



ELSEVIER



CrossMark

Available online at www.sciencedirect.com**ScienceDirect**

Nuclear Physics B 880 (2014) 321–342

**NUCLEAR
PHYSICS B**www.elsevier.com/locate/nuclphysb

Holonomies of gauge fields in twistor space 6: Incorporation of massive fermions

Yasuhiro Abe

Cereja Technology Co. Ltd., 1-13-14 Mukai-Bldg. 3F, Sekiguchi Bunkyo-ku, Tokyo 112-0014, Japan

Received 20 November 2013; received in revised form 8 January 2014; accepted 15 January 2014

Available online 18 January 2014

Abstract

Following the previous paper arXiv:1205.4827, we formulate an S-matrix functional for massive fermion ultra-helicity-violating (UHV) amplitudes, i.e., scattering amplitudes of positive-helicity gluons and a pair of massive fermions. The S-matrix functional realizes a massive extension of the Cachazo–Svrcek–Witten (CSW) rules in a functional language. Mass-dimension analysis implies that interactions among gluons and massive fermions should be decomposed into three-point massive fermion subamplitudes. Namely, such interactions are represented by combinations of three-point UHV and next-to-UHV (NUHV) vertices. This feature is qualitatively different from the massive scalar amplitudes where the number of involving gluons can be arbitrary.

© 2014 The Author. Published by Elsevier B.V. Open access under [CC BY license](http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP³.

1. Introduction

Recent developments on the calculation of scattering amplitudes in gauge theories within a framework of the spinor-helicity formalism are remarkable. Technically and practically these developments can be recapitulated by a field theoretic prescription, either the CSW rules [1] or the BCFW recursion relation [2,3].

Incorporation of massive fermions along the lines of these developments, a main subject of the present paper, has been studied in various methods, e.g., by extension of BCFW-like recursion relations [4–7] (see also recent progress [8,9]), by use of supersymmetric Ward identities

E-mail address: abe@cereja.co.jp.

[10,11], and by developing massive versions of the CSW rules [12,13]. (The massive CSW rules are initially developed in [14,15] for the incorporation of massive scalars into the original CSW rules.) There also exists a more general approach which seems to connect all these methods to one another; this is referred to as the “on-shell constructibility” method [16,17]. In this general approach massive deformation of massless amplitudes is carried out by use of the massive spinor-helicity formalism [18]. The on-shell constructibility method is applicable to massive fermions as well as the other types of massive particles, i.e., scalars and vector bosons [16,17].

In a recent paper [19] we construct an S-matrix functional for massive scalar amplitudes, i.e., amplitudes of an arbitrary number of gluons and a pair of massive scalar particles, in the recently proposed holonomy formalism [20,21]. In [19] we show that the on-shell constructibility of massive scalar amplitudes can naturally be implemented by means of off-shell continuation of Nair’s superamplitude method [22] once a massive holonomy operator is defined. We also show that a careful study of the massive holonomy operator leads to a novel color structure of the massive scalar amplitudes. A main purpose of the present paper is to extend these results to the case of massive fermions in the same framework.

The organization of this paper is as follows. In Section 2 we review essential results of the previous paper [19] on the massive scalar amplitudes in the holonomy formalism. We first write down the definitions of the massless and massive holonomy operators. We then present an S-matrix functional for the massive scalar tree amplitudes, focusing on the amplitudes of positive-helicity gluons and a pair of massive scalars, the so-called ultra-helicity-violating (UHV) massive scalar amplitudes.

In Section 3 we apply these results to the massive fermion amplitudes. We begin with clarifying our notation of fermions in the spinor-helicity formalism. We then apply off-shell continuation of Nair’s superamplitude method such that it lead to tree-level massive fermion UHV amplitudes (scattering amplitudes of an arbitrary number of positive-helicity gluons and a pair of massive fermions) in a form that is consistent with the literature. We also obtain an S-matrix functional for the massive fermion UHV amplitudes.

Mass-dimension analysis on the massive fermion UHV amplitudes implies that the number of gluons involving the amplitudes should be one. This condition means that interactions among gluons and massive fermions should be decomposed into the three-point UHV vertices (with a positive-helicity gluon) and the three-point next-to-UHV (NUHV) vertices (with a negative-helicity gluon). In Section 4, for clarification of our arguments, we first construct the massive fermion NUHV amplitudes for arbitrary number of gluons and see how the above constraint is imposed. Lastly, we present a brief conclusion.

2. S-matrix for massive scalar UHV amplitudes

In this section we review the formulation of an S-matrix functional for the massive scalar UHV amplitudes, i.e., scattering amplitudes of an arbitrary number of positive-helicity gluons and a pair of massive scalar particles, at tree level in the framework of the holonomy formalism, recapitulating the results of the recent study [19].

2.1. The massless holonomy operator

The original “massless” holonomy operator for gluons is defined as [20]

$$\Theta_{R,\gamma}^{(A)}(u) = \text{Tr}_{R,\gamma} \text{P exp} \left[\sum_{m \geq 2} \oint_{\gamma} \underbrace{A \wedge A \wedge \dots \wedge A}_m \right] \tag{2.1}$$

where A is called the comprehensive gluon field. This is expressed as a bialgebraic operator

$$A = g \sum_{1 \leq i < j \leq n} A_{ij} \omega_{ij}, \tag{2.2}$$

$$A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}, \tag{2.3}$$

$$\omega_{ij} = d \log(u_i u_j) = \frac{d(u_i u_j)}{(u_i u_j)} \tag{2.4}$$

where g is a dimensionless coupling constant and u_i ($i = 1, 2, \dots, n$) denotes the two-component spinor momentum for the i -th gluon. $(u_i u_j)$ represents a scalar product of the spinor momenta:

$$(u_i u_j) = \epsilon_{AB} u_i^A u_j^B \equiv u_i \cdot u_j \tag{2.5}$$

where the indices A, B take a value of $(1, 2)$ and ϵ_{AB} denotes the rank-2 Levi-Civita tensor. The physical (creation) operators of positive and negative helicity gluons are given by $a_i^{(+)}$ and $a_i^{(-)}$ in (2.3), respectively. These form the ladder part of the $SL(2, \mathbf{C})$ algebra, satisfying

$$[a_i^{(+)}, a_j^{(-)}] = 2a_i^{(0)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(+)}] = a_i^{(+)} \delta_{ij}, \quad [a_i^{(0)}, a_j^{(-)}] = -a_i^{(-)} \delta_{ij} \tag{2.6}$$

where Kronecker’s deltas show that the non-zero commutators are obtained only when $i = j$. The remaining commutators, those expressed otherwise, all vanish.

The color degree of freedom can be attached to the physical operators $a_i^{(\pm)}$ as

$$a_i^{(\pm)} = t^{c_i} a_i^{(\pm)c_i} \tag{2.7}$$

where t^{c_i} ’s are the generators of the $U(N)$ gauge group in the R -representation. The symbol P in (2.1) denotes an ordering of the numbering indices. The meaning of the action of P on the exponent of (2.1) can be written explicitly as

$$\begin{aligned} & \text{P} \sum_{m \geq 2} \oint_{\gamma} \underbrace{A \wedge \dots \wedge A}_m \\ &= \sum_{m \geq 2} \oint_{\gamma} A_{12} A_{23} \dots A_{m1} \omega_{12} \wedge \omega_{23} \wedge \dots \wedge \omega_{m1} \\ &= \sum_{m \geq 2} \frac{1}{2^{m+1}} \sum_{(h_1, h_2, \dots, h_m)} (-1)^{h_1+h_2+\dots+h_m} a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \dots \otimes a_m^{(h_m)} \\ & \quad \times \oint_{\gamma} \omega_{12} \wedge \dots \wedge \omega_{m1} \end{aligned} \tag{2.8}$$

where $h_i = \pm = \pm 1$ ($i = 1, 2, \dots, m$) denotes the helicity of the i -th gluon.

In (2.1), γ represents a closed path on the physical configuration space

$$\mathcal{C}^{(A)} = \mathbf{C}^n / \mathcal{S}_n \tag{2.9}$$

on which the set of gluon operators are defined. \mathcal{S}_n denotes the rank- n symmetric group. The fundamental homotopy group of $\mathcal{C}^{(A)}$ is given by the braid group, $\Pi_1(\mathcal{C}^{(A)}) = \mathcal{B}_n$. The trace

Tr_γ in (2.1) is taken over the generators of the braid group (or the elements of Iwahori–Hecke algebra, see [21] for details on this point) and is called the braid trace. It can be realized by a sum over the permutations of the numbering indices, i.e.,

$$\text{Tr}_\gamma = \sum_{\sigma \in \mathcal{S}_{n-1}} \tag{2.10}$$

where each of the permutations is denoted by $\sigma = \begin{pmatrix} 23 \dots n \\ \sigma_2 \sigma_3 \dots \sigma_n \end{pmatrix}$. Explicitly, the braid trace Tr_γ over the exponent (2.8) can be expressed as

$$\begin{aligned} \text{Tr}_\gamma \text{P} \sum_{m \geq 2}^\infty \oint_\gamma \underbrace{A \wedge \dots \wedge A}_m \\ = \sum_{m \geq 2} \sum_{\sigma \in \mathcal{S}_{m-1}} \oint_\gamma A_{1\sigma_2} A_{\sigma_2\sigma_3} \dots A_{\sigma_m 1} \omega_{1\sigma_2} \wedge \omega_{\sigma_2\sigma_3} \wedge \dots \wedge \omega_{\sigma_m 1}. \end{aligned} \tag{2.11}$$

A quintessential point in the holonomy formalism is the mathematical fact that a linear representation of a braid group is equivalent to a monodromy representation of the Knizhnik–Zamolodchikov (KZ) equation. Such a monodromy representation can be given by a holonomy of the so-called KZ connection. The comprehensive gauge one-form (2.2) is a variant of the KZ connection in a sense that it satisfies the infinitesimal braid relations to ensure the integrability of the comprehensive gauge one-form. For details on these fundamental issues, see [20,21]; see also [23,24] for recent studies on KZ connections and two-dimensional holonomies.

2.2. The massive holonomy operator

Following these basic ideas, we now construct a holonomy operator for a comprehensive gauge one-form which incorporates the helicity-zero (or spin-zero) scalar operators $a_i^{(0)}$. Such a one-form can be defined as

$$B = \sum_{1 \leq i < j \leq n} B_{ij} \omega_{ij}, \tag{2.12}$$

$$B_{ij} = g(a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}) + a_i^{(0)} \otimes a_j^{(0)} \tag{2.13}$$

where g denotes the dimensionless gauge coupling constant as before. The holonomy operator for B is then defined and calculated as [19]

$$\begin{aligned} \Theta_{R,\gamma}^{(B)}(u) &= \text{Tr}_{R,\gamma} \text{P} \exp \left[\sum_{r \geq 2} \oint_\gamma \underbrace{B \wedge B \wedge \dots \wedge B}_r \right] \\ &= \exp \left[\sum_{r \geq 3} \sum_{(h_2, h_3, \dots, h_{r-1})} g^{r-2} (-1)^{h_2 h_3 \dots h_{r-1}} \text{Tr}(t^{c_2} t^{c_3} \dots t^{c_{r-1}}) \right. \\ &\quad \left. \times \frac{a_1^{(0)} \otimes a_2^{(h_2)c_2} \otimes \dots \otimes a_{r-1}^{(h_{r-1})c_{r-1}} \otimes a_r^{(0)}}{(12)(23) \dots (r-1r)(r1)} \right] \end{aligned} \tag{2.14}$$

where we use the abbreviated notation $(i j) \equiv (u_i u_j)$. In the above expression the massive scalar operators are specified by a pair of the numbering indices $(1, r)$. Accordingly $h_i = \pm = \pm 1$ ($i = 2, 3, \dots, r - 1$) denotes the helicity of the i -th gluon.

For an n -particle system, the physical configuration space is given by

$$\mathcal{C}^{(B)} = \frac{\mathbf{C}^{n-2}}{\mathcal{S}_{n-2}} \otimes \mathbf{C}^2 = \mathbf{C}^n / \mathcal{S}_{n-2} \tag{2.15}$$

as opposed to the purely gluonic case in (2.9). Consequently the braid trace Tr_γ is taken over the numbering elements $\{\sigma_2, \sigma_3, \dots, \sigma_{r-1}, \tau_r\} = \{2, 3, \dots, r\}$, satisfying the P ordering

$$\sigma_2 < \sigma_3 < \dots < \sigma_{r-1}. \tag{2.16}$$

The braid trace Tr_γ in (2.14) can then be represented by a ‘‘homogenous’’ sum

$$\sum_{\{\sigma, \tau\}} = \sum_{\tau_r=2}^r \sum_{\sigma \in \mathcal{S}_{r-2}} \tag{2.17}$$

where $r = 3, 4, \dots, n$.

2.3. Comments on the color structure of the massive holonomy operator

As mentioned in the introduction, the color structure of the massive holonomy operator is qualitatively different from that of the massless holonomy operator in terms of the final realization of the braid trace. In the original massless case, the sum over the permutations of the numbering indices (2.10) is explicit while in the massive case, as shown in (2.14), there appears no explicit sums over permutations. As discussed in [19], this is due to the fact that the calculation of (2.14) is carried out by use of the relation

$$\sum_{\{\sigma, \tau\}} \oint_\gamma \omega_{1\sigma_2} \wedge \omega_{\sigma_2\sigma_3} \wedge \dots \wedge \omega_{\sigma_{r-1}\tau_r} \wedge \omega_{\tau_r 1} = \int_{\gamma_{1r}} \omega_{12} \wedge \dots \wedge \omega_{r-1r} \int_{\gamma_{r1}} \omega_{r1} \tag{2.18}$$

where γ_{1r} and γ_{r1} denote open paths on $\mathcal{C}^{(B)}$ which compose the closed path, $\gamma = \gamma_{1r}\gamma_{r1}$.

The relation (2.18) is a mathematical equation regarding products of iterated integrals over the logarithmic one-forms ω_{ij} ’s. Use of this relation implies that the homogeneous sum (2.17) or the braid trace for $\Theta_{R,\gamma}^{(B)}(u)$ can be absorbed into the expression of (2.14).

2.4. Supersymmetrization

In order to relate the massive holonomy operator to massive scalar amplitudes, we need to define the massive holonomy operator in supertwistor space, as in the case of gluon amplitudes. In the massless case the supersymmetrization is carried out by replacing the physical operators $a_i^{(\pm)}$ in (2.3) with

$$a_i^{(\pm)}(x, \theta) = \int d\mu(p_i) a_i^{(\pm)}(\xi_i) e^{ix_\mu p_i^\mu} \Big|_{\xi_i^\alpha = \theta_A^\alpha u_i^A} \tag{2.19}$$

where $d\mu(p_i)$ is the ordinary Lorentz-invariant measure for the null momentum p_i^μ . This is also called the Nair measure when it is written in terms of the spinor momenta [22]. The operators $a_i^{(\hat{h}_i)}(x, \theta)$ are physical operators that are defined in a four-dimensional $\mathcal{N} = 4$ chiral super-space (x, θ) where $x_{\dot{A}A} = (\sigma^\mu)_{\dot{A}A} x_\mu$ denote coordinates of four-dimensional spacetime and θ_A^α ($A = 1, 2; \alpha = 1, 2, 3, 4$) denote their chiral superpartners with $\mathcal{N} = 4$ extended supersymmetry.

(Here σ^μ is given by $\sigma^\mu = (\mathbf{1}, \sigma^i)$, with σ^i ($i = 1, 2, 3$) being the Pauli matrices.) These coordinates emerge from homogeneous coordinates of the supertwistor space $\mathbf{CP}^{3|4}$, represented by $(u^A, v_{\dot{A}}, \xi^\alpha)$, that satisfy the so-called supertwistor conditions

$$v_{\dot{A}} = x_{\dot{A}A} u^A, \quad \xi^\alpha = \theta_A^\alpha u^A. \tag{2.20}$$

Upon the supersymmetrization, there arise superpartners of the gluon creation operators. The full supermultiplets can be expressed as

$$\begin{aligned} a_i^{(+)}(\xi_i) &= a_i^{(+)}, \\ a_i^{(+\frac{1}{2})}(\xi_i) &= \xi_i^\alpha a_{i\alpha}^{(+\frac{1}{2})}, \\ a_i^{(0)}(\xi_i) &= \frac{1}{2} \xi_i^\alpha \xi_i^\beta a_{i\alpha\beta}^{(0)}, \\ a_i^{(-\frac{1}{2})}(\xi_i) &= \frac{1}{3!} \xi_i^\alpha \xi_i^\beta \xi_i^\gamma \epsilon_{\alpha\beta\gamma\delta} a_i^{(-\frac{1}{2})\delta}, \\ a_i^{(-)}(\xi_i) &= \xi_i^1 \xi_i^2 \xi_i^3 \xi_i^4 a_i^{(-)}. \end{aligned} \tag{2.21}$$

These operators are characterized by the number of $\xi_i^\alpha = \theta_A^\alpha u_i^A$ or the degrees of homogeneities in u_i^A 's. The number is in one-to-one correspondence with the helicity $\hat{h}_i = (0, \pm\frac{1}{2}, \pm 1)$ of the superpartner of interest. This can be easily seen from the definition of the helicity operator

$$\hat{h}_i = 1 - \frac{1}{2} u_i^A \frac{\partial}{\partial u_i^A}. \tag{2.22}$$

Notice that we put the hat of \hat{h}_i simply to distinguish it from the non-supersymmetric version $h_i = \pm$.

Use of the supermultiplets (2.21) enables us to define gluon operators without introducing the conventional polarization/helicity vectors. This method is known as Nair's superamplitude method [22]. In [19] we show that this prescription is also applicable to the system of gluons and a pair of massive scalars. In the following we review such a massive extension.

2.5. Off-shell continuation of Nair's superamplitude method

In carrying out the supersymmetrization of the massive holonomy operator (2.14), we need to consider off-shell version of the scalar operator $a_i^{(0)}(\xi_i)$ in (2.21). In the massive spinor-helicity formalism [18], off-shell continuation of the null spinor-momentum u^A is defined as

$$u^A \longrightarrow \hat{u}^A = u^A + \frac{m}{(u\eta)} \eta^A \tag{2.23}$$

where η^A is a reference null spinor and m denotes the mass of the massive spinor momentum \hat{u}^A . The complex conjugate of (2.23) is also necessary to define the full massive four-momentum $\hat{p}^{A\dot{A}} = \hat{u}^A \hat{u}^{\dot{A}}$. We then introduce another set of supertwistor variables $(w^A, \pi_{\dot{A}}, \zeta^\alpha)$ such that the supertwistor conditions

$$\pi_{\dot{A}} = x_{\dot{A}A} w^A = x_{\dot{A}A} \frac{m}{(u\eta)} \eta^A, \quad \zeta^\alpha = \theta_A^\alpha w^A = \theta_A^\alpha \frac{m}{(u\eta)} \eta^A \tag{2.24}$$

are satisfied ($\alpha = 1, 2, 3, 4$).

Using the new Grassmann variable $\zeta_i^\alpha = \theta_A^\alpha \frac{m}{(u_i \eta_i)} \eta_i^A$, the off-shell continuation of the scalar operator $a_i^{(0)}(\xi_i)$ can be defined as

$$a_i^{(0)}(\xi_i) \longrightarrow a_i^{(0)}(\xi_i, \zeta_i) = \xi_i^1 \xi_i^2 \xi_i^3 \xi_i^4 a_i^{(0)} \tag{2.25}$$

where we shall specify the numbering index to $i = 1, n$ for massive scalars. Notice that the degrees of homogeneities in u_i 's remains the same in the off-shell continuation (2.25). Thus we can naturally interpret $a_i^{(0)}(\xi_i, \zeta_i)$ as massive scalar operators. The chiral superspace representation of the massive operators can be expressed as

$$a_i^{(0)}(x, \theta) = \int d\mu(\hat{p}_i) a_i^{(0)}(\xi_i, \zeta_i) e^{ix_\mu \hat{p}_i^\mu} \Big|_{\xi_i^\alpha = \theta_A^\alpha u_i^A, \zeta_i^\alpha = \theta_A^\alpha w_i^A}, \tag{2.26}$$

$$\hat{p}_i^\mu = p_i^\mu + \frac{m^2}{2(p_i \cdot \eta_i)} \eta_i^\mu, \tag{2.27}$$

$$w_i^A = \frac{m}{(u_i \eta_i)} \eta_i^A. \tag{2.28}$$

A supersymmetric version of the massive holonomy operator can be constructed by use of the expressions (2.19) and (2.26) for physical operators of gluons and massive scalars, respectively. In other words, we can obtain the supersymmetric massive holonomy operator $\Theta_{R,\gamma}^{(B)}(u; x, \theta)$ out of $\Theta_{R,\gamma}^{(B)}(u)$ in (2.14) by replacing $\{a_i^{(\pm)}, a_j^{(0)}\}$ with $\{a_i^{(\pm)}(x, \theta), a_j^{(0)}(x, \theta)\}$ where $i = 2, 3, \dots, r - 1$ and $j = 1, r$.

2.6. Choice of reference spinors and functional derivation of the UHV vertex

We now specify the reference null-vectors for the massive scalars as

$$\eta_1^\mu = p_1^\mu, \quad \eta_n^\mu = p_1^\mu. \tag{2.29}$$

Namely, we choose a specific parametrization of \hat{p}_1 and \hat{p}_n by

$$\hat{p}_1^\mu = p_1^\mu + \frac{m^2}{2(p_1 \cdot p_n)} p_n^\mu, \quad \hat{p}_n^\mu = p_n^\mu + \frac{m^2}{2(p_n \cdot p_1)} p_1^\mu. \tag{2.30}$$

Notice that this parametrization is qualitatively different from off-shell prescription for virtual gluons where we set all reference null-vectors identical.

We now introduce a generating functional

$$\mathcal{F}_{\text{UHV}}^{(\text{vertex})}[a^{(\pm)c}, a^{(0)}] = \exp \left[i \int d^4x d^8\theta \Theta_{R,\gamma}^{(B)}(u; x, \theta) \right]. \tag{2.31}$$

In terms of this generating functional the massive scalar UHV vertex can be obtained as

$$\begin{aligned} &\mathcal{V}_{\text{UHV}}^{(\bar{\phi}_1 g_2^+ \dots g_{n-1}^+ \phi_n)}(x) \\ &= \frac{\delta}{\delta a_1^{(0)}} \otimes \frac{\delta}{\delta a_2^{(+)}} \otimes \frac{\delta}{\delta a_3^{(+)}} \otimes \dots \otimes \frac{\delta}{\delta a_{n-1}^{(+)}} \otimes \frac{\delta}{\delta a_n^{(0)}} \mathcal{F}_{\text{UHV}}^{(\text{vertex})}[a^{(\pm)c}, a^{(0)}] \Big|_{a^{(\pm)c} = a^{(0)} = 0} \end{aligned} \tag{2.32}$$

where $a^{(\pm)c}$ and $a^{(0)}$ denote sets of source functions of gluons and massive scalars, respectively. $\mathcal{V}_{\text{UHV}}^{(\bar{\phi}_1 g_2^+ \dots g_{n-1}^+ \phi_n)}(x)$ denotes the massive scalar UHV vertex in the x -space representation. Explicitly these are expressed as

$$\mathcal{V}_{\text{UHV}}^{(\tilde{\phi}_1 g_2^+ \dots g_{n-1}^+ \phi_n)}(x) \equiv \mathcal{V}_{\text{UHV}}^{(\tilde{\phi}_1 \phi_n)}(x) = \int d\mu(\hat{p}_1) \prod_{i=2}^{n-1} d\mu(p_i) d\mu(\hat{p}_n) \mathcal{V}_{\text{UHV}}^{(\tilde{\phi}_1 \phi_n)}(u, \bar{u}), \tag{2.33}$$

$$\mathcal{V}_{\text{UHV}}^{(\tilde{\phi}_1 \phi_n)}(u, \bar{u}) = -i g^{n-2} (2\pi)^4 \delta^{(4)}\left(\hat{p}_1 + \sum_{i=2}^{n-1} p_i + \hat{p}_n\right) \hat{V}_{\text{UHV}}^{(\tilde{\phi}_1 \phi_n)}(u), \tag{2.34}$$

$$\hat{V}_{\text{UHV}}^{(\tilde{\phi}_1 \phi_n)}(u) = \text{Tr}(t^{c_2} t^{c_3} \dots t^{c_{n-1}}) \frac{m^2 (n1)^2}{(12)(23) \dots (n-1n)(n1)}. \tag{2.35}$$

The integral measures in (2.33) are given by the ordinary Lorentz invariant measures

$$\begin{aligned} d\mu(\hat{p}_1) &= \frac{d^3 \hat{p}_1}{(2\pi)^3} \frac{1}{2\sqrt{|\vec{\hat{p}}_1|^2 + m^2}}, & d\mu(p_i) &= \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2\sqrt{|\vec{p}_i|^2}}, \\ d\mu(\hat{p}_n) &= \frac{d^3 \hat{p}_n}{(2\pi)^3} \frac{1}{2\sqrt{|\vec{\hat{p}}_n|^2 + m^2}}. \end{aligned} \tag{2.36}$$

The momenta of the massive scalars are denoted by \hat{p}_1 and \hat{p}_n , satisfying

$$\hat{p}_1^\mu \hat{p}_{1\mu} = \hat{p}_n^\mu \hat{p}_{n\mu} = m^2. \tag{2.37}$$

In deriving the explicit form (2.35), we use the specified reference spinors (2.29) and the Grassmann integral

$$\begin{aligned} &\int d^8\theta \xi_1^1 \xi_1^2 \xi_1^3 \xi_1^4 \xi_n^1 \xi_n^2 \xi_n^3 \xi_n^4 \\ &= (n1)^3 \int d^2\theta \frac{m}{(u_1 \eta_1)} \eta_1^A \theta_A \frac{m}{(u_n \eta_n)} \eta_n^B \theta_B \\ &= (n1)^3 \frac{m^2}{(u_1 u_n)(u_n u_1)} \underbrace{\int d^2\theta u_n^A \theta_A u_1^B \theta_B}_{=(u_n u_1)} = m^2 (n1)^2. \end{aligned} \tag{2.38}$$

This Grassmann integral guarantees that only the UHV-type vertices survive upon the evaluation of functional derivatives in (2.32). This structure is analogous to the CSW rules, or the MHV rules, in the calculation of gluon amplitudes [1].

2.7. The UHV rules and the massive scalar UHV tree amplitudes

The UHV rules can be understood as a massive extension of the MHV rules to the massive scalar UHV amplitudes. Namely, the UHV rules state that the massive scalar UHV amplitudes can be obtained by connecting the UHV vertices with massive scalar propagators. This is analogous to the MHV rules in structure but a main difference exists, that is, for the massive scalar UHV amplitudes we have freedom to connect an arbitrary number of the UHV vertices since such procedure, contrary to the MHV rules, does not alter the overall helicity configuration of the UHV amplitudes. In compensation, we no longer take a sum over permutations of gluons for the massive UHV amplitudes and non-UHV amplitudes in general. This peculiar nature of the massive amplitudes is due to the definition of massive holonomy operator (2.14).

Probably the easiest way to understand the UHV rules is to visualize the full expression for the massive scalar UHV amplitudes, see Fig. 1. As mentioned above, we have a freedom to

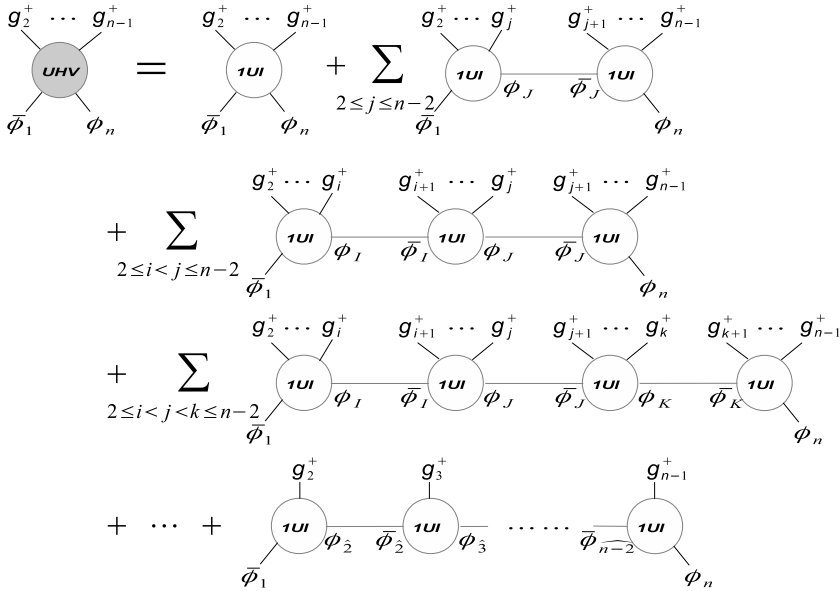


Fig. 1. The UHV rules for the massive scalar UHV amplitudes — “1UI” stands for a 1 UHV irreducible diagram which is equivalent to the massive scalar UHV vertex. The vertices are connected with one another by massive scalar propagators.

arbitrarily add UHV vertices to the UHV amplitudes as long as the total number of the scattering particles is preserved. In this sense, the UHV vertices can be interpreted as 1 UHV irreducible (1UI) subamplitudes. This is why we denote each of the UHV vertices by “1UI” in Fig. 1.

Using the UHV rules, we can then obtain an explicit form of the massive scalar UHV amplitudes as follows.

$$\mathcal{A}_{\text{UHV}}^{(\bar{\phi}_1 g_2^+ \dots g_{n-1}^+ \phi_n)}(x) \equiv \mathcal{A}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(x) = \int d\mu(\hat{p}_1) \prod_{i=2}^{n-1} d\mu(p_i) d\mu(\hat{p}_n) \hat{\mathcal{A}}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u, \bar{u}), \quad (2.39)$$

$$\hat{\mathcal{A}}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u, \bar{u}) = -i g^{n-2} (2\pi)^4 \delta^{(4)}\left(\hat{p}_1 + \sum_{i=2}^{n-1} p_i + \hat{p}_n\right) \hat{A}_{\text{UHV}}^{(\phi_n \bar{\phi}_1)}(u), \quad (2.40)$$

$$\hat{A}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u) = \text{Tr}(t^{c_2} t^{c_3} \dots t^{c_{n-1}}) \hat{C}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u). \quad (2.41)$$

The color-stripped holomorphic massive scalar UHV amplitudes $\hat{C}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u)$ is given by

$$\hat{C}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u) = \hat{C}_{\text{UHV}}^{(\bar{\phi}_1 g_2^+ g_3^+ \dots g_{n-1}^+ \phi_n)}(u) = \frac{m^2(n1)}{(12)(23) \dots (n1)} (\widehat{n1}) \quad (2.42)$$

where

$$\begin{aligned} (\widehat{n1}) &= (n1) + \sum_{j=2}^{n-2} \frac{(J1)}{(jJ)} \frac{m^2(jj+1)}{\hat{p}_J^2 - m^2} \frac{(nJ)}{(Jj+1)} \\ &+ \sum_{2 \leq i < j \leq n-1} \frac{(I1)}{(iI)} \frac{m^2(ii+1)}{\hat{p}_I^2 - m^2} \frac{(JI)}{(Ii+1)(jJ)} \frac{m^2(jj+1)}{\hat{p}_J^2 - m^2} \frac{(nJ)}{(Jj+1)} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{2 \leq i < j < k \leq n-1} \left[\frac{(I1) m^2(ii+1)}{(iI) \hat{P}_I^2 - m^2} \frac{(JI)}{(Ii+1)(jJ)} \frac{m^2(jj+1)}{\hat{P}_J^2 - m^2} \right. \\
 & \times \left. \frac{(KJ)}{(Jj+1)(kK)} \frac{m^2(kk+1)}{\hat{P}_K^2 - m^2} \frac{(nK)}{(Kk+1)} \right] \\
 & + \dots \\
 & = - \left(1 \left| \prod_{j=2}^{n-2} \left[1 - \frac{m^2 |J|(jj+1)(J|)}{(\hat{P}_J^2 - m^2)(jJ)(Jj+1)} \right] \right| n \right). \tag{2.43}
 \end{aligned}$$

In the above expression, $(1|n) = (1n) = \epsilon_{AB} u_1^A u_n^B$ and the uppercase letters label a set of momenta running along each of the propagators. For example, \hat{P}_J^μ is defined as

$$\hat{P}_J^\mu = \hat{p}_1^\mu + p_2^\mu + p_3^\mu + \dots + p_j^\mu = p_j^\mu + \frac{m^2}{2(p_J \cdot \eta_J)} \eta_J^\mu \tag{2.44}$$

where p_j^μ is the on-shell partner of the off-shell momentum \hat{P}_J , η_J^μ is the corresponding reference null-vector, and m denotes the mass of the complex scalar particles $(\bar{\phi}_1, \phi_n)$. The spinor momenta $(u_J^A, \bar{u}_J^{\dot{A}})$ correspond to the null momentum p_j^μ and are defined as usual:

$$p_J^{A\dot{A}} = u_J^A \bar{u}_J^{\dot{A}}. \tag{2.45}$$

2.8. An S-matrix functional for the massive scalar UHV tree amplitudes

It is now straightforward to introduce an S-matrix functional for the massive scalar UHV tree amplitudes (2.39) by use of the supersymmetric massive holonomy operator $\Theta_{R,\gamma}^{(B)}(u; x, \theta)$:

$$\mathcal{F}_{\text{UHV}}[a^{(h)c}, a^{(0)}] = \hat{W}^{(0)}(x) \exp \left[i \int d^4x d^8\theta \Theta_{R,\gamma}^{(B)}(u; x, \theta) \right], \tag{2.46}$$

$$\begin{aligned}
 \hat{W}^{(0)}(x) &= \exp \left[- \int d\mu(\hat{P}_J) \left(\frac{\delta}{\delta a_J^{(0)}} \otimes \frac{\delta}{\delta a_{-J}^{(0)}} \right) e^{-i\hat{P}_J \cdot (x-y)} \right]_{y \rightarrow x(x^0 > y^0)} \\
 &= \exp \left[- \int \frac{d^4 \hat{P}_J}{(2\pi)^4} \frac{i}{\hat{p}_J^2 - m^2} \left(\frac{\delta}{\delta a_J^{(0)}} \otimes \frac{\delta}{\delta a_{-J}^{(0)}} \right) e^{-i\hat{P}_J \cdot (x-y)} \right]_{y \rightarrow x(x^0 > y^0)} \tag{2.47}
 \end{aligned}$$

where in the calculation of $\hat{W}^{(0)}(x)$ we take the limit $y \rightarrow x$, with the time ordering $x^0 > y^0$ being preserved. An explicit derivation of the x -space UHV massive scalar tree amplitudes is given by

$$\begin{aligned}
 & \frac{\delta}{\delta a_1^{(0)}} \otimes \frac{\delta}{\delta a_2^{(+)}} \otimes \frac{\delta}{\delta a_3^{(+)}} \otimes \dots \otimes \frac{\delta}{\delta a_{n-1}^{(+)}} \otimes \frac{\delta}{\delta a_n^{(0)}} \mathcal{F}_{\text{UHV}}[a^{(\pm)c}, a^{(0)}] \Big|_{a^{(\pm)c} = a^{(0)} = 0} \\
 & = \mathcal{A}_{\text{UHV}}^{(\bar{\phi}_1 g_2^+ \dots g_{n-1}^+ \phi_n)}(x) \tag{2.48}
 \end{aligned}$$

where $a^{(\pm)c}$ and $a^{(0)}$ represent generic source functions for gluons and massive scalars, respectively.

3. S-matrix for massive fermion UHV amplitudes

In the following sections, we consider the application of the above results to massive fermion amplitudes. In this section we focus on the massive fermion UHV amplitudes, i.e., the scattering amplitudes of an arbitrary number of positive-helicity gluons and a pair of massive fermions.

3.1. Notation

We first clarify our definition of fermion operators. In the spinor-helicity formalism fermions are represented by two-component spinors. (For recent reviews of two-component spinor techniques, see, e.g., [25,26].) The ordinary Dirac spinor is defined by

$$\Psi = \begin{pmatrix} \psi_L^A \\ \psi_R^{\dot{A}} \end{pmatrix} \tag{3.1}$$

where ψ_L^A ($A = 1, 2$) and $\psi_R^{\dot{A}}$ ($\dot{A} = 1, 2$) are the two-component Weyl spinors. Each of these is analogous to the spinor momentum u^A . Bearing in mind that its conjugate is denoted as $\bar{u}_{\dot{A}} = (u^A)^*$ in our notation, the conjugate of Ψ can be defined as

$$\bar{\Psi} = \Psi^\dagger \gamma^0 = ((\psi_L^A)^* \quad (\psi_R^{\dot{A}})^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\bar{\psi}_{R\dot{A}} \quad \bar{\psi}_{L A}) \tag{3.2}$$

where we use the chiral representation of the gamma matrix γ^0 .

For both massive and massless fermions, one can naturally assign helicity $-\frac{1}{2}$ to ψ_L and helicity $+\frac{1}{2}$ to ψ_R , respectively. (As we shall see in a moment, one can make an off-shell continuation of massless fermions such that the helicity of fermions is not altered; see (3.18)–(3.21). Using such massive deformation, one can and should identify the ‘helicity’ of massive fermions with the helicity of the corresponding massless fermions.) In the helicity-spinor formalism, physical information of any particle is encoded into its helicity and the numbering index. In the framework of the holonomy formalism this is represented by the physical operators $a_i^{(\pm)}(x, \theta)$ in (2.19). With the help of the helicity operator (2.22), these physical operators are essentially described by the $\mathcal{N} = 4$ supermultiplets in (2.21).

Therefore, for *massless* fermions, we can relate the Weyl spinors ψ_L and ψ_R to the following operators:

$$\psi_{Ri} \longleftrightarrow a_{Ri}^{(+\frac{1}{2})}(\xi_i) = \xi_i^\alpha a_{Ri\alpha}^{(+\frac{1}{2})}, \tag{3.3}$$

$$\psi_{Li} \longleftrightarrow a_{Li}^{(-\frac{1}{2})}(\xi_i) = \frac{1}{3!} \xi_i^\alpha \xi_i^\beta \xi_i^\gamma \epsilon_{\alpha\beta\gamma\delta} a_{Li}^{(-\frac{1}{2})\delta} \tag{3.4}$$

where i denotes the numbering index. Similarly the conjugates of ψ_L and ψ_R have the following correspondence:

$$\bar{\psi}_{Li} \longleftrightarrow \bar{a}_{Li}^{(+\frac{1}{2})}(\xi_i) = \xi_i^\alpha \bar{a}_{Li\alpha}^{(+\frac{1}{2})}, \tag{3.5}$$

$$\bar{\psi}_{Ri} \longleftrightarrow \bar{a}_{Ri}^{(-\frac{1}{2})}(\xi_i) = \frac{1}{3!} \xi_i^\alpha \xi_i^\beta \xi_i^\gamma \epsilon_{\alpha\beta\gamma\delta} \bar{a}_{Ri}^{(-\frac{1}{2})\delta}. \tag{3.6}$$

In the chiral-superspace representation, creation operators of these fermions are then defined in the same way as the expression (2.19), with $a_i^{(\pm)}$ being respectively replaced by $a_{Li}^{(-\frac{1}{2})}$, $a_{Ri}^{(+\frac{1}{2})}$,

$\bar{a}_{Ri}^{(-\frac{1}{2})}$ and $\bar{a}_{Li}^{(+\frac{1}{2})}$. In order to incorporate fermions in the holonomy formalism, basically we do not have to use the Weyl spinors but the above creation operators. Notice that these operators do not behave as spinors (which satisfy the ordinary Dirac equation) but as mere Grassmann variables.

We can check the validity of our definitions (3.3)–(3.6) for the massless fermions as follows. From the study of fermionic extension of the CSW rules, we find that the next-to-UHV (NUHV) vertices, i.e., vertices of an arbitrary number of positive-helicity gluon plus one negative-helicity gluon and a pair of fermions, can be calculated as [12,13]:

$$V_{\text{MHV}}(g_i^- g_j^-) = \frac{(ij)^4}{(12)(23)\cdots(n-1n)(n1)}, \tag{3.7}$$

$$V_{\text{NUHV}}(\bar{\psi}_{R1} g_i^- \psi_{Rn}) = \frac{(1i)^3(in)}{(12)(23)\cdots(n-1n)(n1)}, \tag{3.8}$$

$$V_{\text{NUHV}}(\bar{\psi}_{L1} g_i^- \psi_{Ln}) = \frac{(1i)(in)^3}{(12)(23)\cdots(n-1n)(n1)} \tag{3.9}$$

where we include the MHV vertex for clarification of the notation; here the positive-helicity gluons are implicit in the arguments of the left-hand sides, and the color factors are omitted in the right-hand sides. Notation of fermions here is different from the one found in the literature. For example, in [13] light-cone treatment of fermions is used and they are denoted by $\bar{\psi}^\pm, \psi^\pm$; these are related to our chiral notation by $\psi^- \Leftrightarrow \psi_L, \psi^+ \Leftrightarrow \psi_R, \bar{\psi}^- \Leftrightarrow \bar{\psi}_R$ and $\bar{\psi}^+ \Leftrightarrow \bar{\psi}_L$. Carrying out the Grassmann integrals that are analogous to (2.38), we can easily check that the assignments (3.3)–(3.6) indeed lead to the NUHV vertices (3.8) and (3.9) for massless fermions.

3.2. Massive fermion UHV vertices and the choice of reference spinors

We continue our discussion to include massive fermions. The massive fermion UHV vertices, the fermionic analogs of (2.35), are given by [13]:

$$\hat{V}_{\text{UHV}}(\bar{\psi}_{R1} \psi_{Rn}) = \frac{m^2(\eta 1)(1n)}{(12)(23)\cdots(n-1n)(n\eta)}, \tag{3.10}$$

$$\hat{V}_{\text{UHV}}(\bar{\psi}_{L1} \psi_{Ln}) = \frac{m^2(1n)(n\eta)}{(\eta 1)(12)(23)\cdots(n-1n)}, \tag{3.11}$$

$$\hat{V}_{\text{UHV}}(\bar{\psi}_{R1} \psi_{Ln}) = \frac{m(1n)^3}{(12)(23)\cdots(n-1n)(n1)}, \tag{3.12}$$

$$\hat{V}_{\text{UHV}}(\bar{\psi}_{L1} \psi_{Rn}) = 0 \tag{3.13}$$

where η denotes the *identical* reference spinor for the pair of massive fermions labeled by the numbering indices (1, n):

$$u_1^A \longrightarrow \hat{u}_1^A = u_1^A + \frac{m}{(u_1 \eta)} \eta^A, \tag{3.14}$$

$$u_n^A \longrightarrow \hat{u}_n^A = u_n^A + \frac{m}{(u_n \eta)} \eta^A. \tag{3.15}$$

In (3.10)–(3.13) we follow the notations of (3.8) and (3.9). Notice that \hat{V}_{UHV} 's are denoted with hats, meaning that the fermions $\psi_{L/R}$'s and $\bar{\psi}_{L/R}$'s are massive. From here on, we shall use the notation $(\bar{\psi}_{L1}, \psi_{Ln}, \bar{\psi}_{R1}, \psi_{Rn})$ to express *massive* fermions.

The first two vertices (3.10) and (3.11) are explicitly dependent on the identical reference spinor η and would lead to a set of arbitrary expressions, depending on the choice of η 's. These η -dependent vertices are consequences of the parametrization (3.14) and (3.15). This parametrization is useful particularly in defining off-shell momentum transfers running along propagators and indeed it has been utilized in the derivation of the CSW rules [1]. In defining exterior (instead of interior) off-shell momenta, however, we can use alternative parametrization as well.

For example, in [5,6] Rodrigo and others parametrize a pair of massive fermions, utilizing two *distinct* reference spinors. Our choice of the reference spinors for a pair of massive scalars in (2.29) or (2.30) is essentially the same as Rodrigo's. (The difference is that their parametrization imposes an additional condition $\hat{p}_1^\mu + \hat{p}_n^\mu = p_1^\mu + p_n^\mu$ but ours does not; see appendix of [6] for detail of this point.)

The η -dependent vertices (3.10) and (3.11) vanish with suitable choices of η . *In order to properly discuss forms of the massive fermion UHV vertices and amplitudes in general, it is therefore indispensable to use η -independent expressions. Otherwise we can not avoid ambiguities arose from different choices of off-shell parametrizations or reference spinors. The vertices (3.12) and (3.13) are the only such η -independent UHV vertices known in the literature and, hence, we shall focus on these vertices in what follows, keeping our choice of reference spinors for the pair of massive fermions in the form of (2.29), rather than (3.14) and (3.15), as in the case of the massive scalars.*

3.3. Massive fermions in the holonomy formalism

In the massive scalar case, the off-shell continuation of Nair's superamplitude method is carried out by the use of the $\xi\zeta$ -prescription (2.25):

$$a_i^{(0)}(\xi_i) \longrightarrow a_i^{(0)}(\xi_i, \zeta_i) = \xi_i^1 \xi_i^2 \xi_i^3 \zeta_i^4 a_i^{(0)} \tag{3.16}$$

where we have introduced the "massive" Grassmann variable

$$\zeta_i^\alpha = \theta_A^\alpha \frac{m}{(u_i \eta_i)} \eta_i^A. \tag{3.17}$$

Similarly, we can define the off-shell continuation of the massless fermion operators (3.3)–(3.6) such that it leads the η -independent massive UHV vertices (3.12), (3.13). We find that the fermionic off-shell continuation can be carried out by

$$a_{Ri}^{(+\frac{1}{2})}(\xi_i) \longrightarrow a_{Ri}^{(+\frac{1}{2})}(\xi_i) = \xi_i^\alpha a_{Ri\alpha}^{(+\frac{1}{2})}, \tag{3.18}$$

$$a_{Li}^{(-\frac{1}{2})}(\xi_i) \longrightarrow a_{Li}^{(-\frac{1}{2})}(\xi_i) = \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta} \xi_i^\alpha \xi_i^\beta \xi_i^\gamma a_{Li}^{(-\frac{1}{2})\delta}, \tag{3.19}$$

$$\bar{a}_{Li}^{(+\frac{1}{2})}(\xi_i) \longrightarrow \bar{a}_{Li}^{(+\frac{1}{2})}(\xi_i, \zeta_i) = \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta} \xi_i^\alpha \xi_i^\beta \zeta_i^\gamma \bar{a}_{Li}^{(+\frac{1}{2})\delta}, \tag{3.20}$$

$$\bar{a}_{Ri}^{(-\frac{1}{2})}(\xi_i) \longrightarrow \bar{a}_{Ri}^{(-\frac{1}{2})}(\xi_i, \zeta_i) = \frac{1}{4} \xi_i^1 \xi_i^2 \xi_i^3 \xi_i^4 \zeta_i^\alpha \bar{a}_{Ri\alpha}^{(-\frac{1}{2})} \tag{3.21}$$

where $i = 1, n$. Notice that the off-shell continuation by means of the $\xi\zeta$ -prescription is made only for the conjugate fermions $\bar{\psi}_{Li}$ and $\bar{\psi}_{Ri}$. The unbar fermions ψ_{Li} and ψ_{Ri} remain on-shell. This is consistent with the fact that the nontrivial UHV vertex (3.12) is proportional to m . The rest of the UHV vertices vanish upon the Grassmann integrals over the chiral supervariables θ_A^α .

This can easily be seen by counting the by counting the number of Grassmann variables in each pair. The numbers are given by 8, 4, 6 and 6 for $(\bar{\psi}_{R1}, \psi_{Ln})$, $(\bar{\psi}_{L1}, \psi_{Rn})$, $(\bar{\psi}_{R1}, \psi_{Rn})$ and $(\bar{\psi}_{L1}, \psi_{Ln})$, respectively. Thus, upon the Grassmann integral over $d^8\theta$, only the first pair survives.

Our choice (3.18)–(3.21) is also consistent with a recent study on recursion relations for massive fermion currents [9] where a pair of massive fermions are parametrized by an off-shell conjugate fermion and an on-shell fermion.

The chiral superspace representation of the massive fermion operators can then be expressed as

$$\bar{a}_{L1}^{(+\frac{1}{2})}(x, \theta) = \int d\mu(\hat{p}_1) \bar{a}_{L1}^{(+\frac{1}{2})}(\xi_1, \zeta_1) e^{ix_\mu \hat{p}_1^\mu} \Big|_{\xi_1^\alpha = \theta_A^\alpha u_1^A, \zeta_1^\alpha = \theta_A^\alpha w_1^A}, \tag{3.22}$$

$$\bar{a}_{R1}^{(-\frac{1}{2})}(x, \theta) = \int d\mu(\hat{p}_1) \bar{a}_{R1}^{(-\frac{1}{2})}(\xi_1, \zeta_1) e^{ix_\mu \hat{p}_1^\mu} \Big|_{\xi_1^\alpha = \theta_A^\alpha u_1^A, \zeta_1^\alpha = \theta_A^\alpha w_1^A}, \tag{3.23}$$

$$a_{Ln}^{(-\frac{1}{2})}(x, \theta) = \int d\mu(\hat{p}_n) a_{Ln}^{(-\frac{1}{2})}(\xi_n) e^{ix_\mu \hat{p}_n^\mu} \Big|_{\xi_n^\alpha = \theta_A^\alpha u_n^A}, \tag{3.24}$$

$$a_{Rn}^{(+\frac{1}{2})}(x, \theta) = \int d\mu(\hat{p}_n) a_{Rn}^{(+\frac{1}{2})}(\xi_n) e^{ix_\mu \hat{p}_n^\mu} \Big|_{\xi_n^\alpha = \theta_A^\alpha u_n^A} \tag{3.25}$$

where we specify the numbering indices that are relevant to the UHV vertices of our interest. The off-shell momenta \hat{p}_1^μ and \hat{p}_n^μ are defined by (2.30). As in (2.28), w_1^A is given by

$$w_1^A = \frac{m}{(u_1 \eta_1)} \eta_1^A = \frac{m}{(u_1 u_n)} u_n^A \tag{3.26}$$

where our choice of reference spinors is reflected.

3.4. The UHV rules and the massive fermion UHV amplitudes

The basic ingredients of the UHV rules are the UHV vertices and the propagators. In the CSW-type rules, which are originally formulated in twistor space, the vertices correspond to a set of lines in twistor space and they are connected by scalar propagators. Application of the UHV rules to the massive fermion UHV amplitudes is then straightforward because (a) we already know all types of the fermionic UHV vertices and (b) these vertices are connected by massive propagators as in the case of massive scalar amplitudes.

The fermionic UHV vertex $\hat{V}_{\text{UHV}}(\bar{\psi}_1 \psi_n)$ should be expanded by possible pairs of fermions. Under our choice of reference spinors, only a certain type remains nonzero, i.e.,

$$\begin{aligned} \hat{V}_{\text{UHV}}(\bar{\psi}_1 \psi_n) &= \hat{V}_{\text{UHV}}(\bar{\psi}_{R1} \psi_{Ln}) + \hat{V}_{\text{UHV}}(\bar{\psi}_{L1} \psi_{Rn}) + \hat{V}_{\text{UHV}}(\bar{\psi}_{L1} \psi_{Ln}) + \hat{V}_{\text{UHV}}(\bar{\psi}_{R1} \psi_{Rn}) \\ &= \hat{V}_{\text{UHV}}(\bar{\psi}_{R1} \psi_{Ln}) = \frac{(1n)}{m} \hat{V}_{\text{UHV}}(\bar{\phi}_1 \phi_n) \end{aligned} \tag{3.27}$$

where in the last equation we use (2.35) and (3.12), denoting the color-stripped massive scalar UHV vertices by $\hat{V}_{\text{UHV}}(\bar{\phi}_1 \phi_n)$. For the UHV vertices other than $\hat{V}_{\text{UHV}}(\bar{\psi}_{R1} \psi_{Ln})$, the number of Grassmann variables does not reach the saturating number 8. Thus these vertices vanish upon execution the Grassmann integrals unless the pair of fermions couple to other particles, either massive or massless, such that the total number of Grassmann variables becomes 8. We shall consider such interactions in another paper.

In application of the UHV rules, we can obtain the massive fermion UHV amplitudes from the diagrams in Fig. 1 by replacing (ϕ, ϕ) with (ψ_R, ψ_L) . Analytically the massive fermion UHV amplitudes can be written down in the form of (2.42):

$$\hat{C}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u) = \hat{C}_{\text{UHV}}^{(\bar{\psi}_1 g_2^+ g_3^+ \dots g_{n-1}^+ \psi_n)}(u) = \frac{-m(n1)^2}{(12)(23) \dots (n1)} (\widehat{n1})_{\bar{\psi}\psi} \tag{3.28}$$

where

$$\begin{aligned} (\widehat{n1})_{\bar{\psi}\psi} &= (n1) + \sum_{j=2}^{n-2} \frac{(J1)^2}{(jJ)} \frac{-m(jj+1)}{\hat{P}_J^2 - m^2} \frac{(nJ)^2}{(Jj+1)(n1)} \\ &+ \sum_{2 \leq i < j \leq n-1} \frac{(I1)^2}{(iI)} \frac{-m(ii+1)}{\hat{P}_I^2 - m^2} \frac{(JI)^2}{(Ii+1)(jJ)} \frac{-m(jj+1)}{\hat{P}_J^2 - m^2} \frac{(nJ)^2}{(Jj+1)(n1)} \\ &+ \sum_{2 \leq i < j < k \leq n-1} \left[\frac{(I1)^2}{(iI)} \frac{-m(ii+1)}{\hat{P}_I^2 - m^2} \frac{(JI)^2}{(Ii+1)(jJ)} \frac{-m(jj+1)}{\hat{P}_J^2 - m^2} \right. \\ &\times \left. \frac{(KJ)^2}{(Jj+1)(kK)} \frac{-m(kk+1)}{\hat{P}_K^2 - m^2} \frac{(nK)}{(Kk+1)(n1)} \right] \\ &+ \dots \\ &= - \left(1 \left| \prod_{j=2}^{n-2} \left[1 + \frac{|J|(1J)}{(jJ)} \frac{m(jj+1)}{\hat{P}_J^2 - m^2} \frac{(Jn)(J|)}{(Jj+1)} \right] \right| n \right). \end{aligned} \tag{3.29}$$

The full amplitudes in a form holomorphic to the spinor momenta are given, as in (2.41), by

$$\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u) = \text{Tr}(t^{c_2} t^{c_3} \dots t^{c_{n-1}}) \hat{C}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u). \tag{3.30}$$

We should emphasize again that we do not take a sum over permutations of gluons in the above expression; the sum is already taken care of in the definition of the massive holonomy operator. To be more precise, the sum is already included in the computation of the braid trace for the massive holonomy operator; see (2.17) and (2.18) for details.

We should comment on the relation between the massive UHV amplitudes of scalars and fermions. In the literature it is often shown that these are proportional to each other as in the case of the UHV vertices (3.27), i.e., $\hat{C}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u) = \frac{(1n)}{m} \hat{C}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u)$. This relation is derived from the supersymmetric Ward identities [13]; see also [10, 11, 17]. We consider, however, that this relation is contradictory to the UHV rules because in the UHV rules the massive UHV amplitudes are constructed by iterative use of the UHV vertices, see Fig. 1 or the analytic expression (3.29). Therefore the above proportional relation holds only for the 1 UHV irreducible (1UI) amplitudes or the UHV vertices in our framework. This interpretation is natural and would improve our understanding of the massive UHV amplitudes since the notion of 1UI amplitudes and its roles in construction of massive UHV amplitudes have been generally ignored so far in the literature.

3.5. An S-matrix functional for the massive fermion UHV amplitudes

What we have shown so far is that based on the forms of the massive fermion UHV vertices (3.10)–(3.13), one can apply the UHV rules to construct the massive fermion UHV amplitudes in much the same way as the case of massive scalar amplitudes. Motivated by the

results in (2.46)–(2.48), we now formulate field theoretic derivation of the massive fermion UHV amplitudes. Since the massive holonomy operator (2.14) does not contain fermion operators (3.18)–(3.21), for this purpose, we need to incorporate the operators of massive fermions, rather than those of massive scalars ($a_1^{(0)}, a_n^{(0)}$), into the massive holonomy operator. As in (3.27), this can be done by taking a sum over possible pairs of fermions, i.e.,

$$\Theta_{R,\gamma}^{(B)\bar{\psi}\psi}(u) = \exp \left[\sum_{r \geq 3} \sum_{(h_2, h_3, \dots, h_{r-1})} g^{r-2} (-1)^{h_2 h_3 \dots h_{r-1}} \text{Tr}(t^{c_2} t^{c_3} \dots t^{c_{r-1}}) \frac{a_2^{(h_2)c_2} \otimes \dots \otimes a_{r-1}^{(h_{r-1})c_{r-1}}}{(12)(23) \dots (r-1r)(r1)} \right. \\ \left. \otimes (a_{Lr}^{(-\frac{1}{2})} \otimes \bar{a}_{R1}^{(-\frac{1}{2})} + a_{Rr}^{(+\frac{1}{2})} \otimes \bar{a}_{L1}^{(+\frac{1}{2})} + a_{Lr}^{(-\frac{1}{2})} \otimes \bar{a}_{L1}^{(+\frac{1}{2})} + a_{Rr}^{(+\frac{1}{2})} \otimes \bar{a}_{R1}^{(-\frac{1}{2})}) \right] \quad (3.31)$$

where $h_i = \pm = \pm 1$ ($i = 2, 3, \dots, r - 1$) denotes the helicity of the i -th gluon. We obtain the above expression based on the UHV rules, which means that we do not need a knowledge of algebra for the fermionic operators. This is related to the fact that in the holonomy formalism the fermionic operators arise upon supersymmetrization of the gluon operators; see (2.21).

Supersymmetrization of $\Theta_{R,\gamma}^{(B)\bar{\psi}\psi}(u)$ can be carried out by replacing the gluon and fermion operators with (2.19) and (3.22)–(3.25), respectively. As in the massive scalar case, we denote the supersymmetric holonomy operator by $\Theta_{R,\gamma}^{(B)\bar{\psi}\psi}(u; x, \theta)$. The generating functional for the massive fermion UHV vertices is then expressed as

$$\mathcal{F}_{\text{UHV}}^{(\text{vertex})} [a^{(\pm)c}, \bar{a}_{L/R}^{(\pm\frac{1}{2})}, a_{L/R}^{(\mp\frac{1}{2})}] = \exp \left[i \int d^4x d^8\theta \Theta_{R,\gamma}^{(B)\bar{\psi}\psi}(u; x, \theta) \right]. \quad (3.32)$$

This expression provides a fermionic analog of (2.31).

Similarly, in analogy to (2.46) and (2.47), we can straightforwardly obtain an S-matrix functional for the full massive fermion UHV amplitudes by use of $\Theta_{R,\gamma}^{(B)\bar{\psi}\psi}(u; x, \theta)$:

$$\mathcal{F}_{\text{UHV}} [a^{(h)c}, \bar{a}_{L/R}^{(\pm\frac{1}{2})}, a_{L/R}^{(\mp\frac{1}{2})}] = \hat{W}_{\bar{R}L}^{(B)}(x) \hat{W}_{\bar{L}R}^{(B)}(x) \hat{W}_{\bar{L}L}^{(B)}(x) \hat{W}_{\bar{R}R}^{(B)}(x) \\ \times \exp \left[i \int d^4x d^8\theta \Theta_{R,\gamma}^{(B)\bar{\psi}\psi}(u; x, \theta) \right] \quad (3.33)$$

where

$$\hat{W}_{\bar{R}L}^{(B)}(x) = \exp \left[- \int d\mu(\hat{P}_J) \left(\frac{\delta}{\delta \bar{a}_{RJ\alpha}^{(-\frac{1}{2})}} \otimes \frac{\delta}{\delta a_{LJ}^{(-\frac{1}{2})\alpha}} \right) e^{-i\hat{P}_J \cdot (x-y)} \right]_{y \rightarrow x(x^0 > y^0)}, \quad (3.34)$$

$$\hat{W}_{\bar{L}R}^{(B)}(x) = \exp \left[- \int d\mu(\hat{P}_J) \left(\frac{\delta}{\delta \bar{a}_{LJ}^{(+\frac{1}{2})\alpha}} \otimes \frac{\delta}{\delta a_{RJ\alpha}^{(+\frac{1}{2})}} \right) e^{-i\hat{P}_J \cdot (x-y)} \right]_{y \rightarrow x(x^0 > y^0)}, \quad (3.35)$$

$$\hat{W}_{\bar{L}L}^{(B)}(x) = \exp \left[- \int d\mu(\hat{P}_J) \left(\frac{\delta}{\delta \bar{a}_{LJ}^{(+\frac{1}{2})\alpha}} \otimes \frac{\delta}{\delta a_{LJ}^{(-\frac{1}{2})\alpha}} \right) e^{-i\hat{P}_J \cdot (x-y)} \right]_{y \rightarrow x(x^0 > y^0)}, \quad (3.36)$$

$$\hat{W}_{\bar{R}R}^{(B)}(x) = \exp \left[- \int d\mu(\hat{P}_J) \left(\frac{\delta}{\delta \bar{a}_{RJ\alpha}^{(-\frac{1}{2})}} \otimes \frac{\delta}{\delta a_{RJ\alpha}^{(+\frac{1}{2})}} \right) e^{-i\hat{P}_J \cdot (x-y)} \right]_{y \rightarrow x(x^0 > y^0)} \quad (3.37)$$

where off-shell momentum transfer \hat{P}_J^μ is defined by (2.44). The Lorentz invariant measure $d\mu(\hat{P}_J)$ is given by

$$d\mu(\hat{P}_J) = \frac{d^3\hat{P}_J}{(2\pi)^3} \frac{1}{2\sqrt{|\vec{\hat{P}}_J|^2 + m^2}} \tag{3.38}$$

where m denotes the mass of the fermion. As before, in the calculation of (3.34)–(3.37) we take the limit $y \rightarrow x$, while keeping the time ordering $x^0 > y^0$.

The massive fermion UHV amplitudes in the x -space representation are then explicitly derived as

$$\begin{aligned} & \frac{\delta}{\delta a_2^{(+)}} \otimes \frac{\delta}{\delta a_3^{(+)}} \otimes \cdots \otimes \frac{\delta}{\delta a_{n-1}^{(+)}} \otimes \left[\frac{\delta}{\delta a_{Ln}^{(-\frac{1}{2})}} \otimes \frac{\delta}{\delta \bar{a}_{R1}^{(-\frac{1}{2})}} + \frac{\delta}{\delta a_{Rn}^{(+\frac{1}{2})}} \otimes \frac{\delta}{\delta \bar{a}_{L1}^{(+\frac{1}{2})}} \right. \\ & \left. + \frac{\delta}{\delta a_{Ln}^{(-\frac{1}{2})}} \otimes \frac{\delta}{\delta \bar{a}_{L1}^{(+\frac{1}{2})}} + \frac{\delta}{\delta a_{Rn}^{(+\frac{1}{2})}} \otimes \frac{\delta}{\delta \bar{a}_{R1}^{(-\frac{1}{2})}} \right] \mathcal{F}_{\text{UHV}}[a^{(\pm)c}, a^{(0)}] \Big|_{a^{(\pm)c}=\bar{a}^{(\pm\frac{1}{2})}=a^{(\mp\frac{1}{2})}=0} \\ & = \mathcal{A}_{\text{UHV}}^{(\bar{\psi}_1 g_2^+ \cdots g_{n-1}^+ \psi_n)}(x) \end{aligned} \tag{3.39}$$

where $a^{(\pm)c}$ and $(\bar{a}^{(\pm\frac{1}{2})}, a^{(\mp\frac{1}{2})})$ represent generic source functions for gluons and massive fermions, respectively.

3.6. Dimensional analysis and “exclusion rules” for massive fermion amplitudes

Lastly, but most importantly, we consider the m dependence of the massive fermion UHV amplitudes $\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u)$. In the massive scalar case, the UHV amplitudes are proportional to the squared mass, $\hat{A}_{\text{UHV}}^{(\bar{\phi}_1 \phi_n)}(u) \sim m^2$. Owing to the relation (3.27) and the form of the massive propagator, this proportionality does not hold for $\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u)$. Let s be the number of 1 UHV irreducible (1UI) diagrams in the expansion formula of Fig. 1. Then we find that $\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u)$ is expanded by $m^{s-2(s-1)} = m^{-s+2}$ ($s = 1, 2, \dots, n - 2$). This shows that in large m limits we can approximate $\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u)$ to the 1UI subamplitudes or the UHV vertices $\hat{V}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u)$.

The fact that $\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 \psi_n)}(u)$ is expanded by m^{-s+2} ($s = 1, 2, \dots, n - 2$) is unnatural from a perspective that the fermion-mass dependence is irrelevant to the number of gluons involving the scattering processes. The only way to circumvent this problem is to fix the total number of scattering particles by

$$n = 3. \tag{3.40}$$

Naively, this constraint leads to the relation

$$\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 g_2^+ \psi_3)}(u) = \text{Tr}(t^{c2}) \frac{-m(31)^2}{(12)(23)} \tag{3.41}$$

where the color factor $\text{Tr}(t^{c2})$ vanishes unless the gluon propagates along an internal line. As in the case of the MHV rules, internal gluons have the $U(1)$ color factor. But, for external gluons, we have the $SU(N)$ color factor and $\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 g_2^+ \psi_3)}(u)$ vanishes. To be explicit, $\text{Tr}(t^{c2})$ is given by

$$\text{Tr}(t^{c_2}) = \begin{cases} 0 & \text{for } t^{c_2} \in \underline{SU(N)}, \\ \sqrt{\frac{N}{2}} & \text{for } t^{c_2} \in \underline{U(1)} \end{cases} \quad (3.42)$$

where $\underline{SU(N)}$ and $\underline{U(1)}$ denote the algebras of $SU(N)$ and $U(1)$ groups, respectively. We here use the usual normalization $\text{Tr}(t^c t^{c'}) = \frac{\delta^{cc'}}{2}$. The three-point interactions (3.41) vanish unless the involving gluon is a virtual gluon which appears only in internal propagators. Intriguingly, this feature is in accord with the pictures of the quark–gluon interactions in QCD.

In the existing literature there have been no considerations on the number of involving particles in the massive UHV amplitudes. This is mainly because the notion of the 1 UHV irreducible (1UI) diagram in Fig. 1 is not well-recognized in the literature. In our formulation the 1UI diagram corresponds to the massive UHV vertex but it is often interpreted as the massive UHV amplitude; it is so particularly in the case of massive fermions. Once this point is clarified, one can utilize the relation (3.27) to derive the explicit form of the massive fermion UHV amplitudes (3.28). We can then straightforwardly carry out mass-dimension analysis of the UHV amplitudes. For scalar amplitudes a pair of massive scalars can couple to an arbitrary number of gluons. This is essentially due to the fact that the massive scalar UHV vertex (2.35) is proportional to m^2 . Since the massive propagator is proportional to m^{-2} , one can insert as many propagators as possible into the massive scalar UHV amplitudes as far as the mass dimension is concerned. On the other hand, the massive fermion UHV vertex (3.27) is proportional to m . Thus insertion of massive propagators is mass-dimensionally prohibited. This leads to the constraint (3.40) that a pair of massive fermions should couple to a single (internal) gluon.

The constraint $n = 3$ is crucial when we consider the generalization of the massive fermion UHV amplitudes to the non-UHV amplitudes. For example, it means that all the massive fermion amplitudes are classified by the UHV subamplitude and the next-to-UHV (NUHV) subamplitude, the latter containing a single negative-helicity gluon. In this sense, we can interpret the condition $n = 3$ for a sort of exclusion rules for the massive fermion amplitudes. This principle is a consequence of (a) the use of massive scalar propagators for the construction of massive fermion amplitudes in the holonomy formalism and (b) the mass-dimension analysis of the massive fermion UHV amplitudes.

In the following section, for clarification of our argument, we first consider the massive fermion NUHV amplitudes for arbitrary n and then study the consequence of the constraint $n = 3$ afterwards.

4. Massive fermion NUHV amplitudes

4.1. Massive fermion NUHV vertices

For the massive scalar amplitudes there are no vertices other than the massive scalar UHV vertices that contribute to the UHV rules. For the massive fermion amplitudes, however, due to our assignments of Grassmann variables in (3.3)–(3.4), there does exist another type of nonzero vertices that contributes to the UHV rules. This is given by the following NUHV vertices

$$\hat{V}_{\text{NUHV}}(\bar{\psi}_{L1} g_a^- \psi_{Rn}) = \frac{m}{(1n)(12)(23)\cdots(n-1n)(n1)} \frac{(1a)^2 (an)^2}{(1n)(12)(23)\cdots(n-1n)(n1)} \quad (4.1)$$

($2 \leq a \leq n-1$) where we choose the reference spinor $\eta_1 = u_n$ as before for the operator (3.20) of the off-shell fermion $\bar{\psi}_{L1}$. Notice that one can easily check that there are no other non-vanishing

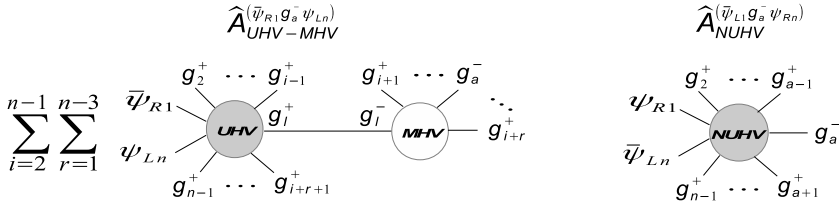


Fig. 2. Diagrams contributing to the massive fermion NUHV amplitudes.

massive fermion vertices, hence, (3.12) and (4.1) are the only vertices that contribute to the massive fermion amplitudes in general.

4.2. Massive fermion NUHV amplitudes

The contribution (4.1) is one of the major differences between the UHV rules of massive scalars and fermions. We here study how it reflects in the construction of the massive fermion NUHV amplitudes. Diagrams contributing to the massive fermion NUHV amplitudes $\hat{A}_{\text{NUHV}}^{(\tilde{\psi}_1 g_a^- \psi_n)}(u)$ are shown in Fig. 2.

There are two types of contributions to $\hat{A}_{\text{NUHV}}^{(\tilde{\psi}_1 g_a^- \psi_n)}(u)$:

$$\hat{A}_{\text{NUHV}}^{(\tilde{\psi}_1 g_a^- \psi_n)}(u) = \hat{A}_{\text{UHV-MHV}}^{(\tilde{\psi}_{R1} g_a^- \psi_{Ln})}(u) + \hat{A}_{\text{NUHV}}^{(\tilde{\psi}_{L1} g_a^- \psi_{Rn})}(u). \tag{4.2}$$

One includes the gluonic MHV amplitudes and the other does not. From the results of $\hat{A}_{\text{NUHV}}^{(\tilde{\phi}_1 g_a^- \phi_n)}(u)$ in [19], we can write down the former contribution $\hat{A}_{\text{UHV-MHV}}^{(\tilde{\psi}_{R1} g_a^- \psi_{Ln})}(u)$ as

$$\begin{aligned} &\hat{A}_{\text{UHV-MHV}}^{(\tilde{\psi}_{R1} g_a^- \psi_{Ln})}(u) \\ &= \sum_{i=2}^{n-1} \sum_{r=1}^{n-3} \hat{A}_{\text{UHV}}^{((i+r+1)_+ \dots \psi_{Ln} \tilde{\psi}_{R1} \dots (i-1)_+ l_+)}(u) \frac{1}{q_{ii+r}^2} \hat{A}_{\text{MHV}}^{(l_-(i)_+ \dots a_- \dots (i+r)_+)}(u) \Big|_{u_l = u_{ii+r}} \\ &= \sum_{i=2}^{n-1} \sum_{r=1}^{n-3} \sum_{\sigma \in S_{r+1}} \text{Tr}(t^{\sigma_i} \dots t^{\sigma_{i+r}} t^{i+r+1} \dots t^{i-1}) \\ &\quad \times \hat{C}_{\text{UHV}}^{((i+r+1)_+ \dots \psi_{Ln} \tilde{\psi}_{R1} \dots (i-1)_+ l_+)}(u) \frac{1}{q_{\sigma_i \sigma_{i+r}}^2} \hat{C}_{\text{MHV}}^{(l_-(i)_+ \dots a_- \dots (i+r)_+)}(u; \sigma) \Big|_{u_l = u_{\sigma_i \sigma_{i+r}}} \end{aligned} \tag{4.3}$$

where we consider the numbering indices in modulo n . The \hat{C} 's are expressed as

$$\hat{C}_{\text{UHV}}^{((i+r+1)_+ \dots \psi_{Ln} \tilde{\psi}_{R1} \dots (i-1)_+ l_+)}(u) = \frac{-m(n1)^2 (\widehat{n1})_{\tilde{\psi}\psi}}{(i+r+1)(i+r+2) \dots (i-1)(li+r+1)}, \tag{4.4}$$

$$\hat{C}_{\text{MHV}}^{(l_-(i)_+ \dots a_- \dots (i+r)_+)}(u; \sigma) = \frac{(la)^4}{(l\sigma_i)(\sigma_i \sigma_{i+1})(\sigma_{i+1} \sigma_{i+2}) \dots (\sigma_{i+r} l)} \tag{4.5}$$

where the off-shell momentum transfer q_{ii+r}^μ is defined in terms of the reference null-vector η_{ii+r}^μ and the associated null momentum p_{ii+r}^μ :

$$q_{ii+r}^\mu = p_i^\mu + p_{i+1}^\mu + \dots + p_{i+r}^\mu \equiv p_{ii+r}^\mu + W \eta_{ii+r}^\mu, \tag{4.6}$$

$$p_{ii+r}^{A\dot{A}} = u_{ii+r}^A \tilde{u}_{ii+r}^{\dot{A}} \equiv u_l^A \tilde{u}_l^{\dot{A}}. \tag{4.7}$$

In (4.7) we use the two-component notation of the null vector p_{i+r}^μ . In (4.3) and (4.5) σ denotes the transposition of the numbering indices for gluons:

$$\sigma = \begin{pmatrix} i & i+1 & \cdots & i+r \\ \sigma_i & \sigma_{i+1} & \cdots & \sigma_{i+r} \end{pmatrix}. \tag{4.8}$$

The other contribution to $\hat{A}_{\text{NUHV}}^{(\bar{\psi}_1 g_a^- \psi_n)}(u)$ is essentially given by the massive fermion NUHV vertices:

$$\hat{A}_{\text{NUHV}}^{(\bar{\psi}_1 g_a^- \psi_{Rn})}(u) = \text{Tr}(t^{c_2} t^{c_3} \cdots t^{c_{n-1}}) \frac{m}{(1n)} \frac{(1a)^2 (an)^2}{(12)(23) \cdots (n-1n)(n1)}. \tag{4.9}$$

This expression is free of the reference spinors.

4.3. Imposition of the $n = 3$ condition

Having written down the massive fermion NUHV amplitudes for arbitrary n , we now impose the condition $n = 3$ of (3.40) on (4.2). Since the MHV vertex needs more than two legs, the first term in (4.2) vanishes for $n = 3$, i.e.,

$$\hat{A}_{\text{NUHV}}^{(\bar{\psi}_1 g_2^- \psi_3)}(u) = \hat{A}_{\text{NUHV}}^{(\bar{\psi}_{L1} g_2^- \psi_{R3})}(u) = \frac{-m(12)(23)}{(31)^2} \tag{4.10}$$

where we omit the color factor. As discussed in (3.41) and (3.42), the gluon labeled by g_2^- is interpreted as a virtual gluon otherwise the above NUHV amplitude becomes zero. In practice, however, one can connect it with the three-point UHV subamplitude (3.41), in which case the color factor becomes nonzero (regardless the choice of the color for each gluon) due to the normalization $\text{Tr}(t^c t^{c'}) = \frac{\delta^{cc'}}{2}$. The color-stripped UHV amplitude can similarly be expressed as

$$\hat{A}_{\text{UHV}}^{(\bar{\psi}_1 g_2^+ \psi_3)}(u) = \hat{A}_{\text{UHV}}^{(\bar{\psi}_{R1} g_2^+ \psi_{L3})}(u) = \frac{-m(31)^2}{(12)(23)}. \tag{4.11}$$

In general non-UHV amplitudes are labeled by N^k UHV amplitudes, specified by the number of negative-helicity gluons, $k = 1, 2, \dots, n - 3$. The massive fermion N^k UHV amplitudes then refer to the scattering amplitudes of a pair of massive fermions and k negative-helicity gluons and $(n - k - 2)$ positive-helicity gluons. Therefore, upon the imposition of $n = 3$, there exist no N^k UHV amplitudes for $k \geq 2$.

4.4. An S-matrix functional for (4.10) and (4.11)

To summarize, we find that the interactions among gluons and a pair of massive fermions are given by (4.10) and (4.11). This result is qualitatively different from the massive scalar amplitudes where we can incorporate an arbitrary number of gluons. The difference arises from the mass dimension analysis which leads to the condition $n = 3$.

Notice that, apart from the $n = 3$ constraint, (4.10) and (4.11) are obtained by application of the UHV rules. Since the UHV rules are automatically realized by defining an S-matrix functional, it is still worth deriving the S-matrix functional for (4.10) and (4.11). Since the massive fermion NUHV vertices (4.1) can be obtained from the generating functional for the UHV vertices (3.32) alone, we find that the amplitudes in (4.10) are derived from (3.33) as well. Therefore, together with (3.34)–(3.37), the S-matrix functional for (4.10) and (4.11) is given by (3.33).

5. Concluding remarks

In this paper we consider incorporation of massive fermions into the holonomy formalism, following the recent study [19] on the massive scalar amplitudes. The upshot of this paper is that the interactions among gluons and a pair of fermions are reduced to the two types of subamplitudes represented by (4.10) and (4.11). This result is simple but there have been several key steps toward it. These steps can be itemized as follows:

1. We clarify our notation of fermions in the spinor-helicity formalism.
2. We carry out off-shell continuation of Nair's superamplitude method for fermions by use of what we call the $\xi\zeta$ -prescription in (3.16)–(3.21). This set of parametrizations leads to the previously known massive fermion UHV vertices [13].
3. We apply the UHV rules, i.e., the massive extension of the CSW rules, to construct massive fermion UHV amplitudes.
4. In the CSW-type rules, vertices correspond to a set of lines in twistor space and they are connected by scalar propagators. In the UHV rules massive fermion amplitudes are therefore obtained by connecting the massive fermion UHV vertices in terms of massive scalar propagators.
5. We formulate an S-matrix functional for the massive fermion UHV amplitudes that realizes the UHV rules in a functional method.
6. Analysis of mass dimension on the massive fermion UHV amplitudes implies that the number of gluons involving the UHV amplitudes should be one.
7. Interactions among gluons and a pair of massive fermions are then decomposed into the two types of subamplitudes (4.10) and (4.11).

The final result 7 means that interactions among gluons and massive fermions should be decomposed into three-point massive fermions vertices. This is consistent with QCD in terms of the gluon–quark interactions. It is intriguing that this result is derived purely from the mass-dimension analysis. In the framework of the holonomy formalism, the result 7 also illustrates a qualitative difference from the massive scalar amplitudes. In the massive scalar amplitudes there are no restrictions on the number of involving gluons. In this sense, we can interpret the result 6 as a sort of “exclusion rules” for the massive fermion amplitudes.

The results 1–5, on the other hand, elucidate the fact that we can naturally incorporate massive fermions into the holonomy formalism. Motivated by these results, in a forthcoming paper, we shall consider massive deformation of gauge bosons in the same framework.

References

- [1] F. Cachazo, P. Svrcek, E. Witten, *J. High Energy Phys.* 0409 (2004) 006, arXiv:hep-th/0403047.
- [2] R. Britto, F. Cachazo, B. Feng, *Nucl. Phys. B* 715 (2005) 499, arXiv:hep-th/0412308.
- [3] R. Britto, F. Cachazo, B. Feng, E. Witten, *Phys. Rev. Lett.* 94 (2005) 181602, arXiv:hep-th/0501052.
- [4] S.D. Badger, E.W.N. Glover, V.V. Khoze, *J. High Energy Phys.* 0601 (2006) 066, arXiv:hep-th/0507161.
- [5] G. Rodrigo, *J. High Energy Phys.* 0509 (2005) 079, arXiv:hep-ph/0508138.
- [6] P. Ferrario, G. Rodrigo, P. Talavera, *Phys. Rev. Lett.* 96 (2006) 182001, arXiv:hep-th/0602043.
- [7] A. Hall, *Phys. Rev. D* 77 (2008) 025011, arXiv:0710.1300 [hep-ph].
- [8] G. Chen, *Phys. Rev. D* 83 (2011) 125005, arXiv:1103.2518 [hep-th].
- [9] R. Britto, A. Ochirov, arXiv:1210.1755 [hep-th].
- [10] C. Schwinn, S. Weinzierl, *J. High Energy Phys.* 0603 (2006) 030, arXiv:hep-th/0602012.
- [11] J.-H. Huang, W. Wang, arXiv:1204.0068 [hep-th].

- [12] J.H. Eittle, T.R. Morris, Z. Xiao, *J. High Energy Phys.* 0808 (2008) 103, arXiv:0805.0239 [hep-th].
- [13] C. Schwinn, *Phys. Rev. D* 78 (2008) 085030, arXiv:0809.1442 [hep-ph].
- [14] R. Boels, C. Schwinn, *Phys. Lett. B* 662 (2008) 80, arXiv:0712.3409 [hep-th].
- [15] R. Boels, C. Schwinn, *J. High Energy Phys.* 0807 (2008) 007, arXiv:0805.1197 [hep-th].
- [16] T. Cohen, H. Elvang, M. Kiermaier, *J. High Energy Phys.* 1104 (2011) 053, arXiv:1010.0257 [hep-th].
- [17] R.H. Boels, C. Schwinn, *Phys. Rev. D* 84 (2011) 065006, arXiv:1104.2280 [hep-th].
- [18] S. Dittmaier, *Phys. Rev. D* 59 (1998) 016007, arXiv:hep-ph/9805445.
- [19] Y. Abe, *Nucl. Phys. B* 865 (2012) 238, arXiv:1205.4827 [hep-th].
- [20] Y. Abe, *Nucl. Phys. B* 825 (2010) 242, arXiv:0906.2524 [hep-th].
- [21] Y. Abe, *Nucl. Phys. B* 825 (2010) 268, arXiv:0906.2526 [hep-th].
- [22] V.P. Nair, *Phys. Lett. B* 214 (1988) 215.
- [23] L.S. Cirio, J.F. Martins, *Differ. Geom. Appl.* 30 (2012) 238, arXiv:1106.0042 [hep-th].
- [24] L.S. Cirio, J.F. Martins, arXiv:1207.1132 [hep-th].
- [25] H.K. Dreiner, H.E. Haber, S.P. Martin, *Phys. Rep.* 494 (2010) 1, arXiv:0812.1594 [hep-ph].
- [26] S.P. Martin, arXiv:1205.4076 [hep-ph].