On ground state solutions for some non-autonomous Schrödinger–Poisson systems

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\textbf{Abstract}
In this paper, we study the Schrödinger–Poisson system
\begin{equation}
\begin{aligned}
-\Delta u + u + K(x)\phi(x)u &= a(x)f(u), & & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= K(x)u^2, & & \text{in } \mathbb{R}^3,
\end{aligned}
\end{equation}
and prove the existence of ground state solutions for system (SP) under certain assumptions on the linear and nonlinear terms. Some recent results from different authors are extended.

\section{Introduction}
Consider the following nonlinear system:
\begin{equation}
\begin{aligned}
-\Delta u + K(x)\phi(x)u &= a(x)f(u), & & \text{in } \mathbb{R}^3, \\
-\Delta \phi &= K(x)u^2, & & \text{in } \mathbb{R}^3.
\end{aligned}
\end{equation}
Such a system, also known as the nonlinear Schrödinger–Maxwell, arises in an interesting physical context. Indeed, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger and the Poisson equations (we refer to [8] for more details on the physical aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve (SP).

Variational methods and critical point theory are powerful tools in studying nonlinear differential equations [4,18,23,24], and in particular Hamiltonian systems and elliptic equations [2,7,12,13,16,17,20–22,25]. In recent years, system (SP) has been studied widely via modern variational methods under the various hypotheses, see [1,3,5,6,19,26] and the references therein. These researches mainly concern either the autonomous case or, the search of the so-called semi-classical states, that is the study of (SP) when the first equation looks like $-\epsilon^2 \Delta u + u + K(x)\phi(x)u = a(x)|u|^{p-1}u$ and the solutions exhibit concentration phenomena as the parameter $\epsilon$ goes to zero.

It is well known that system (SP) can be easily transformed in a nonlinear Schrödinger equation with a non-local term (see Section 2). Briefly, the Poisson equation is solved by using the Lax–Milgram theorem, so, for all $u$ in $H^1(\mathbb{R}^3)$, a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is obtained, such that $-\Delta \phi = K(x)u^2$ and that, inserted into the first equation, gives

$$-\Delta u + u + K(x)\phi_u(x)u = a(x)f(u). \quad (1.1)$$

Moreover, (1.1) is variational and its solutions are the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 \, dx - \int_{\mathbb{R}^3} a(x)F(u) \, dx, \quad (1.2)$$

where $F(u) = \int_0^uf(s) \, ds$.

Very recently, Cerami and Vaira [10] studied system (SP) with $f(u) = |u|^{p-1}u$ and $3 < p < 5$. They established a global compactness lemma to overcome the lack of compactness of the embedding of $H^1(\mathbb{R}^3)$ into the Lebesgue spaces $L^p(\mathbb{R}^3)$, $p \in (2, 6)$, preventing from using the variational techniques in a standard way. They proved the existence of positive ground state and bound state solutions by minimizing $I$ restricted to the Nehari manifold $\mathcal{N}$ when $K$ and $a$ satisfy different assumptions, respectively, but without requiring any symmetry property on them.

Motivated by the above fact, in this paper, our aim is to revisit the system (SP). We consider another case:

- when $f$ is asymptotically linear at infinity, i.e., $f(s)/s \to l$ as $s \to +\infty$, here $l$ is a constant.

In order to obtain our result, we have to solve various difficulties. First, the competing effect of the non-local term with the nonlinear term in the functional $I$ gives rise to some difficulties. Second, since the embedding of $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p \in (2, 6)$, is not compact, in order to recover the compactness, we establish a compactness lemma different from the one in [10]. In fact, this difficulty can be avoided, when autonomous problems are considered, restricting $I$ to the subspace of $H^1(\mathbb{R}^3)$ consisting of radially symmetric functions, or, when one is looking for semi-classical states, by using perturbation methods or a reduction to a finite dimension by the projections method. Third, it is not difficult to find that every $(PS)$ sequence is bounded when $3 < p < 5$ in [10] because a variant of global Ambrosetti–Rabinowitz condition is satisfied when $3 < p < 5$ (see [11]). However, for the asymptotically linear case, we have to find another method to verify the boundedness of $(PS)$ sequence.

**Definition 1.1.** $u \in H^1(\mathbb{R}^3)$ is a ground state of the system (SP) we mean that $u$ is such a solution of (SP) which has the least energy among all solutions of (SP), that is, $I'(u) = 0$ and $I(u) = \inf[I(v) : v \in H^1(\mathbb{R}^3) \setminus \{0\}]$ and $I'(v) = 0$. 
We state our main result.

**Theorem 1.1.** Assume that the following conditions hold:

(F1) $f \in C(\mathbb{R}, \mathbb{R}^+)$, $f(s) \equiv 0$ for all $s < 0$ and $f(s)/s \to 0$ as $s \to 0^+$.

(F2) There exists $l \in (0, +\infty)$ such that $f(s)/s \to l$ as $s \to +\infty$.

(A1) $a(x)$ is a positive continuous function and there exists $R_0 > 0$ such that

$$
\sup\{ f(s)/s : s > 0 \} \leq \inf\{ 1/a(x) : |x| \geq R_0 \}.
$$

(A2) There exists a constant $\beta \in (0, 1)$ such that

$$(1 - \beta)l > \mu^s := \inf\left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx : u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} a(x)F(u) \, dx \geq \frac{l}{2}, \right. \left. \int_{\mathbb{R}^3} K(x)\phi u^2 \, dx < 2\beta l \right\}.
$$

(K1) $K \in L^2(\mathbb{R}^3)$, $K(x) \geq 0$ for all $x \in \mathbb{R}^3$, but $K(x) \neq 0$.

Then system (SP) possesses a ground state solution in $H^1(\mathbb{R}^3)$.

**Remark 1.1.** Indeed, it is not difficult to find some functions $K, a, f$ such that the above conditions are satisfied. For example, for any $R_0 > 0$, let

$$
f(s) = \begin{cases} R_0 s^2/(1 + s), & \text{if } s > 0, \\ 0, & \text{if } s \leq 0. \end{cases}
$$

Clearly, (F1) and (F2) hold. Taking a positive continuous function $a(x)$ such that $a(x) = 1000/(1 + |x|)$, if $|x| \leq \frac{R_0}{2}$, $a(x) = 1/(1 + R_0)$, if $|x| \geq R_0$. Note that

$$\sup\{ f(s)/s : s > 0 \} = R_0 < R_0 + 1 = \inf\{ 1/a(x) : |x| \geq R_0 \},$$

then (A1) holds. Moreover, $l = R_0$ in (F2) and $F(u) = \int_0^u f(s) \, ds = R_0(\frac{1}{2}u^2 - u + \ln(1 + u))$. To verify the condition (A2), we have to choose some special $R_0 > 0$. For any $R > 0$, taking $\psi \in C_0^\infty(\mathbb{R}^3, [0, 1])$ such that $\psi(x) = 1$ if $|x| \leq R$, $\psi(x) = 0$ if $|x| \geq 2R$ and $|\nabla \psi(x)| \leq C/R$ for all $x \in \mathbb{R}^3$, where $C > 0$ is an arbitrary constant independent of $x$, and $K \in L^2(\mathbb{R}^3)$ such that $K(x) \geq 0$ for all $x \in \mathbb{R}^3$, $K(x) \neq 0$ and $|K|^2 \leq \frac{9R_0S^2S_4}{32\pi^2[R^2 + R^2]^2}$, where $\tilde{S}$ and $S$ are also defined in Section 2. Then we have, for $R_0 > 2R$,

$$
\int_{\mathbb{R}^3} a(x)F(\psi) \, dx \geq \int_{|x| \leq R} a(x)F(\psi) \, dx
\geq 1000 \times \left( \ln 2 - \frac{1}{2} \right) \frac{1}{1 + R} |B_R(0)|
= \frac{4000\pi}{3} \left( \ln 2 - \frac{1}{2} \right) \frac{R^3}{1 + R},
$$

and
\[
\int |\nabla \psi|^2 + |\psi|^2 \, dx \leq \int \frac{C^2}{R^2} \, dx + \int \, dx
\]
\[
= \left(1 + \frac{C^2}{R^2}\right) |B_{2R}(0)| = \frac{32\pi}{3} R(R^2 + C^2),
\]
(1.4)

where \(B_{2R}(0), |B_{2R}(0)|\) are also defined in Section 2. Furthermore, by (1.4) and (2.5) in Section 2, one has
\[
\int K(x) \phi \psi^2 \, dx \leq \tilde{S}^{-2} \cdot S^{-4} \|K\|^2_2 \|\psi\|^4
\]
\[
\leq \frac{32^2 \pi^2}{9} \tilde{S}^{-2} \cdot S^{-4} |K|^2_2 \left[ R(R^2 + C^2) \right]^2 \leq R_0.
\]
(1.5)

Taking \(\beta = \frac{1}{2}, R_0 = 1, C = \frac{R}{4}, R = \frac{1}{8} R_0 = \frac{1}{8}\). Then by (1.3) and (1.5) we obtain \( \int_{\mathbb{R}^3} a(x) F(\psi) \, dx > \frac{R_0}{2} \) and \( \int_{\mathbb{R}^3} K(x) \phi \psi^2 \, dx \leq 2 \beta l \). So, in view of the definition of \(\mu^*\) and (1.4), one has \(\mu^* \leq \frac{32^2 \pi R(R^2 + C^2)}{\beta R_0} = (1 - \beta) l\). So condition (A2) holds.

Remark 1.2. Compared to the case when \(f(u) = |u|^{p-1} u\) and \(3 < p < 5\) in [10], in our theorem we need not consider the limit value of \(a(x)\). In addition, also we do not need the assumption \(\lim_{|x| \to \infty} K(x) = 0\).

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.1.

Notation. Throughout the paper we denote by \(C > 0\) various positive constants which may vary from line to line and are not essential to the problem.

2. Preliminaries

In this section, we give a useful theorem. It is the variant version of the mountain pass theorem, which allows us to find a so-called Cerami type (PS) sequence. The properties of this kind of (PS) sequence are very helpful in showing the boundedness of the sequence in the asymptotically linear case.

Theorem 2.1. (See [14], Mountain pass theorem.) Let \(E\) be a real Banach space with its dual space \(E^*\), and suppose that \(I \in C^1(E, \mathbb{R})\) satisfies
\[
\max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\| = \rho} I(u),
\]
for some \(\mu < \eta, \rho > 0\) and \(e \in E\) with \(\|e\| > \rho\). Let \(c \geq \eta\) be characterized by
\[
c = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),
\]
where \(\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}\) is the set of continuous paths joining \(0\) and \(e\), then there exists a sequence \(\{u_n\} \subset E\) such that
\[
I(u_n) \to c \geq \eta \quad \text{and} \quad (1 + \|u_n\|) \left\|I'(u_n)\right\|_{E^*} \to 0, \quad \text{as} \quad n \to \infty.
\]
This kind of sequence is usually called a Cerami sequence. Hereafter we use the following notations:

- \( H^1(\mathbb{R}^3) \) is the usual Sobolev space endowed with the standard scalar product and norm
  \[ (u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) \, dx; \quad \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx. \]

- \( D^{1,2}(\mathbb{R}^3) \) is the completion of \( C_0^\infty(\mathbb{R}^3) \) with respect to the norm
  \[ \|u\|_{D^{1,2}} : = \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}. \]

- \( H^* \) denotes the dual space of \( H^1(\mathbb{R}^3) \).

- \( L^q(\Omega), 1 \leq q \leq +\infty, \Omega \subseteq \mathbb{R}^3, \) denotes a Lebesgue space, the norm in \( L^q(\Omega) \) is denoted by \( |u|_{L^q(\Omega)} \) when \( \Omega \) is a proper subset of \( \mathbb{R}^3 \), by \( |\cdot|_{L^q} \) when \( \Omega = \mathbb{R}^3 \).

- For any \( \rho > 0 \) and for any \( z \in \mathbb{R}^3 \), \( B_\rho(z) \) denotes the ball of radius \( \rho \) centered at \( z \) and \( |B_\rho(z)| \) denotes its Lebesgue measure.

- \( S \) is the best Sobolev constant for the embedding of \( H^1(\mathbb{R}^3) \) in \( L^6(\mathbb{R}^3) \), that is
  \[ S = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_{L^6}}{|u|_6}. \]

- \( \bar{S} \) is the best Sobolev constant for the embedding of \( D^{1,2}(\mathbb{R}^3) \) in \( L^6(\mathbb{R}^3) \), that is
  \[ \bar{S} = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}. \]

It is easy to show that \( (SP) \) can be reduced to a single equation with a non-local term. Actually, considering for all \( u \in H^1(\mathbb{R}^3) \), the linear functional \( L_u \) defined in \( D^{1,2}(\mathbb{R}^3) \) by
\[ L_u(v) = \int_{\mathbb{R}^3} K(x) u^2 v \, dx, \]
the Hölder inequality and the Sobolev inequality imply
\[ |L_u(v)| \leq |k|_2 \cdot |u|^2_3 \cdot |v|_6 = |k|_2 \cdot |u|^2_6 \cdot |v|_6 \leq \bar{S}^{-1} |k|_2 |u|^2_6 \|v\|_{D^{1,2}}. \] (2.1)

Hence, the Lax–Milgram theorem implies that for every \( u \in H^1(\mathbb{R}^3) \), there exists a unique \( \phi_u \in D^{1,2}(\mathbb{R}^3) \) such that
\[ \int_{\mathbb{R}^3} K(x) u^2 v \, dx = \int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v \, dx, \quad \text{for any } v \in D^{1,2}(\mathbb{R}^3). \] (2.2)

Using integration by parts, we get
\[ \int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v \, dx = - \int_{\mathbb{R}^3} v \Delta \phi_u \, dx, \quad \text{for any } v \in D^{1,2}(\mathbb{R}^3), \]
therefore,

$$- \Delta \phi_u = K(x)u^2$$

in a weak sense. We can write an integral expression for $\phi_u$ in the form:

$$\phi_u = \int_{\mathbb{R}^3} \frac{K(y)}{|x - y|} u^2(y) dy,$$

(2.3)

for any $u \in C_0^\infty(\mathbb{R}^3)$ (see [15], Theorem 1); by density it can be extended for any $u \in H^1(\mathbb{R}^3)$ (see Lemma 2.1 of [9]). By (2.1), (2.2) and the Sobolev inequality, the relations

$$\|\phi_u\|_{D^{1,2}} \leq \bar{S}^{-1} \cdot S^{-2} |K|_2 \cdot \|u\|^2, \quad |\phi_u|_6 \leq \bar{S}^{-1} \|\phi_u\|_{D^{1,2}},$$

(2.4)

and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x - y|} u^2(x)u^2(y) dx dy = \int_{\mathbb{R}^3} K(x)u^2 \phi_u(x) dx$$

$$\leq \bar{S}^{-2} \cdot S^{-4} |K|_2^2 \cdot \|u\|^4$$

(2.5)

hold. Substituting $\phi_u$ in $(SP)$, we are led to Eq. (1.1), whose solutions can be obtained looking for critical points of the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined in (1.2). Indeed, it follows from (2.4), (2.5) and the fact of $f$ defined in Theorem 1.1 that $I$ is a well-defined $C^1$ functional, and that

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv + K(x)\phi_u uv - a(x)f(u)v) \, dx.$$  

(2.6)

Hence if $u \in H^1(\mathbb{R}^3)$ is a critical point of $I$, then the pair $(u, \phi_u)$, with $\phi_u$ as in (2.3), is a solution of $(SP)$.

3. Proof of Theorem 1.1

In what follows, we give first Lemmas 2.1 and 2.2 which ensure that the functional $I$ has what is called the mountain pass geometry.

**Lemma 3.1.** Suppose that $(F_1)$, $(F_2)$, $(A_1)$ and $(K_1)$ hold, then there exist $\rho > 0$, $\eta > 0$ such that $\inf\{I(u) : u \in H^1(\mathbb{R}^3) \text{ with } \|u\| = \rho\} > \eta$.

**Proof.** For any $\epsilon > 0$, it follows from $(F_1)$ and $(F_2)$ that there exists $C_\epsilon > 0$ such that

$$|f(s)| \leq \epsilon |s| + C_\epsilon |s|^{2^*-1}, \quad \text{for all } s \in \mathbb{R},$$

(3.1)

where $2^* := \frac{2}{\frac{2}{3} - 2} = 6$, and then,

$$|F(s)| \leq \frac{\epsilon}{2} |s|^2 + \frac{C_\epsilon}{2^*} |s|^{2^*}, \quad \text{for all } s \in \mathbb{R}.$$  

(3.2)
Furthermore, by (F₁), (F₂) and (A₁), there exists $C₁ > 0$ such that

$$a(x) \leq C₁. \quad \text{for all } x \in \mathbb{R}^3. \quad (3.3)$$

So, from (3.2), (3.3) and the Sobolev inequality, we have for all $u \in H^1(\mathbb{R}^3)$,

$$\left| \int_{\mathbb{R}^3} a(x) F(u) \, dx \right| \leq \frac{\epsilon C₁}{2} \int_{\mathbb{R}^3} |u|^2 \, dx + \frac{C₁C_ε}{2^*} \int_{\mathbb{R}^3} |u|^{2^*} \, dx$$

$$\leq \frac{\epsilon C₁}{2} \|u\|^2 + \frac{C₁C_ε}{2^*} \|u\|^{2^*}.$$

Combining this with (K₁) and (2.3), one has

$$I(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x)φ_u(x)u^2 \, dx - \int_{\mathbb{R}^3} a(x) F(u) \, dx$$

$$\geq \frac{1 - \epsilon C₁}{2} \|u\|^2 - \frac{C₁C_ε}{2^*} \|u\|^{2^*}. \quad (3.4)$$

So, by fixing $\epsilon \in (0, C₁^{-1})$ and letting $\|u\| = \rho > 0$ small enough, it is easy to see that there is $\eta > 0$ such that this lemma holds. □

**Lemma 3.2.** Suppose that (F₁), (F₂), (A₁) and (A₂) hold, then there exists $v^* \in H^1(\mathbb{R}^3)$ with $\|v^*\| > \rho$ such that $I(v^*) < 0$, where $\rho$ is given by Lemma 3.1.

**Proof.** By (A₂), in view of the definition of $μ^*$ and $(1 - β)l > μ^*$, there is $v^* \in H^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} a(x) F(v^*) \, dx \geq \frac{1}{2} \int_{\mathbb{R}^3} K(x)φ_{v^*}v^{*2} \, dx < 2βl$, and $μ^* \leq \|v^*\|^2 < (1 - β)l$. Then by (1.2) we obtain

$$I(v^*) = \frac{1}{2} \|v^*\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} K(x)φ_{v^*}v^{*2} \, dx - \int_{\mathbb{R}^3} a(x) F(v^*) \, dx$$

$$\leq \frac{1}{2} \|v^*\|^2 + \frac{1}{4} \times 2βl - \frac{1}{2}$$

$$= \frac{1}{2} \|v^*\|^2 - \frac{1}{2} (1 - β)l$$

$$= \frac{1}{2} \left( \|v^*\|^2 - (1 - β)l \right) < 0.$$

Choosing $\rho > 0$ small enough in Lemma 3.1 such that $\|v^*\| > \rho$, and the lemma is proved. □

By Lemmas 3.1, 3.2 and Theorem 2.1, we obtain that there is a sequence $\{u_n\} \subset H$ such that

$$I(u_n) \to c > 0 \quad \text{and} \quad (1 + \|u_n\|) \|I'(u_n)\|_{H^*} \to 0, \quad \text{as } n \to \infty. \quad (3.5)$$

**Lemma 3.3.** Suppose that (F₁), (F₂), (A₁), (A₂) and (K₁) hold, then $\{u_n\}$ defined in (3.5) is bounded in $H^1(\mathbb{R}^3)$.
Proof. By contradiction, let $\|u_n\| \to +\infty$ as $n \to \infty$. Define $w_n := u_n/\|u_n\|$. Clearly, $w_n$ is bounded in $H^1(\mathbb{R}^3)$ and there is $w \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$w_n \to w, \quad \text{in } H^1(\mathbb{R}^3), \quad w_n \to w \quad \text{a.e. in } \mathbb{R}^3, \quad w_n \to w \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^3) \text{ as } n \to \infty. \quad (3.6)$$

On one hand, we claim that $w \not\equiv 0$. By contradiction, let $w \equiv 0$. By $(A_1)$, there is a constant $\theta \in (0, 1)$ such that

$$\sup \{ f(s)/s: s > 0 \} \leq \theta \inf \{ 1/a(x): |x| \geq R_0 \}. \quad (3.7)$$

This yields, for any $n \in \mathbb{N}$,

$$\int_{|x| \geq R_0} a(x) \frac{f(u_n)}{u_n} |w_n|^2 \, dx \leq \theta \int_{|x| \geq R_0} |w_n|^2 \, dx \leq \theta \|w_n\|^2 = \theta < 1. \quad (3.8)$$

Since the embedding $H^1(B_{R_0}(0)) \hookrightarrow L^2(B_{R_0}(0))$ is compact, $w_n \to w$ strongly in $L^2(B_{R_0}(0))$. Passing to a subsequence, there exists $h \in L^2(B_{R_0}(0))$ such that, for all $n \in \mathbb{N}$,

$$|w_n(x)| \leq h(x) \quad \text{a.e. in } B_{R_0}(0). \quad (3.9)$$

By $(F_1), (F_2)$, there exists $C > 0$ such that

$$\frac{f(t)}{t} \leq C, \quad \text{for all } t \in \mathbb{R}. \quad (3.10)$$

Then, for all $n \in \mathbb{N}$,

$$0 \leq a(x) \frac{f(u_n)}{u_n} w_n^2(x) \leq Ca(x)w_n^2(x) \leq C|a|_{\infty}h^2(x), \quad \text{a.e. in } B_{R_0}(0). \quad (3.11)$$

Noting that $w_n \to w \equiv 0$ a.e. in $\mathbb{R}^3$, we get

$$a(x) \frac{f(u_n)}{u_n} w_n^2 \to 0 \quad \text{a.e. in } \mathbb{R}^3. \quad (3.12)$$

It follows from $(3.9), (3.10)$ and the Dominated Convergence Theorem that

$$\int_{|x| < R_0} a(x) \frac{f(u_n)}{u_n} w_n^2 \, dx = 0. \quad (3.13)$$

So, by $(3.7)$ and $(3.11)$ we obtain that

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3} a(x) \frac{f(u_n)}{u_n} w_n^2 \, dx < 1. \quad (3.14)$$

Since $\|u_n\| \to \infty$, as $n \to \infty$, it follows from $(3.5)$ that...
\[
\{I'(u_n), u_n\}/\|u_n\|^2 = o(1),
\]
that is,
\[
o(1) = \|w_n\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{w_n} u_n^2 dx - \frac{\int_{\mathbb{R}^3} a(x) f(u_n) w_n^2 dx}{\|u_n\|^2}
\]
\[
\geq 1 - \int_{\mathbb{R}^3} a(x) \frac{f(u_n)}{u_n} w_n^2 dx,
\]
where, and in what follows, \(o(1)\) denotes a quantity which goes to zero as \(n \to \infty\). Therefore,
\[
\int_{\mathbb{R}^2} a(x) \frac{f(u_n)}{u_n} w_n^2 dx + o(1) \geq 1,
\]
which contradicts (3.12). So, \(w \neq 0\).

On the other hand, since \(\|u_n\| \to \infty\), as \(n \to \infty\), it follows from (3.5) that
\[
\{I'(u_n), u_n\}/\|u_n\|^4 = o(1),
\]
that is,
\[
o(1) = \frac{1}{\|u_n\|^2} + \int_{\mathbb{R}^3} K(x)\phi_{w_n} w_n^2 dx - \frac{\int_{\mathbb{R}^3} a(x) f(u_n) w_n^2 dx}{\|u_n\|^2}.
\]
(3.13)
Combining this with (3.2) and (3.8), one has
\[
\int_{\mathbb{R}^3} K(x)\phi_{w_n} w_n^2 dx = o(1).
\]
(3.14)
We can easily verify that
\[
\int_{\mathbb{R}^3} K(x)\phi_{w_n} w_n^2 dx = \int_{\mathbb{R}^3} K(x)\phi_w w^2 dx + o(1).
\]
(3.15)
Indeed, in view of the Sobolev embedding theorems and of (3) of Lemma 2.1 in [10], \(w_n \rightharpoonup w\) in \(H^1(\mathbb{R}^3)\) implies that
\[
\begin{align*}
(a) \ w_n & \to w \ in \ L^6(\mathbb{R}^3); \\
(b) \ w_n^2 & \to w^2 \ in \ L^3_{\text{loc}}(\mathbb{R}^3); \\
(c) \ \phi_{w_n} & \to \phi_w \ in \ D^{1,2}(\mathbb{R}^3); \\
(d) \ \phi_{w_n} & \to \phi_w \ in \ L^6_{\text{loc}}(\mathbb{R}^3).
\end{align*}
\]
(3.16)
Thus, given \(\epsilon > 0\), using (3.16)(c), we have, for large \(n\),
\[
\int_{\mathbb{R}^3} K(x) w^2(x)(\phi_{w_n} - \phi_w)(x) dx \leq \epsilon.
\]
(3.17)
Furthermore, considering (3.16)(b), we can assert that, for any choice of \( \epsilon > 0 \) and \( \rho > 0 \), the relation
\[
|w_n^2 - w^2|_{3, B_\rho(0)} < \epsilon
\]  
(3.18)
holds for large \( n \).

Since \( \{w_n\} \) is bounded in \( H^1(\mathbb{R}^3) \), by (2) of Lemma 2.1 in [10] and the continuity of the Sobolev embedding of \( D^{1,2}(\mathbb{R}^3) \) in \( L^6(\mathbb{R}^3) \), then \( \phi_{w_n} \) is bounded in \( D^{1,2}(\mathbb{R}^3) \) and in \( L^6(\mathbb{R}^3) \). Moreover, \( K \in L^2(\mathbb{R}^3) \) implies that \( K w_n^2 \) and \( K w^2 \) belong to \( L^\frac{6}{5}(\mathbb{R}^3) \) and that to any \( \epsilon > 0 \) there corresponds \( \bar{\rho} = \bar{\rho}(\epsilon) \) such that
\[
|K|_{2, \mathbb{R}^3 \setminus B_{\bar{\rho}}(0)} < \epsilon, \quad \forall \rho \geq \bar{\rho}.
\]  
(3.19)
Hence, by using (3.17)–(3.19), we obtain, for large \( n \),
\[
\left| \int_{\mathbb{R}^3} K(x) \phi_{w_n}(x) w_n^2(x) \, dx - \int_{\mathbb{R}^3} K(x) \phi_w(x) w^2(x) \, dx \right|
\leq \left| \int_{\mathbb{R}^3} K(x) \phi_{w_n}(w_n^2 - w^2) \, dx + \int_{\mathbb{R}^3} K(x)(\phi_{w_n} - \phi_w) w^2 \, dx \right|
\leq \left| \int_{\mathbb{R}^3} K(x) \phi_{w_n}(w_n^2 - w^2) \, dx \right| + \left| \int_{\mathbb{R}^3} K(x)(\phi_{w_n} - \phi_w) w^2 \, dx \right|
\leq |\phi_{w_n}|_6 \left( \int_{\mathbb{R}^3} |K(x)(w_n^2 - w^2)|^\frac{6}{5} \, dx \right)^\frac{5}{6} + \epsilon
\leq C \left( \int_{\mathbb{R}^3 \setminus B_{\rho}(0)} |K(x)(w_n^2 - w^2)|^\frac{6}{5} \, dx + \int_{B_{\rho}(0)} |K(x)(w_n^2 - w^2)|^\frac{6}{5} \, dx \right)^\frac{5}{6} + \epsilon
\leq C \left( |K|_{2, \mathbb{R}^3 \setminus B_{\rho}(0)} \cdot |w_n^2 - w^2|_{\frac{6}{5}, 3, B_{\rho}(0)} + |K|_{2, \mathbb{R}^3 \setminus B_{\rho}(0)} \cdot |w_n^2 - w^2|_{\frac{6}{5}, 3, B_{\rho}(0)} \right)^\frac{5}{6} + \epsilon
\leq C \epsilon,
\]
which proves (3.15).

So, by (3.14) and (3.15) we obtain
\[
\int_{\mathbb{R}^3} K(x) \phi_w w^2 \, dx = 0,
\]
which implies that \( w \equiv 0 \). That is a contradiction. Therefore, \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). \( \square \)

To prove that the Cerami sequence \( \{u_n\} \) in (3.5) converges to a nonzero critical point of \( I \), the following compactness lemma is useful.

**Lemma 3.4.** Assume that \( (F_1) \), \( (F_2) \), \( (A_1) \) and \( (K_1) \) hold. Then for any \( \epsilon > 0 \), there exist \( R(\epsilon) > R_0 \) and \( n(\epsilon) > 0 \) such that \( \int_{|x| \geq R} (|\nabla u_n|^2 + u_n^2) \, dx \leq \epsilon \).
Proof. Let \( \xi_R : \mathbb{R}^3 \to [0, 1] \) be a smooth function such that
\[
\xi_R(x) = \begin{cases} 
0 & 0 \leq |x| \leq R, \\
1 & |x| \geq 2R,
\end{cases}
\]  
and, for some constant \( C > 0 \) (independent of \( R \)),
\[
|\nabla \xi_R(x)| \leq \frac{C}{R}, \quad \text{for all } x \in \mathbb{R}^3.
\]  
(3.21)

Then, for all \( n \in \mathbb{N} \) and \( R \geq R_0 \), we have
\[
\int_{\mathbb{R}^3} |\nabla (u_n \xi_R)|^2 \, dx = \int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R^2 \, dx + \int_{\mathbb{R}^3} |u_n|^2 |\nabla \xi_R|^2 \, dx 
\leq \int_{R < |x| < 2R} |\nabla u_n|^2 \, dx + \int_{|x| > 2R} |\nabla u_n|^2 \, dx + \frac{C^2}{R^2} \int_{\mathbb{R}^3} |u_n|^2 \, dx 
\leq \left( 2 + \frac{C^2}{R^2} \right) \|u_n\|^2 \leq \left( 2 + \frac{C^2}{R_0^2} \right) \|u_n\|^2.
\]  

This implies that
\[
\|u_n \xi_R\| \leq \left( 3 + \frac{C^2}{R_0^2} \right)^{\frac{1}{2}} \|u_n\|,
\]  
(3.22)

for all \( n \in \mathbb{N} \) and \( R \geq R_0 \). By (3.5), \( \|I'(u_n)\|_{H^*} \|u_n\| \to 0 \) as \( n \to \infty \). So, for any \( \epsilon > 0 \), there exists \( n(\epsilon) > 0 \) such that
\[
\left\| I'(u_n) \right\|_{H^*} \|u_n\| \leq \frac{\epsilon}{\left( 3 + \frac{C^2}{R_0^2} \right)^{\frac{1}{2}}},
\]  
(3.23)

for all \( n > n(\epsilon) \). Hence, it follows from (3.22) and (3.23) that
\[
\left| \left\langle I'(u_n), u_n \xi_R \right\rangle \right| \leq \left\| I'(u_n) \right\|_{H^*} \|u_n \xi_R\| \leq \epsilon,
\]  
(3.24)

for all \( n > n(\epsilon) \) and \( R > R_0 \). Note that
\[
\left\langle I'(u_n), u_n \xi_R \right\rangle = \int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R \, dx + \int_{\mathbb{R}^3} u_n^2 \xi_R \, dx + \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \xi_R \, dx 
+ \int_{\mathbb{R}^3} K(x) \phi_{u_n}(x) u_n^2 \xi_R \, dx - \int_{\mathbb{R}^3} a(x) f(u_n) u_n \xi_R \, dx.
\]  
(3.25)

For any \( \epsilon > 0 \), there exists \( R(\epsilon) \geq R_0 \) such that
\[
\frac{1}{R^2} \leq \frac{4\epsilon^2}{C^2}, \quad \text{for all } R \geq R(\epsilon),
\]  
(3.26)
By (3.26) and the Young inequality, we get, for all \( n \in \mathbb{N} \) and \( R \geq R(\epsilon) \),
\[
\int_{\mathbb{R}^3} |u_n \nabla u_n \nabla \xi_R| \, dx \leq \epsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + \frac{1}{4\epsilon} \int_{|x| \leq 2R} |u_n|^2 \frac{C^2}{R^2} \, dx \\
\leq \epsilon \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx + \epsilon \int_{|x| \leq 2R} |u_n|^2 \, dx \\
\leq \epsilon \|u_n\|^2.
\]  
(3.27)

By \((F_1), (A_1)\) and \((3.21)\), there exists \( \eta_1 \in (0, 1) \) such that, for all \( n \in \mathbb{N} \) and \( R \geq R_0 \),
\[
\int_{\mathbb{R}^3} |a(x)f(u_n)u_n \xi_R| \, dx \leq \eta_1 \int_{\mathbb{R}^3} u_n^2 \xi_R \, dx.
\]  
(3.28)

Combining this with (3.25) and (3.27), for all \( n \in \mathbb{N} \) and \( R \geq R(\epsilon) \geq R_0 \), we see that
\[
\langle I'(u_n), u_n \xi_R \rangle \geq \int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R \, dx + (1 - \eta_1) \int_{\mathbb{R}^3} u_n^2 \xi_R \, dx \\
+ \int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2 \xi_R \, dx - \epsilon \|u_n\|^2 \\
\geq \int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R \, dx + (1 - \eta_1) \int_{\mathbb{R}^3} u_n^2 \xi_R \, dx - \epsilon \|u_n\|^2.
\]  
(3.29)

Since \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \), it follows from (3.24) and (3.29) that there exists \( C > 0 \) such that, for all \( n \geq n(\epsilon) \) and \( R \geq R(\epsilon) \),
\[
\int_{\mathbb{R}^3} |\nabla u_n|^2 \xi_R \, dx + (1 - \eta_1) \int_{\mathbb{R}^3} u_n^2 \xi_R \, dx \leq C \epsilon.
\]  
(3.30)

From \( \eta_1 \in (0, 1) \) and \((3.20)\), it is easy to see that (3.30) implies the conclusion. \( \square \)

**Theorem 3.1.** Let \((F_1), (F_2), (A_1), (A_2)\) and \((K_1)\) hold. Then \( I \) has a nonzero critical point in \( H^1(\mathbb{R}^3) \).

**Proof.** By Lemma 3.3, the sequence \( \{u_n\} \) in (3.5) is bounded in \( H^1(\mathbb{R}^3) \). We may assume that, up to a subsequence, \( u_n \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^3) \) for some \( u \in H^1(\mathbb{R}^3) \). In order to prove our theorem, it is now sufficient to show that \( \|u_n\| \to \|u\| \) as \( n \to \infty \). Note that, by (3.5),
\[
\langle I'(u_n), u_n \rangle = \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) \, dx + \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 \, dx - \int_{\mathbb{R}^3} a(x)f(u_n)u_n \, dx \\
= o(1),
\] and
\[ \langle I'_{(u_n)}, u \rangle = \int_{\mathbb{R}^3} (\nabla u_n \nabla v + u_n v) \, dx + \int_{\mathbb{R}^3} K(x) \phi u_n u \, dx - \int_{\mathbb{R}^3} a(x) f(u_n) u \, dx = o(1). \]

Since \( u_n \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^3) \), we can see that

\[ \int_{\mathbb{R}^3} (\nabla u_n \nabla v + u_n v) \, dx = \int_{\mathbb{R}^3} (|\nabla v|^2 + v^2) \, dx + o(1). \]

So to show \( \| u_n \| \rightarrow \| u \| \) is equivalent to prove that

\[ \int_{\mathbb{R}^3} K(x) \phi u_n u^2 \, dx = \int_{\mathbb{R}^3} K(x) \phi u_n u \, dx + o(1), \quad (3.31) \]

and

\[ \int_{\mathbb{R}^3} a(x) f(u_n) u \, dx = \int_{\mathbb{R}^3} a(x) f(u_n) u \, dx + o(1). \quad (3.32) \]

First, we prove the equality (3.32). For any \( \epsilon > 0 \), by Lemma 3.4 and for \( n \) large enough, one has

\[ \int_{|x| \geq R(\epsilon)} a(x) f(u_n) u \, dx - \int_{|x| \geq R(\epsilon)} a(x) f(u_n) u \, dx \leq \int_{|x| \geq R(\epsilon)} \left( a^\frac{1}{2} |f(u_n)| \right) \left( a^\frac{1}{2} |u_n - u| \right) \, dx \leq \left( \int_{|x| \geq R(\epsilon)} a(x) |u_n - u|^2 \, dx \right)^\frac{1}{2} \left( \int_{|x| \geq R(\epsilon)} a(x) |f(u_n)|^2 \, dx \right)^\frac{1}{2} \leq C \left( \int_{|x| \geq R(\epsilon)} a(x) |u_n - u|^2 \, dx \right)^\frac{1}{2} \left( \int_{|x| \geq R(\epsilon)} |u_n|^2 \, dx \right)^\frac{1}{2} \leq C \epsilon. \quad (3.33) \]

This and the compactness of embedding \( H^1(\mathbb{R}^3) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3) \) imply (3.32).

Now we verify that the equality (3.31) holds. Since \( u_n \rightharpoonup u \) in \( H^1(\mathbb{R}^3) \), similar to (3.16), we obtain

\[ (a') u_n \rightharpoonup u \quad \text{in} \quad L^6(\mathbb{R}^3); \quad (b') u_n^2 \rightharpoonup u^2 \quad \text{in} \quad L^3_{\text{loc}}(\mathbb{R}^3); \]

\[ (c') \phi u_n \rightharpoonup \phi u \quad \text{in} \quad D^{1,2}(\mathbb{R}^3); \quad (d') \phi u_n \rightharpoonup \phi u \quad \text{in} \quad L^6_{\text{loc}}(\mathbb{R}^3). \quad (3.34) \]

For any choice of \( \epsilon > 0 \) and \( \rho > 0 \), the relation

\[ |u_n - u|_{6,B_{\rho}(0)} < \epsilon \quad (3.35) \]

holds for large \( n \). Hence, by using (3.19) and (3.35), one has, for large \( n \),
\[
\left| \int_{\mathbb{R}^3} K(x) \phi_{u_n}(x) u_n^2(x) \, dx - \int_{\mathbb{R}^3} K(x) \phi_{u_n}(x) u \, dx \right| \\
\leq \int_{\mathbb{R}^3} \left| K(x) \phi_{u_n}(u_n - u) \right| \, dx \\
\leq \left| \phi_{u_n} \right|_6 \left( \int_{\mathbb{R}^3} \left| K(x) u_n(u_n - u) \right|^6 \, dx \right)^{\frac{5}{6}} \\
= \left| \phi_{u_n} \right|_6 \left( \int_{\mathbb{R}^3 \setminus B_{\rho}(0)} \left| K(x) u_n(u_n - u) \right|^6 \, dx + \int_{B_{\rho}(0)} \left| K(x) u_n(u_n - u) \right|^6 \, dx \right)^{\frac{5}{6}} \\
\leq C \left( |K|_{2, \mathbb{R}^3 \setminus B_{\rho}(0)}^{\frac{6}{5}} \cdot |u_n(u_n - u)|_{\frac{6}{5}}^{\frac{6}{3}} + |K|_{2, B_{\rho}(0)}^{\frac{6}{5}} \left[ \int_{B_{\rho}(0)} |u_n|^6 \, dx \right]^{\frac{1}{5}} \left[ \int_{B_{\rho}(0)} |u_n-u|^6 \, dx \right]^{\frac{1}{5}} \right)^{\frac{5}{6}} \\
\leq C \left( \epsilon |u_n(u_n - u)|_{\frac{6}{5}}^{\frac{6}{3}} + |K|_{2, B_{\rho}(0)}^{\frac{6}{5}} \cdot |u_n|_{6, B_{\rho}(0)}^{\frac{6}{5}} \cdot |u_n-u|_{\frac{6}{5}, B_{\rho}(0)}^{\frac{6}{5}} \right)^{\frac{5}{6}} \\
\leq C \left( \epsilon |u_n(u_n - u)|_{\frac{6}{5}}^{\frac{6}{3}} + \epsilon |K|_{2, B_{\rho}(0)}^{\frac{6}{5}} \cdot |u_n|_{6, B_{\rho}(0)}^{\frac{6}{5}} \right)^{\frac{5}{6}} \\
\leq C \epsilon ,
\]
which prove the equality (3.31). So we obtain the result. □

Now we give the proof of Theorem 1.1.

**Proof.** Set the Nehari manifold

\[
\mathcal{N} = \{ u \in H^1(\mathbb{R}^3) \setminus \{0\}: \langle f'(u), u \rangle = 0 \}.
\]  

(3.36)

By Theorem 3.1 \( \mathcal{N} \) is nonempty. For any \( u \in \mathcal{N} \), we have

\[
0 = \langle f'(u), u \rangle = \| u \|^2 + \int_{\mathbb{R}^3} K(x) \phi_u u^2 \, dx - \int_{\mathbb{R}^3} a(x) f(u) u \, dx \\
\geq \| u \|^2 - \int_{\mathbb{R}^3} a(x) f(u) u \, dx.
\]

Now, choose \( \epsilon \in (0, C_1^{-1}) \) as in the proof of Lemma 3.1 and use (3.1)–(3.2) to get

\[
\left| \int_{\mathbb{R}^3} a(x) f(u) u \, dx \right| \leq \int_{\mathbb{R}^3} \left[ C_1 \epsilon u^2(x) + C_1 C_\epsilon u^6(x) \right] \, dx \\
= C_1 \epsilon \| u \|^2 + C_1 C_\epsilon \| u \|^6 \\
\leq C_1 \epsilon \| u \|^2 + \frac{C_1 C_\epsilon}{5^6} \| u \|^6.
\]
Therefore, for every \( u \in \mathcal{N} \) we have
\[
0 \geq \|u\|^2 - C_1 \epsilon \|u\|^2 - \frac{C_1 \epsilon}{S^6} \|u\|^6.
\] (3.37)

We recall that \( u \neq 0 \) whenever \( u \in \mathcal{N} \) and (3.37) implies
\[
\|u\| \geq \frac{\sqrt{S^6(1 - C_1 \epsilon) C_1 \epsilon}}{4} > 0, \quad \forall u \in \mathcal{N}.
\] (3.38)

Hence any limit point of a sequence in the Nehari manifold is different from zero.

We claim that \( I \) is bounded from below on \( \mathcal{N} \), i.e., there exists \( M > 0 \) such that \( I(u) \geq -M \) for all \( u \in \mathcal{N} \). Otherwise, there exists \( \{u_n\} \subset \mathcal{N} \) such that
\[
I(u_n) < -n, \quad \text{for any } n \in \mathbb{N}.
\] (3.39)

It follows from (2.3) and (3.4) that
\[
I(u_n) \geq \frac{1}{4} \|u_n\|^2 - C \|u_n\|^{2^*}.
\] (3.40)

This and (3.39) imply that \( \|u_n\| \to +\infty \). Let \( w_n = u_n/\|u_n\| \), there is \( w \in H^1(\mathbb{R}^3) \) such that (3.6) holds. Note that \( I'(u_n) = 0 \) for \( u_n \in \mathcal{N} \), as in the proof of Lemma 3.3, we obtain that \( \|u_n\| \to +\infty \) is impossible. Then, \( I \) is bounded from below on \( \mathcal{N} \). So, we may define
\[
\bar{c} = \inf\{I(u), \ u \in \mathcal{N}\},
\]
and \( \bar{c} \geq -M \). Let \( \{\bar{u}_n\} \subset \mathcal{N} \) be such that \( I(\bar{u}_n) \to \bar{c} \) as \( n \to \infty \). Following almost the same procedures as the proofs of Lemmas 3.3, 3.4 and Theorem 3.1, we can show that \( \{\bar{u}_n\} \) is bounded in \( H^1(\mathbb{R}^3) \) and it has a convergent subsequence, strongly converging to \( \bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\} \). Thus \( I(\bar{u}) = \bar{c} \) and \( I'(\bar{u}) = 0 \). Therefore \( \bar{u} \in H^1(\mathbb{R}^3) \) is a ground state of system \((SP)\). \( \Box \)

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