*\|_{3,\Omega}\Big) + \delta\|y^* - y_h^*(u)\|_{0,\Omega}\Big\}\|y_F\|_{3,\Omega}.
\end{aligned}$$

Then, it follows from Lemmas 4.2 and 4.5 that

$$\begin{aligned}
&(\nabla p_F, \nabla(y_h^*(u) - y^*)) - (p_h^*(u) - p^* + p_h(u) - p, p_F)\\
&\quad = (\nabla(y_h^*(u) - y^*), \nabla(p_F - R_h p_F)) - (p_h^*(u) - p^*, p_F - R_h p_F) - (p_h(u) - p, p_F - R_h p_F)\\
&\quad = (\nabla(R_h^0 y^* - y^*), \nabla(p_F - R_h p_F)) - (p_h^*(u) - p^*, p_F - R_h p_F) - (p_h(u) - p, p_F - R_h p_F)\\
&\quad \le Ch\Big\{\|y^*\|_{2,\Omega} + \|p_h^*(u) - p^*\|_{0,\Omega} + \|p_h(u) - p\|_{0,\Omega}\Big\}\|p_F\|_{1,\Omega}.
\end{aligned}$$

Combined with the above two estimations and the equality (4.36), we obtained

$$\langle F, y_h^*(u) - y^*\rangle \le C\Big\{h^{1-\varepsilon} + \delta\|y^* - y_h^*(u)\|_{0,\Omega}\Big\}\|y_F\|_{3,\Omega} \le C\Big\{h^{1-\varepsilon} + \delta\|y^* - y_h^*(u)\|_{0,\Omega}\Big\}\|F\|_{-1,\Omega},$$

which leads to

$$\|y_h^*(u) - y^*\|_{1,\Omega} = \sup_{F\in H^{-1}(\Omega)} \frac{\langle F, y_h^*(u) - y^*\rangle}{\|F\|_{-1,\Omega}} \le Ch^{1-\varepsilon} + \delta\|y^* - y_h^*(u)\|_{0,\Omega}$$

such that

$$\|y_h^*(u) - y^*\|_{1,\Omega} \le Ch^{1-\varepsilon}.$$

This is (4.34).

In the case of $k \ge 2$, we have

$$\begin{aligned}
(p_h^*(u) - p^* + p_h(u) - p, p_F) &= (\nabla(R_h p^* - p^*), \nabla(y_F - R_h^0 y_F)) + (\nabla(p_h^*(u) - R_h p^*), \nabla(y_F - R_h^0 y_F))\\
&\quad + (\nabla(R_h p - p), \nabla(y_F - R_h^0 y_F)) + (\nabla(p_h(u) - R_h p), \nabla(y_F - R_h^0 y_F))\\
&\quad + (y_h(u) - y, R_h^0 y_F)\\
&\le C\Big\{h^k\big(\|p\|_{k-1,\Omega} + \|p^*\|_{k-1,\Omega}\big)\|y_F\|_{3,\Omega} + h^{\frac{3}{2}}(\|p_h^*(u) - R_h p^*\|_{0,\Omega}
\end{aligned}$$

$$+\|p_h(u)-R_hp\|_{0,\Omega})\|y_F\|_{3,\Omega}+\|y_h(u)-y\|_{0,\Omega}\|R_h^0y_F\|_{0,\Omega}\Big\}$$
$$\leq C\Big\{h^k+\delta\|y^*-y_h^*(u)\|_{0,\Omega}\Big\}\|y_F\|_{3,\Omega}.$$

Then, it follows from Lemmas 4.2 and 4.5 that

$$\begin{aligned}
&(\nabla p_F,\nabla(y_h^*(u)-y^*))-(p_h^*(u)-p^*+p_h(u)-p,p_F)\\
&\quad=(\nabla(y_h^*(u)-y^*),\nabla(p_F-R_hp_F))-(p_h^*(u)-p^*,p_F-R_hp_F)-(p_h(u)-p,p_F-R_hp_F)\\
&\quad=(\nabla(R_h^0y^*-y^*),\nabla(p_F-R_hp_F))-(p_h^*(u)-p^*,p_F-R_hp_F)-(p_h(u)-p,p_F-R_hp_F)\\
&\quad\leq h\Big\{h^k\|y^*\|_{K+1,\Omega}+h\big(\|p_h^*(u)-p^*\|_{0,\Omega}+\|p_h(u)-p\|_{0,\Omega}\big)\Big\}\|p_F\|_{1,\Omega}.
\end{aligned}$$

Combined with the above two estimations and the equality (4.36), we obtained

$$\langle F,y_h^*(u)-y^*\rangle\ \leq C\Big\{h^k+\delta\|y^*-y_h^*(u)\|_{0,\Omega}\Big\}\|y_F\|_{3,\Omega}\leq C\Big\{h^k+\delta\|y^*-y_h^*(u)\|_{0,\Omega}\Big\}\|F\|_{-1,\Omega},$$

which leads to

$$\|y_h^*(u)-y^*\|_{1,\Omega}=\sup_{F\in H^{-1}(\Omega)}\frac{\langle F,y_h^*(u)-y^*\rangle}{\|F\|_{-1,\Omega}}\leq Ch^k+\delta\|y^*-y_h^*(u)\|_{0,\Omega}$$

such that

$$\|y_h^*(u)-y^*\|_{1,\Omega}\leq Ch^k.$$

The proof of Lemma 4.7 is completed. □

Lemma 4.8. *Let (y_h^*,p_h^*) and $(y_h^*(u),p_h^*(u))$ be the solutions of (3.3) and (4.29), respectively. Then*

$$\|p_h^*(u)-p_h^*\|_{0,\Omega}+\|y_h^*(u)-y_h^*\|_{1,\Omega}\leq C\|u-u_h\|_{0,\Omega}. \tag{4.38}$$

Proof. It is clear that

$$\begin{aligned}
&\text{(a) } (p_h^*(u)-p_h^*,w_h)-(\nabla(y_h^*(u)-y_h^*),\nabla w_h)=(p_h-p_h(u),w_h),\quad\forall\, w_h\in W^h,\\
&\text{(b) } (\nabla(p_h^*(u)-p_h^*),\nabla v_h)=(y_h(u)-y_h,v_h),\quad\forall\, v_h\in V^h.
\end{aligned}\tag{4.39}$$

By taking $w_h=y_h^*(u)-y_h^*$ in (4.39)(a), we have

$$(p_h^*(u)-p_h^*,y_h^*(u)-y_h^*)-(\nabla(y_h^*(u)-y_h^*),\nabla(y_h^*(u)-y_h^*))=(p_h-p_h(u),y_h^*(u)-y_h^*)$$

such that

$$\|\nabla(y_h^*(u)-y_h^*)\|_{0,\Omega}^2\leq C\Big\{\|p_h-p_h(u)\|_{0,\Omega}^2+\|p_h^*-p_h^*(u)\|_{0,\Omega}^2\Big\}.$$

By taking $v_h=y_h^*(u)-y_h^*$ and $w_h=p_h^*(u)-p_h^*$ in (4.39), we see that

$$(p_h^*(u)-p_h^*,p_h^*(u)-p_h^*)=(p_h-p_h(u),p_h^*(u)-p_h^*)+(y_h(u)-y_h,y_h^*(u)-y_h^*)$$

such that

$$\|p_h^*(u)-p_h^*\|_{0,\Omega}^2\leq C\Big\{\|p_h-p_h(u)\|_{0,\Omega}^2+\|y_h(u)-y_h\|_{0,\Omega}^2\Big\}.$$

Summing the results above, we get

$$\|p_h^*(u)-p_h^*\|_{0,\Omega}^2+\|y_h^*(u)-y_h^*\|_{1,\Omega}^2\leq C\Big\{\|p_h-p_h(u)\|_{0,\Omega}^2+\|y_h(u)-y_h\|_{0,\Omega}^2\Big\}. \tag{4.40}$$

Applying (4.28) into (4.40) leads to (4.34). □

Lemma 4.9. *Let (y_h^*,p_h^*) and $(y_h^*(u),p_h^*(u))$ be the solutions of (3.3) and (4.29), respectively. Then under the condition* (H1) *, there hold*

$$\|u-u_h\|_{0,\Omega}\leq\ C\Big\{h_U+h^{1-\varepsilon}\Big\} \tag{4.41}$$

for $k=1$, where $0<\varepsilon\ll 1$ and C depends upon ε and $\hat{C}_1$ but not h and h_U, and

$$\|u-u_h\|_{0,\Omega}\leq\ C\Big\{h_U+h^k\Big\} \tag{4.42}$$

for $k\geq 2$, where C depends upon $\hat{C}_1$ but not h and h_U.

Proof. Noting that

$$\begin{aligned}&\text{(a) } (p_h^*(u)-p_h^*, w_h) - (\nabla(y_h^*(u)-y_h^*), \nabla w_h) = (p_h - p_h(u), w_h), \quad \forall\, w_h \in W^h,\\ &\text{(b) } (\nabla(p_h^*(u)-p_h^*), \nabla v_h) = (y_h(u)-y_h, v_h), \quad \forall\, v_h \in V^h,\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}&\text{(a) } (p_h(u)-p_h, w_h) - (\nabla(y_h(u)-y_h), \nabla w_h) = 0, \quad \forall\, w_h \in W^h,\\ &\text{(b) } (\nabla(p_h(u)-p_h), \nabla v_h) = (u-u_h, v_h), \quad \forall\, v_h \in V^h,\end{aligned} \tag{4.44}$$

and by taking $v_h = y_h^*(u) - y_h^*$ in (4.44), we have

$$\begin{aligned}(u-u_h, y_h^*(u)-y_h^*) &= (\nabla(p_h(u)-p_h), \nabla(y_h^*(u)-y_h^*))\\ &= (p_h^*(u)-p_h^*, p_h(u)-p_h) + (p_h(u)-p_h, p_h(u)-p_h).\end{aligned} \tag{4.45}$$

Then by taking $w_h = p_h^*(u) - p_h^*$ in (4.44) and $v_h = y_h(u) - y_h$ in (4.40), we have

$$(p_h(u)-p_h, p_h^*(u)-p_h^*) - (\nabla(y_h(u)-y_h), \nabla(p_h^*(u)-p_h^*)) = 0$$

and

$$(\nabla(p_h^*(u)-p_h^*), \nabla(y_h(u)-y_h)) = (y_h(u)-y_h, y_h(u)-y_h)$$

such that

$$(p_h(u)-p_h, p_h^*(u)-p_h^*) = (y_h(u)-y_h, y_h(u)-y_h). \tag{4.46}$$

We have

$$\begin{aligned}&\alpha(u-u_h, u-u_h) + (p_h(u)-p_h, p_h(u)-p_h) + (y_h(u)-y_h, y_h(u)-y_h)\\ &= (u-u_h, \alpha(u-u_0) + y^* + y_h^*(u) - y^*) - (u-u_h, \alpha(u_h-u_0) + y_h^*)\\ &\le (u-u_h, y_h^*(u)-y^*) - (u-\mathcal{P}_h u, \alpha(u_h-u_0) + y_h^*),\end{aligned} \tag{4.47}$$

where we use the fact $\mathcal{P}_h u \in K^h$ since $u \geq 0$ such that its local averaging also is non-negative. So we derive

$$\|u-u_h\|_{0,\Omega}^2 + \|y_h(u)-y_h\|_{0,\Omega}^2 + \|p_h(u)-p_h\|_{0,\Omega}^2 \le C\left\{\|y_h^*(u)-y^*\|_{0,\Omega}^2 + h_U^2\left(\|u\|_{1,\Omega}^2 + \|u_0\|_{1,\Omega}^2 + \|y^*\|_{1,\Omega}^2\right)\right\}.$$

The proof of Lemma 4.9 is completed. □

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. *From Lemmas 4.4, 4.7 and 4.8 and under the condition* (H1) , *we derive*

$$\|y_h(u)-y_h\|_{1,\Omega} + \|y_h^*(u)-y_h^*\|_{1,\Omega} \le C\left\{h_U + h^{1-\varepsilon}\right\}$$

and

$$\|p_h(u)-p_h\|_{0,\Omega} + \|p_h^*(u)-p_h^*\|_{1,\Omega} \le C\left\{h_U + h^{\frac{1}{2}-\varepsilon}\right\}$$

in the case of $k=1$, *and*

$$\|y_h(u)-y_h\|_{1,\Omega} + \|y_h^*(u)-y_h^*\|_{1,\Omega} \le C\left\{h_U + h^{k}\right\}$$

and

$$\|p_h(u)-p_h\|_{0,\Omega} + \|p_h^*(u)-p_h^*\|_{1,\Omega} \le C\left\{h_U + h^{k-1}\right\}$$

in the case of $k \geq 2$. *Furthermore, under the condition* (H2), *we have*

$$\|p_h(u)-p_h\|_{0,\Omega} + \|p_h^*(u)-p_h^*\|_{1,\Omega} \le C\left\{h_U + h^{k-\frac{1}{2}}\right\}$$

for $k \geq 1$. *By using*

$$\begin{aligned}&\|y-y_h\|_{1,\Omega} + \|y^*-y_h^*\|_{1,\Omega} \le \|y-y_h(u)\|_{1,\Omega} + \|y_h(u)-y_h\|_{1,\Omega} + \|y^*-y_h^*(u)\|_{1,\Omega} + \|y_h^*(u)-y_h^*\|_{1,\Omega},\\ &\|p-p_h\|_{0,\Omega} + \|p^*-p_h^*\|_{0,\Omega} \le \|p-p_h(u)\|_{0,\Omega} + \|p_h(u)-p_h\|_{0,\Omega} + \|p^*-p_h^*(u)\|_{0,\Omega} + \|p_h^*(u)-p_h^*\|_{0,\Omega},\end{aligned}$$

we can derive (4.1)–(4.2). The proof of Theorem 4.1 is completed. □

Table 1
Example 1 with $k = 1$.

		u piecewise constant, y, p, y^*, p^* piecewise linear			
		$h = 0.08$	$h = 0.04$	$h = 0.02$	$h = 0.01$
Mesh	# nodes	224	783	3007	11 800
	# edges	617	2246	8818	34 997
	# elements	394	1464	5812	23 198
DOFs of	control	394	1464	5812	23 198
	state & co-state	224	783	3007	11 800
Error	$\|u-u_h\|_{0,\Omega}$	4.876e−2	2.530e−2	1.275e−3	6.373e−3
	$\|y-y_h\|_{1,\Omega}$	1.098e−3	5.711e−4	2.847e−4	1.425e−4
	$\|p-p_h\|_{0,\Omega}$	8.286e−4	2.000e−4	7.343e−5	5.810e−5
	$\|y^*-y_h^*\|_{1,\Omega}$	2.197e−3	1.142e−3	5.694e−4	2.849e−4
	$\|p^*-p_h^*\|_{0,\Omega}$	8.201e−4	1.956e−4	7.276e−5	5.799e−5

Table 2
Example 1. Convergent rate with $k = 1$.

	u piecewise constant, y, p, y^*, p^* piecewise linear			
	$h = 0.08$	$h = 0.04$	$h = 0.02$	$h = 0.01$
$\|u-u_h\|_{0,\Omega}$		0.96	0.99	1.00
$\|y-y_h\|_{1,\Omega}$		0.94	1.00	1.00
$\|p-p_h\|_{0,\Omega}$		2.05	1.07	0.69
$\|y^*-y_h^*\|_{1,\Omega}$		0.94	1.00	1.00
$\|p^*-p_h^*\|_{0,\Omega}$		2.07	1.43	0.69

5. Numerical experiments

In this section, we carry out some numerical experiments to demonstrate the a priori error estimates developed in Section 4. As the model problem, we investigate the optimal control problem (2.1) in $\Omega = (0, 1)^2$:

$$\min_{u\in K} J(y, u) = \frac{1}{2}\int_\Omega (y - y_d)^2 + \frac{1}{2}\int_\Omega (\Delta y)^2 + \frac{1}{2}\int_\Omega (u - u_0)^2 \tag{5.1}$$

subject to

$$\begin{cases} \Delta^2 y = f + u, & \text{in } \Omega, \\ y = \dfrac{\partial y}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

We perform two numerical experiments. We compute the state and co-state with piecewise linear approximation in the first example, and then with piecewise quadratic approximation in the second numerical experiment. For the approximation of the control variable, we only use piecewise constant elements. In computing the solutions, we used the software package: AFEpack, see [18] for the details.

Example 1. In the first numerical experiment, the date and the exact solution are as follows:

$$\begin{aligned}
&y = x_1^2(1-x_1)^2 x_2^2(1-x_2)^2, \qquad y^* = 2y\\
&p = -\Delta y, \qquad p^* = p\\
&u_0 = \sin(2\pi x_1)\sin(2\pi x_2)\\
&u = \max(u_0 - y^*, 0)\\
&f = \Delta^2 y - u\\
&y_d = y - f - u.
\end{aligned}$$

Firstly, we approximate the control u by using piecewise constant elements and both the state and the co-state by using the piecewise linear elements on the same meshes, i.e., $h = h_U$. The error estimates of the control, the state and the co-state are in the following Table 1.

The convergent rates are put into the Table 2.

From the results shown in Table 2, we may clearly see that the convergence rate of the control, the state and co-state are order 1, which coincide with our analysis, i.e.,

(a) $\|u-u_h\|_{0,\Omega} + \|y-y_h\|_{1,\Omega} + \|y^*-y_h^*\|_{1,\Omega} \le C\left\{h_U + h^{1-\varepsilon}\right\}$,

(b) $\|p-p_h\|_{0,\Omega} + \|p^*-p_h^*\|_{0,\Omega} \le C\left\{h_U + h^{\frac{1}{2}}\right\}$.

Table 3
Example 1 with $k = 2$.

		u piecewise constant, y, p, y^*, p^* piecewise quadratic			
		$h = 0.08$	$h = 0.04$	$h = 0.02$	$h = 0.01$
mesh	# nodes	224	783	3007	11 800
	# edges	617	2246	8818	34 997
	# elements	394	1464	5812	23 198
DOFs of	control	394	1464	5812	23 198
	state & co-state	841	3029	11 825	46 797
Error	$\|u-u_h\|_{0,\Omega}$	4.882e−2	2.539e−2	1.275e−2	6.374e−3
	$\|y-y_h\|_{1,\Omega}$	8.026e−5	2.146e−5	5.307e−6	1.331e−6
	$\|p-p_h\|_{0,\Omega}$	6.242e−4	2.215e−4	7.534e−5	2.647e−5
	$\|y^*-y_h^*\|_{1,\Omega}$	1.603e−4	4.284e−5	1.060e−5	2.660e−6
	$\|p^*-p_h^*\|_{0,\Omega}$	6.216e−4	2.209e−4	7.526e−5	2.646e−5

Table 4
Example 1. Convergent rate with $k = 2$.

	u piecewise constant, y, p, y^*, p^* piecewise quadratic			
	$h = 0.08$	$h = 0.04$	$h = 0.02$	$h = 0.01$
$\|u-u_h\|_{0,\Omega}$		0.94	0.99	1.00
$\|y-y_h\|_{1,\Omega}$		1.90	2.02	2.00
$\|p-p_h\|_{0,\Omega}$		1.49	1.56	1.51
$\|y^*-y_h^*\|_{1,\Omega}$		1.93	2.02	1.99
$\|p^*-p_h^*\|_{0,\Omega}$		1.49	1.55	1.51

Secondly, we approximate the control u by piecewise constant elements and both the state and co-state by using the piecewise quadratic elements on the same meshes. The error estimates of the control, the state and the co-state are in the following Table 3.

The convergent rates are put into the Table 4.

From the results shown in Tables 3 and 4, we may clearly see that the convergence rate of the control is order 1, which coincide with our analysis, i.e.,

$$\text{(a) } \|u-u_h\|_{0,\Omega} + \|y-y_h\|_{1,\Omega} + \|y^*-y_h^*\|_{1,\Omega} \le C\left\{h_U + h^2\right\},$$

$$\text{(b) } \|p-p_h\|_{0,\Omega} + \|p^*-p_h^*\|_{0,\Omega} \le C\left\{h_U + h^{\frac{3}{2}}\right\}.$$

here $h_U = h$. However the error of the co-state is order 3/2 and the error of the state is order 2. It seems that $\|y-y_h\|_{1,\Omega}$, $\|y^*-y_h^*\|_{1,\Omega}$, $\|p-p_h\|_{0,\Omega}$ and $\|p^*-p_h^*\|_{0,\Omega}$ is not effected by h_U. These maybe the super-convergence. Similar super-convergent results of finite element approximations for other optimal control problems governed by the second order PDEs have been observed and proved by Meyer and Rösch firstly and then others in [19–24]. We will try to prove the super-convergence for C–R mixed finite element methods of the control problem governed by the first bi-harmonic equation in the further work.

Example 2. In this example, the date and the exact solutions are as follows:

$$y = \sin^2(\pi x_1)\sin^2(\pi x_2), \qquad y^* = 2y$$
$$p = -\Delta y, \qquad p^* = p$$
$$u_0 = 8\pi^2 \sin(\pi x_1) * \sin(\pi x_2)$$
$$u = \max(u_0 - y^*, 0)$$
$$f = \Delta^2 y - u$$
$$y_d = y - f - u.$$

Firstly, we approximate the control u by piecewise constant elements and both the state and co-state by using the piecewise linear elements on the same meshes, i.e., $h = h_U$. The error estimates of the control, the state and the co-state are in the following Table 5.

The convergent rates are put into the Table 6.

From the results shown in Table 6, we may clearly see that the convergence rate of the control and the state are order 1, which coincide with our analysis.

Secondly, we approximate the control u by piecewise constant elements, and the state and co-state by using the piecewise quadratic elements on the different meshes, in which $h \approx \sqrt{h_U}$. The error estimates of the control, the state and the co-state are in the following Table 7.

Table 5
Example 2 with $k = 1$.

		u piecewise constant, y, p, y^*, p^* piecewise linear			
		$h = 0.08$	$h = 0.04$	$h = 0.02$	$h = 0.01$
Mesh	# nodes	224	782	3005	11 817
	# edges	617	2243	8812	35 048
	# elements	394	1462	5808	23 232
DOFs of	control	394	1462	5808	23 232
	state & co-state	224	782	3005	11 817
Error	$\|u-u_h\|_{0,\Omega}$	2.709e−0	1.404e−0	7.006e−1	3.507e−1
	$\|y-y_h\|_{1,\Omega}$	2.724e−1	1.398e−1	6.964e−2	3.484e−2
	$\|p-p_h\|_{0,\Omega}$	3.239e−1	8.625e−2	2.344e−2	1.062e−2
	$\|y^*-y_h^*\|_{1,\Omega}$	5.447e−1	2.796e−1	1.393e−1	6.968e−2
	$\|p^*-p_h^*\|_{0,\Omega}$	3.203e−1	8.525e−2	2.321e−2	1.059e−2

Table 6
Example 2. Convergent rate with $k = 1$.

	u piecewise constant, y, p, y^*, p^* piecewise linear			
	$h = 0.08$	$h = 0.04$	$h = 0.02$	$h = 0.01$
$\|u-u_h\|_{0,\Omega}$		0.95	1.00	1.00
$\|y-y_h\|_{1,\Omega}$		0.96	1.01	1.00
$\|p-p_h\|_{0,\Omega}$		1.91	1.88	1.14
$\|y^*-y_h^*\|_{1,\Omega}$		1.31	1.00	1.00
$\|p^*-p_h^*\|_{0,\Omega}$		1.91	1.88	1.13

Table 7
Example 2 with $k = 2$ on different meshes.

		u piecewise constant, y, p, y^*, p^* piecewise quadratic			
		Mesh1	Mesh2	Mesh3	Mesh4
u mesh	h_U	0.08	0.04	0.02	0.01
	# nodes	305	985	3257	11 897
	# DOFs	544	1856	6304	23 392
$y-p$ mesh	h	0.32	0.16	0.08	0.04
	# nodes	26	73	224	782
	# DOFs	85	261	841	3025
Error	$\|u-u_h\|_{0,\Omega}$	2.358e−0	1.274e−0	6.781e−1	3.511e−1
	$\|y-y_h\|_{1,\Omega}$	2.045e−01	5.943e−2	1.176e−2	4.612e−3
	$\|p-p_h\|_{0,\Omega}$	1.078e−0	2.024e−1	4.055e−2	6.339e−3
	$\|y^*-y_h^*\|_{1,\Omega}$	4.090e−1	1.189e−1	3.432e−2	9.225e−3
	$\|p^*-p_h^*\|_{0,\Omega}$	1.077e−0	2.023e−1	4.054e−2	6.337e−3

In many practical applications, one cares much more about the control. Comparing Table 5 with Table 7, the accuracy of the two numerical tests are almost equal. But the number of the global freedom of the first test is almost 10 times of the number of the global freedom of the second test, so that much computational work is saved in the second test. One may use a very coarse grid to solve the state and the co-state.

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