J. Math. Anal. Appl. 366 (2010) 310–318

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On fractional iterates of a Brouwer homeomorphism embeddable in a flow

Zbigniew Leśniak

Institute of Mathematics, Pedagogical University, Podchorążych 2, 30-084 Kraków, Poland

ARTICLE INFO

Article history:
Received 5 May 2009
Available online 29 December 2009
Submitted by Y. Huang

Keywords:
Fractional iterates
Brouwer homeomorphism
Flow
Parallelizable region
First prolongational limit set

ABSTRACT

We present a method for finding continuous (and consequently homeomorphic) orientation preserving iterative roots of a Brouwer homeomorphism which is embeddable in a flow. To obtain the roots we use a countable family of maximal parallelizable regions of the flow which is a cover of the plane. The maximal parallelizable regions are unions of equivalence classes of an appropriate equivalence relation. We show that if an equivalence class is invariant under the nth iterate of a Brouwer homeomorphism g, then it is invariant under g. We use this fact to prove that each maximal parallelizable region of the flow must be invariant under all homeomorphic orientation preserving iterative roots of the given Brouwer homeomorphism.

1. Introduction

In this paper we give the form of all homeomorphic orientation preserving solutions g of the functional equation

$$g^n = f, \tag{1}$$

where n is a positive integer and f is a given Brouwer homeomorphism of $\mathbb{R}^2$ (i.e. it is a homeomorphism of the plane onto itself without fixed points which preserves orientation) embeddable in a flow $\{f^t\colon t \in \mathbb{R}\}$.

Difficulties in dealing with the problem of determining solutions of Eq. (1), where f is a given function, were described in the survey paper [3] by Karol Baron and Witold Jarczyk. All continuous solutions of Eq. (1) were constructed by Marek Kuczma [14] in the case where the given function f is a homeomorphism of a real interval. The existence and properties of iterative roots of continuous piecewise monotone (and piecewise linear) maps of an interval were studied by Alexander Blokh, Ethan Coven, Michał Misiurewicz and Zbigniew Nitecki [4], by Jingzhong Zhang and Lu Yang [34], by Lin Li, Dilian Yang and Weinian Zhang [22] and by Weinian Zhang [36]. The existence of globally smooth iterative roots were investigated by Weinian Zhang [35].

Wanxiong Zhang and Weinian Zhang [37] gave an algorithm to compute piecewise linear solutions of Eq. (1) in the case where f is a piecewise linear function with finitely many non-monotone points. For linear fractional transformation f an algorithm to compute iterative roots was given by Richard J. Martin [23]. Under the assumption that f is a homeomorphism of the unit circle, Eq. (1) was studied by Krzysztof Ciepliński and Marek C. Zdun [6], Witold Jarczyk [10], Paweł Solarz [30,31,33,32] and by Marek C. Zdun [38].

Eq. (1) has been extensively studied in the class of complex analytic functions. Carl C. Cowen [7] considered this equation for an analytic mapping f of the open unit disc into itself and showed that such a map can be intertwined with a linear fractional transformation. Mark Elin, Victor Goryainov, Simeon Reich and David Shoikhet [8] gave criteria for the existence of

E-mail address: zlesniak@ap.krakow.pl.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.12.033

fractional iterates for a holomorphic self-mapping of the open unit disc. All meromorphic iterative roots of a linear fractional transformation on $\mathbb{C}$ were constructed by YongGuo Shi and Li Chen [29].

Tony Narayaninsamy [26,27] studied Eq. (1) for some non-bijective continuous mapping $f : M \to \mathbb{R}^n$, where M is a bounded convex subset of $\mathbb{R}^n$. Fractional iterates of mappings defined on some neighbourhood of $U \subset \mathbb{R}^n$ of zero which are given by formal series with zero free coefficient were investigated by Semeon Bogatyi [5]. Witold Jarczyk and Weinian Zhang [11] considered the existence of iterative roots of set-valued functions defined on an arbitrary nonempty set.

The general form of a continuous solution of Eq. (1) under the assumption that the given function f is a Sperner homeomorphism (i.e. a homeomorphism of the plane which is topologically conjugate with a translation) was given in [15,19]. The construction is based on the idea of Marek Kuczma [14]. Such a construction was possible since Sperner homeomorphisms have a similar properties to homeomorphisms of an interval. However, the behaviour of Brouwer homeomorphisms is much more complicated. The structure of an arbitrary Brouwer homeomorphism was described by Tatsuo Homma and Hidetaka Terasaka [12]. Hiromichi Nakayama [25] and Yoon Hoe Goo [9] gave an example of a Brouwer homeomorphism which have no iterative roots of order 2. In the present paper we restrict our attention to Brouwer homeomorphisms which are embeddable in a flow. We find all orientation preserving homeomorphic solutions of Eq. (1). In the construction we use the structure theorem for Brouwer homeomorphisms that are embeddable in a flow given in [18].

2. Maximal parallelizable regions

Throughout this section we assume that f is a Brouwer homeomorphism which is embeddable in a flow $\{f^t\colon t \in \mathbb{R}\}$. It follows from the Jordan theorem that each orbit C of $\{f^t\colon t \in \mathbb{R}\}$ divide the plane into two simply connected regions, since $f^t(p) \to \infty$ as $t \to \pm\infty$ for each $p \in \mathbb{R}^2$. Note that each of them is invariant under f^t for $t \in \mathbb{R}$. Thus two different orbits C_p and C_q of points p and q, respectively, divide the plane into three simply connected invariant regions, one of which contains both C_p and C_q in its boundary. We will call this region by the *strip* between C_p and C_q and denote by D_{pq}.

For any distinct orbits C_{p_1}, C_{p_2}, C_{p_3} of $\{f^t\colon t \in \mathbb{R}\}$ one of the following two possibilities must be satisfied: exactly one of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two or each of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two. In the first case if C_{p_j} is the orbit which lies in the strip between C_{p_i} and C_{p_k} we will write $C_{p_i}|C_{p_j}|C_{p_k}$ ($i, j, k \in \{1, 2, 3\}$ and i, j, k are different). In the second case we will write $|C_{p_i}, C_{p_j}, C_{p_k}|$ (see [13]).

Put

$$J^+(q) := \big\{p \in \mathbb{R}^2\colon \text{there exist a sequence } (q_n)_{n\in\mathbb{Z}^+} \text{ and a sequence } (t_n)_{n\in\mathbb{Z}^+} \text{ such that } q_n \to q, t_n \to +\infty, f^{t_n}(q_n) \to p \text{ as } n \to +\infty\big\},$$

$$J^-(q) := \big\{p \in \mathbb{R}^2\colon \text{there exist a sequence } (q_n)_{n\in\mathbb{Z}^+} \text{ and a sequence } (t_n)_{n\in\mathbb{Z}^+} \text{ such that } q_n \to q, t_n \to -\infty, f^{t_n}(q_n) \to p \text{ as } n \to +\infty\big\}.$$

The set $J(q) := J^+(q) \cup J^-(q)$ is called the *first prolongational limit set* of q. Let us observe that $p \in J(q)$ if and only if $q \in J(p)$ for any $p, q \in \mathbb{R}^2$. For a subset $H \subset \mathbb{R}^2$ we define

$$J(H) := \bigcup_{q\in H} J(q).$$

One can observe that for each $p \in \mathbb{R}^2$ the set $J(p)$ is invariant.

An invariant region $M \subset \mathbb{R}^2$ is said to be *parallelizable* if there exists a homeomorphism ψ mapping M onto $\mathbb{R}^2$ such that

$$f^t(x) = \psi^{-1}\big(\psi(x) + (t, 0)\big) \quad \text{for } x \in M.$$

The homeomorphism ψ occurring in this equality will be called a *parallelizing homeomorphism* of M. It is known that a region M is parallelizable if and only if there exists a homeomorphic image K of a straight line which is a closed set in M such that K has exactly one common point with every orbit of $\{f^t\colon t \in \mathbb{R}\}$ contained in M (see [2, p. 49] and e.g. [21]). Such a set K we will call a *section* in M.

It is known that a region M is parallelizable if and only if $J(M) \cap M = \emptyset$ (see [2, pp. 46 and 49]). Hence for every parallelizable region M we have $J(M) \subset \operatorname{fr} M$. If M is a maximal parallelizable region (i.e. M is not contained properly in any parallelizable region), then $J(M) = \operatorname{fr} M$ (see [24]).

Let $\alpha = (p_1, \ldots, p_n)$ be a sequence of integers. Then, for any integer k by $\alpha * k$ will be denoted the concatenation of the sequences α and the one-element sequence k (for one-element sequences we omit parentheses), i.e. the sequence $(p_1, \ldots, p_n, k)$.

A class A^+ of finite sequences α of positive integers will be termed *admissible* if the following conditions hold:

(1) A^+ contains the sequence: 1, and no other one-element sequence;
(2) if $\alpha * k$ is in A^+ and $k > 1$, then so also is $\alpha * (k-1)$;
(3) if $\alpha * 1$ is in A^+, then so also is α.

A class A^- of finite sequences α of negative integers will be termed *admissible* if the following conditions hold:

(1) A^- contains the sequence: -1, and no other one-element sequence;
(2) if $\alpha * k$ is in A^- and $k < -1$, then so also is $\alpha * (k+1)$;
(3) if $\alpha * -1$ is in A^-, then so also is α.

The set $A := A^+ \cup A^-$, where A^+, A^- are some admissible classes of finite sequences of positive, negative integers, respectively, will be said to be *admissible class of finite sequences*.

Lemma 2.1. *(See [18].) Let $\{f^t\colon t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let $p \in \mathbb{R}^2$. Then there exists an at most countable family of maximal parallelizable regions $\{M_j\colon j \in J\}$, where J is the set of all positive integers or $J = \{1, \ldots, N\}$ for some positive integer N, such that $p \in M_1$ and for each positive integer n the set* $\operatorname{cl} B(p,n)$, *where $B(p,n)$ is the ball centered at p with radius n, is covered by a finite subfamily $\{M_1, \ldots, M_{j_n}\}$ of $\{M_j\colon j \in J\}$. Moreover, $j_n \leqslant j_{n+1}$ for every n.*

Theorem 2.2. *(See [18].) Let $\{f^t\colon t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then there exist a family of orbits $\{C_\alpha\colon \alpha \in A\}$ and a family of maximal parallelizable regions $\{M_\alpha\colon \alpha \in A\}$, where $A = A^+ \cup A^-$ is an at most countable admissible class of finite sequences, such that*

$$C_\alpha \subset M_\alpha \quad \text{for } \alpha \in A,$$
$$\bigcup_{\alpha \in A} M_\alpha = \mathbb{R}^2$$

and

$$f^t(x) = \psi_\alpha^{-1}\bigl(\psi_\alpha(x) + (t,0)\bigr) \quad \text{for } x \in M_\alpha,\ t \in \mathbb{R}$$

for arbitrarily chosen parallelizing homeomorphism ψ_α of M_α. Moreover, the families can be constructed in such a way that

$$M_\alpha \cap M_{\alpha * i} \neq \emptyset \quad \text{for } \alpha * i \in A,$$
$$C_{\alpha * i} \subset J(M_\alpha) \quad \text{for } \alpha * i \in A,$$
$$|C_\alpha, C_{\alpha * i_1}, C_{\alpha * i_2}| \quad \text{for } \alpha * i_1, \alpha * i_2 \in A,\ i_1 \neq i_2,$$
$$C_\alpha | C_{\alpha * i} | C_{\alpha * i * l} \quad \text{for } \alpha * i * l \in A.$$

The construction of the families occurring in Theorem 2.2 starts from the orbit $C_1 = C_{-1}$ of an arbitrary point $p \in \mathbb{R}^2$ and the maximal parallelizable region M_1 occurring in Lemma 2.1 (we take $M_{-1} = M_1$ and the same parallelizing homeomorphisms $\psi_1 = \psi_{-1}$ of M_1). Having constructed an $\alpha \in A$ and C_α, M_α, we index bijectively the set of all orbits contained in

$$\operatorname{fr} M_\alpha \cap H_\alpha,$$

where H_1, H_{-1} are components of $\mathbb{R}^2 \setminus C_1$ and for all $\alpha = \beta * l \in A$ the set H_α is the components of $\mathbb{R}^2 \setminus C_\alpha$ which has no common point with M_β, by sequences of the form $\alpha * k$ starting from $k = 1$ and taking subsequent positive integers k if $\alpha \in A^+$ and starting from $k = -1$ and taking subsequent negative integers k if $\alpha \in A^-$. We enlarge the set A by all sequences $\alpha * k$ and for each such a sequence $\alpha * k$ we denote by $C_{\alpha * k}$ the orbit indexed by $\alpha * k$ and take as $M_{\alpha * k}$ an element of the subfamily $\{M_j\colon j = 1, \ldots, j_{m_{\alpha * k}}\}$ of the family occurring in Lemma 2.1, where $m_{\alpha * k}$ is the smallest integer which is greater or equal to the distance of the orbit $C_{\alpha * k}$ from p. Moreover, we only consider such parallelizing homeomorphism $\psi_\alpha : M_\alpha \to \mathbb{R}^2$ that $\psi_\alpha(C_\alpha) = \mathbb{R} \times \{0\}$ and

$$\psi_\alpha(M_\alpha \cap H_\alpha) = \mathbb{R} \times (0, +\infty) \quad \text{if } \alpha \in A^+$$

and

$$\psi_\alpha(M_\alpha \cap H_\alpha) = \mathbb{R} \times (-\infty, 0) \quad \text{if } \alpha \in A^-.$$

Corollary 2.3. *(See [18].) Let $\{f^t\colon t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then there exists a family of connected subsets of the plane $\{U_\alpha\colon \alpha \in A\}$, where $A = A^+ \cup A^-$ is the admissible class of finite sequences occurring in Theorem 2.2, such that*

$$\bigcup_{\alpha \in A} U_\alpha = \mathbb{R}^2,$$

$U_\alpha = C_\alpha \cup N_\alpha$, where $N_\alpha = M_\alpha \cap H_\alpha$ and C_α, M_α and H_α are those occurring in Theorem 2.2, and $\operatorname{fr} U_\alpha = C_\alpha \cup \bigcup_{\alpha * i \in A} C_{\alpha * i}$ for $\alpha \in A$, $N_{\alpha_1} \cap N_{\alpha_2} = \emptyset$ for distinct $\alpha_1, \alpha_2 \in A$, $C_\alpha \subset J(\mathbb{R}^2)$ for $\alpha \in A \setminus \{-1, 1\}$, $C_1 = C_{-1}$, $C_{\alpha_1} \neq C_{\alpha_2}$ for distinct $\alpha_1, \alpha_2 \in A$ satisfying at least one of the conditions $\alpha_1 \notin \{-1, 1\}$, $\alpha_2 \notin \{-1, 1\}$ and

$$f^t(x) = \varphi_\alpha^{-1}\big(\varphi_\alpha(x) + (t, 0)\big), \quad x \in U_\alpha,\ t \in \mathbb{R}$$

for some homeomorphisms

$$\varphi_\alpha : U_\alpha \xrightarrow{\text{onto}} \mathbb{R} \times [0, +\infty) \quad \text{for } \alpha \in A^+,$$
$$\varphi_\alpha : U_\alpha \xrightarrow{\text{onto}} \mathbb{R} \times (-\infty, 0] \quad \text{for } \alpha \in A^-.$$

3. Invariance of equivalence classes

Now we proceed to the functional equation (1). It turns out that under our assumption on f every continuous solution of (1) is a homeomorphism. To prove this fact one can use the invariance of domain theorem, since directly from Eq. (1) we obtain that g is a bijective map.

Proposition 3.1. *(See [19].) If f is a homeomorphism of $\mathbb{R}^2$ onto itself, g is continuous and $g^n = f$ for some positive integer n, then g is also a homeomorphism of $\mathbb{R}^2$ onto itself. Moreover, if f has no fixed points, then g has no fixed points.*

Proposition 3.2. *(See [19].) Let f be a homeomorphism of $\mathbb{R}^2$ onto itself which preserves orientation. Let g be a continuous function defined on $\mathbb{R}^2$ such that $g^n = f$ for some odd positive integer n. Then g is a homeomorphism of $\mathbb{R}^2$ onto itself which preserves orientation.*

For even n Eq. (1) can have also orientation reversing solutions. In this paper we are interested only in orientation preserving homeomorphisms g, so from now on we will assume that g is a Brouwer homeomorphism.

For an arbitrary Brouwer homeomorphism g we consider an equivalence relation in $\mathbb{R}^2$ defined in the following way:

$$p \sim q \quad \text{if } p = q \text{ or } p \text{ and } q \text{ are endpoints of some arc } K \text{ for which } g^m(K) \to \infty \text{ as } m \to \pm\infty$$

(see [1,16]). By an *arc* K with endpoints p and q we mean the image of a homeomorphism $c : [0, 1] \to c([0, 1])$ satisfying conditions $c(0) = p$, $c(1) = q$, where the topology on $c([0, 1])$ is induced by the topology of $\mathbb{R}^2$.

Proposition 3.3. *Let g be a Brouwer homeomorphism and n be a positive integer. Then the Brouwer homeomorphisms g and g^n have the same equivalence classes.*

Proof. Put $f = g^n$. Let $x, y \in \mathbb{R}^2$ belong to an equivalence class of g. Then there exists an arc K with endpoints x and y such that $g^m(K) \to \infty$ as $m \to \pm\infty$. Hence $g^{kn}(K) \to \infty$ as $k \to \pm\infty$. Since $g^n = f$, we have $g^{kn}(K) = f^k(K)$. Thus $f^k(K) \to \infty$ as $k \to \pm\infty$, and consequently x, y belong to an equivalence class of f.

Fix $x, y \in \mathbb{R}^2$ belonging to an equivalence class of f. Then there exists an arc K with endpoints x and y such that $f^m(K) \to \infty$ as $m \to \pm\infty$. We will show that $g^k(K) \to \infty$ as $k \to \pm\infty$. Since g is a homeomorphism of the plane onto itself, it can be prolonged to a homeomorphism of S^2 onto itself by putting $g(\infty) = \infty$. Fix a ball $B(p, R)$ centered at a $p \in \mathbb{R}^2$ with radius $R > 0$. Take an $R_0 > R$. Then

$$\mathbb{R}^2 \setminus \operatorname{cl} B(p, R_0) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R).$$

By continuity of $g, g^2, \ldots, g^{n-1}$ at ∞, for every $i \in \{1, 2, \ldots, n-1\}$ there exists a real $R_i > 0$ such that

$$g^i\big(\mathbb{R}^2 \setminus \operatorname{cl} B(p, R_i)\big) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R).$$

Put $R_n = \max\{R_0, R_1, \ldots, R_{n-1}\}$. Then for every $i \in \{0, 1, \ldots, n-1\}$

$$\mathbb{R}^2 \setminus \operatorname{cl} B(p, R_n) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R_i)$$

and consequently

$$g^i\big(\mathbb{R}^2 \setminus \operatorname{cl} B(p, R_n)\big) \subset g^i\big(\mathbb{R}^2 \setminus \operatorname{cl} B(p, R_i)\big).$$

Thus for every $i \in \{0, 1, \ldots, n-1\}$

$$g^i\big(\mathbb{R}^2 \setminus \operatorname{cl} B(p, R_n)\big) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R).$$

From the fact that $f^m(K) \to \infty$ as $m \to \pm\infty$ it follows that there exists a positive integer N_0 such that

$$f^m(K) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R_n)$$

for all $m \in \mathbb{Z}$ such that $|m| \geqslant N_0$. Then for all $i \in \{0, 1, \ldots, n-1\}$ and m such that $|m| \geqslant N_0$ we have

$$g^i\big(f^m(K)\big) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R).$$

Hence

$$g^{i+nm}(K) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R)$$

for all $i \in \{0, 1, \ldots, n-1\}$ and m such that $|m| \geqslant N_0$, since $f = g^n$. Thus

$$g^k(K) \subset \mathbb{R}^2 \setminus \operatorname{cl} B(p, R)$$

for every integer k such that $k \geqslant nN_0$ and $k \leqslant -nN_0 + n - 1$. Hence $g^k(K) \to \infty$ as $k \to \pm\infty$, and consequently x, y belong to the same equivalence class of g. □

Proposition 3.4. *Let $\{G_i\}_{i\in I}$ be the family of all equivalence classes of the relation $\sim$ defined for a Brouwer homeomorphism g. Then for every $i \in I$ there exists a $j \in I$ such that $g(G_i) = G_j$.*

Proof. Fix an $i \in I$ and a $p_0 \in G_i$. Then there exists a $j \in I$ such that $g(p_0) \in G_j$. Take a $p \in G_i$. We will show that $g(p) \in G_j$. From the definition of $\sim$ we obtain that there exists an arc K_1 having p_0 and p as its endpoints such that

$$g^m(K_1) \to \infty \quad \text{as } m \to \pm\infty.$$

Put $L_1 = g(K_1)$. Then $g(p_0)$ and $g(p)$ are endpoints of L_1 and

$$g^m(L_1) \to \infty \quad \text{as } m \to \pm\infty.$$

Hence $g(p) \in G_j$, and consequently

$$g(G_i) \subset G_j.$$

Take a $q \in G_j$. Then there exists an arc L_2 such that $g(p_0)$ and q are its endpoints and

$$g^m(L_2) \to \infty \quad \text{as } m \to \pm\infty.$$

Put $K_2 = g^{-1}(L_2)$. Then p_0 and $g^{-1}(q)$ are endpoints of K_2 and

$$g^m(K_2) \to \infty \quad \text{as } m \to \pm\infty.$$

Thus $g^{-1}(q) \in G_i$, and hence $q \in g(G_i)$. Consequently

$$G_j \subset g(G_i). \qquad \square$$

To prove a sufficient condition for invariance of an equivalence class we will use the following proposition.

Proposition 3.5. *(See [1].) Let g be a Brouwer homeomorphism of $\mathbb{R}^2$ and assume that $\{E_i\}_{i\in I}$ is a finite collection of disjoint arcwise connected sets. If each $g(E_i)$ is equal to some E_j, then*

$$g(E_i) = E_i \quad \text{for } i \in I.$$

Proposition 3.6. *Let g be a Brouwer homeomorphism of $\mathbb{R}^2$ and n be a positive integer. Then for every equivalence class G_0 of the relation $\sim$ the equality $g^n(G_0) = G_0$ implies that $g(G_0) = G_0$.*

Proof. Let G_0 be an equivalence class of the relation $\sim$ such that $g^n(G_0) = G_0$. Put $G_m = g^m(G_0)$ for all $m \in \mathbb{Z}$. Then, by Proposition 3.4, G_m is an equivalence class for every $m \in \mathbb{Z}$. Then $G_r = G_{r+kn}$ for all $k \in \mathbb{Z}$ and $r \in \{1, \ldots, n-1\}$, since $G_n = G_0$. Thus the family $\{G_m\colon m \in \mathbb{Z}\}$ contains at most n distinct equivalence classes. Consequently, by Proposition 3.5, we have that G_m is invariant under g for each $m \in \mathbb{Z}$ (each equivalence class is arcwise connected directly from the definition of the relation). In particular, $g(G_0) = G_0$. □

From the above proposition we obtain the following result.

Corollary 3.7. *If f is a Brouwer homeomorphism of $\mathbb{R}^2$. Let G_0 be an equivalence class such that $f(G_0) = G_0$. Assume that g is a Brouwer homeomorphism of $\mathbb{R}^2$ such that $g^n = f$ for some positive integer n. Then $g(G_0) = G_0$.*

Proof. Since G_0 is invariant under f and $g^n = f$, we have that G_0 is invariant under g^n. Thus by Proposition 3.6, G_0 is invariant under g. □

To obtain the next corollary we will use Corollary 3.7 and the following proposition.

Proposition 3.8. *(See [16].) Let $\{f^t\colon t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then each equivalence class of the relation $\sim$ defined for f^1 is invariant under f^t for $t \in \mathbb{R}$.*

Corollary 3.9. *If f is a Brouwer homeomorphism of $\mathbb{R}^2$ which is embeddable in a flow, g is a Brouwer homeomorphism of $\mathbb{R}^2$ and $g^n = f$ for some positive integer n, then $g(G_0) = G_0$ for every equivalence class G_0 of the relation $\sim$.*

To obtain the next corollary we use the following result.

Proposition 3.10. *(See [20].) A maximal parallelizable region M of $\{f^t\colon t \in \mathbb{R}\}$ is a union of equivalence classes of the relation $\sim$.*

Corollary 3.11. *If f is a Brouwer homeomorphism of $\mathbb{R}^2$ which is embeddable in a flow $\{f^t\colon t \in \mathbb{R}\}$, g is a Brouwer homeomorphism of $\mathbb{R}^2$ and $g^n = f$ for some positive integer n, then $g(M) = M$ for every maximal parallelizable region M of $\{f^t\colon t \in \mathbb{R}\}$.*

Moreover, using the next two results we will prove the invariance of the boundary of each equivalence class.

Proposition 3.12. *(See [21].) Let $\{f^t\colon t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Let G be an equivalence class and $p \in G \cap \operatorname{fr} G$. Then the whole orbit of p is contained in $G \cap \operatorname{fr} G$.*

Proposition 3.13. *(See [21].) Let $\{f^t\colon t \in \mathbb{R}\}$ be a flow of Brouwer homeomorphisms. Then the boundary of each equivalence class is a union of a family of orbits and each equivalence class can contain at most two orbits that are contained in its boundary.*

Proposition 3.14. *Let f be a Brouwer homeomorphism of $\mathbb{R}^2$ which is embeddable in a flow $\{f^t\colon t \in \mathbb{R}\}$. Let g be a Brouwer homeomorphism of $\mathbb{R}^2$ such that $g^n = f$ for some positive integer n. Then for each $p \in \mathbb{R}^2$ contained in the boundary of an equivalence class of the relation $\sim$ defined for f the orbit C_p of p is invariant under g.*

Proof. Let $p \in \operatorname{fr} G_1$, where G_1 is an equivalence class of the relation $\sim$. Denote by G_2 the equivalence class which contains p (it may happen that $G_2 = G_1$). Then $p \in \operatorname{fr} G_2$. By Proposition 3.12, the orbit C_p of p is contained $G_2 \cap \operatorname{fr} G_2$. On account of Corollary 3.7, we have $g(G_2) = G_2$ and consequently $g(\operatorname{int} G_2) = \operatorname{int} G_2$. Hence $g(G_2 \cap \operatorname{fr} G_2) = G_2 \cap \operatorname{fr} G_2$.

By Proposition 3.13 the set $G_2 \cap \operatorname{fr} G_2$ is either an orbit or a union of two orbits. In case $G_2 \cap \operatorname{fr} G_2 = C_p$, we have $g(C_p) = C_p$, since $g(G_2 \cap \operatorname{fr} G_2) = G_2 \cap \operatorname{fr} G_2$. Now, we consider the case where $G_2 \cap \operatorname{fr} G_2 = C_p \cup C_q$ for some $q \in (G_2 \cap \operatorname{fr} G_2) \setminus C_p$. Then either $g(C_p) \subset C_p$ or $g(C_p) \subset C_q$, since $g(C_p)$ is a connected set. Similarly, either $g(C_q) \subset C_q$ or $g(C_q) \subset C_p$. Hence either $g(C_p) = C_p$ and $g(C_q) = C_q$ or $g(C_p) = C_q$ and $g(C_q) = C_p$. Using Proposition 3.5 to the family $\{C_p, C_q\}$ we get that the second possibility cannot hold. Thus $g(C_p) = C_p$ and $g(C_q) = C_q$. □

4. Form of fractional iterates

In this section we give the form of all solutions of Eq. (1) which are Brouwer homeomorphisms. We start from recalling such a result for the case where the given function f is a Sperner homeomorphism (i.e. a Brouwer homeomorphism having exactly one equivalence class). It follows from Proposition 3.3 that in this case every Brouwer homeomorphism satisfying Eq. (1) is a Sperner homeomorphism.

Theorem 4.1. *(See [19].) Let f be a Sperner homeomorphism of $\mathbb{R}^2$. Then for every positive integer n the Brouwer homeomorphism g is a solution of Eq.* (1) *if and only if it can be expressed in the form*

$$g = \varphi^{-1} \circ T_{\frac{1}{n}} \circ \varphi,$$

where φ is a homeomorphic solution of the Abel equation

$$\varphi\bigl(f(x)\bigr) = \varphi(x) + (1, 0), \quad x \in \mathbb{R}^2$$

and

$$T_{\frac{1}{n}}(x_1, x_2) := \left(x_1 + \frac{1}{n}, x_2\right) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2. \tag{2}$$

An immediate consequence of Theorem 4.1 is the following result.

Corollary 4.2. *Let f be a Sperner homeomorphism of $\mathbb{R}^2$. Then for every positive integer n and every Brouwer homeomorphism g satisfying Eq.* (1) *there exists a flow $\{f^t\colon t \in \mathbb{R}\}$ such that $f = f^1$ and $g = f^{\frac{1}{n}}$.*

From Theorem 4.1 we can also obtain the following form of the iterative roots of f on each of the maximal parallelizable regions of the cover occurring in Theorem 2.2.

Corollary 4.3. *Let f be a Brouwer homeomorphism of $\mathbb{R}^2$ which is embeddable in a flow and g be a Brouwer homeomorphism satisfying Eq.* (1). *Let $\{M_\alpha\colon \alpha \in A\}$ be a family of maximal parallelizable regions occurring in Theorem* 2.2. *Then for each $\alpha \in A$ there exists a homeomorphism $\psi_\alpha : M_\alpha \to \mathbb{R}^2$ such that*

$$g(x) = \left(\psi_\alpha^{-1} \circ T_{\frac{1}{n}} \circ \psi_\alpha\right)(x), \quad x \in M_\alpha.$$

Proof. Fix an $\alpha \in A$. Then f restricted to M_α is a Sperner homeomorphism of M_α. Since M_α is simply connected region, it is homeomorphic to the whole plane. Therefore we can use Theorem 4.1 which gives the form of g. □

Proposition 4.4. *(See [13].) Let F be a family of homeomorphic images of a straight line which are closed sets in the plane such that for every $p \in \mathbb{R}^2$ there exists exactly one $C \in F$ such that $p \in C$ and for every $p \in \mathbb{R}^2$ there exists an open set U_p containing p which can be mapped homeomorphically on the open square $\{(x, y) \in \mathbb{R}^2\colon |x| < 1,\ |y| < 1\}$ in such a way that the images of the intersections of elements of F with U_p are the sets $\{(x, y) \in \mathbb{R}^2\colon |x| < 1,\ y = c\}$ for some $c \in \mathbb{R}$ such that $|c| < 1$. Assume that S is an infinite subfamily of F such that for all distinct $C_1, C_2, C_3 \in S$ the relation $|C_1, C_2, C_3|$ holds. Then S is countable and every compact set K has a common point with a finite number of elements of the family S.*

Proposition 4.5. *(See [17].) Let M be a parallelizable region of $\{f^t\colon t \in \mathbb{R}\}$. Let $r \in M$ and H be a component of $\mathbb{R}^2 \setminus C_r$. Then for all distinct orbits C_{p_1}, C_{p_2} contained in* $\operatorname{fr} M \cap H$ *the relation $|C_{p_1}, C_{p_2}, C_r|$ holds.*

Lemma 4.6. *Let f be a Brouwer homeomorphism of $\mathbb{R}^2$ which is embeddable in a flow $\{f^t\colon t \in \mathbb{R}\}$. Let $\{C_\alpha\colon \alpha \in A\}$ and $\{M_\alpha\colon \alpha \in A\}$ be families occurring in Theorem* 2.2. *Then for every $\alpha \in A$ and every $p \in C_\alpha$ there exists an $\varepsilon > 0$ such that the ball $B(p, \varepsilon)$ centered at p with radius ε has common points with exactly two elements of the family $\{U_\alpha\colon \alpha \in A\}$ occurring in Corollary* 2.3 *(i.e. with U_1 and U_{-1} for $\alpha \in \{1, -1\}$ and with U_β and $U_{\beta*k}$ for $\alpha = \beta * k$).*

Proof. Let F be the family of all orbits of the flow $\{f^t\colon t \in \mathbb{R}\}$. Then the family F satisfies the assumptions of Proposition 4.4, since $\{f^t\colon t \in \mathbb{R}\}$ is a flow of Brouwer homeomorphisms. Put $S_1 = \{C_1\} \cup \{C_{1*i}\colon 1 * i \in A\}$, $S_{-1} = \{C_{-1}\} \cup \{C_{-1*i,}\colon -1 * i \in A\}$, $S_\beta = \{C_\beta\} \cup \{C_{\beta*i}\colon \beta * i \in A\}$ and $S_{\beta*k} = \{C_{\beta*k}\} \cup \{C_{\beta*(k,l)}\colon \beta * (k, l) \in A\}$. By Proposition 4.5, the subfamilies S_1, S_{-1}, S_β and $S_{\beta*k}$ of F satisfy the assumptions of Proposition 4.4.

Let $p \in C_1$. Then $p \in C_{-1}$, since $C_1 = C_{-1}$. On account of Proposition 4.4 the closure of the ball $B(p, 1)$ centered at p with radius 1 has common points with finitely many elements of the families S_1 and S_{-1}. Hence there exists an $\varepsilon > 0$ such that $B(p, \varepsilon) \cap (\bigcup(S_1 \cup S_{-1}) \setminus C_1) = \emptyset$, since $\operatorname{cl} B(p, 1) \cap (\bigcup(S_1 \cup S_{-1}) \setminus C_1)$ is a compact set which does not contain p. Since $B(p, \varepsilon) \cap (\bigcup S_1 \setminus C_1) = \emptyset$, the set $B(p, \varepsilon) \cap H_1$ is contained in U_1, where H_1 is the region occurring in Theorem 2.2. Similarly, since $B(p, \varepsilon) \cap (\bigcup S_{-1} \setminus C_{-1}) = \emptyset$, the set $B(p, \varepsilon) \cap H_{-1}$ is contained in U_{-1}. Thus $B(p, \varepsilon) \subset M_1$.

Let $p \in C_\alpha$, where $\alpha = \beta * k$. On account of Proposition 4.4 the closure of the ball $B(p, 1)$ has common points with finitely many elements of the families S_β and $S_{\beta*k}$. Hence there exists an $\varepsilon > 0$ such that $B(p, \varepsilon) \cap (\bigcup(S_\beta \cup S_{\beta*k}) \setminus C_{\beta*k}) = \emptyset$, since $\operatorname{cl} B(p, 1) \cap (\bigcup(S_\beta \cup S_{\beta*k}) \setminus C_{\beta*k})$ is a compact set which does not contain p. Since $B(p, \varepsilon) \cap (\bigcup S_{\beta*k} \setminus C_{\beta*k}) = \emptyset$, the set $B(p, \varepsilon) \cap H_{\beta*k}$ is contained in $U_{\beta*k}$. Denote by $\tilde{H}_{\beta*k}$ the component of $\mathbb{R}^2 \setminus C_{\beta*k}$ which has no common points with $U_{\beta*k}$. Then $B(p, \varepsilon) \cap \tilde{H}_{\beta*k}$ is contained in U_β, since $B(p, \varepsilon) \cap (\bigcup S_\beta \setminus C_{\beta*k}) = \emptyset$. Consequently $B(p, \varepsilon)$ has common points with exactly two elements of the family $\{U_\alpha\colon \alpha \in A\}$, namely with U_β and $U_{\beta*k}$. □

Theorem 4.7. *Let f be a Brouwer homeomorphism of $\mathbb{R}^2$ which is embeddable in a flow and n be a positive integer. Let $\{C_\alpha\colon \alpha \in A\}$ and $\{M_\alpha\colon \alpha \in A\}$ be families occurring in Theorem* 2.2. *For each $\alpha \in A$ let $\{f_\alpha^t\colon t \in \mathbb{R}\}$ be a flow such that $f_\alpha^t : M_\alpha \to M_\alpha$ for $t \in \mathbb{R}$ and $f_\alpha^1(x) = f(x)$ for all $x \in M_\alpha$. Assume that $f_1^{\frac{1}{n}}(x) = f_{-1}^{\frac{1}{n}}(x)$ for every $x \in C_1 = C_{-1}$ and*

$$\lim_{k\to\infty} f_\alpha^{\frac{1}{n}}(x_k) = f_{\alpha*i}^{\frac{1}{n}}(x)$$

for each $x \in C_{\alpha * i}$ and every sequence $(x_k)_{k\in\mathbb{Z}^+}$ of elements of M_α such that $\lim_{k\to\infty} x_k = x$. Then the function g such that for every $\alpha \in A$

$$g(x) = f_\alpha^{\frac{1}{n}}(x), \quad x \in U_\alpha,$$

where $\{U_\alpha\colon \alpha \in A\}$ is the family occurring in Corollary 2.3, is a Brouwer homeomorphism of $\mathbb{R}^2$ satisfying Eq. (1). Moreover, each Brouwer homeomorphism of $\mathbb{R}^2$ satisfying Eq. (1) can be obtained in this way.

Proof. From the construction of the family $\{U_\alpha\colon \alpha \in A\}$ it follows that $\{U_\alpha\colon \alpha \in A^+\}$ and $\{U_\alpha\colon \alpha \in A^-\}$ consist of pairwise disjoint sets and

$$\bigcup_{\alpha\in A^+} U_\alpha = C_1 \cup H_1 \quad \text{and} \quad \bigcup_{\alpha\in A^-} U_\alpha = C_{-1} \cup H_{-1}.$$

The assumption that $f_1^{\frac{1}{n}}(x) = f_{-1}^{\frac{1}{n}}(x)$ for every $x \in C_1 = C_{-1}$, guaranties that the function g is well defined on the whole plane. From the definition of g and the assumption that $f_\alpha^1(x) = f(x)$ for all $x \in U_\alpha$ we obtain that g satisfies Eq. (1).

Now we will prove that g is continuous. Fix $x \in C_1 = C_{-1}$. Let $(x_m)_{m\in\mathbb{Z}^+}$ be a sequence such that $\lim_{m\to\infty} x_m = x$. On account of Lemma 4.6 there exists an $\varepsilon > 0$ such that the ball $B(p, \varepsilon)$ has common points with exactly two elements of the family $\{U_\alpha\colon \alpha \in A\}$, namely with U_1 and U_{-1}. Thus without loss of generality we can assume that $x_m \in U_1 \cup U_{-1}$ for all $m \in \mathbb{Z}^+$. If there is a subsequence $(x_{m_l})_{l\in\mathbb{Z}^+}$ of the sequence $(x_m)_{m\in\mathbb{Z}^+}$ such that each element of the subsequence belongs to U_1, then by the continuity of $f_1^{\frac{1}{n}}$ we have

$$\lim_{l\to\infty} g(x_{m_l}) = \lim_{l\to\infty} f_1^{\frac{1}{n}}(x_{m_l}) = f_1^{\frac{1}{n}}(x) = g(x).$$

Similarly, for a subsequence $(x_{m_l})_{l\in\mathbb{Z}^+}$ which consists of elements belonging to U_{-1} we have

$$\lim_{l\to\infty} g(x_{m_l}) = \lim_{l\to\infty} f_{-1}^{\frac{1}{n}}(x_{m_l}) = f_{-1}^{\frac{1}{n}}(x) = g(x).$$

Fix an $\alpha = \beta * k \in A$ and a point $p \in C_\alpha$. Let $(x_m)_{m\in\mathbb{Z}^+}$ be a sequence such that $\lim_{m\to\infty} x_m = x$. On account of Lemma 4.6 there exists an $\varepsilon > 0$ such that the ball $B(p, \varepsilon)$ has common points with exactly two elements of the family $\{U_\alpha\colon \alpha \in A\}$, namely with U_β and $U_{\beta*k}$. Thus without loss of generality we can assume that each elements of the sequence $(x_m)_{m\in\mathbb{Z}^+}$ belongs to $U_\beta \cup U_{\beta*k}$. For a subsequence $(x_{m_l})_{l\in\mathbb{Z}^+}$ of the sequence $(x_m)_{m\in\mathbb{Z}^+}$ which consists of elements belonging to $U_{\beta*k}$, by the continuity of $f_{\beta*k}^{\frac{1}{n}}$, we have

$$\lim_{l\to\infty} g(x_{m_l}) = \lim_{l\to\infty} f_{\beta*k}^{\frac{1}{n}}(x_{m_l}) = f_{\beta*k}^{\frac{1}{n}}(x) = g(x).$$

However, for a subsequence $(x_{m_l})_{l\in\mathbb{Z}^+}$ such that $x_{m_l} \in U_\beta$ for all $l \in \mathbb{Z}^+$ we use our assumption. Therefore

$$\lim_{l\to\infty} g(x_{m_l}) = \lim_{l\to\infty} f_\beta^{\frac{1}{n}}(x_{m_l}) = f_{\beta*k}^{\frac{1}{n}}(x) = g(x).$$

On account of Proposition 3.1 we have that g is a homeomorphism of the plane onto itself without fixed points. Now we use the fact that each homeomorphism of the plane either preserves orientation or reverses orientation and to check to which of the two classes a homeomorphism belongs it is sufficient take one arbitrary Jordan curve (see [28, p. 197]). Let J be a Jordan curve contained in M_1. Then J and $g(J) = f_1^{\frac{1}{n}}(J)$ have the same orientation, since every element of a flow preserves orientation. Thus g preserves orientation, and consequently g is a Brouwer homeomorphism.

Now let g be a Brouwer homeomorphism satisfying Eq. (1). Then for each $\alpha \in A$ the restrictions of f and g to M_α are Brouwer homeomorphisms on M_α. Since M_α is homeomorphic to $\mathbb{R}^2$, we can use Corollary 4.2. We obtain then that for each $\alpha \in A$ there exists a flow $\{f_\alpha^t\colon t \in \mathbb{R}\}$ such that $f_\alpha^t : M_\alpha \to M_\alpha$ for $t \in \mathbb{R}$ and $f_{|M_\alpha} = f_\alpha^1$, $g_{|M_\alpha} = f_\alpha^{\frac{1}{n}}$. Let $x \in C_1$. Then $f_1^{\frac{1}{n}}(x) = g(x) = f_{-1}^{\frac{1}{n}}(x)$. For all $\alpha * i \in A$ and $x \in C_{\alpha*i}$ the condition

$$\lim_{k\to\infty} f_\alpha^{\frac{1}{n}}(x_k) = f_{\alpha*i}^{\frac{1}{n}}(x)$$

is satisfied for every sequence $(x_k)_{k\in\mathbb{Z}^+}$ of elements of M_α such that $\lim_{k\to\infty} x_k = x$, since $f_{\alpha*i}^{\frac{1}{n}}$ is continuous and $f_\alpha^{\frac{1}{n}}(z) = g(z) = f_{\alpha*i}^{\frac{1}{n}}(z)$ for $z \in M_\alpha \cap M_{\alpha*i}$ (without loss of generality we can assume that each element of the sequence $(x_k)_{k\in\mathbb{Z}^+}$ belongs to $M_{\alpha*i}$, since $M_{\alpha*i}$ is a neighbourhood of x and $\lim_{k\to\infty} x_k = x$). $\square$

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