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SUBJECTIVE GEOMETRY AND GEOMETRIC PSYCHOLOGY

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Abstract—"Subjective geometry" is a term coined by Weintraub and Krantz to describe the distortion imposed upon geometric patterns by the visual system itself—so-called optical illusions. The latter are widely regarded as being generated by misplaced "constancy" effects, i.e., they are regarded as stemming from the invariance of an object's appearance under wide variations in viewing conditions, such as obliquity, rotations, etc. The invariances represented by these constancies—shape constancy, size constancy, etc.—are spatiotemporal invariants of certain Lie subgroups of $P_4(R) \oplus CO(1,3) \oplus GL(4,R)$ that govern Euclidean and non-Euclidean geometry. The Euclidean subgroups describe a Cyclopean visual world; the non-Euclidean, a binocular (bipolar) world of hyperbolic nature, according to the work of Luneburg, Blank, Indow, and others. The visual field of view is itself a geometric object involving not only "figure" and "ground" but also visual contours (orbits of the Lie groups involved), linear perspective, interposition, and contact and symplectic structures. The retina and "cortical retina" are both covered by a family of "circular-surround" cellular response fields (of a "Mexican hat" nature) which constitute an atlas for the visual manifold S. Upon this manifold are defined certain equivariant vector bundles that account for constancy phenomena and certain jet bundles, arising out of the vector bundles by prolongation, that generate the differential invariants characterizing higher form perception. The resultant theory of perceptual-cognitive processing has been termed "geometric psychology," in analogy to MacLane's "geometrical mechanics" and Brockett–Hermann–Mayne's "geometry of systems," the mathematical structure being very similar in all three instances. Functorial maps from the category $G_V FB(S)$ of equivariant fibre bundles to the simplicial category and the category of simplicial objects complete the theory by extending the perceptual system to cognitive phenomena and information-processing psychology.

1. INTRODUCTION

Galileo, in his *Saggiatore* (Opere VI, p. 232), put it this way:

The great book of Nature lies ever open before our eyes and the true philosophy is written in it . . . But we cannot read it unless we have first learned the language and the alphabet in which it is written. . . . It is written in the language of mathematics and its elements are triangles, circles, and other geometrical figures . . .

Change "triangles" and "circles" in the foregoing to invariants of the full conformal group [1] and "other geometrical figures" to "geometric objects" and recast the statement in terms of contact and symplectic manifolds and fibre bundles, and it would be no less true today. In physics, with its Hamiltonian and Lagrangean structures and gauge transformations, this is an old story [1,2]. The burden of this paper is that geometry is no less important in psychology—in fact, that a "geometric psychology" is the key to an understanding of perceptual and cognitive phenomena. The mathematical structures involved appear to be the same as those in the physical sciences, i.e., modern differential geometry and topology, or more specifically, vector fields and manifolds, Lie transformation groups, the contact and symplectic structures induced thereby, and the category $GFB(M)$ of equivariant fibre bundles. Indeed, it is hard to see how it could be otherwise: The visual system evolves to adjust to the nature of the geometrico-physical world it lives in. The projection of the visual image on the retina is a surface. But the retina then imposes its own microscopic "charts" and macroscopic "atlas" on the projection of the field of view through dissection of the visual image by rods and cones (with typical "Mexican hat" response fields) which interact at deeper levels via horizontal, bipolar, amacrine, and ganglion cells. This same sort of manifold structure—a covering of the visual field of view by locally Euclidean circular-surround response fields—persists as far along the visual pathway as the visual cortex. At this level, however, something new makes its appearance. In addition to the circular-surround type of single-cell response, one now finds the so-called "orientation response," i.e., a directional response characteristic of a cortical vector field, as well as a binocular single-cell response, which appears to be the infinitesimal generator of a hyperbolic geometry.

Vector fields are actually an old story in perceptual psychology, having been invoked for the first time as far back as 1864 by Lotze and Hering in their concepts of visual direction and local sign. They make their appearance again in the work of Orbison [3] on the determination of shape by vector fields and the theory of Gibson [4] on visual flow fields. Here, based on the electrohistological results of Hubel and Wiesel [5], vector fields and covector fields will be used to describe the microscopic ("local") elements of visual contours. In effect, a visual contour will be viewed as a polyhedral arc approximation to the actual physical contour, at or beyond the limit of visual acuity.

Now suppose that our perception of these contours was not equivariant, i.e., not independent of the deformations of the visual image imposed by such viewing conditions as rotation, being up-down or right-left in the visual field, obliquity, and the well known binocular distortion close up. The classic example of shape constancy is the circular dinner plate upon the table, which, though its projection upon the retina is an ellipse, we perceive as round. And, unlike the small infant and the catatonic schizophrenic, we recognize someone down at the end of the hall just as well as close up. Lacking such "constancy," our perceptual world would, in Von Fieandt's phrase, be filled with perpetually deforming, rubbery objects, very like a surrealist painting. The economy for memory storage, in which only invariants need be stored, is obvious.

Such "constancies" are, however, not an unmixed blessing, for on occasion the perceptual system misapplies them, thus leading to misperceptions. The classical instance is that of so-called visual (or "optical") illusions, although apparent motion effects (MAE) and the properties of the visual Gestalt—closure, symmetry, "common fate," transposition, continuity, similarity, proximity, etc.—also evidence such constancy influences. The term "subjective geometry" was coined by Weintraub and Krantz [6], in connection with their study of one of these illusions, the Poggendorf illusion (Fig. 1), to describe such phenomena. Weintraub and Krantz claim to have detected two visual effects involved in the Poggendorf illusion: subjective deformation of vertex angles and "tilt assimilation" toward the nearest subjective reference direction, horizontal or vertical. These tendencies are regarded as bringing about a change in the orientation of the transverse line segment, i.e., a tilt between right and left halves. It is worthy of note that Weintraub concluded in this connection that figural perception is a property of *directed* line segments. Ross Day [7] and the present author [8,9] found, on the other hand, a parallel displacement between right and left halves of the transversal. The author's treatment of the Poggendorf, Hering, and Zollner illusions, was based on an orthogonal pairing of constancy effects, the so-called Principle of Transverse Control [10]. Theories of how such subjective geometric effects are induced run the gamut from retinal mechanisms to adaptation theory [11] and misplaced constancy effects [8].

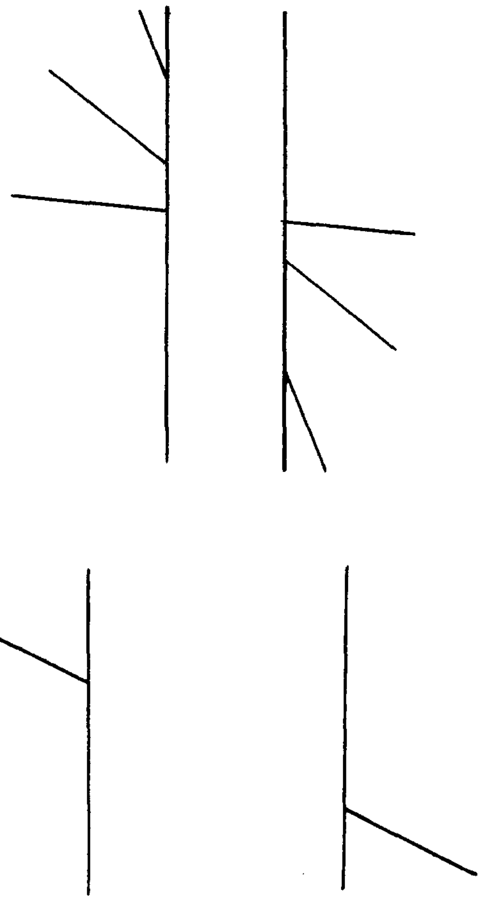

Fig. 1. The Poggendorf illusion, illustrating the effect of varying the slope of the oblique line segments and the width between the parallels.

In mathematical systems theory a discipline termed the "geometry of systems" [12, 13] has recently come to the fore. This is a generalization of systems theory in terms of modern differential geometry that takes proper account of nonlinear and "global" phenomena. As we have suggested above, the geometry of systems also has application to perceptual psychology, for the invariances embodied in the perceptual constancies are actually spatiotemporal invariants of certain Lie transformation groups that occur in Euclidean and non-Euclidean geometry. The whole situation is very reminiscent of Klein's *Erlanger Program*. That these invariances and pseudogroups are present is, of course, no great surprise, for, as mentioned above, evolution demands that we properly adjust to the geometrico-physical nature of our environment if we are to survive the hazards of that environment and live to reproduce our kind.

These transformation groups act upon the perceptual manifold of seen, heard, and haptically sensed objects, as well as, perhaps, chemoreceptor stimuli. In the visual case the submanifold involved is the visual field of view. The latter is a geometric object, consisting of the visual contours of the Figure–Ground Relation, visual textures and gradients, all of which may be in color or in motion. As such, this "cortical retina" is accessible to modelling and analysis in terms of the (differential) geometry of systems, based on the invariances of form perception: the constancies and form memory. To this aspect of the matter we now proceed.

2. THE CONSTANCIES AS SPATIOTEMPORAL INVARIANCES OF THE CONFORMAL GROUP

We live in "spacetime" S, which is a connected, oriented, time-oriented Lorentzian 4-manifold (M,g) [14]. "Lorentzian" in this context means that for each $x \in M$, there exists a basis in T_xM relative to which the matrix of the (nondegenerate) Lorentzian metric tensor g is of the form $(+,-,-,-)$ at x. M here thus has the character of Minkowski space, i.e., a flat space with metric

$$ds^2 = (dx^0)^2 - \sum_{1}^{n-1} (dx^i)^2, \; n = \dim M.$$

The fourth coordinate x^0 in S is ct, where t is time in the observer's physical frame and c is the velocity of light, or more generally, the signal propagation velocity [15, Chap 2]. Since neuronal signals propagate with only a finite velocity (about 2 m/s for the fine neurons of the Central Nervous System) we thus have to do with two space–times, one physical and external and the other internal and subjective (cortical space–time).

Denote the maximum signal velocity for the CNS by $c^{\#}$ and subjective spacetime by S. Then if causality, regarded as preservation of the temporal order of events, is to be preserved, the transformation group appropriate to subjective as well as physical spacetime is the full conformal group [1], whose transformations comprise the following subgroups:

(i) The affine group $\mathscr{A}$: $x \to {}'x = \Lambda x + b$, where $\Lambda \in SO_0(1,3)$ and $b \in S$;
(ii) Dilations $x \to {}'x = \lambda x$, $\lambda \in \mathbf{R}$; (1)
(iii) Special conformal transformations ("accelerations") $x \to {}'x = \omega(a,x)^{-1}(x+ax^2)$, where $a \in S$, $\omega(a,x) = 1+2ax+a^2x^2$ and $a^2 = \sum_{\mu,\nu} g_{\mu\nu}a^\mu a^\nu$, $ax = \sum_{\mu,\nu} g_{\mu\nu}a^\mu x^\nu$;
(iv) Space reflections $x = (x^0, x^1, x^2, x^3) \to {}'x = (x^0,-x^1,-x^2,-x^3)$ and time reversal $x \to {}'x = (-x^0, x^1, x^2, x^3)$.

The transformations (i), (ii), and (iii) generate the connected component C_0 of the identity in the full conformal group $CO(1,3)$. C_0 is a 15-dimensional Lie group with infinitesimal generators:

General Lorentz transformations:

$$\mathscr{L}_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu, \; (-x^0 \text{ for } \nu = 0, \; \partial_\nu = \frac{\partial}{\partial x^\nu}, \text{ etc.}), \; (\mu, \nu = 0, 1, 2, 3)$$

Translations: $T_\mu = \partial_\mu$

Dilations: $\mathscr{D} = \sum_\rho x^\rho\partial_\rho$ (2)

Special conformal transformations: $K_\mu = \left(\sum_\nu x^{\nu^2}\right) \partial_\mu - 2\, x^\mu \sum_\rho x^\rho\partial_\rho.$

Now the general Lorentz transformation consists of two kinds of transformations: (i) a rotation which yields the *standard configuration*, wherein the observer's frame is aligned with the moving frame, and (ii) the special Lorentz transformation appropriate to the metric of Minkowski space. Just so, too, is there an affine transformation group acting on the observer's space with origin at his egocenter (presumably the interocular point in the visual case) which aligns his direction of regard with some portion of the field of view of immediate interest. This direction of regard lies in a plane of regard (Fig. 2) of the Luneburg–Blank–Hoffman–Indow theory of binocular vision [16,17], which is a hyperbolic space H^2. This space is the factor space $H^2 \cong SL(2,R)/O(2)$ [18, p. 167] of pseudo-Euclidean rotations. But there must also be a Cyclopean, or egocentered, space, into which

$$H^2 \times \Theta, \quad \Theta = \left(-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon' \right)$$

maps in bijective fashion. This space is projective 3-space (Fig. 3) together with a projective group that contains the affine group above. See in this connection Proposition 7.1

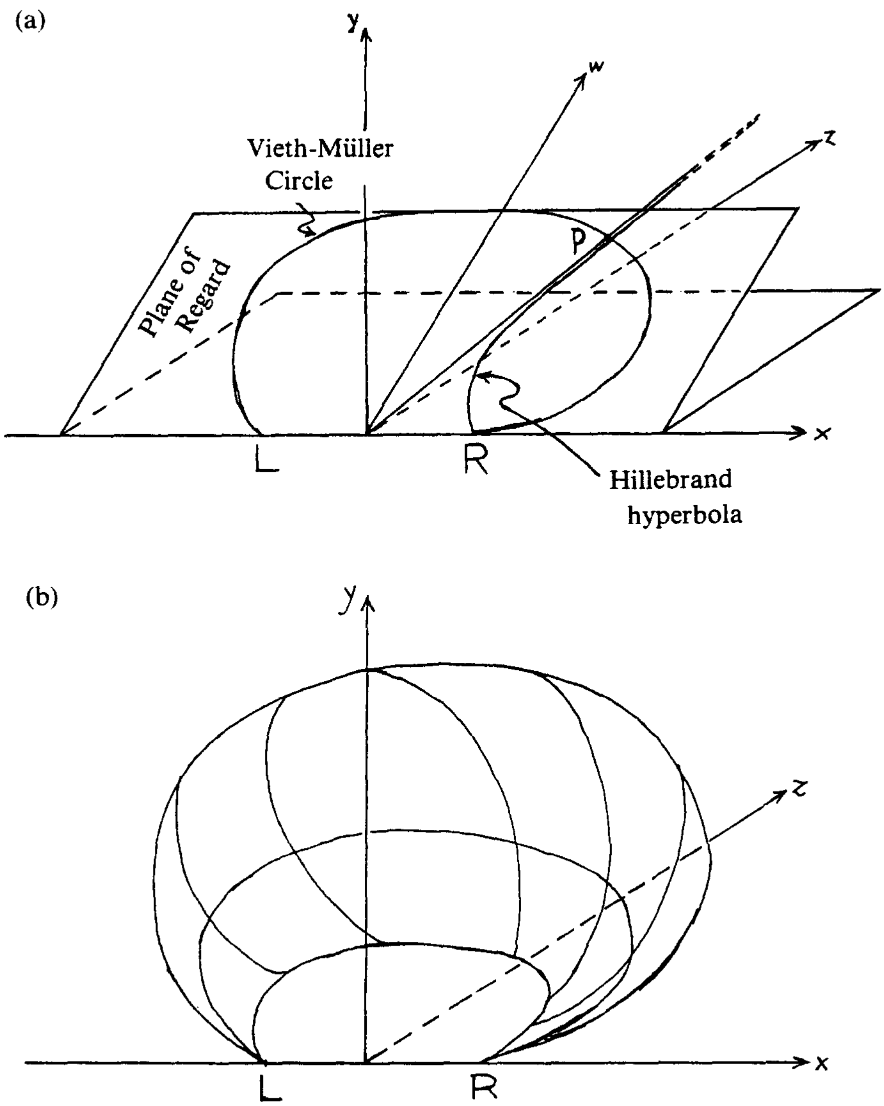

Fig. 2. (a) Hillebrand hyperbolas and Vieth–Müller circles in the plane of regard. (b) The Cyclopean image of (a): An egocentered Vieth–Müller "torus."

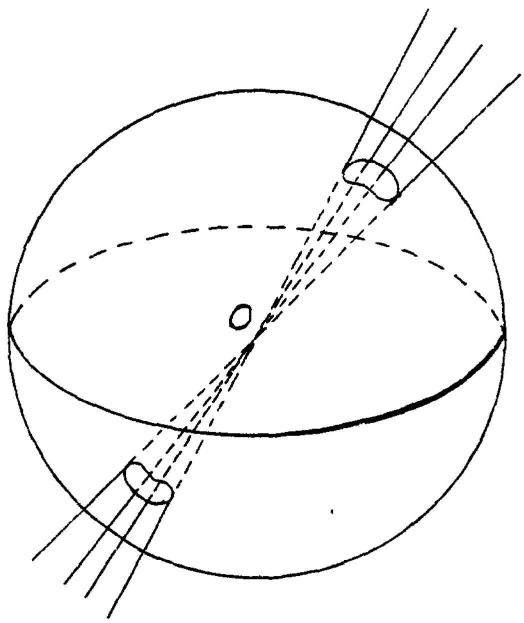

Fig. 3. Real projective space as a collection of equivalence classes of lines through the origin in the subjective egocentered bali [equivalent to Fig. 2(b)].

and Theorem 7.3 of Chap. IV of [19]. This type of projective transformation is fundamental to Gunnar Johansson's [20] theory of visual space perception. His second postulate (appropriate to Cyclopean perceivers) asserts that

All information about space inherent in (the) optical flow (field), is given in a continuous projective transformation at the nodal point of the eye.

At the same time our visual system responds to a different type of projective space centered at the vanishing point on the horizon. This phenomenon is provided either by perspectivities consisting of combined affine transformations and dilations or, in the case of moving objects, by the causal transformation $x^\mu \to x^\mu \div |x|^2$ called conformal inversion. The latter, together with the proper conformal transformations, maps ∞ (which is *not* a point of Minkowski space) into a finite point (the vanishing point) and vice versa. The conformal group is thus defined over conformally compacted Minkowski space.

It is well known, however, that conformal spaces and projective spaces are different in essential ways, especially when it comes to their behavior over flag manifolds and jet bundles ([19, Sec. IV.7]. In effect, it is the difference between isometries and general affine transformations (ones containing the group of dilations). Yet, the projective conformal group [21] is an important subgroup of both $P(4)$ and $CO(1,3)$ responsible for major components of shape and size constancy (see below).

What is the significance for perceptual psychology of the preceding spaces and groups? As noted above, causality requires the conformal group [1,22], in particular the Lorentz transformations therein [23]. The presence of the Lorentz transformation group also makes possible motion-invariant perception and the subjective Fitzgerald contraction effects that have been observed. ("The bullet travels too fast for the eye to follow.") In turn, the Lorentz group contains the orthogonal group $O(3)$, invariance under which gives that component of shape constancy governing invariant perception of a rotated object or one viewed obliquely.

Size constancy is governed by the dilation group in space–time, i.e, size constancy exhibits invariance under homotheties, or similarity transformations, and perspectivities consist of the direct product of $SL(1,3)$ and the dilation group. It is known [24] that the conformal transformations in S^3 constitute a Lie transformation group generated by similarities and inversions. An equivalent statement is that the projective conformal group [21] consists of translations, rotations, and homotheties. In addition to rotation-invariant perception, shape constancy involves translation invariance (right–left, up-or-down in the field of view) and the pseudo-Euclidean rotations of afferent and efferent binocular perception [10, 16, 25]. (This last conclusion is forced upon one by the necessity for closure—in the sense of Lie's second fundamental theorem—of the associated Lie algebra. So, too, is the interpretation as form memory of perceptions invariant under time translations.) Another important aspect of $CO(1,3)$ for perceptual phenomena is that it is angle preserving, i.e., its mappings induce (nonoriented) homotheties of the local tangent spaces. Further, any C^3 surface is conformal to the plane, which legitimatizes the replacement of the curved actual field of view by its projection on the frontal plane in the analysis of perceptual effects, much as in an ordinary painting. Finally, there is the matter of the geometric nature of subjective space–time, which, as we have seen above, is both Euclidean and non-Euclidean. This latter aspect is in keeping with the known fine-structure of the visual cortex in the brain, which consists of two types of crossed "slabs," one responsive to ocular dominance—that is, binocular, and hence associated with a hyperbolic geometry—and the other to "orientation" (direction of the stimulus element), and hence Euclidean [5]. In this connection we recall the following theorem [26, p. 532]:

> If, in a connected, locally compact metric space, there exists, for any two (sufficiently small) congruent triangles in the space, an isometry taking one into the other, then the space is a real Euclidean, elliptic, spherical, or hyperbolic space.

Again we see the importance of conformal properties. All of the foregoing types of geometry are encountered within the perceptual systems, which therefore contain, somewhere within them, the capabilities for invariant transformations of the corresponding kind. Lest one doubt the role of invariant triangles in perception, we recall Uttal's [27] finding that in form perception, triangles are more characterized by sides than vertices. On the other hand, there is the salient comment of Hubel [28] that, for the cells of the visual cortex,

> . . . this is the way nature looks at things by the directions of the tangents. The cells are not responding to lines, they are responding to directions and sizes.

This leads us naturally into the other, *local* phase of the description of the constancies in terms of transformation groups. In this connection we have the Bochner–Montgomery Theorem: If G is a locally compact group of differentiable transformations of a manifold M, then

$$G \times M \xrightarrow{\gamma} M$$

is a Lie transformation group. As such, there exists a smooth vector field $X \subset TM$ on M, the orbits of which will constitute the visual contours of *basic* form perception in the perceptual application. In such a perceptual context the group G will be denoted by G_V, and the latter will be whichever one of $P(4)$, $CO(1,3)$, or $GL(4)$ (or its invariant subgroups) may be appropriate to the phenomenon at hand.

By duality there also exists a contact structure on $T^*M \times R$ [29, p. 11], where T^*M

is the cotangent bundle of M, R denotes a time interval, and $\dim(T^*M \times R) = (2 \cdot 3) + 1 = 7$. Such a contact structure is determined by a collection of the differential forms

$$\eta = \mathrm{d}t - \sum_{1}^{3} p_i \, \mathrm{d}x^i. \tag{3}$$

These differential forms in space–time are closely related to the classical idea of "line element" in R^2, which consists of a point with an associated direction field element. The tangents to a plane curve (such as a visual contour projected onto the frontal plane) determine the curve itself as an integral curve of a corresponding contact distribution D on $T^*R \times R$, and consequently transformations mapping integral curves of D into integral curves of D are termed "contact transformations." The Boothby–Wang Theorem [29], to the effect that such a contact manifold is the bundle space of a principal fibre bundle $\pi : M^{2n+1} \rightarrow M^{2n}$ over a symplectic manifold M^{2n}, with structure group the circle group S^1, has an important further consequence in the present context. The symplectic manifold in the visual case is the surface projected on the retina. M^3 is its representation in the visual cortex, and the cyclic nature of S^1 governs the continual flow and interchange between afferent (incoming) and efferent (outgoing) volleys of nerve impulses.

The contact structures and associated Lie derivatives of Eqs. (2) and the pseudo-Euclidean rotations involved in binocular vision [16] generate the invariant curves of basic form perception—in other words, the orbits of the constancies: lines, circles, stars of radial lines, hyperbolas, and curves resulting from linear combinations of these, such as spirals. However, this is of course not enough for higher form perception of such complicated forms as we perceive in the visual field of view about us at this very moment. It has been shown [16, 30] that the way such higher forms are perceived follows readily in terms of their differential invariants, of arbitrarily high order, superimposed upon the basic constancy orbit spaces. In this connection we recall the following empirical finding [31]: An electrode or strychnine pad applied to the exposed pial surface in area 17 (the visual cortex) evokes such visual hallucinations as sparks, stars, balls, disks, wheels, and moving lines . . . (which are the orbits of the basic Lie algebra of visual perception). On the other hand, if the cortical surface of areas 18 and 19 (the so-called psychovisual cortex) is similarly stimulated, conscious subjects report seeing "complete and well-defined objects, but without definite size or position, much as in ordinary mental imagery." This latter aspect suggests a hierarchy of visual processing, in which size and location in the field of view are washed out at the first stage of cortical visual processing, a feature consistent with the known cytoarchitecture of the visual cortex, wherein an afferent volley of nerve impulses first encounters the stellate cells of layers III and IV [16, 32], which may be regarded as germs of the dilation group for size constancy.

A (total) flag [26] for a differentiable n-dimensional manifold M comprises, for any point x in the manifold, a direction $u^{(1)}$ through x, a 2-dimensional direction $u^{(2)}$ through $u^{(1)}$, and so on up to an $(n-1)$-dimensional direction $u^{(n-1)}$ through $u^{(n-2)}$. These are of course nothing but the differential invariants generated by prolongation of the basic Lie derivatives of the constancies and constitute the contact structure for the associated 1-form

$$\eta = \mathrm{d}y - \sum_{i=1}^{n} p_i \, \mathrm{d}x^i \text{ on } M.$$

As a corollary of the theorem on isometries of triangles cited above, we have the following [26]:

If there is a transformation group $G \times M \to M$ such that for any two total flags on M there is exactly one transformation of G taking one into the other, then M is one of the Euclidean or non-Euclidean spaces listed above: Euclidean, elliptic, spherical, or hyperbolic space of appropriate dimension, and G is isomorphic to the corresponding group of motions.

Let us now examine the hierarchy of perceptual geometries in terms of their infinitesimal transformations. The most general situation among those described above is that of the projective group, whose infinitesimal generators are [33]

$$\partial_i,\ x^i\partial_j,\ x^i\sum_0^3 x^j\partial_j,\ (i,j=0,1,2,3;\ x_0=t). \tag{4}$$

$P_4(R)$ is simple and hence cannot contain $CO(1,3)$ or any of the other groups listed above as invariant subgroups. Even so, the infinitesimal generators (2) of $CO(1,3)$ are to be found among those in (4), possibly as linear combinations, so that one can expect that the neural structures governed by $CO(1,3)$ are nested within and generated by those for (4). In turn $CO(1,3)$ consists, in terms of its infinitesimal generators, of the three invariant subgroups [33] (i) ∂_i (translations), (ii) $x^i\partial_j$ (dilations and rotations), and (iii)

$$x^i\partial_i - x^j\partial_j \tag{5}$$

(pseudo-Euclidean rotations) of the General Linear Group $GL(4,R)$, together with

$$K_\mu = \left(\sum_\nu x^{\nu^2}\right)\partial_\mu - 2x^\mu\sum_\rho x^\rho\partial_\rho.$$

Here, the infinitesimal generators of $GL(4,R)$ [33] are a subset of (4), viz.,

$$\partial_i \text{ and } x^i\partial_j \qquad (i,j = 0,1,2,3). \tag{6}$$

K_μ follows from the infinitesimal generators of (4) by forming a linear combination of

$$\left(\sum_0^3 x^{i^2}\right)\partial_j \text{ and } -2\left(x^i\sum_{j=0}^3 x^j\partial_j\right).$$

Superimposed on all of these are the prolonged Lie derivatives [30] appropriate to the contact transformations of S.

Since the subgroup situation outlined above indicates no interaction between $P_4(R)$ and $CO(1,3)$, nor between $CO(1,3)$ and $GL(4,R)$, each system must be represented separately in the structure of the brain, for otherwise the integrability of vector fields into the integral curves of the visual contours implied by the closure in the sense of the Lie algebra relation

$$[\mathscr{L}_\alpha,\mathscr{L}_\beta] = \sum_\gamma c^\gamma_{\alpha\beta}\mathscr{L}_\gamma$$

of the corresponding Lie algebra would not be possible. Considerable support for such a view is to be found in the known cytoarchitecture of the visual cortex and the other posterior perceptual regions of the brain [10, 16, 32, 34, 35].

A word, in the context of the constancies, about a competitor of the cortical vector-field theory outlined above, viz., the spatial frequency theory of form perception, is appropriate here. The spatial frequency theory holds that, in effect, form perception takes place via computations in the spatial frequency domain of the Fourier components of a perceived form. There may be some merit, according to the Peter–Weyl Theorem, to such a view with respect to the parameter groups of the Lie transformation groups involved in the perceptual invariances of the constancies—at least for size and location—and for the symplectic structure of the perceptual manifold, but there is no way that such a Fourier approach could ever account for invariance under transformations other than translations and dilations, that is, for anything other than size constancy and one component of shape constancy. In other words, the invariances represented by subgroups (ii) and (iii) of (4) above would be irretrievably lost. Hence, the kind of equivariant perceptual transformations that we carry out constantly as a matter of course—rotations, obliquities, pseudo-Euclidean rotations, and reduction of moving percepts to our subjective frame of reference—are impossible in the spatial frequency theory. Hopefully, the avid proponents of this red herring will someday come to appreciate this fact and so clear the air for more relevant lines of experimental investigation.

In any case, spatial frequency phenomena are readily subsumed under the contact structure associated with the infinitesimal generator $\mathscr{L} = y\partial_y$ of the general linear and projective groups in R^2. The first and second prolongations of this Lie derivative are, respectively,

$$\mathscr{L}^{(1)} = y\partial_y + y'\partial_{y'} \text{ and } \mathscr{L}^{(2)} = y\partial_y + y'\partial_{y'} + y''\partial_{y''},$$

and it is clear that the differential equation of simple harmonic motion $y'' + y = 0$ is invariant under $\mathscr{L}^{(2)}$. The most general differential equation invariant under $\mathscr{L}$ is [36, p. 92]

$$y'' = y\, f(x, y'/y), \tag{7}$$

where f is an arbitrary function representative of the particular form stimulus then present. As such (7) contains all the generality needed for the most general spatial frequency interpretation under the cortical vector-field theory. Further, the general linear second order differential equation

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{8}$$

is a special case of (7). Hence, on the basis of Occam's razor, the cortical vector field theory displaces the spatial frequency theory in any case.

3. THE CATEGORIES OF THE POSTERIOR PERCEPTUAL SYSTEMS

We have seen above that certain Lie transformation groups acting over the visual manifold suffice to explain constancy phenomena and form memory. When pitch and loudness constancies are adjoined in the form of nonuniform translation operators in their respective variables, together with binaural localization of a sound source in space–time, auditory perception is also explainable in the Lie transformation group format. Sounds are perceived as relaxation oscillations upon the Cochlea [10] in the presence of the same sort of transformation from hyperbolic space to Cyclopean, spherical space as that discussed above [see 16]. Higher perception takes place in both the visual and auditory cases via prolongation of the basic Lie groups.

Now it is a truism, mathematically speaking, that where there are Lie transformation groups and differentiable manifolds, fibre bundles—coset bundles, tangent and cotangent bundles—cannot be far away. We recall the general nature of a fibre bundle in Fig. 4. Two types of fibre bundles will prove important in the perceptual context. One is the vector bundle, wherein the structure group is either $GL(n,R)$ itself or some subgroup thereof; the other is the jet bundle(s) defined over the vector bundle. The archetypal example of vector bundle is the tangent bundle TM to a manifold M. The jet bundles [37] over M, obtained from the flag sequence

$$M \rightarrow T^*M \rightarrow T^*T^*M = T^{*2}M \rightarrow T^{*3}M \rightarrow \cdots \rightarrow T^{*k}M,$$

which is defined by the hierarchy of differential forms

$$\mathrm{d}y^{(j-1)} - \sum_i p_{ji}\ \mathrm{d}x^i,$$

represent the higher differential invariants of $G_V \times M \rightarrow M$, and as such, generate higher form perceptions out of the basic invariants registered by the constancies.

The gross neuroanatomical structure of the brain has the nature of such a fibre bundle. Figure 5, modified from [38], shows the basic forebrain plan. One discerns a total space consisting of the neocortex and limbic ring (the emotional brain) over a base space in the midbrain region, together with projections from the former and cross-sections (inverse

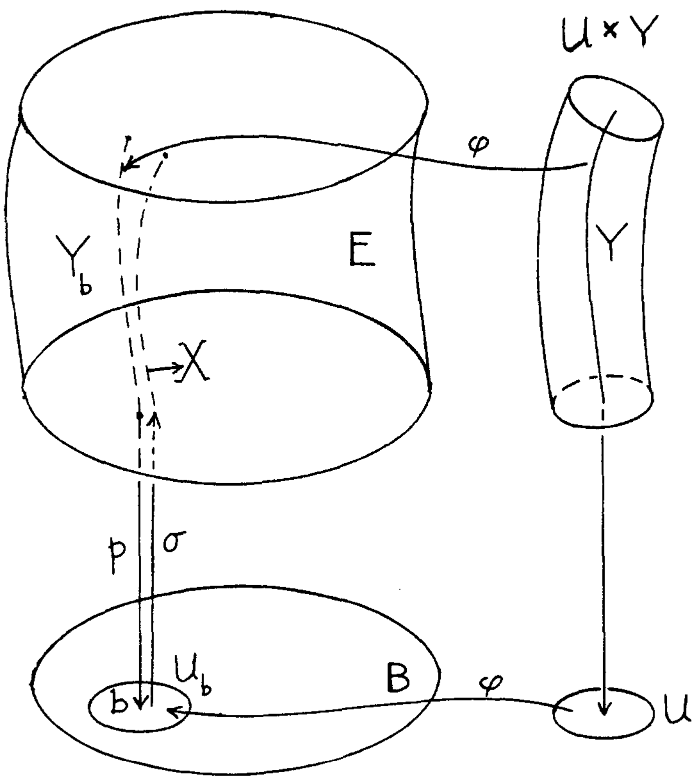

Fig. 4. The structure of a fibre bundle $\beta = \{E, p, B, Y, G\}$. The total space is $E = G_V \times M$ in the application to the visual manifold. ϕ is a homeomorphism in the topology of E that takes the tubular fibre neighborhood $U \times Y$, Y being the fibre space, into the tubular neighborhoods $U_b \times Y_b$ of fibres over points (Y_b over $b \in B$). The projection p: $E \rightarrow B$ has the cross-section σ: $B \rightarrow E$ (a vector field X on the visual manifold M): $\sigma \cdot p = id = p \cdot \sigma$. The base space B is the visual manifold M in the perceptual application.

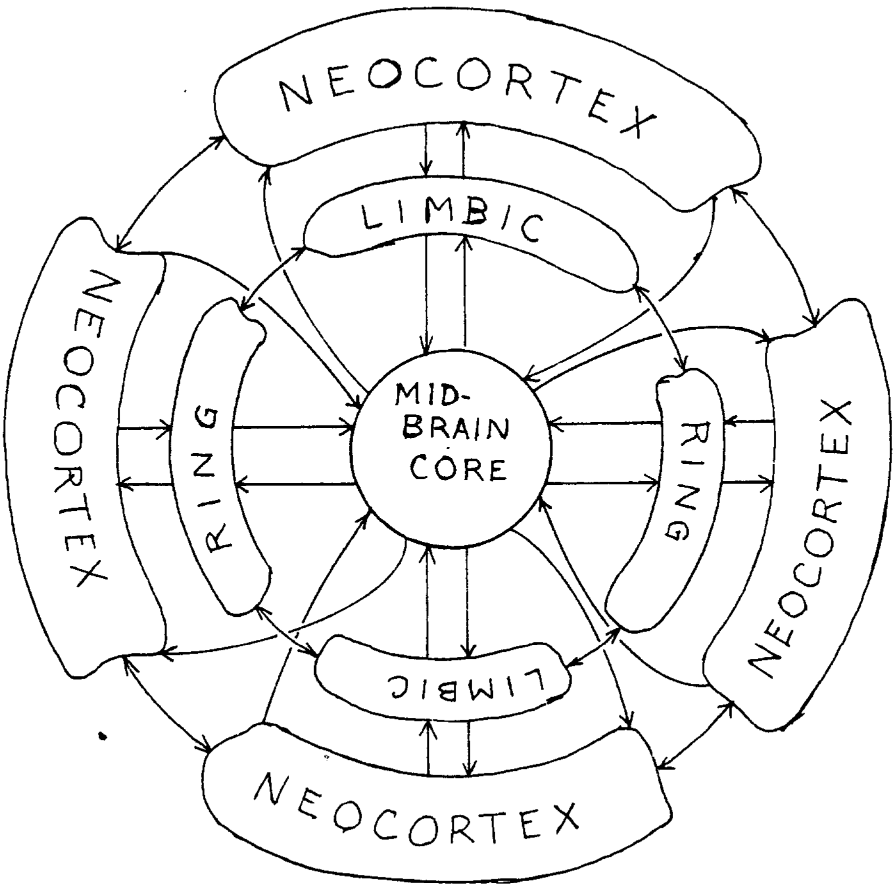

Fig. 5. The neuropsychological fibre bundle (adapted from [38]).

projections) from the midbrain region to higher brain regions. Such a configuration has also been noted by Evarts *et al.* [39] for the haptic (sensorimotor–kinesthetic) perceptual system.

In the present instance such fibre bundles are also equivariant, and determine corresponding categories $G_V VB(S)$ of equivariant vector bundles and $G_V FB(S)$ of equivariant fibre bundles [40]. We recall that S is a G_V space if $G_V \times S \to S$, and for such a G_V space, a G_V vector bundle on S is a G_V space E coupled with a G_V map p: $E \to S$ such that

$$p(g \cdot x) = g \cdot p(x), \qquad x \in S,\ g \in G_V, \tag{9}$$

with finite-dimensional vector spaces as fibres-over-points: $E_x = p^{-1}(x)$. The definition for $G_V FB(S)$ runs similarly. The essential feature is that p and g commute in their actions on $x \in S$. TS belongs to $G_V VB(S)$ in a natural way, and the space of equivariant sections of $E \in G_V VB(S)$ or $G_V FB(S)$ is

$$C^r_{G_V}(E) = \{X \in C^r(E) \mid gXg^{-1}(x) = X(x), \quad g \in G_V, \quad x \in S\}. \tag{10}$$

Thus, in the presence of equivariance, perception of visual contours and the transformation group actions of the constancies are interchangeable. It is immaterial whether or not the visual field of view is distorted by the viewing conditions; the primal object contour will be perceived in any case. According to (10), this will carry over also to the vector fields that generate the visual contours as integral curves.

We note some further important consequences of equivariance for perceptual phenomena. First of all, there is Lemma 4.4 of [41] to the effect that if S is a separable metrizable G space and $U_1, U_2, \ldots, U_n$ a covering of S by invariant open subsets, and if each U_i admits an equivariant imbedding in a Euclidean G space, then so does S itself. Dissection of the visual field of view S into subimages, each invariant under some Lie group $G_V \subset P^4(S) \oplus CO(1,3) \oplus GL(4)$ or the prolongations thereof, is thus possible, and the decomposition hypothesized in the preceding section is legitimatized, mathematically at least.

Secondly, the perceptual manifold S admits a splitting of the tangent bundle in the case of equivariance. If S' is a compact submanifold of a Riemannian manifold S and $\phi \in \mathrm{Diff}^r_{G_V}(S)$, i.e., S' is both G_V invariant and ϕ invariant, then

$$T_{S'}S = TS \oplus N^s \oplus N^u,$$

where TS, N^s, and N^u are sub-bundles invariant under the differential $T\phi$ of ϕ such that $T\phi$ expands N^u more rapidly than TS and $T\phi$ contracts N^s more sharply than TS. The normal (to S) action of $T\phi$ is thus hyperbolic (in the sense of global analysis) and dominates the tangential behavior of S. This fact apparently has important implications for the cytoarchitecture of the neuronal net [32].

Equivariance also appears to have important implications for the visual Gestalt, in particular the well known properties of closure, similarity, symmetry, continuity ("good continuation"), and "common fate" (grouping of subimages that move or flow in a common direction is perceptually favored). These Gestalt properties apparently follow readily from the proposition [42] that an equivariant section of a G_V vector sub-bundle $E|_{E'}$, where E' is a closed G_V subspace of the compact G_V space S, can be extended to an equivariant section of E itself by averaging an arbitrarily generated extension of $C^r_{G_V}(E|_{E'})$ over G_V.

Finally, there is the matter of how the visual system constructs visual contours from the microscopic action of ensembles of neurons, the action of the latter being that of a cortical vector field generating the "constancies." It is well known [36] that the invariance of a differential equation (and hence differential form) under a single group yields an integrating factor. Invariance under two groups, such as those of shape and size constancy [$SL(3)$ and the group of dilations] yields *without quadrature* the corresponding integral curve [36].

We now revert to the contact structure on S. A theorem of E. Cartan guarantees that the orientable portion M^{2n+1} of S has a global 1-form η such that $\eta \wedge d\eta^n \neq 0$ throughout M^{2n+1}. In the planar projection $\mathrm{proj}(M^3) = \overline{M}^2$, $\eta = \mathrm{d}x^2 - p\,\mathrm{d}x^1$. In $M^3 = R^3$ itself, $\eta = \mathrm{d}x^3 - p_1\,\mathrm{d}x^1 - p_2\,\mathrm{d}x^2$, and in $M^4 = R^3 \times T$ (T a time interval),

$$\eta = \mathrm{d}t - \sum_1^3 p_i\,\mathrm{d}x^i.$$

In each case the 1-form defines a family of "contact hyperplanes" that determine a hyperplane field on M^{2n+1}. The latter is a $2n$-dimensional sub-bundle of TM^{2n+1}.

The relation of such a hyperplane field with the objects (affine simplices in one interpretation) of the simplicial category $\mathbf{\Delta}$ is immediate: "Included among the polyhedra are such spaces as the compact differential manifolds" [43]. Modern information-processing psychology regards long-term memory (LTM) as a network of concepts, each node representing a concept. The functorial relationship between such networks and objects and the simplicial category **Δ** and the category of simplicial objects **Σ** leads naturally to the next stage, that of cognitive psychology. In other words, form perceptions and constancies, embodied in the category $G_V FB(S)$ of equivariant fibre bundles, are given meaning(s) by functorial maps from $G_V FB(S)$ to the category **Σ** of simplicial objects, indexed and analyzed in the many ways that we shall encounter below by the simplicial category **Δ**.

4. CATEGORY THEORY AND COGNITIVE INFORMATION PROCESSING

Figure 6 depicts in outline the essential nature of the next stage of neuropsychological processing, the transition, via appropriate functors from the category $G_V FB(S)$ characteristic of the posterior perceptual systems to the categories **Δ** and **Σ** of cognitive information processing—the so-called higher faculties. **Δ** is the simplicial category, whose objects, Ob**Δ** $= \{[n];\ n \in Z^+\}$, are finite total orders—i.e., finite ordinals—and whose morphisms Mor**Δ** are order-preserving maps: Mor**Δ** $= \{\mu|\mu\colon [m] \to [n]\}$. Another way of looking at **Δ** is that it is generated by the ensemble of all increasing injections $\delta_n^i\colon [n-1] \to [n]$ with $i \notin$ image δ^i, for $n > 0$ and $0 \leq i \leq n$ and all nondecreasing surjections σ_n^i:

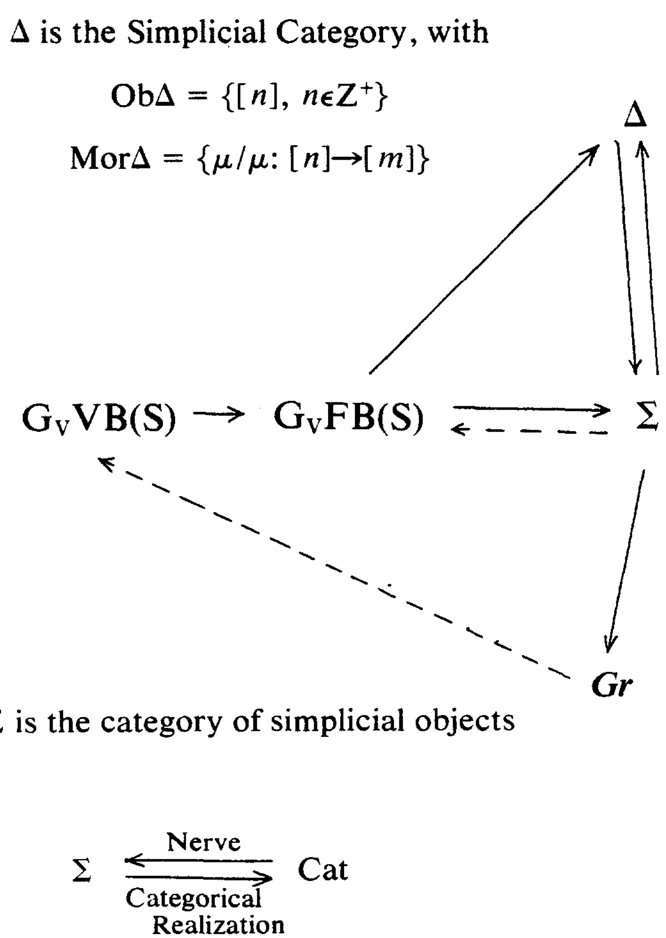

Fig. 6. Outline of the functorial relationships between the category $G_V FB(S)$, characteristic of the posterior perceptual systems, and the categories and groupoids characteristic of cognitive phenomena: **Δ**, **Σ**, and *Gr*.

$[n+1] \to [n]$, with $n \geq 0$, in which i, $0 \leq i \leq n$, occurs twice. $\mathbf{\Delta}$, as a functorial image of $G_V FB(S)$, orders, reorders, and either deletes or extends the bundle maps (the morphisms) of the fibre bundle category. $\mathbf{\Delta}$ has other important properties in the psychological application as well, to which we shall return shortly, but first we take up the category $\mathbf{\Sigma}$ of simplicial objects. Referred to an arbitrary category **C**, $\mathbf{\Sigma}$ is defined as follows [44]:

$$\text{Ob}\mathbf{\Sigma} = \{\Sigma : \mathbf{\Delta}^{op} \to \mathbf{C}\},$$

i.e., the objects of $\mathbf{\Sigma}$ are actually contravariant functors, and the morphisms of $\mathbf{\Sigma}$ are natural transformations

$$\text{Mor}\mathbf{\Sigma} = \{\dot{\theta} : \Sigma \to \Sigma'\}.$$

A simplicial object for the category **C** is thus a list $\Sigma_0, \Sigma_1, \ldots, \Sigma_n, \ldots$ of objects of **C**. The corresponding morphisms are the "face operators" d_k: $\Sigma_n \to \Sigma_{n-1}$ and "degeneracies" s_j: $\Sigma_n \to \Sigma_{n+1}$. Regarding Ob$\mathbf{\Sigma}$ as the orderings of Ob**C** by $\mathbf{\Delta}$, one sees that if **C** represents some particular collection of objects and arrows of thought or higher semiotic memory that is induced by form elements from $G_V FB(S)$, then we have a means for relating perception of forms to their associated meanings, learned or otherwise, in the category $\mathbf{\Sigma}$ of simplicial objects. The morphisms of $\mathbf{\Sigma}$ then permit "trains of thought" of the most abstruse kind, particularly when adjoint functors between two different categories, **C** and **C**′, are invoked.

A category all of whose morphisms are invertible is called a groupoid and denoted by ***Gr***. Piaget's *groupements* ("groupings") are actually groupoids [45, 46]. Other psychological phenomena also exhibit the groupoid property, notably Piaget's "reversibility" and even recall and recollection types of memory. The order of temporal events flows in one direction only, past to future, but memory permits the reversal of this flow to recapture events long past. ***Gr*** also applies to problem-solving behavior, which may be regarded as the finding of a cognitive path—a composition of morphisms, forward and backward—through a cognitive quotient groupoid: $\mathbf{Gr}^{op} \times \mathbf{Gr} \to \mathbf{Gr}$. In the present context, then, there exist close functorial relationships among $\mathbf{\Sigma}$, $\mathbf{\Sigma}^{op}$, and ***Gr***. The functor category $\mathbf{B}^{\mathbf{C}}$, whose objects consist of all functors from **C** to **B**, for arbitrary categories **B** and **C**, and whose morphisms comprise the natural transformations between such pairs of functors, is known [44, p. 45] to be an object function of $\mathbf{\Sigma}^{op} \times \mathbf{\Sigma} \to \mathbf{\Sigma}$, when $\mathbf{\Sigma} \in$ **Cat**, the category of all small categories. As such, the functor category appears to possess all the generality needed to represent the most far-ranging and protean sort of mental processes.

We summarize our postulated hierarchy of perceptual–cognitive information processing in Fig. 7. The elements of the figure are defined above, except for **Rel**($\mathbf{\Sigma},\mathbf{\Sigma}'$), which denotes the category with small sets as objects and binary relations as morphisms [44], and so can be related to first-order logic. By use of pullbacks, **Rel** can be made, for any fixed $\mathbf{\Sigma}'$, into the functor **Rel**($-,\mathbf{\Sigma}'$).

The identifications with ITM (Intermediate Term, or "Working," Memory) and LTM (Long Term Memory—a conceptual network, the concepts being nodes) indicated in Fig. 7 are suggested by current views of information-processing psychology [47]. Short Term Memory (STM), which has only limited information capacity and which is aided by "rehearsal," may involve the circle bundles of the Boothby–Wang Theorem, different ones for each modality. Certainly STM requires the preservation of temporal order that is offered by $\mathbf{\Delta}$. "Chunking," the representation in STM of incoming stimuli in terms of symbolic "chunks" of information, involves $\mathbf{\Delta}$, $\mathbf{\Sigma}$, appropriate quotient categories and groupoids, the Noether isomorphism theorems, etc.

Memory, of whatever type, must represent an invariance of some particular concept,

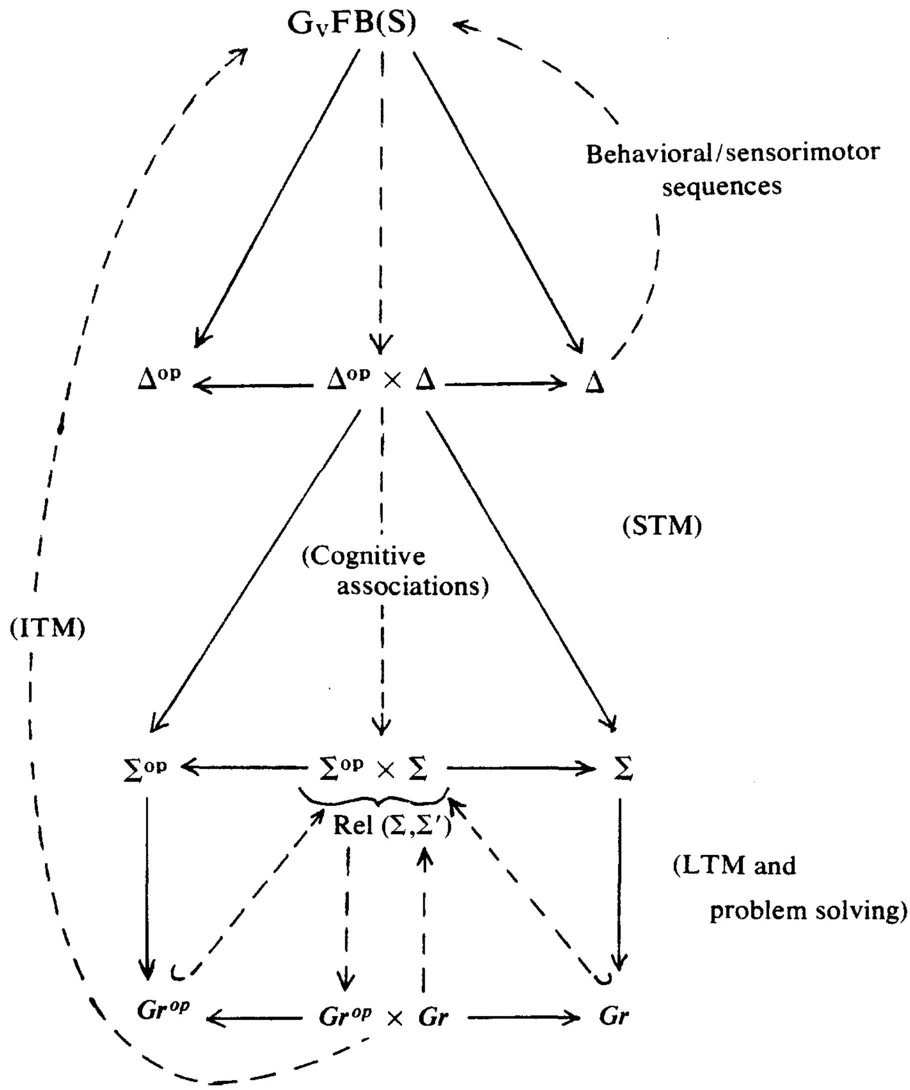

Fig. 7. The postulated hierarchy of perceptual-cognitive processing.

just as the invariances of the constancies constituted the key to basic form perception. In this connection we have "The Theorem" of Peter Freyd [48]: An elementary property on categories is invariant within equivalence types of categories if and only if it is a diagrammatic property, defined by a finite rooted tree of C graphs, ordered by extension, and each graph labeled ∀ or ∃. Diagrammatic sentences of such properties are closed under the usual Boolean operations of negation, conjunction, and disjunction, thus providing a bridge to "logical" thought. Hence formal thought processes, such as those of Piaget's Formal Operations Period, are accessible within the format of category theory in terms of diagrammatic properties, even though they are not as fundamental as the basic categorical structure itself. And of course, with respect to the most abstruse forms of logic, there is always topos theory (see below).

We return now to the key element of the above cognitive information processing structure, i.e., the simplicial category Δ itself. MacLane [44] has termed the various aspects of $\mathbf{\Delta}$ "protean," and we can but underline the point here. First of all, there is the relation of $\mathbf{\Delta}$ to the precategory of directed graphs. Any directed graph constitutes a *diagram scheme* which may be made into a category by adjoining appropriate loops (identity maps) and strings of composable edges. The simplicial complexes of $\mathbf{\Delta}$ are thus graphs in an obvious way. The (pre-) category **Grph** of all small directed graphs follows from the category **Cat** of all small categories by means of the forgetful functor $\mathcal{U}$. Since we view thought processes as "chasing around the diagrams" of $\mathbf{\Delta}$ and $\mathbf{\Sigma}$, the directed flowgraphs of information-processing psychology play a key role: Thought processes are structured by graphs and related by natural transformations and adjoint functors. This comes about via the natural isomorphism $\mathbf{Cat}(F\Gamma,\mathbf{B}) \simeq \mathbf{Grph}(\Gamma,\mathcal{U}\mathbf{B})$ [44]. The meaning of this relationship is as follows: If $\mathbf{B}$ is some arbitrary category in **Cat**, the forgetful functor $\mathcal{U}$: **Cat** $\rightarrow$ **Grph** will be defined by a diagram of the shape of the graph Γ in $\mathbf{B}$:

$$F': \Gamma \rightarrow \mathcal{U}\mathbf{B}.$$

The left adjoint to the F' functor is F: **Grph** $\rightarrow$ **Cat**, defined by $F_\Gamma = F\Gamma$ for Γ; this adjoint functor is equivalent to the above natural isomorphism.

All this may well be in contact with the current doctrines of information processing psychology, but the mind boggles when it comes to the seat of $\mathbf{\Delta}$ and **Grph** in the brain itself. Yet the geometrical realization functor [43] that expresses $\mathbf{\Delta}$ in terms of geometry–topology has a cortical counterpart in the triangulation of "the manifold of consciousness" embodied in the network of neuronal arborescences. The latter is known to proliferate rapidly during childhood development [49] and indeed all through life [50]. It is, in fact, the growth and ramification of the neuronal arborescence that keeps pace with memory and learning. "Memory molecules," on the other hand, are continually conveyed to the periphery by the neuroflow process and become ammonialike decay products in about three weeks.

Another aspect of $\mathbf{\Delta}$ worthy of note is the precedence [51] in the normal course of children's development of "numerosity" of ordinal numbers over cardinal numbers. The boundary of the finite ordinal $[n]$ is simply n itself, and the boundary operator must await the establishment of the set to which it applies before it can become operative.

The preceding discussion has dealt with the numerical aspects of $\mathbf{\Delta}$. But $\mathbf{\Delta}$ also has geometrico–topological aspects that are no less important in the context of psychology. The geometrical realization functor

$$\Delta: \mathbf{\Delta} \rightarrow \mathbf{Top},$$

where **Top** denotes the category of topological spaces, admits a geometric interpretation in terms of affine simplices. The functor Δ takes the ordinal $[n+1]$ to the standard n-dimensional simplex Δ_n. The objects Δ_n of the (sub-) category so mapped are the standard $(n-1)$-dimensional affine simplices determined by barycentric coordinates, and the morphisms are, for μ: $[n] \rightarrow [m]$, the order-preserving affine maps Δ_μ that take each vertex, say the ith, of Δ_{n+1} to the corresponding [the $\mu(i)$th] vertex of Δ_{m+1}. This representation provides the cognitive basis for perspective phenomena and Piaget's conservation of length, area, and volume [46].

The important functorial relationship between simplicial category and category of simplicial objects, which admits higher meanings and the thought processes associated with the perception of particular objects has already been discussed. Another important aspect of $\mathbf{\Delta}$ is that it is a full subcategory of the category **Ord** of all linearly ordered sets. This feature makes possible such ordinal behavior as the sequencing, in their proper order, of a complicated chain of behavioral actions.

Finally, there is the matter of logical thought. In actuality no one thinks, digital-computer style, in the manner of Bertrand Russell's *Principia Mathematica*. As Lawvere [52] has put it,

> . . . most mathematicians feel that a logical presentation of a theory is an absurd machine strangely unrelated to the theory or its subject matter. . . .

But once a proper "train of thought" has been established by other means (presumably Freyd's diagrammatic sentences), then one can *do* logic, *compose* music, *write* literary works, in a most formal and conventionally structured way. As we have mentioned above, Freyd's diagrammatic sentences admit the operations of Boolean logic. In addition, one can always call on the theory of topoi [52], which lies on the boundary between logic and category theory. Lawvere noted several years ago a considerable degree of formal match-up between the rules of logic and the calculus of adjoint functors, and this connection has been under active development since.

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