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A fixed point theorem for the infinite-dimensional simplex ☆

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Abstract

We define the infinite-dimensional simplex to be the closure of the convex hull of the standard basis vectors in $\mathbb{R}^\infty$, and prove that this space has the *fixed point property*: any continuous function from the space into itself has a fixed point. Our proof is constructive, in the sense that it can be used to find an approximate fixed point; the proof relies on elementary analysis and Sperner's lemma. The fixed point theorem is shown to imply Schauder's fixed point theorem on infinite-dimensional compact convex subsets of normed spaces.

Keywords: Schauder fixed point theorem; Brouwer fixed point theorem; Sperner's lemma; Infinite-dimensional simplex

1. Introduction

In finite dimensions, one of the simplest methods for proving the Brouwer fixed point theorem is via a combinatorial result known as Sperner's lemma [7], which is a statement about labelled triangulations of a simplex in $\mathbb{R}^n$. In this paper, we use Sperner's lemma to prove a fixed point theorem on an infinite-dimensional simplex in $\mathbb{R}^\infty$. We also show that this theorem implies the infinite-dimensional case of Schauder's fixed point theorem on normed spaces.

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Since $\mathbb{R}^\infty$ is locally convex, our theorem is a consequence of Tychonoff's fixed point theorem [6]. However, some notable advantages of our approach are: (1) the constructive nature of Sperner's lemma provides a method for producing approximate fixed points for functions on the infinite-dimensional simplex, (2) the proof is based on elementary methods in topology and analysis, and (3) our proof provides another route to Schauder's theorem.

Fixed point theorems and their constructive proofs have found many important applications, ranging from proofs of the Inverse Function Theorem [5], to proofs of the existence of equilibria in economics [9,10], to the existence of solutions of differential equations [2,6].

2. Working in $\mathbb{R}^\infty$

Let $\mathbb{R}^\infty$ and $I^\infty = \prod[0,1]$ be the product of countably many copies of $\mathbb{R}$, and $I = [0,1]$, respectively. We equip $\mathbb{R}^\infty$ with the standard product topology, which is metrizable [1] by the complete metric

$$\bar{d}(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1+|x_i - y_i|)}.$$

In $\mathbb{R}^n$, a k-dimensional simplex, or *k-simplex*, σ^k is the convex hull of $k+1$ affinely independent points. The *standard n-simplex* in $\mathbb{R}^{n+1}$, denoted Δ^n, is the convex hull of the $n+1$ standard basis vectors of $\mathbb{R}^n$.

The natural extension of this definition to $\mathbb{R}^\infty$ is to consider Δ^∞, the convex hull of the standard basis vectors $\{e_i\}$ in $\mathbb{R}^\infty$, where $(e_i)_j = \delta_{ij}$, the Kronecker delta function. As convex combinations are finite sums, this convex hull is:

$$\Delta^\infty = \left\{ x \in \mathbb{R}^\infty \;\middle|\; \sum_{i=1}^{\infty} x_i = 1,\ 0 \leqslant x_i \leqslant 1,\ \text{and only finitely many } x_i \text{ are nonzero} \right\}.$$

Unfortunately, Δ^∞ is not closed; under the metric $\bar{d}$ the sequence $\{e_i\}$ converges to $\mathbf{0}$, which is not in Δ^∞. So consider, instead Δ_0^∞, the closure of Δ^∞, which can be shown to be:

$$\Delta_0^\infty = \left\{ x \in \mathbb{R}^\infty \;\middle|\; \sum_{i=1}^{\infty} x_i \leqslant 1 \text{ and } 0 \leqslant x_i \leqslant 1 \right\}.$$

It is easy to see that Δ_0^∞ is convex. It is also the closure of the convex hull of the standard basis vectors $\{e_i\}$ and $\mathbf{0}$. It is also compact because it is a closed subset of I^∞, which is compact by Tychonoff's theorem.[1] We call Δ_0^∞ the *standard infinite-dimensional simplex.*

It will be important for our purposes later to consider F^n, the n-dimensional face of Δ_0^∞ given by $F^n = conv\{e_1, e_2, \ldots, e_{n+1}\}$. Notice that each F^n is closed and thus compact.

[1] If one would like to avoid the Axiom of Choice, which is equivalent to Tychonoff's theorem, it is not difficult to show that I^∞ is a closed and totally bounded subset of the complete space $\mathbb{R}^\infty$, which implies compactness.

3. Some preliminary machinery

Let $\sigma^k = conv(x_0, \dots, x_k)$ be a k-simplex in $\mathbb{R}^n$. Let T be a triangulation of σ^k and V be the set of vertices of T (i.e., the vertices of simplices in T). A *Sperner labelling* of the triangulation T is a labelling function $\ell : V \to \{0, \dots, k\}$ such that

$$\text{if} \quad J \subseteq \{0, \dots, k\} \quad \text{and} \quad v \in conv\{x_j \mid j \in J\}, \quad \text{then} \quad \ell(v) \in J.$$

A k-simplex τ of T is called a *fully-labelled simplex* (or *full*) if the image of the vertices of τ under ℓ maps onto $\{0, \dots, k\}$. Note that τ has exactly $k+1$ vertices, so all the vertices have distinct labels.

Sperner's lemma. *Let σ^k be a k-simplex in $\mathbb{R}^n$ with triangulation T and let ℓ be a Sperner-labelling of T. Then the number of full simplices of T is odd* (*and hence, nonzero*).

Though we will not prove this theorem here, proofs can be found in [8]. In particular, there are constructive "path-following" proofs that locate the full simplex by tracing a path of simplices through the triangulation. Such path-following proofs have formed the basis of algorithms for locating fixed points of functions in finite-dimensional spaces, e.g., see [9] for a nice survey. In Section 4, we show how to use Sperner's lemma for a fixed point theorem in the infinite-dimensional space Δ_0^∞.

Another crucial theorem for our purposes states that, under appropriate hypotheses, the existence of approximate fixed points implies the existence of fixed points. On the metric space (X, d), we can quantify the notion of an approximate fixed point by defining an *ϵ-fixed point*, which for a given function f is a point $x \in X$ such that $d(x, f(x)) < \epsilon$. Versions of the following lemma may be found in, e.g., [3,6].

Lemma 1 *(Epsilon fixed point theorem). Suppose that A is a compact subset of the metric space (X, d) and that $f : A \to A$ is continuous. If f has an ϵ-fixed point for every $\epsilon > 0$ then f has a fixed point.*

Proof. Let $\{a_n\}$ be a sequence of $1/n$-fixed points. That is, $d(a_n, f(a_n)) < 1/n$ for all n. Since A is compact it is sequentially compact and thus $\{a_n\}$ has a convergent subsequence, which we denote $\{a'_n\}$ with $a'_n \to x \in A$. Let $\epsilon > 0$. Since $a'_n \to x$ there exists N_1 such that $n \geqslant N_1$ implies that $d(a'_n, x) < \epsilon/2$. Let $N = \max(N_1, 2/\epsilon)$. Then $n \geqslant N$ implies that

$$d\big(x, f\big(a'_n\big)\big) \leqslant d\big(x, a'_n\big) + d\big(a'_n, f\big(a'_n\big)\big) < \epsilon,$$

so that $f(a'_n) \to x$. However, since f is continuous, we also know that $f(a'_n) \to f(x)$. Since limits are unique, we conclude that $f(x) = x$, which completes the proof. □

Later it will be desirable to have an isometry between Δ^{n-1}, the standard $(n-1)$-simplex in $\mathbb{R}^n$, and F^{n-1}. The easiest way to do this is to consider $\mathbb{R}^n$ as a subspace of $\mathbb{R}^\infty$ by projection onto the first n factors, and restricting the metric on $\mathbb{R}^\infty$ to $\mathbb{R}^n$. Call this metric $\bar{d}_n$ and consider Δ^{n-1} in the metric space $(\mathbb{R}^n, \bar{d}_n)$. It is worthwhile to ensure that $(\mathbb{R}^n, \bar{d}_n)$ has a rich supply of continuous functions. Before proceeding, recall that all norms on $\mathbb{R}^n$ are equivalent and thus essentially interchangeable; we now prove that $\bar{d}_n$ is interchangeable with norm-induced metrics on bounded sets.

Lemma 2. *Let A be a bounded subset of the normed space $(\mathbb{R}^n, \|\cdot\|_\infty)$. On A, the metric $\bar{d}_n$ is equivalent to the metric induced by the norm $\|\cdot\|_\infty$.*

Proof. Suppose that $x, y \in \mathbb{R}^n$. We see that

$$\bar{d}_n(x,y) = \sum_{i=1}^{n} \frac{|x_i - y_i|}{2^i(1+|x_i - y_i|)} \leqslant n\|x-y\|_\infty.$$

Now, since A is bounded, there is some M such $\|x-y\|_\infty \leqslant M$ for $x, y \in A$. Thus we see that

$$\frac{\|x-y\|_\infty}{2^n(1+M)} \leqslant \frac{\|x-y\|_\infty}{2^n(1+\|x-y\|_\infty)} \leqslant \bar{d}_n(x,y),$$

which implies that

$$\|x-y\|_\infty \leqslant 2^n(1+M)\bar{d}_n(x,y). \tag{1}$$

Thus $\bar{d}_n$ is equivalent to the metric induced by the norm on A. □

Lemma 2 tells us that bounded subsets of $\mathbb{R}^n$ have the same continuous functions regardless of whether they are considered as subsets of a normed space or as subsets of $(\mathbb{R}^n, \bar{d}_n)$. Importantly, notice that Δ^{n-1} is bounded. Furthermore, the isometry $f : \Delta^{n-1} \to F^{n-1}$ between Δ^{n-1} in $(\mathbb{R}^n, \bar{d}_n)$ and F^{n-1} in $\mathbb{R}^\infty$ is clearly given by $f(x) = f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. This is important because it implies that F^{n-1} has an arbitrarily small barycentric subdivision. Recall that the diameter of a set X is $d(X) = \sup_{x,y\in X} d(x,y)$ and if $\mathscr{T}$ is a family of sets, then $size(\mathscr{T}) = \sup_{\sigma\in\mathscr{T}} d(\sigma)$. Thus, given $\epsilon > 0$, F^{n-1} has a barycentric subdivision $\mathscr{T}$ with $size(\mathscr{T}) < \epsilon$.

Now we are ready to prove a fixed point theorem for Δ_0^∞.

4. A fixed point theorem for Δ_0^∞

Theorem 1. *Suppose that $f : \Delta_0^\infty \to \Delta_0^\infty$ is continuous. Then f has a fixed point.*

Proof. Since Δ_0^∞ is compact, by Lemma 1, it is sufficient to show that f has an ϵ-fixed point for each $\epsilon > 0$. Let $\epsilon > 0$ be given. Choose $N \geqslant \log_2(2/\epsilon) + 1$. Notice that for $x, y \in \Delta_0^\infty$, this implies that

$$\sum_{i=N+1}^{\infty} \frac{|x_i - y_i|}{2^i(1+|x_i - y_i|)} \leqslant \sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}. \tag{2}$$

Since f maps between countably infinite-dimensional spaces, we can write f in terms of its components: $f(x) = (f_1(x), f_2(x), \ldots)$. Since f is continuous, f_i is continuous for each i. Consider the function

$$g(x) = \big(g_1(x), g_2(x), \ldots\big) = \left(f_1(x), f_2(x), \ldots, f_N(x), 1 - \sum_{i=1}^{N} f_i(x), 0, 0, 0, \ldots\right).$$

Since each f_i is continuous and finite sums of continuous function are continuous, g_i is continuous for each i. Consequently, g is continuous. Furthermore, we see that $g : F^N \to F^N$.

Let $\epsilon_0 = \frac{\epsilon}{8(N+1)}$ and $\epsilon_1 = \frac{\epsilon}{2^{N+5}(N+1)}$. Since g is continuous on a compact set, it is uniformly continuous. Thus there exists $\delta_1 > 0$ such that $\bar{d}(x,y) < \delta_1$ implies that $\bar{d}(g(x), g(y)) < \epsilon_1$. Let

$\delta = \min(\delta_1, \epsilon_1)$. Since F^N can be triangulated with an arbitrarily small triangulation, let $\mathscr{T}$ be a triangulation with $size(\mathscr{T}) < \delta$. Label the vertices of $\mathscr{T}$ with the map

$$\ell(x) = \operatorname{argmax}_i \big(x_i - g_i(x)\big).$$

Recall that the *argmax* function returns the index of the largest element of the argument, and if there are multiple indices that give the maximum value, the argmax function returns the least of these indices.

Observe that $\ell(x)$ produces a Sperner labeling on the vertices of $\mathscr{T}$. Thus by Sperner's lemma, there exists a fully-labeled simplex in $\mathscr{T}$. This simplex can be found using the path-following method described in [8]. Let $\{x^1, x^2, \dots, x^{N+1}\}$ be the vertices of this simplex where the index of each vertex is its Sperner label. From this, we see that for all j,

$$x_i^i - g_i\big(x^i\big) \geqslant x_j^i - g_j\big(x^i\big).$$

Furthermore, since for each x in F^N we have

$$\sum_{j=1}^{N+1} x_j = \sum_{j=1}^{N+1} g_j(x) = 1,$$

there is at least one j such that $g_j(x) \leqslant x_j$. In particular, since $\ell(x^i) = i$, this implies that for each x^i,

$$x_i^i - g_i\big(x^i\big) = \max_j \big(x_j^i - g_j\big(x^i\big)\big) \geqslant 0.$$

Since $size(\mathscr{T}) < \delta$ we have that, for all i, $\bar{d}(x^1, x^i) < \delta$. From the bound (1) in Lemma 2 (note in this case $M = 1$ and $n = N + 1$), we find that for all i, j,

$$\big|x_j^1 - x_j^i\big| < 2^{N+2}\delta \leqslant 2^{N+2}\epsilon_1 \leqslant \epsilon_0. \tag{3}$$

By the same logic, we have that for all i, j,

$$\big|g_j\big(x^1\big) - g_j\big(x^i\big)\big| < 2^{N+2}\epsilon_1 \leqslant \epsilon_0. \tag{4}$$

Consequently, we have that

$$x_j^1 + \epsilon_0 > x_j^i \quad \text{and} \quad -g_j\big(x^i\big) < \epsilon_0 - g_j\big(x^1\big)$$

which, in turn, implies that

$$2\epsilon_0 + x_j^1 - g_j\big(x^1\big) > x_j^i - g_j\big(x^i\big)$$

for all i and j. In particular, this implies that the following list of inequalities hold (simply let $i = j$ and run through all i):

$$\begin{array}{lll} 2\epsilon_0 + x_1^1 - g_1\big(x^1\big) & > x_1^1 - g_1\big(x^1\big) & \geqslant 0, \\ 2\epsilon_0 + x_2^1 - g_2\big(x^1\big) & > x_2^2 - g_2\big(x^2\big) & \geqslant 0, \\ \quad\vdots & \quad\vdots & \\ 2\epsilon_0 + x_{N+1}^1 - g_{N+1}\big(x^1\big) & > x_{N+1}^{N+1} - g_{N+1}\big(x^{N+1}\big) & \geqslant 0. \end{array}$$

Summing down each column yields the following inequality:

$$2\epsilon_0(N+1) + \sum_{i=1}^{N+1} x_i^1 - \sum_{i=1}^{N+1} g_i\big(x^1\big) > \sum_{i=1}^{N+1} \big(x_i^i - g_i\big(x^i\big)\big) \geqslant 0.$$

Now we recall that for all i, $x_i^i - g_i(x^i) \geqslant 0$ and

$$\sum_{i=1}^{N+1} x_i^1 - \sum_{i=1}^{N+1} g_i\big(x^1\big) = 1 - 1 = 0.$$

Consequently,

$$\begin{aligned}
2\epsilon_0(N+1) &= 2\epsilon_0(N+1) + \sum_{i=1}^{N+1} x_i^1 - \sum_{i=1}^{N+1} g_i\big(x^1\big) \\
&> \sum_{i=1}^{N+1} \big(x_i^i - g_i\big(x^i\big)\big) \\
&= \sum_{i=1}^{N+1} \big|x_i^i - g_i\big(x^i\big)\big|.
\end{aligned}$$

Using (3) and (4) and the continuity of g, for all i, we have that:

$$\big|x_i^1 - g_i\big(x^1\big)\big| \leqslant \big|x_i^1 - x_i^i\big| + \big|x_i^i - g_i\big(x^i\big)\big| + \big|g_i\big(x^i\big) - g_i\big(x^1\big)\big| < 2\epsilon_0 + \big|x_i^i - g_i\big(x^i\big)\big|.$$

Hence,

$$\begin{aligned}
\bar{d}\big(x^1, g\big(x^1\big)\big) = \sum_{i=1}^{N+1} \frac{|x_i^1 - g_i(x^1)|}{2^i(1 + |x_i^1 - g_i(x^1)|)} &\leqslant \sum_{i=1}^{N+1} \big|x_i^1 - g_i\big(x^1\big)\big| \\
&< \sum_{i=1}^{N+1} \big(2\epsilon_0 + \big|x_i^i - g_i\big(x^i\big)\big|\big) \\
&< 4(N+1)\epsilon_0 \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Let $y = (x_1^1, x_2^1, \ldots, x_N^1, 0, 0, 0, \ldots)$. We see that

$$\begin{aligned}
\sum_{i=1}^{N} \frac{|y_i - f_i(y)|}{2^i(1 + |y_i - f_i(y)|)} &= \sum_{i=1}^{N} \frac{|y_i - g_i(y)|}{2^i(1 + |y_i - g_i(y)|)} \\
&= \sum_{i=1}^{N} \frac{|x_i^1 - g_i(x^1)|}{2^i(1 + |x_i^1 - g_i(x^1)|)} \\
&\leqslant \sum_{i=1}^{N+1} \frac{|x_i^1 - g_i(x^1)|}{2^i(1 + |x_i^1 - g_i(x^1)|)} \\
&< \frac{\epsilon}{2}.
\end{aligned} \tag{5}$$

From (2) and (5), we have

$$\bar{d}\big(y, f(y)\big) = \sum_{i=1}^{\infty} \frac{|y_i - f_i(y)|}{2^i(1 + |y_i - f_i(y)|)}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \frac{|y_i - f_i(y)|}{2^i(1 + |y_i - f_i(y)|)} + \sum_{i=N+1}^{\infty} \frac{|y_i - f_i(y)|}{2^i(1 + |y_i - f_i(y)|)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

Therefore, y is the desired ϵ-fixed point. □

Notice that the construction of the $\epsilon/2$ fixed point in F^N in the proof above is identical to the construction of an $\epsilon/2$ fixed point for an arbitrary continuous function on Δ^N, because of the isometry between the two sets. This construction, in conjunction with Lemmas 1 and 2, provides a proof of the Brouwer Fixed Point Theorem on the finite-dimensional simplex, which is similar to constructions found in, e.g., [9].

5. Schauder's theorem

A well-known infinite-dimensional fixed point theorem that holds for normed spaces is Schauder's theorem [3,6]:

Schauder's theorem. *Suppose that X is a compact convex subset of the normed space G. If $f : X \to X$ is continuous, then f has a fixed point.*

In this section we show how our proof of Theorem 1 can be used to prove Schauder's theorem for the case where X is infinite-dimensional. (The finite-dimensional version of Schauder's theorem reduces to the Brouwer Fixed Point Theorem.)

Recall that a space X has the *fixed point property* if every continuous function $f : X \to X$ has a fixed point. Note that this is a topological property, so if X is homeomorphic to Y then Y also has the fixed point property. We will establish Schauder's theorem by noting that Δ_0^∞ is homeomorphic to every infinite-dimensional compact convex subset of a normed space.

Define the vector space H to be

$$H = \left\{ x \in \mathbb{R}^\infty \;\middle|\; \sum_{i=1}^{\infty} \frac{|x_i|}{2^i} < \infty \right\}.$$

It is not difficult to see that H is indeed a vector space. Furthermore, we see that $\|x\| = \sum_{i=1}^{\infty} \frac{|x_i|}{2^i}$ defines a norm on this space and the closure of the standard simplex in H is

$$\Delta_0^H = \left\{ x \in H \;\middle|\; \sum_{i=1}^{\infty} x_i \leqslant 1 \text{ and } 0 \leqslant x_i \leqslant 1 \right\}.$$

Proposition 1. *Δ_0^∞ is homeomorphic to Δ_0^H.*

The proof of this proposition is trivial using the homeomorphism $g : \Delta_0^\infty \to \Delta_0^H$ being $g(x) = x$. Note that Δ_0^H is an infinite-dimensional compact convex subset of a normed space H. Now consider the following proposition [4]:

Proposition 2. *Every infinite-dimensional compact convex subset of a normed space is homeomorphic to the Hilbert Cube.*

The significance of these propositions is that *every* infinite-dimensional compact convex subset of a normed space is homeomorphic to Δ_0^∞. Thus Theorem 1 implies the infinite-dimensional case of Schauder's theorem.

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