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There Is No Tame Triangulation of the Infinite Real Grassmannian[1]

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We show that there is no triangulation of the infinite real Grassmannian $G(k, \mathbb{R}^\infty)$ nicely situated with respect to the coordinate axes. In terms of matroid theory, this says there is no triangulation of $G(k, \mathbb{R}^\infty)$ subdividing the matroid stratification. This is proved by an argument in projective geometry, considering a specific sequence of configurations of points in the plane. © 2001 Academic Press

The Grassmannian $G(k, \mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ is a smooth manifold, hence can be triangulated. Identify $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$, and let $\mathbb{R}^\infty$ be the union (colimit) of the $\mathbb{R}^n$'s. The Grassmannian $G(k, \mathbb{R}^\infty)$ is infinite dimensional; it is unclear whether it can be triangulated for $k \geq 3$. We are interested in triangulations which are nicely situated with respect to the coordinates axes. Such triangulations are of interest in combinatorics in the context of matroid theory; see Section 4.

DEFINITION 0.1. A triangulation of $G(k, \mathbb{R}^n)$ or $G(k, \mathbb{R}^\infty)$ is *tame* if for every simplex σ, for every pair of k-planes $V, W \in \operatorname{int}\sigma$, and for every vector $v \in V$, there is a vector $w \in W$ so that for all of the standard basis vectors e_i,

$$v \cdot e_i = 0 \Leftrightarrow w \cdot e_i = 0.$$

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Using triangulation theorems from real algebraic geometry, it is not difficult to prove the following theorem (see [3, 5]).

THEOREM 0.2. *For every k and n, there is a tame triangulation $T_{n,k}$. Furthermore for $n' \leq n$ and $k' \leq k$, the triangulation $T_{n,k}$ restricts to a subdivision of $T_{n',k'}$*

This theorem does not lead to a triangulation of $G(k, \mathbb{R}^\infty)$, because perhaps one would have to infinitely subdivide $G(k, \mathbb{R}^n) \subset G(k, \mathbb{R}^\infty)$.

MAIN THEOREM. *There is no tame triangulation of $G(3, \mathbb{R}^\infty)$.*

It follows immediately that there is no tame triangulation of $G(k, \mathbb{R}^\infty)$ for $k \geq 3$.

We first rephrase the main theorem in terms of matroids and oriented matroids and give some very basic context. Section 2 of the paper gives the proof of the main theorem, Section 3 gives generalizations of the main theorem to more general subdivisions and more general stratifications. Section 4 discusses matroid bundles and the MacPhersonian, which gives the context for this paper. The last three sections can be read independently of one another.

The Grassmannians $G(1, \mathbb{R}^\infty)$ and $G(2, \mathbb{R}^\infty)$ do have tame triangulations. In the case of $G(1, \mathbb{R}^\infty)$, this is easily seen by viewing $G(1, \mathbb{R}^\infty)$ as the quotient space of $\{v \in \mathbb{R}^\infty : \|v\| = 1\}$ by the antipodal action. The proof of Proposition 3.1 in [1] constructs triangulations of all $G(2, \mathbb{R}^n)$ in such a way that if $n' > n$ then the triangulation of $G(2, \mathbb{R}^n)$ is a restriction of the triangulation of $G(2, \mathbb{R}^{n'})$. Thus the limit of these triangulations is a triangulation of $G(2, \mathbb{R}^\infty)$.

1. MATROID STRATIFICATIONS

The motivation for tameness comes from matroids and oriented matroids, which are combinatorial abstractions of linear algebra. We don't give the definition here (see [3] for the definition, and [4] for the full story), but simply state that an oriented matroid on a set E is a subset of all functions from E to the three-element set $\{+, -, 0\}$, where the subset satisfies certain axioms. A similar definition for a matroid can be given as a subset of all functions from E to $\{1, 0\}$. Any oriented matroid determines a matroid by identifying $+$ and $-$ with 1. The functions in the (oriented) matroid are called covectors. An (oriented) matroid has a rank associated to it, and the MacPhersonian $\mathrm{MacP}(k, n)$ is the set of all rank k oriented matroids on $E = \{1, 2, \ldots, n\}$. (The MacPhersonian has a natural partial order, but this does not play a role in the proof of our main theorem.) Let $\{e_1, e_2, \ldots, e_n\}$ denote the unit coordinate vectors in $\mathbb{R}^n$.

For any $V \in G(k, \mathbb{R}^n)$ there is an associated rank k oriented matroid on $E = \{1, 2, \ldots, n\}$ whose set of covectors is the set of all

$$\{i \mapsto \operatorname{sign}\langle v, e_i\rangle\}_{v \in V}.$$

There is also a matroid associated to the arrangement of vectors, where $i \mapsto 1$ if and only if $\langle v, v_i\rangle$ is non-zero.

DEFINITION 1.1. The *(oriented) matroid stratification* of $G(k, \mathbb{R}^n)$ is the partition in which V, W are in the same block of the partition if and only if they determine the same (oriented) matroid. Blocks of the partition are called (oriented) matroid strata.

We now see that a tame triangulation is nothing more than a triangulation which refines the matroid stratification, that is, every stratum is a union of interiors of simplices. If M_1 and M_2 are distinct rank k oriented matroids on $\{1, 2, \ldots, n\}$ with the same underlying matroid, then the strata of M_1 and M_2 lie in disjoint open subsets of $G(k, \mathbb{R}^n)$. Thus any connected subset of a matroid stratum is contained in an oriented matroid stratum, and so any tame triangulation must refine the oriented matroid stratification. Our main proof will show that no triangulation of $G(3, \mathbb{R}^\infty)$ refines the oriented matroid stratification.

The (oriented) matroid stratification is interesting geometrically and combinatorially and has been studied extensively [4]. We note two results. First is the observation of Gelfand, Goresky, MacPherson, and Serganova that the matroid stratification is precisely the coarsest common refinement of all of the Schubert cell decompositions given by the standard basis and permutations of the standard basis. The second result is the theorem of Mnëv [6] that the oriented matroid strata can have arbitrarily ugly homotopy type; i.e., for any semialgebraic set S, there is an n and an oriented matroid stratum of $G(3, \mathbb{R}^n)$ for some n having the homotopy type of S.

Our interest in this question arose in considering the theory of *matroid bundles* (cf. [2]), a combinatorial model for real vector bundles. The relationship between $G(k, \mathbb{R}^\infty)$ and $\mathrm{MacP}(k, \infty)$ is a critical question in the theory. The results in this paper arose as a revelation that extending arguments from the finite Grassmannians to the infinite Grassmannian is harder than one might anticipate.

2. PROOF OF THE MAIN THEOREM

The proof involves constructing a sequence of specific elements of $G(3, \mathbb{R}^\infty)$ which, assuming a tame triangulation, leads to an infinite number of simplices in a compact space $G(3, \mathbb{R}^8)$, and hence to a contradiction.

However, it is rather difficult to visualize 3-planes in $\mathbb{R}^n$, so instead we work with arrangements of vectors in $\mathbb{R}^3$. Let $\mathrm{Arr}(k, n)$ be the space of all spanning n-tuples $(v_1, \ldots, v_n)$ of vectors in $\mathbb{R}^k$. Include $\mathrm{Arr}(k, n) \subset \mathrm{Arr}(k, n+1)$ by adding the zero vector.

LEMMA 2.1 [4, *Proposition* 2.4.4]. *There is a homeomorphism* $\phi : G(k, \mathbb{R}^n) \to \mathrm{Arr}(k, n)/GL_k$ *with* $\phi(V) = [(\alpha \circ \pi_V(e_1), \ldots, \alpha \circ \pi_V(e_n))]$, *where* $\pi_V : \mathbb{R}^n \to V$ *is orthogonal projection and* $\alpha : V \to \mathbb{R}^k$ *is any isomorphism*.

To aid in visualization, we recall in detail the standard map $m : \mathrm{Arr}(k, n) \to \mathrm{MacP}(k, n)$. Given an arrangement $(v_1, v_2, \ldots, v_n)$, the nonzero covectors of the associated oriented matroid are given as follows. Any oriented $(k-1)$-dimensional subspace L of $\mathbb{R}^k$ determines a covector $i \mapsto +, -,$ or 0 depending on whether v_i is above, is below, or is on L. This map m is invariant under the action of GL_k on $\mathrm{Arr}(k, n)$, and $m[\phi(V)]$ is precisely the oriented matroid associated to each point in $G(k, \mathbb{R}^n)$ mentioned earlier.

For a rank k oriented matroid M on $\{1, 2, \ldots, n\}$, let $U_M \subset G(k, \mathbb{R}^n)$ be the associated stratum.

LEMMA 2.2. (1) *For* $M \in \mathrm{MacP}(k, n)$, $\phi(U_M) = m^{-1}(M)/GL_k$.

(2) *For* $M \in \mathrm{MacP}(k, n)$, $\phi(\overline{U_M}) = \overline{m^{-1}(M)}/GL_k$.

The first statement is clear from the definitions. The second follows from the first and the fact that $\mathrm{Arr}(k, n) \to \mathrm{Arr}(k, n)/GL_k$ is a principal bundle and hence a closed map.

We will construct a particular oriented matroid $M \in \mathrm{MacP}(3, 8)$ and an infinite family of oriented matroids M_i whose properties force any triangulation of $G(3, \mathbb{R}^\infty)$ refining the matroid stratification to have infinitely many simplices in U_M.

To further help our visualization, consider the set of affine arrangements $\mathrm{AArr}(2, n)$, i.e., the set of n-tuples of points in the plane, not all collinear. Consider $\mathrm{AArr}(2, n) \subset \mathrm{Arr}(3, n)$ by identifying $((x_1, y_1), \ldots, (x_n, y_n))$ with $((x_1, y_1, 1), \ldots, (x_n, y_n, 1))$. Given an affine arrangement $(v_1, \ldots v_n)$, the nonzero covectors of the associated oriented matroid are obtained by considering oriented lines in the affine plane and the positions of the v_i with respect to these lines. For any $v \in \mathbb{R}^3$ with positive z-coordinate v_z, let $v' = v_z^{-1} v$. Then the oriented matroid associated to an arrangement $(v_1, v_2, \ldots, v_n)$ of vectors with positive z-coordinate is identical to the oriented matroid associated to the affine arrangement $(v'_1, v'_2, \ldots, v'_n)$.

The affine arrangement has the advantage that collinearity, convexity, and intersection properties are determined by the oriented matroid.

It will be convenient to write elements of $\mathbb{N}$ as

$$\{\alpha, \beta, \gamma, \omega, \nu, a\} \cup \{b_1, b_2, \ldots\} \cup \{c_1, c_2, \ldots\} \cup \{d_1, d_2, \ldots\}.$$

We will define inductively a sequence $A_0 \subset A_1 \subset A_2 \subset \cdots$ of affine arrangements. Consider the affine arrangement A_0 pictured in Fig. 1.

Given an arrangement A_{n-1} with elements $\{\alpha, \beta, \gamma, \omega, \nu, a\} \cup \{b_1, b_2, \ldots, b_n\} \cup \{c_1, c_2, \ldots, c_{n-1}\} \cup \{d_1, d_2, \ldots, d_{n-1}\}$, define A_n by adding points $\{d_n, b_{n+1}, c_n\}$ to A_n as follows:

(1) Add d_n at the intersection of the lines $\overline{\omega\gamma}$ and $\overline{\alpha b_n}$.

(2) Add b_{n+1} at the intersection of the lines $\overline{\omega\beta}$ and $\overline{a d_n}$.

(3) Add c_n at the intersection of the lines $\overline{\alpha\beta}$ and $\overline{a b_{n+1}}$.

For instance, A_1 is pictured in Fig. 2 and A_2 is pictured in Fig. 3.

An induction on n shows that the c_i are all distinct in each A_n.

For each positive integer i, let M_i be the oriented matroid associated to the arrangement obtained from A_i by changing the name of c_i to δ. Note that for any realization of M_i in $\mathbb{R}^3$, the corresponding realization in affine space is determined by the positions of $\{\alpha, a, b_1, \beta, \nu\}$.

Finally, let M be the oriented matroid associated to the affine arrangement shown in Fig. 4.

The following lemma says that the stratum U_M intersects the closure of each U_{M_i}, and these intersections are disjoint.

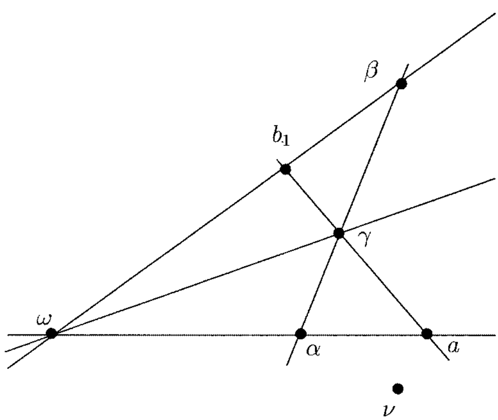

FIG. 1. The arrangement A_0.

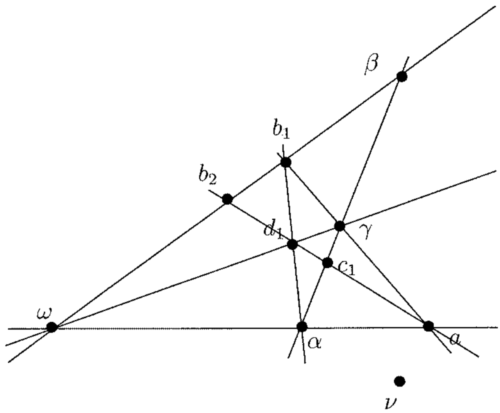

FIG. 2. The arrangement A_1.

LEMMA 2.3. (1) *$\overline{m^{-1}(M_i)} \cap m^{-1}(M) \neq \emptyset$ for every $i > 0$, and*

(2) *$\overline{m^{-1}(M_i)} \cap \overline{m^{-1}(M_j)} \cap m^{-1}(M) = \emptyset$ for every $i \neq j$ where $i, j > 0$.*

Proof. For the first statement, we need a convergent sequence $A_i^1, A_i^2, \ldots$ of elements of $\mathrm{Arr}(3, \infty)$, each of which represents the oriented matroid M_i, and whose limit is in $\mathrm{Arr}(3, 8)$ and represents M. This sequence is defined by closing up the angle $\angle \beta\omega a$, leaving the points $\{\alpha, \beta, \gamma, \delta, \omega, \nu, a, b_1\}$ all at height 1 in $\mathbb{R}^3$ and in the right order in the limit. Meanwhile, the realizations of the remaining points are obtained by letting each z-coordinate be $1/n$ while maintaining the collinearity and intersection properties determined by M_i.

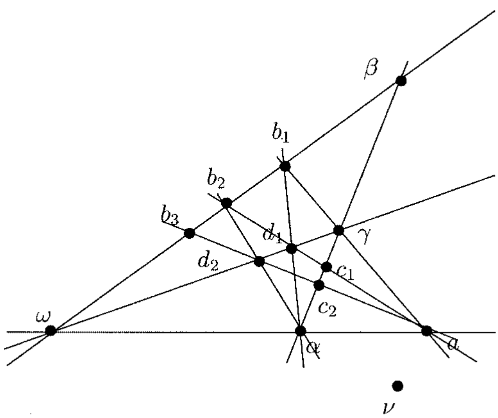

FIG. 3. The arrangement A_2.

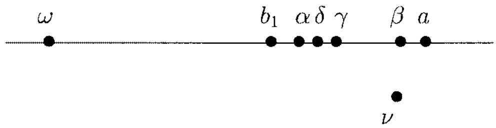

FIG. 4. A realization of M.

The second statement is proven using two facts from elementary projective geometry:

(1) For every two $(n+1)$-tuples $\{x_0, \ldots, x_n\}$ and $\{y_0, \ldots, y_n\}$ of points in general position in affine n-dimensional space, there exists a unique affine automorphism[2] taking each x_i to y_i.

(2) If (a, b, c, d) are four points on a line in the affine plane (in the order given), their *cross-ratio*

$$\frac{\|a-c\|}{\|b-c\|}\frac{\|b-d\|}{\|a-d\|}$$

is invariant under affine automorphism.

Now assume by way of contradiction there exists an affine arrangement $B \in \overline{m^{-1}(M_i)} \cap \overline{m^{-1}(M_j)} \cap m^{-1}(M)$ for some $i \neq j$. We will compute the cross-ratio of the points $(\alpha, \delta, \gamma, \beta)$ in B in two different ways (an i-way and a j-way) and come up with a contradiction.

Construct a sequence $B_i^1, B_i^2, \ldots$ of arrangements in $m^{-1}(M_i)$ such that

- the sequence converges to B,
- the elements $\{\alpha, \beta, \gamma, \delta, \omega, \nu, a, b_1\}$ all have z-coordinate 1 in each B_i^n, and
- the subarrangements $(\alpha, \beta, \gamma, \delta, a, b_1)$ in each B_i^n are all projectively equivalent. (That is, for every n_1, n_2 there is an affine automorphism of the plane taking the subarrangement in $B_i^{n_1}$ to the subarrangement in $B_i^{n_2}$.)

Fix a small ϵ. Define B_i^n to be the unique realization of M_i with

(1) α, a, ω, and ν in the same positions as in B,

(2) β the point at distance ϵ/n above the position of β in B,

(3) b_1 determined by requiring the 1-dimensional affine arrangement $\{\omega, b_1, \beta\}$ in B_i^n to be projectively equivalent to the corresponding arrangement in B, and

[2] An affine automorphism is the composite of a linear automorphism with a translation. These are the bijections of affine space which take lines to lines.

(4) All other points determined by the collinearity and convexity conditions of M_i, together with the condition that the z-coordinates of γ and δ are 1 and the z-coordinates of all remaining points are $1/n$.

That all elements of the arrangements B_i^n except γ and δ converge to the corresponding elements of B is clear. We get convergence of γ and δ by noting that there exists some sequence $C_i^1, C_i^2, \ldots$ in $m^{-1}(M_i)$ converging to B. As n increases, the elements $\{\alpha, \beta, \omega, \nu, a, b_1\}$ in B_i^n converge to the corresponding elements of C_i^n. Since the positions of γ and δ are determined by the positions of $\{\alpha, \beta, a, b_1\}$, γ and δ converge as well.

Note that the subarrangements $\{\alpha, \beta, \gamma, \delta, a, b_1\}$ in the B_i^n are all projectively equivalent (by the affine automorphism fixing ω and a and mapping the corresponding β to each other). Thus the cross-ratio $\operatorname{cr}(i)$ of $(\alpha, \delta, \gamma, \beta)$ is the same in all the B_i^n, and so $\operatorname{cr}(i)$ is the cross-ratio of $(\alpha, \delta, \gamma, \beta)$ in B.

Similarly, we get a sequence $B_j^1, B_j^2, \ldots$ in $m^{-1}(M_j)$ and calculate the cross-ratio of $(\alpha, \delta, \gamma, \beta)$ in B to be $\operatorname{cr}(j)$. Thus $\operatorname{cr}(i) = \operatorname{cr}(j)$. On the other hand, consider the affine automorphism of the plane fixing the points ω and a in B and taking the point β in B_i^1 to the point β in B_j^1. This sends the subarrangement (α, β, a, b_1) in B_i^1 to the corresponding subarrangement of B_j^1, hence sends the point γ in B_i^1 to the point γ in B_j^1. But it does *not* send the point δ in B_i^1 to the point δ in B_j^1, since $c_i \neq c_j$ in $A_{\max\{i,j\}}$. Hence $\operatorname{cr}(i) \neq \operatorname{cr}(j)$, a contradiction. ∎

Proof of Main Theorem. We assume that there is a tame triangulation of $G(3, \mathbb{R}^\infty)$ and reach a contradiction. By Lemma 2.3 and Lemma 2.2, $U_M \cap \overline{U_{M_i}} \neq \emptyset$ and $U_M \cap \overline{U_{M_i}} \cap \overline{U_{M_j}} = \emptyset$ for every i and j. Choose a sequence of simplices $\sigma_1, \sigma_2, \ldots$ so that

$$\operatorname{int} \sigma_i \cap U_M \cap \overline{U_{M_i}} \neq \emptyset.$$

Then there exists a sequence of simplices $\tau_1, \tau_2, \ldots$ so that $\operatorname{int} \tau_i \subset U_{M_i}$ and σ_i is a face of τ_i. Then $\operatorname{int} \sigma_i \subset U_M \cap \overline{U_{M_i}}$ so, by part (2) of the above lemma, the σ_i's are distinct. Thus there are an infinite number of simplices in a compact set $\overline{U_M} \subset G(3, \mathbb{R}^8)$, which is a contradiction. ∎

3. GENERALIZATIONS

Our main theorem can be generalized in two different ways: generalize to partitions more general than a triangulation, and to stratifications more general than the matroid stratification.

DEFINITION 3.1. A *weak subdivision* of a partition $P = \{U_i : i \in I\}$ of a space X is a partition Q of X such that

- Q *refines* P, i.e., every block U of P is the union of blocks of Q,
- each block of Q is *connected*,
- Q is *locally finite*, i.e., every compact set K of X intersects only a finite number of blocks of Q,
- Q is *normal*, i.e., if U and V are blocks of Q and $U \cap \overline{V} \neq \emptyset$ then $U \subseteq \overline{V}$.

If Q consists of the interiors of simplices in a triangulation of X, and Q refines a partition P, then Q is a weak subdivision of P. However, a CW decomposition of X refining P need not be a weak subdivision; normality may not hold.

If M and M' are (oriented) matroids on a set E and if every covector of M' is a covector of M, then one says that there is a *strong map* $M \to M'$. Define the *combinatorial Grassmannian*

$$\Gamma(k, M) = \{M' : \text{rank } M' = k \text{ and } M \to M'\}.$$

If M is the (oriented) matroid associated to a collection of vectors $\{v_1, v_2, \dots, v_m\}$ spanning $\mathbb{R}^n$, define

$$\mu_M : G(k, \mathbb{R}^n) \to \Gamma(k, M)$$

by sending V to the (oriented) matroid with covectors

$$\{i \mapsto \ \operatorname{sign}\langle v, v_i\rangle\}_{v \in V}.$$

The point inverse images are called the *generalized (oriented) matroid strata*. This stratification comes up in the study of extension spaces of oriented matroids (cf. [7]).

To get a stratification of the infinite Grassmannian, one needs some compatibility between the matroids. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subset \mathbb{R}^\infty$ be a sequence of finite sets, such that for each i, $\operatorname{span}(A_i) = \mathbb{R}^{n_i}$ for some n_i, and every element of $A_{i+1} - A_i$ has all of its first n_i coordinates zero. Then the associated oriented matroids $M_1, M_2, \dots$ satisfy

- $M_{i+1} \to M_i$ for all i, and so there are inclusions $\Gamma(k, M_i) \to \Gamma(k, M_{i+1})$, and
- the associated maps $\mu_{M_i} : G(k, \mathbb{R}^{n_i}) \to \Gamma(k, M_i)$ commute with the inclusions of real resp. combinatorial Grassmannians.

Thus the maps μ_{M_i} give a generalized (oriented) matroid stratification of $G(k, \mathbb{R}^\infty)$. By choosing $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ so that the $B_i \subseteq A_i$ and the B_i are linearly independent, we see that the generalized (oriented) matroid stratification is essentially a refinement of the oriented matroid stratification. Thus the proof of our main theorem showed the following result.

THEOREM 3.2. *There is no weak subdivision of any generalized* (*oriented*) *matroid stratification of* $G(k, \mathbb{R}^\infty)$ *for* $k \geq 3$.

4. TAME TRIANGULATIONS AND MATROID BUNDLES

The point of this section is to give a context for tame triangulations and to show how they allow the process of *combinatorialization*, the passage from topological structures to combinatorial ones. This occurs in two related ways: in constructing maps from real to combinatorial Grassmannians, and in passing from vector bundles to matroid bundles. For more on this see [2, 3]. Our main theorem forces delicate constructions in going from the finite to the infinite dimensional case in [3].

Let $\pi : E \to B$ be a rank k vector bundle over a simplicial complex. Assume the fibers $F_b = \pi^{-1}(b)$ are equipped with a continuously varying inner product. If B is finite-dimensional, there is a set of spanning sections $S = \{s_1, s_2, \ldots, s_n\}$. Then we have a map

$$\mathcal{M} : B \to \mathrm{MacP}(k, n)$$

sending $b \in B$ to the oriented matroid associated to $\{s_1(b), s_2(b), \ldots, s_n(b)\} \subset F_b$. (The reader is strongly urged to work out the case of the open Möbius strip mapping to the circle.) The set of sections S is said to be *tame* when $\mathcal{M}$ is constant on the interior of simplices, in which case we have a true combinatorial gadget, a *matroid bundle*. The existence of a tame triangulation of $G(k, \mathbb{R}^n)$ shows that every rank k vector bundle over a finite-dimensional complex has a tame set of sections after subdividing the base. This is accomplished by applying the simplicial approximation theorem to a classifying map $B \to G(k, \mathbb{R}^n)$, and pulling back the canonical sections.

One can think of the MacPhersonian $\mathrm{MacP}(k, n)$ of rank k oriented matroids on $\{1, 2, \ldots, n\}$ as a classifying space for matroid bundles. It has a partial order given by $M_1 \geq M_2$ if there is a weak map from M_1 to M_2. If $U_{M_1} \cup \overline{U_{M_2}} \neq \emptyset$, then $M_1 \geq M_2$. Let $\|\mathrm{MacP}(k, n)\|$ be the geometric realization (= order complex) of this poset. Let $\mu : G(k, \mathbb{R}^n) \to \mathrm{MacP}(k, n)$ be the realization map. Given a tame triangulation of $G(k, \mathbb{R}^n)$, then one can construct a simplicial map $\tilde{\mu} : G(k, \mathbb{R}^n) \to \|\mathrm{MacP}(k, n)\|$ from the barycentric subdivision of the tame triangulation agreeing with μ on the vertices. The main result of [3] shows that $\tilde{\mu}$ carries the Stiefel–Whitney classes, and hence Stiefel–Whitney classes can be defined purely combinatorially.

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