“Réseaux réguliers” or regular graphs—Georges Brunel as a French pioneer in graph theory

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Abstract

The early research on regular graphs of the French mathematician Georges Brunel (1856–1900) is discussed. Brunel developed early graph terminology and started the French “applied” approach towards graph theory.

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1. Introduction

1.1. Motivation

It is 100 years ago that Georges Brunel died in 1900. Hence, it seems worth in the last year of this millennium to look back and discuss the work of a French pioneer of early graph theory.

Brunel was a professor of mathematics in Bordeaux from 1884 until his death in 1900. Hence, it seems reasonable to publish this paper in the Proceedings volume of the International Graph Theory Conference in Marseille in Southern France.

1.2. Definitions

Certainly the definition of a regular graph is known to everybody who works in graph theory. Let me define it here in an unusual but historical way. The following definition of a configuration is due to Th. Reye (1876) [15]. Hence, configurations will celebrate

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their 125th anniversary as combinatorial structures in their own right in 2001, the first year of the third millennium.

**Definition 1.1.** A configuration \((v_r, b_k)\) is a finite incidence structure of \(v\) points and \(b\) lines such that there are \(k\) points on each line, there are \(r\) lines through each point, and through two different points there is at most one common line. For such a configuration \(vr = bk\).

In modern terminology, such a configuration can be regarded as a linear \(r\)-regular \(k\)-uniform hypergraph. This implies the following definition of a regular graph.

**Definition 1.2.** A regular graph of degree \(r\) with \(v\) vertices and \(b\) edges is a configuration \((v_r, b_2)\).

**Definition 1.3.** If through two different points of a configuration there is exactly one line, the configuration \((v_r, b_k)\) is called a Steiner system \(S(2, k, v)\). For such a configuration \(r = (v - 1)/(k - 1)\) and \(b = v(v - 1)/k(k - 1)\).

2. Georges Brunel (1856–1900)

There is a paper on the life and work of Brunel by Duhem [5] which was published soon after Brunel’s death. It is very detailed and the basis for my following short summary of his life. It consists of 30 pages concerning Brunel’s life and a report of more than 50 pages on the work of Brunel. Some of his papers which are of interest here will be discussed below. Finally, the paper of Duhem contains a list of the 97 published papers of Brunel.

2.1. Brunel’s life

Georges Édouard Auguste Brunel was born in Abbeville near Amiens on September 17, 1856 and died in Bordeaux on July 24, 1900.

After school education in Abbeville, Lille, and Paris, Brunel studied mathematics at the Ecole Normale from 1877 until 1880. For 1 year he studied in Leipzig (Germany) with Felix Klein and published his first paper [1] in a German journal in 1882. In the following year, he was an assistant (agrégé-préparateur de math.) at the Ecole Normale. From 1882 until 1884, Brunel was a lecturer (chargé du cours de mécanique) at the Ecole des Sciences in Alger (Algeria). In 1883 he wrote his thesis [2].

In October 1884, he became the successor of Houël in Bordeaux and held a chair of pure mathematics in the faculty of sciences which in 1886 was changed into a chair of calculus. He held this chair until his death in 1900 at the age of 43. In November 1884, Brunel became a member of the Société des sciences physiques et naturelles de Bordeaux. In this society, he played a very active role as archiviste, as speaker in many sessions, and in 1898 as vice president. Brunel published the vast majority of his papers in the Mémoires or in the Procès-verbaux of this society.
2.2. Brunel’s work

Following Duhem’s classification [5] of Brunel’s papers into 10 sections around 30 of his papers belong to geometry, number theory, algebra and its extensions. There are four sections called Analysis situs which cover more than half of Brunel’s 97 papers. These are related to graphs (or réseaux in Brunel’s language), combinatorial analysis, structures on surfaces and knot theory. Some of Brunel’s papers will be discussed in further detail below.

3. Brunel's graph theory

3.1. General remarks

In order to keep the list of references reasonably small I shall refer to those papers of Brunel which are published in the Procès-verbaux des séances de la société des sciences physiques et naturelles de Bordeaux as follows: [PV year, page]. In some cases the paper is bound together with the corresponding Mémoires volume.

In this paper, I shall focus on some aspects of Brunel’s work in graph theory. Most of them, but not all are related to regular graphs. In these papers, early notation of graph theory, in particular in French language, is introduced. For further details see [9]. The application of graphs in chemistry plays an important role in Brunel’s work. Configurations and Steiner systems are discussed in several of Brunel’s papers. One year after his early death, his result on the enumeration of certain Steiner systems was published in [4]. The text is a manuscript which he wrote already in 1895. For the general historical background of this result see [6].

3.1.1. Regular graphs

In [PV 1894/95, p. 3, Réseaux réguliers], a regular graph is defined as follows. The talk was given on December 13, 1894.

Nous appelons réseau régulier un réseau tel que de chaque sommet parte le même nombre d’arêtes.

In a consecutive paper [PV 1894/95, p. 8, Polymérisation du carbone], he explains the use of these regular graphs in chemistry (see below).

A few years earlier Petersen [14] had defined regular graphs. As Petersen explained, he took the English word “graph” in his German text. For further background on this development see [8].

Englische Verfasser haben für ähnliche Figuren den Namen graph eingeführt; ich werde diesen Namen beibehalten und nenne den graph regulär, weil in jedem Punkte gleich viele Linien zusammenlaufen.
Moreover, Jan de Vries had already enumerated small regular graphs in 1889/1891, but in the language of configurations (see [7]).

3.1.2. Graphs and chemistry

Brunel’s interest in chemistry and its relation to graph theory was certainly an important motivation.

As already mentioned above in [PV 1894/95, p. 8, *Polymérisation du carbone*], Brunel gave a talk on this relation, only 2 weeks later, and between Christmas and New Year, on December 27, 1894.

Dans une communication précédente, nous nous sommes occupés des propriétés générales des réseaux réguliers, c’est-à-dire des réseaux tels que de chacun de leurs sommets parte un même nombre d’arêtes. …

… la représentation graphique des composés chimiques de formule donnée peut être faite à l’aide d’un réseau déterminé. …

En un mot, il s’agit alors de déterminer les différents polymères du carbone ou, en d’autres termes, la construction et l’enumération des réseaux réguliers à sommets quadrilatéraux.

In [PV 1898/99, p. 108, *Sur la représentation graphique des isomères*], Brunel continues this approach. He transforms the chemical molecule into a graph as usual, and then he computes the skeleton by removing vertices of degree 1 and degree 2 in a suitable manner.

Furthermore, Brunel obtains 919 possible isomers for C₆H₆ one of them being benzene. In other cases he improves results of Cayley.

3.1.3. Adjacency matrices

In my paper [9], I mentioned that in 1895 Brunel [3] introduced a matrix to describe a graph. The idea even occurred 2 years earlier in [PV 1894, p. IX] and was presented to the society on January 12, 1893.

En désignant par $n$ le nombre des sommets, on forme un tableau à double entrée formé de $n$ lignes et de $n$ colonnes. Si le point d’indice $p$ est relié au point d’indice $q$ par une arête, on insère dans la $p^{ième}$ ligne et dans la $q^{ième}$ colonne le symbole $(pq)$ ou plus simplement $pq$, et dans la $q^{ième}$ ligne et la $p^{ième}$ colonne le symbole $(qp)$ ou $qp$, … Une case vide indiquera l’absence de liaison.

Here Brunel defines the adjacency matrix of a graph. If the points $p$ and $q$ are adjacent, i.e. joined by an edge, the corresponding entry is $(pq)$ or $pq$. Otherwise the cell is empty. The obtained matrix is symmetric. Later in his paper, he gives the example of a $K₄$ which he calls tetrahedron.

Brunel extends his matrix concept also to digraphs which contain *arêtes irréversibles*. In this case the matrix is no longer symmetric. Such a directed edge can perhaps mean: $p$ est enfant de $q$ [PV 1894, p. XXV].
3.2. Graphs and configurations

In [PV 1894/95, p. 28, *Sur le problème des alignements*], Brunel refers to earlier work of Cayley and Sylvester on “alignments of trees”. He also mentions the work of Reye, Kantor and his colleagues on configurations. Since he uses the French word *configuration* in a more general sense, he calls them *alignements réguliers*. The date of the oral presentation was February 21, 1895.

Nous nous proposons ici de montrer la liaison du problème des alignements réguliers et de la question des systèmes de $v$-ades, … Si l’on considère 9 éléments que nous représenterons par les chiffres 1,2,3,…,9, on peut former un synthème de triades contenant toutes les duades, et chacune d’elles une seule fois; … Considérons, en effet, les chiffres 1,2,3,…,9 comme représentant des points, et l’existence d’une triade comme indiquant que les trois points qui y apparaissent sont en ligne droite.

Here, Brunel very clearly describes the relation between configurations of points and lines (in geometric language) and the partition of edges (duades) of a complete graph into triangles (triades). This comes very close to the idea of a hypergraph (at the end of the last century).

3.2.1. Steiner systems $S(2,3,13)$

Already in [PV 1895, p. XLVII], Brunel presents some detailed remarks on the arrangement of pairs in triples. The session of the society was held on July 19, 1894. Brunel discusses the earlier work of Netto and Hastings Moore and continues.

Il n’est pas inutile de signaler à ce sujet les travaux bien antérieurs de Kirkman parus dans le *Cambridge and Dublin Mathematical Journal*. En 1847, cet auteur avait déjà montré que pour $x = 6n + 1$ ou $6n + 3$ on peut construire un système de triades comprenant toutes les duades. Il s’était alors également occupé de déterminer pour les autres formes du nombre $x$ le nombre maximum de triades possible que l’on peut construire sans répéter aucune duade. …

It is quite remarkable that Brunel knows about Kirkman’s work, while it is unknown to Netto and his colleagues at this time (compare [6]). In his report on the life and work of Brunel, P. Duhem remarks the following concerning an unpublished manuscript which he found after Brunel’s death in 1900.

[5, p. XXV] En feuilletant les papiers laissés par notre Doyen, j’y trouvai le manuscrit d’un mémoire achevé et préparé pour l’impression; ce mémoire donnait la solution d’un problème de la théorie des substitutions, solution que Kirkman, Reiss, Netto, Hastings-Moore, Jan de Vries n’avaient pu obtenir. Or, ce manuscrit était daté de décembre 1895 et, depuis ce temps, il attendait l’imprimeur.
Hence, Brunel’s manuscript of December 1895 contains the construction of all Steiner systems $S(2,3,13)$ and was only published after his death in 1901 [4], but twice: in the Mémoires of the Bordeaux society as well as in the famous journal of Liouville.

In the mean time, however, a solution of the Italian de Pasquale had been published in 1899. For further background see [6]. Hence, Brunel was probably the first who proved and de Pasquale the first who published that there are no further Steiner systems $S(2,3,13)$ than the two which were already known.

3.2.2. Partitions of complete graphs $K_n$ into smaller $K_k$

The existence of Steiner systems $S(2,3,n)$ for only certain values of $n$ (if $n \equiv 1$ or $3 \mod 6$) leads to the following problem.

In [PV 1895/96, p. 40, Sur les triades formées avec $6n-1$ et $6n-2$ éléments], Brunel refers to Kirkman’s work of 1847 [12] concerning the maximum number $Q_n$ of edge-disjoint triangles in a complete graph $K_n$ and mentions that the results of Kirkman in the cases of $n \equiv 4$ or $5 \mod 6$ are not correct.

Dans le Cambridge and Dublin Mathematical Journal (t.II, p. 191–204, 1847), Kirkman a énoncé la proposition suivante:

Si l’on désigne par $Q_x$ le nombre maximum de triades que l’on peut former avec $x$ éléments, en sorte qu’il n’y ait aucune répétition des duades, le nombre $Q_x$ est donné par l’expression

$$3Q_x = x \frac{x-1}{2} - V_x,$$

... (1) $V_x = \cdots$ pour $x = 6n - 1$,

(2) $V_x = \cdots$ pour $x = 6n - 2$,

(3) $V_x = 0$ pour $x = 6n + 1$ et $x = 6n + 3$,

(4) $V_x = \cdots$ pour $x = 6n$ et $x = 6n + 2$.

Les propositions relatives aux cas (1) et (2) ne sont pas exactes.

Brunel discusses Kirkman’s arguments as follows.

Il donne quelques raisons ... ajoute qu’elles “manquent de rigueur mathématique”, et conclut en disant qu’il croit à la vérité de la proposition jusqu’à ce qu’on lui ait montré des cas où elle ne s’applique pas.

C’est ce que nous nous proposons de faire ici.

... On n’a donc pas $Q_{16} = 36$, mais bien $Q_{16} = 37$.

By using a solution of 35 triangles in a $K_{15}$ (a Steiner system $S(2,3,15)$) Brunel obtains 37 triangles in a $K_{16}$ and 44 triangles in a $K_{17}$ where Kirkman’s solution gives 36 and 42, respectively, as the maximal numbers of triangles. Moreover, Brunel also improves Kirkman’s results for $K_{28}$ and $K_{29}$.

Whether Brunel also proved the optimal numbers for all cases mod4 and 5(mod 6) is not clear. At least he gives the correct solutions and states that the proof would be too long to be published in this paper.
La modification à apporter à l’énoncé de Kirkman est la suivante:
on a
\[ V_x = 4 \text{ pour } x = 6n - 1, \]
\[ V_x = \frac{1}{2} x + 1 \text{ pour } x = 6n - 2. \]
La démonstration doit être conduite comme celle de Kirkman-Reiss, mais demande
trop d’espace pour trouver place ici.

In “modern times” Schönheim [18] determined these maximal numbers of triangles in
\( K_n \) neither mentioning Kirkman nor Brunel. I do not know of any remark (other than
in Duhem’s report [5]) on Brunel’s correction of Kirkman’s paper anywhere in the
literature.

3.2.3. Partitions of complete graphs into other graphs
In [PV 1895/96, p. 58. Sur la construction des systèmes de quadricycles de \( 8n + 1 \)
éléments], Brunel discusses a generalization of his earlier work. Instead of partitioning
the complete graph into edge-disjoint triangles he considers cycles.
Again Brunel describes the relation between graph theoretical and set theoretical
language. He uses the words \textit{duade} and \textit{arête} nearly equivalently.

\[ \ldots \text{ la triade } abc \text{ est considérée comme comprenant les trois duades } ab, \ ac \text{ et } bc. \]
Si nous figurons les éléments par des points et les duades par des lignes joignant
ces points deux à deux, le système de duades considéré est représenté par un
réseau, et le problème de la construction d’un système de triades revient à la
construction d’un ensemble de triangles dont les côtés sont des arêtes du réseau,
une arête apparaissant dans l’un des triangles, mais dans un seul triangle.

In the following, Brunel discusses graphs whose edges consist of the sides of a polygon,
in modern language cycles \( C_4 \).

Si l’on considère le réseau des duades de \( N \) éléments, peut-on le considérer comme
obtenu en traçant, sans répétition d’arêtes, des polygones à quatre, cinq, six, etc.,
côtés. \ldots
Nous appellerons quadricycle une figure contenant quatre côtés consécutifs du
réseau, \ldots symbole \( abcd \). Un quadricycle contient quatre arêtes, correspond aux
quatre duades \( ab, bc, cd, da \ldots \)

Brunel determines that \( N \) must be of the form \( 8n + 1 \) if a partition into 4-cycles is
possible, that their number is \( n(8n + 1) \), and he finds examples for \( N = 9, 17, 25 \).

3.3. A recreational problem

In [PV 1895, p. XIV], Brunel discusses the problem of three vases of 8, 5, and
3 l which can contain wine. The possible states are denoted by vertices of a graph;
two vertices are joined by a (directed) edge, if one state is reachable by another one
performing a \textit{transvasement}. 
On a trois vases de 8, 5, et 3 litres. Le premier est plein de vin. Effectuer dans
ces trois vases des transvasements en sorte qu'il y ait 4 litres dans le premier et
4 litres dans le second.

The initial position is 8,0,0, i.e. 8 l in the first vase. The final position should be 4,4,0,
i.e. 4 l in the first vase and 4 l in the second vase. Brunel gives 16 different solutions,
the shortest of which contains seven edges: 800, 350, 323, 620, 602, 152, 143, 440.
Brunel remarks that for solving this problem he can use his graph-theoretical
methods.

La représentation graphique ainsi obtenue permet de montrer sous une forme frapp-
pante les solutions obtenues …

The corresponding graph can be found in the first book [13] on graph theory, written
by the Hungarian Dénes König in 1936. Altogether, this book contains several refer-
ences to the work of Bruné as well as to the work of André Sainte-Laguë, a French
mathematician, who continued the research in graph theory in the tradition of Brunel.
In 1926, he wrote the zeroth book [16] on graph theory.

Sainte-Laguë’s contributions to graph theory are discussed in two papers [9,10].
The only book of Sainte-Laguë which is easily available today is [17], since it was
reprinted in 1994. It contains the nice river-crossing problem involving a wolf, a goat
and a cabbage, also presented in the 1 year older book by Dénes König [13]. The
problem itself is more than 1200 years old (see [11]).

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