JOURNAL OF ALGEBRA 23, 88–99 (1972)

On Fully Left Bounded Left Noetherian Rings

GÜNTER KRAUSE

University of Manitoba, Winnipeg 19, Manitoba, Canada

Communicated by A. W. Goldie

Received March 28, 1971

TO PROFESSOR REINHOLD BAER, TO COMMEMORATE HIS SEVENTIETH BIRTHDAY, JULY 22, 1972

INTRODUCTION

All rings R considered in this note are associative and have an identity. A module M is a unitary left R-module, $E(M) = E_R(M)$ denotes its injective hull, and $\text{Ass}(M)$ is the set of its associated prime ideals. A prime ring is said to be left bounded if every essential left ideal contains a nonzero two-sided ideal, and we call the ring R fully left bounded, if R/P is left bounded for each prime ideal P.

It is well-known that for an injective, indecomposable module E over a left noetherian ring the set $\text{Ass}(E)$ consists of a uniquely determined prime ideal. The map

$$\beta : E \to \text{Ass}(E)$$

induces a surjective map from the set of isomorphism classes of injective, indecomposable left R-modules onto the set $\text{spec}(R)$ of all prime ideals of R. The main purpose of this paper is to show that this induced map is bijective if and only if R is fully left bounded (Theorem 3.5). It may be worth noting that this characterization provides some justification for the terminology "rings with sufficiently many two-sided ideals" used by Gabriel in [1].

If the classical Krull-dimension and the left Krull-dimension of the ring R are defined as in [5], then these dimensions are equal for every fully left bounded, left noetherian ring (Theorem 2.4). Together with Theorem 3.5 this shows that the restriction to rings with finite left Krull-dimension in similar theorems of Gabriel ([1], p. 426, Corollaire 3) Gabriel and Rentschler ([2], p. 713), and Michler ([7], p. 136, Theorem 3.4) is not necessary.

Concerning the terminology we refer to Goldie [3] and Matlis [6]. For a subset S of the left R-module M the left annihilator of S in R is denoted by $\ell_R(S)$. A prime ideal P of R is *associated* with the module M if there exists a submodule $N \neq 0$ of M with $P = \ell_R(N^*)$ for all nonzero submodules N^* of N. A ring is *left Goldie* if it satisfies the maximum condition for left annihilators and does not contain an infinite direct sum of nonzero left ideals. The *singular submodule* $Z(M)$ is the set of all elements m in M for which $\ell_R(m)$ is an essential left ideal. A module M is *homogeneous* (*isotypic*) if $E(M)$ is a direct sum of injective, indecomposable modules.

1. Classical Krull-Dimension

The purpose of this section is to provide some straightforward results concerning the classical Krull-dimension. For the convenience of the reader we repeat the relevant definitions of [5].

Definition 1.1. Let spec(R) be the set of all prime ideals of R, let spec$_0(R)$ denote the set of all maximal ideals of R, and for ordinals $\alpha > 0$ let

$$\operatorname{spec}_\alpha(R) = \left\{ P \in \operatorname{spec}(R) \mid P \subsetneq Q \in \operatorname{spec}(R) \text{ implies } Q \in \bigcup_{\beta<\alpha} \operatorname{spec}_\beta(R) \right\}.$$

The smallest ordinal α (if it exists) for which spec$(R) =$ spec$_\alpha(R)$ is called the *classical Krull-dimension* cl.K-dim(R) of R.

It is clear that cl.K-dim(R) need not always be defined. It was shown in [5] that the maximum condition for two-sided ideals of R implies the existence of an ordinal α such that spec$_\alpha(R) =$ spec(R). The following provides a necessary and sufficient condition.

Proposition 1.2. *The following properties of the ring R are equivalent*:

(1) *R satisfies the maximum condition for prime ideals.*

(2) *There is an ordinal $\alpha \geqslant 0$ with* spec$_\alpha(R) =$ spec(R).

Proof. (1) $\to$ (2): Let spec$_\beta(R) \neq$ spec(R) for some $\beta \geqslant 0$. It suffices to show that spec$_\beta(R) \subsetneq$ spec$_{\beta+1}(R)$. If P is maximal in spec(R) $-$ spec$_\beta(R)$ and Q is a prime ideal containing P properly, then

$$Q \in \operatorname{spec}_\beta(R) = \bigcup_{\gamma<\beta+1} \operatorname{spec}_\gamma(R)$$

which implies $P \in$ spec$_{\beta+1}(R)$.

(2) $\to$ (1): Assume spec$_\alpha(R) =$ spec(R) for some ordinal $\alpha \geqslant 0$, and let S be a nonempty set of prime ideals of R. Let β be the smallest one among the ordinals $\gamma \leqslant \alpha$ for which $S \cap$ spec$_\gamma(R)$ is not empty. Let $P \in S \cap$ spec$_\beta(R)$ and assume there is an element Q in S which contains P

properly. Then $Q \in \bigcup_{\gamma<\beta} \operatorname{spec}_\gamma(R)$ and hence $Q \in \operatorname{spec}_\gamma(R) \cap S$ for some $\gamma < \beta$, contradicting the minimality of β. Thus P is maximal in S.

LEMMA 1.3. *Let R be a ring and $\alpha \geqslant 0$ an ordinal. Then*:

(a) *If I is an ideal contained in the prime ideal P, then $P \in \operatorname{spec}_\alpha(R)$ if and only if $P/I \in \operatorname{spec}_\alpha(R/I)$.*

(b) cl.K-dim$(R) = \alpha$ *implies* cl.K-dim$(R/I) \leqslant \alpha$ *for every ideal I of R.*

(c) *If R is a prime ring with* cl.K-dim$(R) = \alpha$ *and $P \neq 0$ is a prime ideal, then* cl.K-dim$(R/P) < \alpha$.

Proof. (a) The statement is obvious for $\alpha = 0$. Let $\alpha > 0$ and assume (a) holds for all $\beta < \alpha$. From the definition of $\operatorname{spec}_\alpha(R)$ and the induction hypothesis we get: $P \in \operatorname{spec}_\alpha(R)$ iff $P \subsetneq Q \in \operatorname{spec}(R)$ implies $Q \in \operatorname{spec}_\beta(R)$ for some $\beta < \alpha$, iff $P/I \subsetneq Q/I \in \operatorname{spec}(R/I)$ implies $Q/I \in \operatorname{spec}_\beta(R)$ for some $\beta < \alpha$, iff $P/I \in \operatorname{spec}_\alpha(R/I)$.

(b) If $P/I \in \operatorname{spec}(R/I)$, then $I \subseteq P \in \operatorname{spec}(R) = \operatorname{spec}_\alpha(R)$, so $P/I \in \operatorname{spec}_\alpha(R/I)$ by (a).

(c) Clearly $P \in \operatorname{spec}_\beta(R)$ for some $\beta < \alpha$. If $Q/P \in \operatorname{spec}(R/P)$, then $Q \in \operatorname{spec}_\gamma(R)$ for some $\gamma \leqslant \beta$ and hence $Q/P \in \operatorname{spec}_\gamma(R/P)$ by (a). Thus

$$\operatorname{spec}(R/P) \subseteq \bigcup_{\gamma \leqslant \beta} \operatorname{spec}_\gamma(R/P) = \operatorname{spec}_\beta(R/P)$$

which implies cl.K-dim$(R/P) \leqslant \beta < \alpha$.

LEMMA 1.4. *Let R be a ring with* cl.K-dim$(R) \geqslant \alpha \geqslant 0$. *If* cl.K-dim$(R/I) < \alpha$ *for every ideal $I \neq 0$, then R is a prime ring with* cl.K-dim$(R) = \alpha$.

Proof. Let P and Q be prime ideals with $P \subsetneq Q$. Since cl.K-dim$(R/Q) = \beta < \alpha$ we get $Q \in \operatorname{spec}_\beta(R)$ by 1.3 (a). Thus $P \in \operatorname{spec}_\alpha(R)$ for all $P \in \operatorname{spec}(R)$, and this yields cl.K-dim$(R) \leqslant \alpha$, so cl.K-dim$(R) = \alpha$. Assume R is not a prime ring and let A and B be nonzero ideals with $AB = 0$. Let $\beta =$ max(cl.K-dim(R/A), cl.K-dim(R/B)) and let P be a prime ideal of R. We may assume $A \subseteq P$, and it follows from 1.3 (b) that

$$\text{cl.K-dim}(R/P) = \text{cl.K-dim}(R/A/P/A) \leqslant \text{cl.K-dim}(R/A) \leqslant \beta < \alpha,$$

so $P \in \operatorname{spec}_\beta(R)$ by 1.3 (a). Thus $\operatorname{spec}(R) = \operatorname{spec}_\beta(R)$ with $\beta < \alpha$, which contradicts cl.K-dim$(R) = \alpha$.

PROPOSITION 1.5. *The following properties of the ring R with maximum condition for two-sided ideals are equivalent*:

(1) *R is a prime ring.*

(2) cl.K-dim$(R/P) <$ cl.K-dim(R) *for every prime ideal* $P \neq 0$.

(3) cl.K-dim$(R/I) <$ cl.K-dim(R) *for every ideal* $I \neq 0$.

Proof. By 1.2 it is clear that cl.K-dim(S) is defined for every epimorphic image S of R. By 1.3 (c) condition (1) implies (2), and Lemma 1.4 asserts that (1) follows from (3). Assume (2) and let I be an ideal which is maximal with respect to cl.K-dim$(R/I) =$ cl.K-dim$(R) = \alpha$. If K/I is a nonzero ideal of R/I, then

$$\text{cl.K-dim}(R/I/K/I) = \text{cl.K-dim}(R/K) < \alpha = \text{cl.K-dim}(R/I)$$

by the maximality of I, so R/I is a prime ring by Lemma 1.4. Thus $I = 0$ by (2), and this implies (3).

2. Krull-Dimension of Fully Left Bounded Rings

Definition 2.1 (See [5]). For the left R-module M let $G(M)$ denote the set of all pairs (K, N) of submodules K and N of M with $N \subseteq K$. Let

$$G_0(M) = \{(K, N) \in G(M) \mid K/N \text{ artinian}\}$$

and

$$G_\alpha(M) = \left\{(K, N) \in G(M) \mid K \supseteq K_1 \supseteq \cdots \supseteq K_i \supseteq K_{i+1} \supseteq \cdots \supseteq N \right.$$
$$\left. \text{implies} \quad (K_i, K_{i+1}) \in \bigcup_{\beta<\alpha} G_\beta(M) \quad \text{for almost all } i \right\}$$

for ordinals $\alpha > 0$. If $G_\alpha(M) = G(M)$ for some ordinal α, then the smallest such ordinal is the *Krull-dimension* K-dim(M) of the module M. For the ring R the ordinal ℓ.K-dim$(R) =$ K-dim$(_R R)$ is the *left Krull-dimension* of R.

For finite ordinals Definition 2.1 coincides with the definition of the Krull-dimension as given by Gabriel and Rentschler in [2]. In [5] it was shown, that the existence of an ordinal with $G_\alpha(M) = G(M)$ implies that M is finite-dimensional in the sense of Goldie. The existence of such an ordinal is guaranteed in case M is noetherian.

Proposition 2.2. *Let R be a fully left bounded ring whose left Krull-dimension is defined. Then*

(a) *The classical Krull-dimension of R is defined.*

(b) cl.K-dim$(R) \leqslant \ell$.K-dim(R).

Proof. (a) Assume ℓ.K-dim$(R) = \alpha$ for some ordinal $\alpha \geqslant 0$. It is clear that ℓ.K-dim$(R/I) \leqslant \alpha$ for every ideal I of R. By Proposition 1.2 we have to verify the maximum condition for prime ideals of R. Let $\mathfrak{M} = \{P_k, k \in K\}$ be a nonempty set of prime ideals, and let $\beta_k = \ell.\text{K-dim}(R/P_k)$. Let β_0 be the smallest one among the ordinals β_k, and assume that $P_k \supsetneq P_0$ for some $0 \neq k \in K$. By Proposition 4 of [5] the ring $S = R/P_0$ has finite left Goldie-dimension. Since S is left bounded, the singular ideal $Z({}_S S)$ is zero, so S possesses a simple artinian quotient ring by Goldie's theorem for prime rings. The ideal $Q = P_k/P_0$ is an essential left ideal of S, so it contains a regular element c. Obviously

$$\text{K-dim}(S/Sc) \geqslant \text{K-dim}(S/Q) = \text{K-dim}(R/P_k) = \beta_k \geqslant \beta_0 .$$

Since $Sc^i/Sc^{i+1} \simeq S/Sc$ for $i = 1, 2,\ldots$, and since the left ideals Sc^i form a properly descending chain, this contradicts $\ell.\text{K-dim}(S) = \beta_0$. Thus P_0 is maximal in $\mathfrak{M}$.

(b) By (a) R satisfies the maximum condition for prime ideals. Furthermore, as observed in the proof of (a), R/P is a left Goldie ring for every prime ideal P. Therefore, the proof of Lemma 12 in [5] can be used, and we omit the details.

Remark 2.3. The converse of Proposition 2.2 is not true, there are even fully left bounded prime rings whose classical Krull-dimension is defined, but their left Krull-dimension is not. Let S be the simple ring of all $\aleph_0 \times \aleph_0$-matrices over a field K which have only a finite number of nonzero entries in each row and column. Let $R = \{(s, k) \mid s \in S, k \in K\}$ with addition defined componentwise and multiplication according to the rule

$$(s_1, k_1)(s_2, k_2) = (s_1 s_2 + k_1 s_2 + k_2 s_1, k_1 k_2).$$

It is easy to verify that the only prime ideals of R are $\{0\}$ and S, so $\text{cl.K-dim}(R) = 1$. Since $R/S \simeq K$, the ring R/S is left bounded. Since S is the left socle of R, every essential left ideal of R contains S, so R is left bounded as well. But $\ell.\text{K-dim}(R)$ is not defined because of Proposition 4 in [5], since S is a direct sum of infinitely many minimal left ideals.

THEOREM 2.4. *If R is a fully left bounded, left noetherian ring, then* $\ell.\text{K-dim}(R) = \text{cl.K-dim}(R)$.

Proof. Since R is left noetherian, $\text{cl.K-dim}(R) \leqslant \ell.\text{K-dim}(R)$ by Lemma 12 of [5]. Let $\text{cl.K-dim}(R) = \alpha$ and assume first that R is a prime ring. If $\alpha = 0$, then R is a simple ring with no essential left ideals other than R, so it is artinian. Let $\alpha > 0$ and assume that for any $\beta < \alpha$ every fully left bounded, left noetherian prime ring S with $\text{cl.K-dim}(S) = \beta$ satisfies $\ell.\text{K-dim}(S) = \beta$. Let L be an essential left ideal of R and let $I \neq 0$ be an ideal contained in L. Since R is left noetherian, there exist prime ideals $P_1, \ldots, P_n \supseteq I$ with $P_1P_2 \cdots P_n \subseteq I$. Since cl.K-dim$(R/P_i) < \alpha$ by Lemma 1.3, ℓ.K-dim$(R/P_i) < \alpha$ by induction hypothesis. Thus the Krull-dimension of each of the finitely generated left R/P_i-modules $P_{i+1} \cdots P_n/P_iP_{i+1} \cdots P_n$ is less than α by Lemma 7 of [5], so

$$\text{K-dim}(R/L) \leqslant \max(\text{K-dim}(P_{i+1} \cdots P_n/P_i \cdots P_n)) < \alpha$$

by Lemma 7 of [5], and this implies ℓ.K-dim$(R) \leqslant \alpha$ by Proposition 8 of [5].

If R is not a prime ring, then $0 = P_1P_2 \cdots P_n$ with prime ideals $P_i \neq 0$ because R is left noetherian. For each i the module $P_{i+1} \cdots P_n/P_iP_{i+1} \cdots P_n$ is a finitely generated left R/P_i-module, and since the assertion has already been proved for prime rings, it follows from Lemma 7 in [5] that

$$\text{K-dim}(P_{i+1} \cdots P_n/P_i \cdots P_n) \leqslant \ell.\text{K-dim}(R/P_i) = \text{cl.K-dim}(R/P_i) \leqslant \alpha$$

and hence

$$\ell.\text{K-dim}(R) = \max(\text{K-dim}(P_{i+1} \cdots P_n/P_i \cdots P_n)) \leqslant \text{cl.K-dim}(R).$$

3. Rings with Sufficiently Many Two-Sided Ideals

In this section we give an ideal-theoretic characterization of those left noetherian rings for which the map

$$\beta : E \to \text{Ass}(E)$$

induces a bijective map from the set of isomorphism classes of injective, indecomposable left R-modules onto the set spec(R).

Lemma 3.1. *Let R be a ring whose prime ideals are finitely generated left ideals. Then:*

(a) *For every ideal $I \neq R$ there exists a family of prime ideals $P_1, P_2, \ldots, P_n$ which contain I and for which $P_1P_2 \cdots P_n \subseteq I$.*

(b) *R satisfies the ascending chain condition for prime ideals.*

Proof. (a) Since all finite products of prime ideals are also finitely generated left ideals, this follows easily by Zorn's lemma.

(b) Let $P_1 \subseteq P_2 \subsetneq \cdots \subseteq P_n \subsetneq \cdots$ be an ascending chain of prime ideals, and let P be their set theoretic union. By (a) there are prime ideals $Q_1, Q_2, \ldots, Q_m$ containing P with $Q_1Q_2 \cdots Q_m \subseteq P$. Since $Q_1Q_2 \cdots Q_m$ is a finitely generated left ideal, $Q_1Q_2 \cdots Q_m \subseteq P_n$ for some n and hence

$$Q_i \subseteq P_n \subseteq P \subseteq Q_i \qquad \text{for some } i, \qquad 1 \leqslant i \leqslant m.$$

Thus $P = P_n$, and the chain becomes stationary after finitely many steps.

LEMMA 3.2. *The class of all rings R with* $\mathrm{Ass}(E_1) \neq \mathrm{Ass}(E_2)$ *for any two nonisomorphic injective, indecomposable left R-modules* E_1 *and* E_2 *is closed under epimorphic images.*

Proof. Let $S = R/I$ be an epimorphic image of R, and let F_1 and F_2 be two nonisomorphic injective, indecomposable left S-modules. Clearly F_1 and F_2 are nonisomorphic and uniform as left R-modules. If $E_R(F_1) \simeq E_R(F_2) \simeq X$, then the injective, indecomposable left R-module X contains copies of both F_1 and F_2 and hence their nonzero intersection N. Since N is clearly an S-module, it follows from Proposition 2.2 in [6] that $F_i \simeq E_S(N)$, $i = 1, 2$. This contradiction shows that $E_R(F_1)$ and $E_R(F_2)$ are not isomorphic. Since $I \subseteq \ell_R(F_i)$, it is easily verified that the set $\mathrm{Ass}_S(F_i)$ is empty (a case which is conceivable since R is not assumed to be left noetherian) if and only if $\mathrm{Ass}_R(E_R(F_i))$ is empty and that $\mathrm{Ass}_S(F_i) = P/I$ for some prime ideal P of R if and only if $\mathrm{Ass}_R(E_R(F_i)) = P$. Thus $\mathrm{Ass}_S(F_1) \neq \mathrm{Ass}_S(F_2)$ by hypothesis.

PROPOSITION 3.3. *Let R be a ring which satisfies the following conditions:*

(a) *Prime ideals of R are finitely generated left ideals.*

(b) *R/P is a left Goldie ring for every prime ideal P.*

(c) $\mathrm{Ass}(E) \neq \mathrm{Ass}(F)$ *for any two nonisomorphic injective, indecomposable left R-modules E and F.*

Then R is a fully left bounded, left noetherian ring.

Proof. First we prove that R is fully left bounded. For this R may be assumed to be a prime ring, since condition (c) is inherited by epimorphic images (Lemma 3.2). Assume R is not left bounded. Then there exists an essential left ideal $L \neq R$ which does not contain a product of finitely many nonzero prime ideals. Because of (a), Zorn's lemma allows to choose L maximal with respect to this property. By Lemma 3.1 the left ideal L does not contain a nonzero two-sided ideal, so $\ell_R(R/L) = 0$. Therefore

$$\ell_R(K/L) \cdot \ell_R(R/K) \subseteq \ell_R(R/L) = 0$$

for every left ideal K which contains L properly. Since R is a prime ring, it follows that $\ell_R(K/L) = 0$, because by the maximality of L the left ideal K contains a product of nonzero prime ideals, so $\ell_R(R/K)$ cannot be zero. Thus $\mathrm{Ass}(R/L)$ consists of the zero ideal which is also the unique associated prime ideal of the left R-module ${}_RR$. As a left Goldie ring R is left finite-dimensional, so

$$E({}_RR) = X_1 \oplus \cdots \oplus X_n$$

with injective, indecomposable left R-modules X_i. Since clearly $\mathrm{Ass}(X_i) = \{0\}$ for $i = 1, 2, \ldots, n$, it follows from (c) that $E(R/L) \simeq X_i$ for each i, because R/L is uniform by the maximality of L. If φ denotes the canonical R-homomorphism from R onto R/L and σ denotes an R-isomorphism from $E(R/L)$ onto X_1, then $\sigma\varphi(R) \subseteq X_1 \subseteq E(_RR)$. Now

$$x(\sigma\varphi(1)) = \sigma\varphi(x \cdot 1) = \sigma\varphi(x) = 0$$

for all x in L, so $L \subseteq \ell_R(\sigma\varphi(1))$. Because of $L \neq R$ the element $\sigma\varphi(1)$ is not zero, so $Z(E(_RR) \neq 0$. But this implies $Z(_RR) \neq 0$ which is impossible because R is a left Goldie prime ring.

Now we turn to the proof of the maximum condition for left ideals. Let $P_1, P_2, \ldots, P_n$ be a set of prime ideals and assume the rings R/P_i are left noetherian. The i-th factor module of the chain

$$P_1P_2 \cdots P_n \subseteq P_2 \cdots P_n \subseteq \cdots \subseteq P_{n-1}P_n \subseteq P_n \subseteq R$$

is a finitely generated and hence noetherian left R/P_i-module. Therefore it is also a noetherian left R-module, so the left R-module $R/P_1P_2 \cdots P_n$ is noetherian. By Lemma 3.1 the zero ideal is the product of finitely many prime ideals, so it suffices to show that R/P is left noetherian for each prime ideal P. Assume this is not true. By Lemma 3.1 there is a prime ideal P which is maximal with respect to the property that R/P is not left noetherian. Let L be an essential left ideal of $S = R/P$. Since S is left bounded, L contains a two-sided ideal $I \neq 0$, and this in turn contains a product $P_1P_2 \cdots P_n$ of prime ideals $P_i \supseteq I$ of S by Lemma 3.1. By the argument above, $S/P_1P_2 \cdots P_n$ is a noetherian left S-module, so the submodule $L/P_1P_2 \cdots P_n$ is finitely generated. Since $P_1P_2 \cdots P_n$ is a finitely generated left ideal by (a), it follows that L is finitely generated. But every left ideal is a direct summand of an essential left ideal. Contradiction!

Remark. Lemma 3.1 and the fact that a fully left bounded ring whose prime ideals are finitely generated left ideals is left noetherian have independently been proved also by G. Michler and L. Small.

COROLLARY 3.4. *The ring R is semisimple artinian if and only if it is a semiprime left Goldie ring whose prime ideals are maximal, and for which* $\mathrm{Ass}(E) \neq \mathrm{Ass}(F)$ *for any two nonisomorphic injective, indecomposable left R-modules E and F.*

Proof. A semisimple artinian ring is clearly left Goldie and its prime ideals are maximal. It follows from Gabriel ([1], p. 423, Lemma 2) that two nonisomorphic injective, indecomposable left R-modules have different associated prime ideals. For the converse we note that a semiprime left

Goldie ring has only a finite number of maximal annihilator ideals and that their intersection is zero (see Lemma 4.7 and Lemma 4.8 of [4]). Furthermore, these maximal annihilator ideals $P_1, \ldots, P_n$ are prime ([4], Lemma 4.6) and hence maximal by assumption. By Lemma 4.10 of [4], each of the rings R/P_i is left Goldie. Since the only prime ideal of R/P_i is trivially finitely generated as left ideal and since $\mathrm{Ass}(E) \neq \mathrm{Ass}(F)$ for any two nonisomorphic injective, indecomposable left R/P_i-modules by Lemma 3.2, Proposition 3.3 applies. Thus the simple rings R/P_i are left bounded and hence artinian. Since

$$\begin{aligned}(P_{i+1} \cap \cdots \cap P_n)/(P_i \cap P_{i+1} \cap \cdots \cap P_n) \\ \underset{R}{\simeq} (P_i + (P_{i+1} \cap \cdots \cap P_n))/P_i = R/P_i\end{aligned}$$

for each $i = 1, 2, \ldots, n$, the factor modules of the chain

$$0 = P_1 \cap \cdots \cap P_n \subsetneq \cdots \subsetneq P_{n-1} \cap P_n \subsetneq P_n \subsetneq R$$

are artinian, and therefore R is left artinian. Thus R is also semisimple because it is semiprime.

THEOREM 3.5. *The following properties of the left noetherian ring R are equivalent*:

(1) *The map $\beta : E \to \mathrm{Ass}(E)$ induces a bijective map from the set of isomorphism classes of injective, indecomposable left R-modules onto the set* $\mathrm{spec}(R)$.

(2) *R is fully left bounded.*

(3) *If E is an injective, indecomposable left R-module with* $\mathrm{Ass}(E) = P$, *then there exists an element* $0 \neq e \in E$ *with* $\text{K-dim}(Re) = \ell.\text{K-dim}(R/P)$.

(4) *Each finitely generated tertiary left R-module is Goldman-primary.*

Proof. The equivalence of (1) and (4) is due to Michler ([7], Theorem 2.4) and has only been added for completeness. The implication (1) $\to$ (2) follows immediately from Proposition 3.3.

(2) $\to$ (3): Let E be an injective, indecomposable left R-module with associated prime ideal P, so that $P = \ell_R(Re)$ for some element $e \neq 0$ in E. Assume first that

$$\text{K-dim}(R/L) < \ell.\text{K-dim}(R/P) \qquad \text{for all left ideals } L \supsetneq P.$$

Then R/P is a left Öre domain by Theorem 10 of [5]. If $\ell_R(e) \supsetneq P$, then $\ell_R(e) \supseteq I \supsetneq P = \ell_R(Re)$ for some ideal I by (2), contradicting the fact that $\ell_R(Re)$ is the largest two-sided ideal contained in $\ell_R(e)$. Therefore $\ell_R(e) = P$, whence $E \simeq E(Re) \simeq E(R/\ell_R(e)) = E_R(R/P)$, so E contains a cyclic submodule which is even isomorphic to the left R-module R/P. Assume now that there is a left ideal $L \supsetneq P$ with $\text{K-dim}(R/L) = \ell.\text{K-dim}(R/P)$. It will suffice to show that $E \simeq E(R/L)$. Let L be represented as an irredundant intersection of irreducible left ideals L_i, $i = 1, 2,..., n$. Since R/L can be imbedded in $R/L_1 \oplus \cdots \oplus R/L_n$, it follows from Lemma 7 of [5] that

$$\text{K-dim}(R/L) \leqslant \max(\text{K-dim}(R/L_i), i = 1, 2,..., n).$$

Since each L_i contains L, we also have $\text{K-dim}(R/L_i) \leqslant \text{K-dim}(R/L)$ for each i, so $\text{K-dim}(R/L) = \text{K-dim}(R/L_i)$ for some i. Thus we may assume that L is irreducible. Assume next that L/P is an essential left ideal of R/P. Since R/P is left bounded, there exists an ideal I with $P \subsetneq I \subseteq L$. By Proposition 1.5 we have $\text{cl.K-dim}(R/I) < \text{cl.K-dim}(R/P)$, so it follows from Theorem 2.4 that

$$\begin{aligned} \text{K-dim}(R/L) &\leqslant \ell.\text{K-dim}(R/I) = \text{cl.K-dim}(R/I), \\ &< \text{cl.K-dim}(R/P) = \ell.\text{K-dim}(R/P). \end{aligned}$$

This contradiction shows that L/P is not an essential left ideal of R/P. Let X be a left ideal which is maximal with respect to the properties $X \supsetneq P$ and $X \cap L = P$, and let X be represented as an irredundant intersection of irreducible left ideals X_i, $i = 1, 2,..., n$. Then $P = L \cap X_1 \cap \cdots \cap X_n$ is an irredundant intersection and hence

$$E_R(R/P) \simeq E(R/L) \oplus E(R/X_1) \oplus \cdots \oplus E(R/X_n)$$

by Theorem 2.3 of [6]. On the other hand, $\ell_R(e)/P$ is not an essential left ideal of R/P, and since the submodule $Re \simeq R/\ell_R(e)$ of the injective, indecomposable module E is left uniform, the argument above can be applied again to yield

$$E_R(R/P) \simeq E(R/\ell_R(e)) \oplus E(R/Y_1) \oplus \cdots \oplus E(R/Y_m)$$

with irreducible left ideals Y_j, $j = 1, 2,..., m$. By Gabriel ([1], p. 421) the left R-module R/P is homogeneous, and this finally implies

$$E \simeq E(Re) \simeq E(R/\ell_R(e)) \simeq E(R/L).$$

(3) $\to$ (1): Since R is left noetherian the map $\beta : E \to \text{Ass}(E)$ certainly induces a surjective map from the set of isomorphism classes of injective, indecomposable left R-modules onto the prime spectrum spec(R) of R. Let E be an injective, indecomposable left R-module with associated prime ideal P. By Gabriel ([1], p. 421) any two injective, indecomposable sub-

modules of $E_R(R/P)$ are isomorphic, so (1) will follow if it can be shown that E is isomorphic to a submodule of $E_R(R/P)$. By assumption, there exists an element $e \neq 0$ in E with K-dim$(Re) = \ell$.K-dim(R/P). Assume $L = \ell_R(e)/P$ is an essential left ideal of $S := R/P$. By Theorem 3.9 of Goldie [3], L contains a regular element c, so L contains the properly descending chain

$$Sc \supsetneq Sc^2 \supsetneq \cdots \supsetneq Sc^i \supsetneq Sc^{i+1} \supsetneq \cdots \supsetneq 0.$$

It follows from $Sc^i/Sc^{i+1} \simeq S/Sc$ that

$$\begin{aligned} \text{K-dim}(Sc^i/Sc^{i+1}) &\geqslant \text{K-dim}(S/L) = \text{K-dim}(R/\ell_R(e)), \\ &= \ell.\text{K-dim}(R/P) = \ell.\text{K-dim}(S). \end{aligned}$$

Contradiction! Therefore L is not essential, so there exist irreducible left ideals X_i, $i = 1, 2, \ldots, n$ such that the intersection $P = \ell_R(e) \cap X_1 \cap \cdots \cap X_n$ is irredundant. By Theorem 2.3 of [6] this implies

$$E_R(R/P) \simeq E(R/\ell_R(e)) \oplus E(R/X_1) \oplus \cdots \oplus E(R/X_n),$$

so the assertion follows from $E \simeq E(Re) \simeq E(R/\ell_R(e))$.

In [5] a left noetherian ring R was defined to be a *left Matlis ring* if for every injective, indecomposable left R-module E there exists a prime ideal P of R with $E \simeq E_R(R/P)$. Theorem 3.5 gives the following characterization of these rings.

COROLLARY 3.6. *A left noetherian ring R is a left Matlis ring if and only if every nonzero left ideal of every prime epimorphic image of R contains a nonzero two-sided ideal.*

Proof. Assume R is a left Matlis ring. For every prime ideal P of R there exists an injective, indecomposable left R-module E with Ass$(E) = P$. Obviously $E \simeq E_R(R/P)$, so the ring R/P is left uniform and the assertion follows from Theorem 3.3. Conversely, let E be an injective, indecomposable left R-module and P its associated prime ideal. Then $P = \ell_R(Re)$ for some element $e \neq 0$ of E, and it follows that $P = \ell_R(e)$, for otherwise there would be an ideal I with $P \subsetneq I \subsetneq \ell_R(e)$, contradicting the fact that $\ell_R(Re)$ is the largest ideal contained in $\ell_R(e)$. Thus $E \simeq E(Re) \simeq E(R/\ell_R(e)) = E_R(R/P)$.

ACKNOWLEDGMENT

The author gratefully acknowledges support from the National Research Council of Canada.

References

1. P. Gabriel, Des catégories abéliennes, *Bull. Soc. Math. France* **90** (1962), 323–448.
2. P. Gabriel and R. Rentschler, Sur la dimension des anneaux et ensembles ordonnés, *C. R. Acad. Sci. Paris* **265** (1967), 712–715.
3. A. W. Goldie, Semi-prime rings with maximum condition, *Proc. London Math. Soc.* (3) **10** (1960), 201–220.
4. I. N. Herstein, "Topics in Ring Theory," Mathematics Lecture Notes, University of Chicago, 1965.
5. G. Krause, On the Krull-dimension of left noetherian left Matlis-rings, *Math. Z.* **118** (1970), 207–214.
6. E. Matlis, Injective modules over noetherian rings, *Pacific J. Math.* **8** (1958), 511–528.
7. G. Michler, Goldman's primary decomposition and the tertiary decomposition, *J. Algebra* **16** (1970), 129–137.