MATHEMATICS

ON ALMOST DIFFEOMORPHIC BANACH SPACES

BY

A. PEŁCZYŃSKI

Warsaw

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Two Banach spaces X and Y are said to be *homeomorphic* if there is a homeomorphism, say h, from X onto Y. Moreover if both h and h^{-1} are uniformly continuous (with respect to the uniformities induced by the linear topological structures of X and Y), then X and Y are said to be *uniformly equivalent*. If h and h^{-1} are differentiable, then X and Y are said to be *diffeomorphic*. Finally, if the homeomorphism h is linear, then X and Y are said to be *linearly homeomorphic* or *isomorphic*. Each of the above equivalence relations determines a classification of Banach spaces; they are: topological, uniform, differentiable, and linear topological classifications respectively. For finite dimensional Banach spaces all these classifications coincide with the classification by (real) dimension. Clearly if two Banach spaces are isomorphic, then they are also equivalent in the sense of any of the other relations. On the other hand the topological classification seems to be "trivial". All infinite dimensional separable Banach spaces are homeomorphic (KADEC [6], cf. also [10]). It is conceivable that two infinite dimensional Banach spaces are homeomorphic, if and only if they are of the same density character (cf. [3] for the recent results). In [9] it is shown that there exist homeomorphic Banach spaces which are not uniformly equivalent. For instance no reflexive space is uniformly equivalent to any space $C(S)$ of all scalar valued continuous functions on an infinite compact Hausdorff space S. We do not know whether the uniform classification coincides with the linear topological classification of Banach spaces. However it is well known that the differentiable classification coincides with the linear topological classification (if h is a (local) diffeomorphism from X onto Y, then the differential $(Dh)_{x=0}$ is an isomorphism from X onto Y).

In this note we consider the almost diffeomorphic classification of Banach spaces. We show that this classification "lies essentially in between" the topological and the linear topological classifications. We consider only real Banach spaces. For the definitions and basic properties of differentiable and analytic maps the reader is reffered to the books [5] and [8].

DEFINITION. Two Banach spaces X and Y are said to be *almost C^k-diffeomorphic* for $k=1, 2, \ldots, +\infty$ (resp. *almost analytically homeomorphic*) provided there are homeomorphisms $f: X \xrightarrow[\text{onto}]{} Y$ and $g: Y \xrightarrow[\text{onto}]{} X$, such that f and g are k-time continuously differentiable maps (resp. real analytic maps).

PROPOSITION 1. *There exist non isomorphic separable Banach spaces, which are almost analytically homeomorphic. Hence they are almost C^k-diffeomorphic for every $k=1, 2, \ldots, +\infty$.*

Proof. If (E_j) is a sequence of Banach spaces and if $1 \leqslant p < +\infty$, then we denote by $(E_1 \times E_2 \times \ldots)_p$ the Banach space of all sequences $e=\{e_j\}$ such that $e_j \in E_j$ for $j=1, 2, \ldots$ and $\|e\| = (\sum_{j=1}^{\infty} \|e_j\|^p)^{p^{-1}} < +\infty$. If all E_j are one dimensional spaces, then we denote the space $(E_1 \times E_2 \times \ldots)_p$ by l_p. For $e=\{e_j\} \in l_p$ we write $e(j)$ instead of e_j $(j=1, 2, \ldots)$.

Let $F_k = l_{3k}$. Now we introduce

$$X = (F_1 \times F_3 \times F_5 \times \ldots)_p; \qquad Y = (F_1 \times F_2 \times F_4 \times F_6 \times \ldots)_p.$$

Also define $h: F_k \to F_{k-1}$ $(k=2, 3, \ldots)$ by

$$(h(x))(n) = (x(n))^3,$$

and $u: F_1 \times F_1 \to F_1$ by

$$(u(a, b))(n) = \begin{cases} a(m) & \text{for } n=2m-1 \\ b(m) & \text{for } n=2m \end{cases}.$$

Then $f: X \to Y$ is defined by

$$(f(x))_j = \begin{cases} x_j & \text{for } j=1 \\ h(x_j) & \text{for } j \geqslant 2 \end{cases},$$

$g: Y \to X$ is defined by

$$(g(y))_j = \begin{cases} u(y_1, h(y_2)) & \text{for } j=1 \\ h(y_{j+1}) & \text{for } j \geqslant 2 \end{cases}.$$

Clearly both f and g are polynomial operators of degree 3. Hence they are real analytic maps. Since all maps $(f(\cdot))_j$ are homeomorphisms (cf. [2; 7.1]), f is a one to one map from X onto Y. The continuity of f and f^{-1} is an immediate consequence of [2; 6.3]. Thus f is a real analytic homeomorphism from X onto Y. The same argument shows that g is a real analytic homeomorphism from Y onto X. Hence X and Y are almost analytically homeomorphic.

The non-isomorphism of the spaces X and Y is a simple consequence of the following lemma

LEMMA. *Let $1 \leqslant q_i < +\infty$ $(i=0, 1, \ldots)$. Let E be an infinite dimensional closed linear subspace of the space $B = (l_{q_1} \times l_{q_2} \times \ldots)_{q_0}$. Then for some $i=0, 1, \ldots$ E contains a closed linear subspace isomorphic to l_{q_i}.*

Indeed, let $v: Y \to X$ be an arbitrary linear operator and let $E = v(F_2)$ for $p \neq 3^2$, and $E = v(F_4)$ for $p = 3^2$ (We identify F_{2k} with the subspace $\{y = \{y_i\} \in Y: y_i = 0 \text{ for } i \neq k\}$). If v were an isomorphism, then E would be isomorphic to l_r (actually either to l_9 or to l_{81}) with $r \neq p$ and $r \neq 3^{2i-1}$ for $i = 1, 2, \ldots$. By the Lemma (applied in the case $B = X$, $q_0 = p$ and $q_i = 3^{2i-1}$ for $i = 1, 2, \ldots$) E would contain a closed linear subspace isomorphic either to l_p or to $l_{3^{2i-1}}$ for some $i = 1, 2, \ldots$. But this leads to a contradiction with the following result due to Banach [1; Chap. XII].

(B). *If* $1 \leqslant p_1 \neq p_2 < +\infty$, *then no infinite dimensional closed linear subspace of* l_{p_1} *is isomorphic to a subspace of* l_{p_2}.

Proof of the Lemma. Let $P_0 = 0$, and let for $n \geqslant 1$, $P_n: B \to B$ be a projection defined by

$$P_n(x) = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\} \text{ for } x = \{x_j\} \in B.$$

Clearly

$$(1) \qquad \|x\| \geqslant \|P_n(x)\|; \qquad \lim_n \|x - P_n(x)\| = 0 \text{ for } x \in B.$$

There are two possibilities:

1°. $\inf_{e \in E;\ \|e\|=1} \|P_n(e)\| = 0$ for $n = 1, 2, \ldots$.

2°. There are an index m and a constant c with $0 < c \leqslant 1$ such that

$$(2) \qquad \|P_m(e)\| \geqslant c\|e\| \text{ for } e \in E.$$

If 1° holds, then one can define inductively a sequence of elements $(e^{(k)})$ in E and an increasing sequence of indices (n_k) so that

$$(3) \qquad \begin{cases} \|e^{(k)}\| = 1 \text{ for } k = 1, 2, \ldots, \\ \|\pi_{k-1}(e^{(k)})\| < 4^{-k} \text{ for } k = 2, 3, \ldots, \\ \|e^{(k)} - \pi_k(e^{(k)})\| < 4^{-k-1} \text{ for } k = 1, 2, \ldots, \end{cases}$$

where $\pi_0 = 0$ and $\pi_k = P_{n_k}$ for $k = 1, 2, \ldots$.
Clearly (3) implies

$$1 = \|e^{(k)}\| \geqslant \|\pi_k(e^{(k)}) - \pi_{k-1}(e^{(k)})\| \geqslant \|e^{(k)}\| - \|\pi_{k-1}(e^{(k)})\| - \|e^{(k)} - \pi_k(e^{(k)})\| \geqslant \tfrac{2}{3}.$$

Hence, by definition of the norm in B, for arbitrary real $t_1, t_2, \ldots, t_N$ $(N = 1, 2, \ldots)$ we get

$$\Big(\sum_{k=1}^{N} |t_k|^{q_0}\Big)^{q_0^{-1}} \geqslant \Big\| \sum_{k=1}^{N} t_k(\pi_k(e^{(k)}) - \pi_{k-1}(e^{(k)}))\Big\|$$
$$= \Big(\sum_{k=1}^{N} \|t_k\|^{q_0} \|\pi_k(e^{(k)}) - \pi_{k-1}(e^{(k)})\|^{q_0}\Big)^{q_0^{-1}} \geqslant \tfrac{2}{3}\Big(\sum_{k=1}^{N} |t_k|^{q_0}\Big)^{q_0^{-1}}.$$

Thus

$$(4)\quad \begin{cases} \|\sum_{k=1}^{N} t_k e^{(k)}\| \leqslant \|\sum_{k=1}^{N} t_k(\pi_k(e^{(k)})-\pi_{k-1}(e^{(k)}))\| \\ \quad + \sum_{k=1}^{N} |t_k|\, \|e^{(k)}-\pi_k(e^{(k)})\| + \sum_{k=1}^{N} |t_k|\, \|\pi_{k-1}(e^{(k)})\| \\ \quad \leqslant (\sum_{k=1}^{N} |t_k|^{q_0})^{q_0^{-1}} + \sup_{1\leqslant k\leqslant N} |t_k|(2\sum_{m=2}^{\infty} 4^{-m}+4^{-1}) \leqslant \frac{4}{3}(\sum_{k=}^{N} |t_k|^{q_0})^{q_0^{-1}}. \end{cases}$$

Similarly

$$(5)\quad \begin{cases} \|\sum_{k=1}^{N} t_k e^{(k)}\| \geqslant \frac{2}{3}(\sum_{k=1}^{N} |t_k|^{q_0})^{q_0^{-1}} - \sup_{1\leqslant k\leqslant N} |t_k|\,(2\sum_{m=2}^{\infty} 4^{-m}+4^{-1}) \\ \quad \geqslant \frac{1}{3}(\sum_{k=1}^{N} |t_k|^{q_0})^{q_0^{-1}}. \end{cases}$$

It follows from (4) and (5) that the linear operator $u: l_{q_0} \to E$ defined by,

$$u(s) = \sum_{k=1}^{\infty} s(k)\, e^{(k)} \qquad \text{for } s = \{s(k)\} \in l_{q_0},$$

satisfies the inequality

$$\frac{4}{3}\|s\| \geqslant \|u(s)\| \geqslant \frac{1}{3}\|s\| \text{ for } s \in l_{q_0}.$$

Hence u is an isomorphism from l_{q_0} into E. Therefore if 1° holds, then E contains a closed linear subspace isomorphic to l_{q_0}.

Let us consider the second possibility. Clearly 2° implies that there are an infinite dimensional closed linear subspace F of E, an index $M \leqslant m$, and a constant C with $0 < C \leqslant 1$ such that

$$(6)\qquad \|P_M(e)\| \geqslant C\|e\| \text{ for } e \text{ in } F,$$

(7) for every infinite dimensional closed linear subspace $G \subset F$

$$\inf_{e\in G;\ \|e\|=1} \|P_{M-1}(e)\| = 0.$$

Let

$$G_k = \{e = \{e_i\} \in F : e_M(j) = 0 \text{ for } j = 1, 2, \ldots, k\}.$$

Clearly all G_k are infinite dimensional closed linear subspaces of F. Since (by (6)) $\|e\| \leqslant C^{-1}(\|e_M\| + \|P_{M-1}(e)\|)$ for $e \in F$, it follows from (7) that

$$(8)\qquad \inf_{e\in G_k;\ \|e_M\|=1} \|P_{M-1}(e)\| = 0 \qquad \text{for } k = 1, 2, \ldots.$$

Using (8) we define inductively a sequence $(e^{(k)})$ in F and an increasing sequence of indices (n_k) such that

$$(9)\qquad \|e_M^{(k)}\| = 1,$$

$$(10)\qquad e^{(k)} \in G_{n_k},$$

$$(11)\qquad \Big(\sum_{j=n_{k+1}+1}^{\infty} |e_M^{(k)}(j)|^{q_M}\Big)^{q_M^{-1}} < 4^{-k},$$

$$(12)\qquad \|P_{M-1}(e^{(k)})\| < 7^{-k} \qquad (k=1, 2, \ldots).$$

By the similar computation as in the first case we obtain from formulas (9)–(11)

$$(13)\qquad \tfrac{4}{3}\Big(\sum_{k=1}^{N} |t_k|^{q_M}\Big)^{q_M^{-1}} \geqslant \Big\| \sum_{k=1}^{N} t_k e_M^{(k)} \Big\| \geqslant \tfrac{1}{3} \Big(\sum_{k=1}^{N} |t_k|^{q_M}\Big)^{q_M^{-1}}$$

for arbitrary real $t_1, t_2, \ldots, t_N$ $(N=1, 2, \ldots)$.
From (12) we get

$$(14)\qquad \Big\| \sum_{k=1}^{N} t_k P_{M-1}(e^{(k)}) \Big\| \leqslant \max_{1\leqslant k\leqslant N} |t_k| \Big(\sum_{k=1}^{N} 7^{-k}\Big) \leqslant \tfrac{1}{6}\Big(\sum_{k=1}^{N} |t_k|^{q_M}\Big)^{q_M^{-1}}.$$

Combining (1), (6), (13), and (14) we get

$$(15)\qquad \begin{cases} \Big\| \sum_{k=1}^{N} t_k e^{(k)} \Big\| \geqslant \Big\| \sum_{k=1}^{N} t_k P_M(e^{(k)}) \Big\| \\ \geqslant \Big\| \sum_{k=1}^{N} t_k e_M^{(k)} \Big\| - \Big\| \sum_{k=1}^{N} t_k P_{M-1}(e^{(k)}) \Big\| \geqslant \tfrac{1}{6}\Big(\sum_{k=1}^{N} |t_k|^{q_M}\Big)^{q_M^{-1}}, \end{cases}$$

$$(16)\qquad \begin{cases} \Big\| \sum_{k=1}^{N} t_k e^{(k)} \Big\| \leqslant C^{-1} \Big\| \sum_{k=1}^{N} t_k P_M(e^{(k)}) \Big\| \\ \leqslant C^{-1} \Big\| \sum_{k=1}^{N} t_k e_M^{(k)} \Big\| + C^{-1} \Big\| \sum_{k=1}^{N} t_k P_{M-1}(e^{(k)}) \Big\| \\ \leqslant \dfrac{3}{2\cdot C} \Big(\sum_{k=1}^{N} |t_k|^{q_M}\Big)^{q_M^{-1}}. \end{cases}$$

It follows from (15) and (16) that the linear operator $u: l_{q_M} \to E$ defined by

$$u(s) = \sum_{k=1}^{\infty} s(k)\, e^{(k)} \qquad \text{for } s=\{s(k)\} \in l_{q_M}$$

satisfies the inequality

$$\frac{3}{2\cdot C}\|s\| \geqslant \|u(s)\| \geqslant \tfrac{1}{6}\|s\| \qquad \text{for } s \in l_{q_M}.$$

Hence u is an isomorphism from l_{q_M} onto a closed linear subspace of E. That completes the proof.

PROPOSITION 2. *There are separable infinite dimensional Banach spaces which are not almost C^1-diffeomorphic.*

This Proposition is an immediate consequence of the much stronger Proposition 2a stated below. We recall that a linear operator $u: X \to Y$

(X, Y Banach spaces) is *strictly singular* (cf. Kato [7]) if for every infinite dimensional linear subspace E of X, $\inf_{x \in E;\ \|x\|=1} \|u(x)\| = 0$.

PROPOSITION 2a. *Let X and Y be Banach spaces such that*: 1) *X contains a closed linear subspace isomorphic to l_1,* 2) *every linear operator from X into Y is strictly singular. Then there is no* 1*-differentiable homeomorphism from X onto Y.*

Proof. According to BONIC and FRAMPTON [4, Theorem 5] the conditions 1) and 2) imply that if $h : X \to Y$ is a differentiable map, then $h(0) \in cl W$, where $W = h(\{x \in X : \|x\| = 1\})$ and $cl W$ denotes the norm closure of W. Choose y_n in W so that $\lim_n y_n = h(0)$ and let $y_n = h(x_n)$ with $\|x_n\| = 1$ $(n = 1, 2, \ldots)$. If h were a homeomorphism, then we would have $\lim_n x_n = \lim_n h^{-1}(y_n) = h^{-1}(\lim_n y_n) = h^{-1}(h(0)) = 0$. But this leads to a contradiction with the condition $\|x_n\| = 1$ $(n = 1, 2, \ldots)$. That completes the proof.

COROLLARY. *Let either* a) *X be an infinite dimensional abstract L-space and let Y be an arbitrary Banach space no subspace of which is isomorphic to l_1, or* b) *$X = C(S)$ for some uncountable compact metric space S and let Y be an arbitrary Banach space no subspace of which is isomorphic to the space c_0 of all convergent to zero scalar-valued sequences. Then X and Y satisfy the conditions* 1) *and* 2) *of Proposition* 2a. *Hence X and Y are not almost C^1-diffeomorphic.*

This Corollary is an immediate consequence of the results of [11]. Observe (to verify condition 2) in the case b)) that if S is an uncountable compact metric space, then every separable Banach space, in particular l_1, is isomorphic to a closed linear subspace of $C(S)$).

In particular we have

The Hilbert space l_2 is not almost C^1-diffeomorphic neither to l_1 nor to $C(S)$ for any uncountable compact metric space S.

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