Discrete Mathematics 96 (1991) 101–110
North-Holland

Some perpendicular arrays for arbitrarily large t

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Received 6 January 1989
Revised 22 May 1990

Abstract

Kramer, E.S., Q.-R. Wu, S.S. Magliveras and T. van Trung, Some perpendicular arrays for arbitrarily large t, Discrete Mathematics 96 (1991) 101–110.

We show that perpendicular arrays exist for arbitrarily large t and with $\lambda = 1$. In particular, if d divides $(t+1)$ then there is a $PA_1(t, t+1, t+((t+1)/d))$. If $v \equiv 1$ or $2 \pmod 3$ then there is a $PA_\lambda(3, 4, v)$ for any λ. If 3 divides λ then there is a $PA_\lambda(3, 4, v)$ for any v. If $n \geq 2$ there is a $PA_1(4, 5, 2^n + 1)$. Using recursive constructions we exhibit several infinite families of perpendicular arrays with $t \geq 3$ and relatively small λ. We finally discuss methods of constructing perpendicular arrays based on automorphism groups. These methods allow the construction of PA's with $(k - t) > 1$.

1. Introduction

A *perpendicular array* $PA_\lambda(t, k, v)$ is a $\lambda\binom{v}{t}$ by k array, A, using the symbols from a v-set X, which has the properties: (i) every row of A contains k distinct symbols; and (ii) for any t columns of A and for any t-set T of X there are exactly λ rows of A that contain the symbols of T in the chosen t columns. Note that (ii) implies (i) when $t > 1$. Also observe that (i) implies that $k \leq v$ in a perpendicular array. It is also immediate that a $PA_\lambda(t, k, v)$ produces a $PA_\lambda(t, k', v)$, for $k' \leq k$, by simply removing columns.

Perpendicular arrays have recently been examined by several researchers (see [7, 11, 13–15, 18]). Some of the results found in [11] will be obtained more efficiently in what follows.

Some necessary conditions for perpendicular arrays are provided in [11] by the following theorem.

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Theorem 1.1. *Suppose* $0 \leq t' \leq t$ *and* $\binom{k}{t} \geq \binom{k}{t'}$. *Then a* $\mathrm{PA}_\lambda(t, k, v)$ *is also a* $\mathrm{PA}_{\lambda'}(t', k, v)$, *where* $\lambda' = \lambda\binom{v-t'}{t-t'}/\binom{t}{t'}$. *Hence* $\lambda\binom{v-t'}{t-t'} \equiv 0 \bmod \binom{t}{t'}$.

Thus, a $\mathrm{PA}_1(2, 3, v)$ has v odd; a $\mathrm{PA}_1(3, 4, v)$ has $v \equiv 1$ or $2 \pmod 3$; and a $\mathrm{PA}_1(3, 5, v)$ has $v \equiv 2 \pmod 3$.

One of the standard techniques (see [11]) for constructing perpendicular arrays uses t-wise balanced designs. Let v and t be positive integers and let K be a subset of $\{t, \ldots, v-1\}$. A *t-wise balanced design*, with parameters t-(v, K, λ), sometimes also called a tBD, is a pair $(X, \mathscr{B})$, where X is a v-set and $\mathscr{B}$ is a collection of subsets (called *blocks*) of X, with sizes from K, such that any t-set from X is contained in precisely λ blocks of $\mathscr{B}$. If $K = \{k\}$ then our tBD is called a t-design with parameters t-(v, k, λ).

Theorem 1.2 (tBD Construction). *Suppose* $(X, \mathscr{B})$ *is a* t-(v, K, λ) *such that for every n in K there exists a* $\mathrm{PA}_{\lambda_1}(t, k, n)$. *Then we can construct a* $\mathrm{PA}_{\lambda\lambda_1}(t, k, v)$ *by taking a* $\mathrm{PA}_{\lambda_1}(t, k, |B|)$, *on symbol set B, for every B in* $\mathscr{B}$.

2. A matrix theorem

The following is a nice application of P. Hall's Theorem [3] which also provides a tool in constructing perpendicular arrays.

Theorem 2.1 (Kramer, Wu). *Let A be a matrix with n columns and integer entries from* $S = \{1, 2, \ldots, k\}$ *where integer i appears exactly* nr_i, r_i *an integer, times in A. By permuting the entries within each row we can transform A to a matrix in which each integer i appears* r_i *times in each column.*

Proof. We apply subscripts to the occurrences of i in A where subscript j, for $1 \leq j \leq r_i$, will appear exactly n times as a subscript of i. In our new matrix, call it A', there will be exactly $m = (r_1 + r_2 + \cdots + r_k)$ distinct entries. Note also that A' has precisely m rows and that each of the m entries appears n times in A'. Let $S_i = \{a_{i,j'}: 1 \leq j \leq n\}$ be the set of distinct elements that appear in row i of A'. There may, of course, be repetition of elements in any given row of A or A'. Now we claim that P. Hall's condition applies to the sets $S_1, \ldots, S_m$. For if the union of any t of these sets contained fewer than t elements it would clearly imply that some element appeared more than n times in the corresponding t by n submatrix of A'. This obviously does not happen so we can select a system of distinct representatives for the m sets. We arrange these into the first column via appropriate permutations within each row of A'. Clearly we can apply P. Hall's theorem to the remaining $n-1$ columns and produce a matrix B' from A' where each entry appears exactly once in each column of B' and where B' is obtained from A' by permuting each row of A'. By removing subscripts we get our result. □

There are strong connections between t-designs and perpendicular arrays. Note, for example that any $PA_\lambda(t, k, v)$ yields a t-$(v, k, \lambda\binom{k}{t})$ design by taking as the ith block the set of elements in the ith row of the array. Conversely, the next result shows that a perpendicular array can be manufactured from *any* t-design where $k = t + 1$.

Theorem 2.2. *If there exists a t-$(v, t+1, \lambda)$ design then there is a* $PA_{\lambda_1}(t, t+1, v)$, *where* $\lambda_1 = \lambda/(\lambda, t+1)$.

Proof. Repeat the blocks of the t-$(v, t+1, \lambda)$ design to produce a t-$(v, t+1, \lambda_1(t+1))$ design $(X, \mathscr{B})$. Let $B = \{B_1, B_2, \ldots, B_b\}$ and let M be the matrix whose ith row contains B_i. Let the t-subsets of X be $\{T_k: 1 \leq k \leq \binom{v}{t}\}$ and let N be the b by $(t+1)$ matrix whose (i, j) entry $n_{ij} = k$ if $B_i \setminus \{m_{ij}\} = T_k$. Clearly each k, $1 \leq k \leq \binom{v}{t}$, appears $\lambda_1(t+1)$ times in N. By Theorem 2.1 we can transform N, via permutations within each of the rows, into a matrix N', such that each symbol in N' appears exactly λ_1 times in each column of N'. Performing the exact same permutations within each of the rows of M as on N produces a matrix M' where M' is clearly a $PA_{\lambda_1}(t, t+1, v)$. □

Note that the special case of $t = 2$, $\lambda = 3$ of Theorem 2.2 was proved in [7].

As an immediate application of Theorem 2.2 we can use some known t-designs to easily produce some families of perpendicular arrays (part (i) of this next theorem was done in [17] but here we do it with ease).

Theorem 2.3. (i) *For odd $v \geq 3$ and any λ there exists a* $PA_\lambda(2, 3, v)$.

(ii) *For all $v \geq 3$ and even $\lambda > 0$ there exists a* $PA_\lambda(2, 3, v)$.

Proof. There exists a 2-$(v, 3, 3)$ design, see [3], for all odd $v \geq 3$ and so there exists a $PA_1(2, 3, v)$ for such v's. Taking copies yields (i). Now there exists a 2-$(v, 3, 6)$ design for any $v \geq 3$, see [3], so we get a $PA_2(2, 3, v)$ and (ii) is clear. □

Theorem 2.4. (i) *If $v \equiv 1, 2 \pmod 3$ there is a $PA_\lambda(3, 4, v)$ for any $\lambda > 0$.*

(ii) *If $\lambda \equiv 0 \pmod 3$ there is a $PA_\lambda(3, 4, v)$ for any v.*

Proof. For any v not divisible by 3 there exists a 3-$(v, 4, 4)$ design, see [4], and (i) is clear. For any integer $v \geq 4$ there is a 3-$(v, 4, 12)$ design, see [5], and (ii) follows from Theorem 2.2. Note that part (ii) was first proved in [11], but here our proof is quick. □

Theorem 2.5. *For all $n \geq 2$ there exists a* $PA_5(5, 6, 2^n + 2)$.

Proof. By [9] there is a 5-$(2^n + 2, 6, 15)$ design and applying Theorem 2 gives our result. This improves the result in [11]. □

Theorem 2.6. *For all $n \geq 2$ there is a* $PA_1(4, 5, 2^n + 1)$.

Proof. In [8] Hubaut constructs $4\text{-}(2^n + 1, 5, 5)$ designs for $n \geq 3$. □

By using the trivial design $t\text{-}(v, t + 1, v - t)$ the following result is immediate.

Theorem 2.7. *For any t and v with $1 \leq t < v$ there is* $PA_\lambda(t, t + 1, v)$ *where* $\lambda = (v - t)/(v - t, t + 1)$.

Corollary 2.8. *For any integer $t > 0$ and any divisor d of $(t + 1)$ there exists a perpendicular array* $PA_1(t, t + 1, v)$ *where* $v = t + ((t + 1)/d)$.

The following result is useful.

Theorem 2.9. *If there is a* $PA_\lambda(t, v, v)$ *which is also a* $PA_{\lambda_{t-1}}(t - 1, v, v)$ *then there is a* $PA_{\lambda(v-t+1)}(t, v + 1, v + 1)$.

Proof. First note that $\lambda_{t-1} = \lambda(v - t + 1)/t$. Let A be our $PA_\lambda(t, v, v)$ using symbols from a v-set X and let y not be in X. Let A' be the new array obtained from A by replacing each row of A, say $a_1a_2 \cdots a_v$, by the $(v + 1)$ by $(v + 1)$ matrix, which we later call a stack:

$$\begin{matrix} y & a_1 & a_2 & \cdots & a_v \\ a_1 & y & a_2 & \cdots & a_v \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_1 & a_2 & a_3 & \cdots & y. \end{matrix}$$

Let T be a set of t elements from $X \cup \{y\}$. Select any t columns of A', which without loss of generality, we can take to be the first t columns of A'. If T is a subset of X and T appears in the first t columns of a stack it will appear in the first t columns in exactly $(v + 1 - t)$ rows of that stack. But T will be in the first t columns in exactly λ of these stacks and hence in $\lambda(v + 1 - t)$ rows of A'. Suppose y is in the t-set T. Easily, T will be in $t\lambda_{t-1} = \lambda(v - t + 1)$ rows of A' and our result is proved. □

As an application of this result we get the following.

Theorem 2.10. *If q is a prime power then there is a* $PA_{q-1}(2, q + 1, q + 1)$.

Proof. In [16] $PA_1(2, q, q)$ are shown to exist for all prime powers q. Our result then follows by the previous theorem. □

As we have already seen, Theorem 2.2 is one main method of constructing $PA_\lambda(t, k, v)$. After a discussion with Stinson the following two theorems became evident.

Theorem 2.11. *If there is a* $PA_\lambda(3, k, q+1)$, *q a prime power, then there is a* $PA_\lambda(3, k, q^n+1)$, $n \geq 1$.

Proof. In [19] it is shown that a $3\text{-}(q^n+1, q+1, 1)$ design exists for all $n \geq 1$. By Theorem 1.2 we get our result. □

Corollary 2.12. *There exists a* $PA_{30}(3, 33, 32^n+1)$ *for all* $n \geq 1$.

Proof. There exists a $PA_1(33, 32, 32)$ [18]. Using Theorem 2.9 we obtain a $PA_{30}(3, 33, 33)$ and so there exists a $PA_{30}(3, 33, 32^n+1)$ by Theorem 2.11. □

Theorem 2.13. *For any integer* $v \geq 5$ *there is a* $PA_{30}(3, 5, v)$.

Proof. Hanani [5] proves the existence of a $3\text{-}(v, 5, 30)$ design for all $v \geq 5$ and since a $PA_1(3, 5, 5)$ [16] exists we get our result. □

Theorem 2.14 (Hanani (see [5])). *If there exists a* $3\text{-}(v+1, q+1, \lambda)$ *design, q a prime power, then there exists a* $3\text{-}(vq^n+1, q+1, \lambda)$ *design for all* $n \geq 0$.

Theorem 2.15. *For any* $m \geq 0$, $n \geq 1$, *there exists a* $PA_3(3, 6, 5^m(4^{n+1}-1)/3)$.

Proof. By [1] there is a $3\text{-}((4^{n+1}-1)/3+1, 6, 1)$ design, $n \geq 1$, so by Theorem 2.14 there is $3\text{-}(5^m(4^{n+1}-1)/3+1, 6, 1)$ design, $m \geq 0$, $n \geq 1$. Using the $PA_3(3, 6, 6)$ [11] and Theorem 1.2 we get our result. Note that 3-designs with the same parameters as in [1] can be obtained from a $3\text{-}(6, 6, 1)$ using the recursion: If there is a $3\text{-}(v+1, 6, 1)$ then there is a $3\text{-}(4v+2, 6, 1)$ (see Hanani [5]). □

Corollary 2.16. *There exists a* $PA_{15}(3, 32, 31^n63+1)$ *for all* $n \geq 0$.

Proof. There exists a $3\text{-}(63+1, 31+1, 15)$ design (a Hadamard design). So we have a $3\text{-}(63.31^n+1, 32, 15)$ design for $n \geq 0$. By Theorem 1.2 we get the result from $PA_1(3, 32, 32)$ [18]. □

As in the proof, we can construct families of $PA_\lambda(t, k, v)$'s with $t=3$ from smaller perpendicular arrays and smaller 3-designs by Theorem 1.2 and Theorem 2.14. For example there exists a $PA_1(3, 5, 5)$ [16] and $PA_1(3, 8, 8,)$ [18]. Also, by Theorem 2.9 there exists a $PA_3(3, 6, 6)$ and $PA_6(3, 9, 9)$. Using these perpendicular arrays and the small 3-designs in [8] we can get many families of $PA_\lambda(t, k, v)$ with $t=3$. We list some of these arrays in Table 1 for $t=3$; for $k=5, 6, 8$, or 9; for $n \geq 0$; and for $\lambda \leq 6$.

Table 1

No.	Family of PA's	Parameters of Small PA used	Parameters of Small 3-design
1	$PA_1(3,5,4^{n+1}+1)$	$PA_1(3,5,5)$	3-(5, 5, 1)
2	$PA_1(3,5,4^n25+1)$	$PA_1(3,5,5)$	3-(26, 5, 1)
3	$PA_2(3,5,4^n31+1)$	$PA_1(3,5,5)$	3-(32, 5, 2)
4	$PA_3(3,5,4^n5+1)$	$PA_1(3,5,5)$	3-(21, 5, 3)
5	$PA_3(3,5,4^{n+1}6+1)$	$PA_1(3,5,5)$	3-(25, 5, 3)
6	$PA_3(3,5,4^n9+1)$	$PA_1(3,5,5)$	3-(10, 5, 3)
7	$PA_3(3,5,4^n21+1)$	$PA_1(3,5,5)$	3-(22, 5, 3)
8	$PA_3(3,5,4^n29+1)$	$PA_1(3,5,5)$	3-(30, 5, 3)
9	$PA_4(3,5,4^n10+1)$	$PA_1(3,5,5)$	3-(11, 5, 4)
10	$PA_4(3,5,4^n19+1)$	$PA_1(3,5,5)$	3-(20, 5, 4)
11	$PA_5(3,5,4^n13+1)$	$PA_1(3,5,5)$	3-(14, 5, 5)
12	$PA_5(3,5,4^{n+1}7+1)$	$PA_1(3,5,5)$	3-(29, 5, 5)
13	$PA_6(3,5,4^n10+1)$	$PA_1(3,5,5)$	3-(11, 5, 6)
14	$PA_6(3,5,4^n11+1)$	$PA_1(3,5,5)$	3-(12, 5, 6)
15	$PA_6(3,5,4^n14+1)$	$PA_1(3,5,5)$	3-(15, 5, 6)
16	$PA_6(3,5,4^n15+1)$	$PA_1(3,5,5)$	3-(16, 5, 6)
17	$PA_6(3,5,4^n19+1)$	$PA_1(3,5,5)$	3-(20, 5, 6)
18	$PA_6(3,5,4^n26+1)$	$PA_1(3,5,5)$	3-(27, 5, 6)
19	$PA_6(3,5,4^n30+1)$	$PA_1(3,5,5)$	3-(31, 5, 6)
20	$PA_3(3,6,5^{n+1}+1)$	$PA_3(3,6,6)$	3-(6, 6, 1)
21	$PA_3(3,6,5^n21+1)$	$PA_3(3,6,6)$	3-(22, 6, 1)
22	$PA_6(3,6,5^n11+1)$	$PA_3(3,6,6)$	3-(12, 6, 2)
23	$PA_1(3,8,7^{n+1}+1)$	$PA_3(3,6,6)$	3-(8, 8, 1)
24	$PA_3(3,8,7^n15+1)$	$PA_1(3,8,8)$	3-(16, 8, 3)
25	$PA_6(3,9,8^{n+1}+1)$	$PA_6(3,9,9)$	3-(9, 9, 1)

3. Groups and perpendicular arrays

Groups provide a powerful tool in the construction of perpendicular arrays. We need some definitions.

Assume G is a permutation group acting on a v-set X so G has *degree* v and we then write $G \mid X$. If S is a subset of X we denote by $S^G = \{S^g : g \in G\}$ the *orbit* of S under G. The group action $G \mid X$ is *semiregular* if the only element of G fixing an element of X is the identity. This means that all orbits have length equal to $|G|$, since in general we have $|G| = |x^G|\,|G_x|$ where G_x is the stabilizer in G of x. The group G is said to be *t-homogeneous* if for any t-sets T_1, T_2 of X, there is a $g \in G$ such that $(T_1)^g = T_2$. Now G is an automorphism group of a t-design $(X, \mathscr{B})$ if G acts on X and preserves $\mathscr{B}$. Also G is an automorphism group of a perpendicular array A if G preserves the multiset of rows of A. We will let $X^{(i)}$ be the set of all i-subsets of X.

As noted in [18] if the permutations of a t-homogeneous group of degree v form the rows of an array A, then A is a $PA_\lambda(t, v, v)$, where $\lambda = |G|/\binom{v}{t}$. This perpendicular array has $\lambda = 1$ if the group is *sharply* t-homogeneous. Unfortunately, if $t>5$ there are no t-homogeneous groups other than the symmetric and alternating groups. When $t=4$ or 5 there are only the Mathieu groups M_{24}, M_{23}, M_{12}, M_{11} and the groups $P\Gamma L_2(32)$, $P\Gamma L_2(8)$, $PGL_2(8)$ in their natural representations. There are infinitely many t-homogeneous groups for $2 \leqslant t \leqslant 3$ (see [2]). From each of these groups corresponding $PA_\lambda(t, k, v)$'s arise.

Without the assumption of t-homogeneity, groups are still useful in the direct construction of PA's and in the construction of tBD's from which PA's can be constructed by means of Theorem 1.2. A general method for constructing t-wise balanced designs admitting a particular group of automorphisms is described in [10]. Very briefly, if G acts on X, then let $A_{t,k}$ be the matrix whose (i, j)th entry is the number of members in the jth orbit of k-sets containing a fixed member of the ith orbit of t-sets. A t-(v, K, λ) design with G as automorphism group then exists iff there is a nonnegative integral solution U to the matrix equation $A_{t,K}U = \lambda J$, where $A_{t,K}$ is the catenation of the matrices $A_{t,k}$, for $k \in K$, and J is the all 1's vector. Let $B_{t,k}$ be the matrix whose (i, j)th entry is the number of members of the ith orbit of t-sets contained in a fixed member of the jth orbit of k-sets. If $\rho(t)(\rho(k))$ is the number of G-orbits on t-subsets (k-subsets) of X, then $A_{t,k}$ and $B_{t,k}$ are $\rho(t) \times \rho(k)$ nonnegative integral matrices. Note that $A_{t,k}$ has constant row sums equal to $\binom{v-t}{k-t}$, and $B_{t,k}$ has constant column sums equal to $\binom{k}{t}$. The matrices $A_{t,k}$ and $B_{t,k}$ are related as follows (see [12]): Let T_i be the ith orbit of t-sets and K_j the jth orbit of k-sets. Then $a_{i,j}\,|T_i| = b_{i,j}\,|K_j|$. The reader is referred to [12] for a more extensive discussion of these matrices and related properties.

If $(X, \mathscr{B})$ is a t-(v, k, λ) design with automorphism group G, then $\mathscr{B}$ is the union of certain G-orbits of k-sets. Let $A_{\mathscr{B}}$, $(B_{\mathscr{B}})$ be the submatrix of $A_{t,k}(B_{t,k})$ with columns corresponding to the orbits of k-sets occurring in $\mathscr{B}$. Note that $A_{\mathscr{B}}$ has row sums equal to λ, and $B_{\mathscr{B}}$ has column sums equal to $\binom{k}{t}$. We define a third matrix $OR_{\mathscr{B}}$ which relates directly to $B_{\mathscr{B}}$. Let the orbit representatives (G-starters) for $\mathscr{B}$ be $B_1, \ldots, B_s$. For each B_i let OB_i be an ordered k-tuple whose elements are the k elements in B_i. The rows of $OR_{\mathscr{B}}$ are indexed by $OB_1, \ldots, OB_s$ and its $\binom{k}{t}$ columns are indexed by the t-subsets of $\{1, 2, \ldots, k\}$ arranged in a particular fixed order. If $OR_{\mathscr{B}} = (r_{i,j})$ has its ith row indexed by $(x_1, \ldots, x_k)$ and its jth column indexed by $\{j_1, \ldots, j_t\}$ then $r_{i,j} = m$ if $\{x_{j_1}, \ldots, x_{j_t}\}$ is in the mth orbit of t-sets. Note that there are $(k!)^s$ such $OR_{\mathscr{B}}$'s for this design $(X, \mathscr{B})$. Observe that $B_{\mathscr{B}}$ can be computed from $OR_{\mathscr{B}}$ by setting the (i, j)th entry of $B_{\mathscr{B}}$ equal to the number of occurrences of orbit index i in jth row of $OR_{\mathscr{B}}$. A direct consequence of this observation is the following theorem.

Theorem 3.1. *Let $(X, \mathscr{B})$ be a t-(v, k, λ) design admitting an automorphism group G which is semiregular on $X^{(t)} \cup \mathscr{B}$. Then each orbit index i of t-sets appears exactly λ times in $OR_{\mathscr{B}}$.*

Proof. G preserves $X^{(t)}$, and separately $\mathscr{B}$ but is semiregular on $X^{(t)} \cup \mathscr{B}$. Hence, all orbits on t-sets and on $\mathscr{B}$ have the same length. Because $a_{i,j}\,|T_i| = b_{i,j}\,|K_j|$ it follows that $B_{\mathscr{B}} = A_{\mathscr{B}}$. Thus $B_{\mathscr{B}}$ has constant row sums equal to λ. We now note that the number of times that i appears in $\mathrm{OR}_{\mathscr{B}}$ is $\sum_j B_{\mathscr{B}}(i,j) = \sum_j A_{\mathscr{B}}(i,j) = \lambda$. □

The point of Theorem 3.1 is that the uniformity with which each orbit index i occurs in $\mathrm{OR}_{\mathscr{B}}$ makes it conceivable (in lieu of Theorem 2.1) that there exist reorderings of the blocks $B_1, \ldots, B_s$ which would yield a corresponding $\mathrm{OR}_{\mathscr{B}}$ with the property that each of its columns has all orbit indices for t-sets appearing the same number of times. If such a rearrangement of the blocks exists, then, clearly, using the ordered blocks as starters under G we could develop a perpendicular array. In any case we clearly have the following.

Theorem 3.2. *Let $(X, \mathscr{B})$ be a t-$(v, k, \lambda\binom{k}{t})$ design admitting a group of automorphisms G which acts semiregularly on $X^{(t)} \cup \mathscr{B}$. If there exists an $\mathrm{OR}_{\mathscr{B}}$ where each integer i, $1 \leq i \leq \rho(t)$, appears λ times in each column of $\mathrm{OR}_{\mathscr{B}}$ then there is a $\mathrm{PA}_{\lambda}(t, k, v)$ with G as an automorphism group.*

We illustrate this theorem by the following example.

Example of our procedure. For a $\mathrm{PA}_1(3,5,11)$ to exist it necessarily requires the existence of a 3-(11, 5, 10). Let G be the Frobenious group of order 55 acting on a set X of cardinality 11. We generate G using $(0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,10)$ and $(0)(1\,3\,9\,5\,4)(2\,6\,7\,10\,8)$. A 3-(11, 5, 10) exists using starting blocks $\{\{0,1,2,3,4\}, \{0,1,2,3,5\}, \{0,1,2,3,9\}\}$. There are exactly 3 orbits of 3-sets with representatives $\{0,1,2\}$, $\{0,1,3\}$ and $\{0,1,5\}$ each having length 55. An $\mathrm{OR}_{\mathscr{B}}$ that works is:

	123	124	125	134	135	145	234	235	245	345
(0, 1, 2, 3, 4)	1	2	2	3	1	3	1	2	3	1
(0, 2, 10, 1, 4)	2	1	1	1	3	2	3	3	2	2
(0, 3, 5, 2, 4)	3	3	3	2	2	1	2	1	1	3

where 123 represents $\{1,2,3\}$, etc. Clearly, we get a $\mathrm{PA}_1(3,5,11)$ by letting G act on the rows of the array:

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 10 & 1 & 4 \\ 0 & 3 & 5 & 2 & 4. \end{matrix}$$

This $\mathrm{PA}_1(3,5,11)$ is unique given this particular group.

Many perpendicular arrays used in the tBD-type construction were obtained by a procedure similar to the preceeding example (see [11] for example). Also, it should be noted that known automorphisms of a design or array can be used to provide an economical listing of the structure. In passing, we should mention the following theorem.

Theorem 3.3. *Let $(X, \mathscr{B})$ be a t-$(v, t+1, \lambda)$ design where G acts semiregularly on $X^{(t)} \cup \mathscr{B}$. With $\lambda_1 = \lambda/(\lambda, t+1)$ there is a perpendicular array* $\mathrm{PA}_{\lambda_1}(t, t+1, v)$ *with G as an automorphism group.*

Proof. Let s be the number of orbits in $\mathscr{B}$. Choose some $\mathrm{OR}_{\mathscr{B}}$ where column j, $1 \leqslant j \leqslant t+1$, is indexed by $\{1, \ldots, t+1\} \setminus \{j\}$. Construct a matrix A of size m by $(t+1)$, with $m = s(t+1)/(\lambda, t+1)$, by replicating each row of the $\mathrm{OR}_{\mathscr{B}}(t+1)/\lambda, t+1)$ times. By Theorem 3.1 each i, $1 \leqslant i \leqslant \rho(t)$, appears exactly $\lambda(t+1)/(\lambda, t+1)$ times in A. By Theorem 2.1 we can transform A, via permutations within the rows of A to a matrix A' where each i, from 1 to $\rho(t)$, appears exactly $\lambda/(\lambda, t+1)$ times in each column of A'. But these permutations are clearly bijections between the rows of A and the k-tuples indexing the rows. Clearly Theorem 3.2 applies and we have our result. □

Observe that if G is the identity group and $(X, \mathscr{B})$ is a t-(v, k, λ) design then Theorem 3.3 yields Theorem 2.2.

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