A new fourth-order compact finite difference scheme for the two-dimensional second-order hyperbolic equation

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\textbf{A B S T R A C T}

In this paper, we propose a three level compact difference scheme of \(O(\tau^4 + h^4)\) for the difference solution of a two-dimensional second-order non-homogeneous linear hyperbolic equation \(u_{tt} + 2\alpha u_t + \beta^2 u = u_{xx} + u_{yy} + f(x, y, t), 0 < x, y < 1, t > 0\), where \(\alpha > \beta \geq 0\). Stability analysis of the method has been carried out. Finally, numerical examples are used to illustrate the efficiency of the new difference scheme.

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\section{1. Introduction}

In recent years, much attention has been given in the literature to the development, analysis, and implementation of finite difference schemes for the numerical solution of second-order hyperbolic equations; see, for example, [1–4]. Recently in [5–8], Mohanty has proposed some three level implicit unconditionally stable difference schemes of \(O(\tau^2 + h^2)\) for the one-, two- and three-dimensional linear hyperbolic equations with constant and variable coefficients: however, these schemes are parameter dependent. Rashidinidia et al. [9] developed two conditionally stable three level difference schemes of \(O(\tau^2 + h^2)\) and \(O(\tau^2 + h^4)\) based on non-polynomial cubic spline functions for the solution of a one-dimensional second-order non-homogeneous hyperbolic equation. Meanwhile, Gao and Chi [10] developed two difference schemes of \(O(\tau^4 + h^6)\) and \(O(\tau^2 + h^2)\) for the special one-dimensional second-order hyperbolic equation. In this paper, by using a new method, we construct a new difference scheme of \(O(\tau^4 + h^4)\) to solve a two-dimensional second-order non-homogeneous hyperbolic equation.

Consider the second-order linear hyperbolic partial differential equation in two-space dimensions

\begin{equation}
\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \quad \alpha > \beta \geq 0,
\end{equation}

over a region \(\Omega = [0 < x < 1] \times [0 < y < 1] \times [t > 0]\), with the initial conditions

\begin{align}
  u(x, y, 0) &= u_0(x, y), & 0 \leq x, y \leq 1, & (1.2a) \\
  \frac{\partial u(x, y, 0)}{\partial t} &= u_1(x, y), & 0 \leq x, y \leq 1, & (1.2b)
\end{align}
and boundary conditions
\[
\begin{align*}
  u(x, 0, t) &= g_0(x, t), & u(x, 1, t) &= g_1(x, t), & 0 \leq x \leq 1, t > 0, \\
  u(0, y, t) &= p_0(y, t), & u(1, y, t) &= p_1(y, t), & 0 \leq y \leq 1, t > 0,
\end{align*}
\]
where \(\alpha\) and \(\beta\) are constants. For \(\alpha > 0, \beta = 0\), and \(\alpha > \beta > 0\), the above equation represents a damped wave equation and a telegraph equation, respectively, see [4]. We assume that \(u_0(x, y)\), \(u_1(x, y)\) and their derivatives are continuous functions of \(x\) and \(y\).

This paper is organized as follows. In Section 2, we propose a high accuracy difference scheme for solving Eq. (1.1). In Section 3, we study the stability of the new difference scheme. In Section 4, some numerical examples are used to illustrate the efficiency of the new difference scheme.

2. Proposition of the difference scheme

For a difference solution of the above initial boundary value problems, let us assume that the solution domain \(\Omega = [0 < x < 1] \times [0 < y < 1] \times [t > 0]\) is covered by a rectilinear grid with \(h > 0, \tau > 0\) in the space and time coordinates, respectively, where we treat \(x, y\) as space and \(t\) as time coordinate, respectively. The grid points \((x_i, y_j, t_k)\) are given by \(x_i = ih, y_j = jh(0 \leq i, j \leq M)\) with \(Mh = 1\) and \(t_k = k\tau (k = 0, 1, 2, \ldots)\). Again let \(u_{i,j}^k\) be the approximate value of \(u\) at the grid point \((x_i, y_j, t_k)\).

At first, let us consider the following equation by using the ideas in [11,12]
\[
\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} = q, \tag{2.1}
\]
where \(\alpha > 0\) is constant and \(q\) is a function of \(t\).

We derive two fourth-order difference formulas by Taylor series expansions as follows:
\[
\begin{align*}
  \frac{\partial w}{\partial t} &= \mu_i \delta_t w - \frac{\tau^2}{6} \frac{\partial^3 w}{\partial t^3} + O(\tau^4), \tag{2.2} \\
  \frac{\partial^2 w}{\partial t^2} &= \delta^2_t w - \frac{\tau^2}{12} \frac{\partial^4 w}{\partial t^4} + O(\tau^4), \tag{2.3}
\end{align*}
\]
where \(w\) is representative of \(q\) or \(u\).

From (2.1)-(2.3), we easily get
\[
\begin{align*}
  \frac{\partial^3 u}{\partial t^3} &= \frac{\partial q}{\partial t} - 2\alpha \frac{\partial^2 u}{\partial t^2} \\
  &= \mu_i \delta_t q - \frac{\tau^2}{6} \frac{\partial^3 q}{\partial t^3} - 2\alpha \delta^2_t u + \frac{\alpha\tau^2}{6} \frac{\partial^4 u}{\partial t^4} + O(\tau^4), \tag{2.4} \\
  \frac{\partial^4 u}{\partial t^4} &= \frac{\partial^2 q}{\partial t^2} - 2\alpha \frac{\partial^3 u}{\partial t^3} \\
  &= \delta^2_t q - 2\alpha \mu_i \delta_t q + 4\alpha^2 \delta^2_t u - \frac{\tau^2}{12} \frac{\partial^4 q}{\partial t^4} + \frac{\alpha\tau^2}{3} \frac{\partial^3 q}{\partial t^3} - \frac{\alpha^2\tau^2}{3} \frac{\partial^4 u}{\partial t^4} + O(\tau^4). \tag{2.5}
\end{align*}
\]
Substituting (2.4) and (2.5) into (2.2) and (2.3), respectively, yields
\[
\begin{align*}
  \frac{\partial u}{\partial t} &= \mu_i \delta_t u - \frac{\tau^2}{6} (\mu_i \delta_t q - 2\alpha \delta^2_t u) + O(\tau^4), \tag{2.6} \\
  \frac{\partial^2 u}{\partial t^2} &= \delta^2_t u - \frac{\tau^2}{12} (\delta^2_t q - 2\alpha \mu_i \delta_t q + 4\alpha^2 \delta^2_t u) + O(\tau^4). \tag{2.7}
\end{align*}
\]
Then, we substitute (2.6) and (2.7) into (2.1) and get a three-point fourth-order compact difference scheme for solving Eq. (2.1) at grid point \((x_i, y_j, t_k)\) as follows
\[
\left(1 + \frac{\alpha^2\tau^2}{3}\right) \delta^2_t u_{i,j}^k + 2\alpha \mu_i \delta_t u_{i,j}^k = \left(1 + \frac{\tau^2}{12} \delta^2_t + \frac{\alpha \tau^2}{6} \mu_i \delta_t\right) q_{i,j}^k + O(\tau^4), \tag{2.8}
\]
where \(\delta^2_t\) and \(\mu_i \delta_t\) are the second- and first-order center difference operators with respect to the \(t\)-direction, respectively. For convenience, we define two finite difference operators
\[
L_t = 1 + \frac{\tau^2}{12} \delta^2_t + 2\alpha \mu_i \delta_t, \quad A_t = \left(1 + \frac{\alpha^2\tau^2}{3}\right) \delta^2_t + 2\alpha \mu_i \delta_t.
\]
Eq. (2.8) can then be formulated symbolically as
\[
L_t^{-1}A_t u^k_{i,j} = q^k_{i,j} + O(\tau^4).
\] (2.9)
So, a fourth-order semi-discrete approximation to Eq. (1.1) can be obtained by replacing \( q^k_{i,j} \) in (2.9):
\[
L_t^{-1}A_t u^k_{i,j} = \frac{\partial^2 u^k_{i,j}}{\partial x^2} + \frac{\partial^2 u^k_{i,j}}{\partial y^2} - 2u^k_{i,j} + f^k_{i,j} + O(\tau^4).
\] (2.10)
For the terms \( \frac{\partial^2 u^k_{i,j}}{\partial x^2} \) and \( \frac{\partial^2 u^k_{i,j}}{\partial y^2} \), we use the following compact difference approximation, respectively
\[
\frac{\partial^2 u^k_{i,j}}{\partial x^2} = \frac{\delta^2 u^k_{i,j}}{1 + \frac{h^2}{12} \delta^2 x} + O(h^4), \quad \frac{\partial^2 u^k_{i,j}}{\partial y^2} = \frac{\delta^2 u^k_{i,j}}{1 + \frac{h^2}{12} \delta^2 y} + O(h^4),
\] (2.11)
where \( \delta^2_x \) and \( \delta^2_y \) are the second-order center difference operators with respect to the x-direction and y-direction, respectively.

Combining (2.10) and (2.11) and neglecting high-order term \( O(\tau^4 + h^4) \), then we can get a fourth-order compact difference scheme for solving Eq. (1.1) as follows:
\[
\left( \delta^2_x + \delta^2_y \right) L_t u^k_{i,j} = \left( 1 + \frac{h^2}{12} \delta^2_x \right) \left( 1 + \frac{h^2}{12} \delta^2_y \right) \left( A_t u^k_{i,j} + \beta^2 L_t u^k_{i,j} - L_t f^k_{i,j} \right).
\] (2.12)

3. Stability analysis

In this section, we study the stability of the difference scheme (2.12) by using the method in [9]. At first, we write (2.12) in component form as follows:

\[
\begin{align*}
A_1 u^{k+1}_{i-1,j+1} + B_1 u^{k+1}_{i+1,j+1} + A_1 u^{k+1}_{i+1,j-1} + B_1 u^{k+1}_{i-1,j-1} + C_1 u^{k+1}_{i,j} + B_1 u^{k+1}_{i,j+1} + B_1 u^{k+1}_{i,j-1} + C_1 u^{k+1}_{i+1,j} + B_1 u^{k+1}_{i-1,j} + C_1 u^{k+1}_{i,j+1} + B_1 u^{k+1}_{i,j-1} + C_1 u^{k+1}_{i,j} + B_1 u^{k+1}_{i+1,j-1} + C_1 u^{k+1}_{i,j+1} & = f^{k+1}_{i,j} + f^{k+1}_{i+1,j} + f^{k+1}_{i,j+1} + f^{k+1}_{i+1,j+1} + f^{k+1}_{i-1,j} + f^{k+1}_{i,j-1} + f^{k+1}_{i-1,j-1} + f^{k+1}_{i,j+1} + f^{k+1}_{i,j-1} + f^{k+1}_{i,j} + f^{k+1}_{i+1,j+1} + f^{k+1}_{i-1,j+1} + f^{k+1}_{i,j+1} + f^{k+1}_{i+1,j} + f^{k+1}_{i,j} + f^{k+1}_{i+1,j-1} + f^{k+1}_{i-1,j} + f^{k+1}_{i,j+1} + f^{k+1}_{i-1,j-1} + f^{k+1}_{i,j}.
\end{align*}
\] (3.1)

The coefficients of the above formula are given by
\[
\begin{align*}
A_1 &= \frac{1}{144} \left[ \frac{1}{\tau^2} + \frac{\alpha^2}{3} + \frac{\alpha}{\tau} - \frac{2(\alpha \tau + 1)}{h^2} + \frac{\beta^2(\alpha \tau + 1)}{12} \right], \\
B_1 &= \frac{1}{72} \left[ 5 \left( \frac{1}{\tau^2} + \frac{\alpha^2}{3} + \frac{\alpha}{\tau} \right) - \frac{4(\alpha \tau + 1)}{h^2} + \frac{5\beta^2(\alpha \tau + 1)}{12} \right], \\
C_1 &= \frac{1}{36} \left[ 25 \left( \frac{1}{\tau^2} + \frac{\alpha^2}{3} + \frac{\alpha}{\tau} \right) - \frac{10(\alpha \tau + 1)}{h^2} + \frac{25\beta^2(\alpha \tau + 1)}{12} \right], \\
A_2 &= \frac{1}{72} \left[ - \left( \frac{1}{\tau^2} + \frac{\alpha^2}{3} \right) - \frac{10}{h^2} + \frac{5\beta^2}{12} \right], \\
B_2 &= \frac{1}{36} \left[ -5 \left( \frac{1}{\tau^2} + \frac{\alpha^2}{3} \right) - \frac{20}{h^2} + \frac{25\beta^2}{12} \right], \\
C_2 &= \frac{1}{18} \left[ -25 \left( \frac{1}{\tau^2} + \frac{\alpha^2}{3} \right) - \frac{50}{h^2} + \frac{125\beta^2}{12} \right].
\end{align*}
\]
\[
A_3 = \frac{1}{144} \left[ \frac{1}{\tau^2} + \frac{\alpha^2}{3} - \frac{\alpha}{\tau} - \frac{2(1 - \alpha \tau)}{h^2} + \frac{\beta^2(1 - \alpha \tau)}{12} \right],
\]
\[
B_3 = \frac{1}{72} \left[ 5 \left( \frac{1}{\tau^2} + \frac{\alpha^2}{3} - \frac{\alpha}{\tau} \right) - \frac{4(1 - \alpha \tau)}{h^2} + \frac{5\beta^2(1 - \alpha \tau)}{12} \right],
\]
\[
C_3 = \frac{1}{36} \left[ 25 \left( \frac{1}{\tau^2} + \frac{\alpha^2}{3} - \frac{\alpha}{\tau} \right) + \frac{10(1 - \alpha \tau)}{h^2} + \frac{25\beta^2(1 - \alpha \tau)}{12} \right],
\]
\[
A_4 = \frac{\beta^2(\alpha \tau + 1)}{1728}, \quad B_4 = \frac{5\beta^2(\alpha \tau + 1)}{864}, \quad C_4 = \frac{25\beta^2(\alpha \tau + 1)}{432},
\]
\[
A_5 = \frac{5\beta^2}{864}, \quad B_5 = \frac{25\beta^2}{432}, \quad C_5 = \frac{25\beta^2}{216},
\]
\[
A_6 = \frac{\beta^2(1 - \alpha \tau)}{1728}, \quad B_6 = \frac{5\beta^2(1 - \alpha \tau)}{864}, \quad C_6 = \frac{25\beta^2(1 - \alpha \tau)}{432}.
\]

For analysis of stability of the difference scheme (3.1), we assume that the solution of (3.1) at the grid point \((x_i, y_j, t_k)\) is of the form
\[
u^i_{j} = \xi^i e^{i\phi} e^{j\psi}, \quad (3.2)
\]
where \(I = \sqrt{-1}, \phi \) and \(\psi \) are reals, and \(\xi \) is in general, complex. Substituting (3.2) into the homogeneous part of (3.1), we obtain a characteristic equation
\[
Q\xi^2 + M\xi + N = 0, \quad (3.3)
\]

where
\[
Q = 4A_1 \cos(i\phi) \cos(j\psi) + 2B_1 \cos(i\phi) + 2B_1 \cos(j\psi) + C_1,
M = 4A_2 \cos(i\phi) \cos(j\psi) + 2B_2 \cos(i\phi) + 2B_2 \cos(j\psi) + C_2,
N = 4A_3 \cos(i\phi) \cos(j\psi) + 2B_3 \cos(i\phi) + 2B_3 \cos(j\psi) + C_3.
\]

Under the transformation \(\xi = \frac{1 + x}{r^2}, \) we can rewrite (3.3) as
\[
(Q - M + N)z^2 + 2(Q - N)x + (Q + M + N) = 0. \quad (3.4)
\]

The necessary and sufficient condition for \(|\xi| < 1\), is that \(Q - M + N > 0, Q - N > 0\) and \(Q + M + N > 0\).

From the coefficients of the characteristic Eq. (3.3), we easily get
\[
Q - N = \frac{1}{36} \left[ \frac{2\alpha}{\tau^4} + \frac{4\alpha \tau}{h^2} + \frac{\beta^2 \alpha \tau}{6} \right] \cos(i\phi) \cos(j\psi) + \frac{1}{18} \left[ \frac{5\alpha}{\tau^4} - \frac{4\alpha \tau}{h^2} + \frac{5\beta^2 \alpha \tau}{12} \right] \cos(i\phi) \cos(j\psi)
= \frac{1}{9} \left( \frac{2\alpha}{\tau^2} + \frac{\beta^2 \alpha \tau}{6} \right) \left( \cos^2 \frac{i\phi}{2} \cos^2 \frac{j\psi}{2} + 2 \cos^2 \frac{i\phi}{2} + 2 \cos^2 \frac{j\psi}{2} + 4 \right)
= \frac{1}{9} \left( \frac{2\alpha}{\tau^2} + \frac{\beta^2 \alpha \tau}{6} \right) \left( \cos^2 \frac{i\phi}{2} \cos^2 \frac{j\psi}{2} + 2 \cos^2 \frac{i\phi}{2} + 2 \cos^2 \frac{j\psi}{2} + 4 \right)
> 0,
\]
In order to test the efficiency and viability of the new difference scheme (2.12), we have solved the proposed differential Eq. (1.1) for different values of $\alpha > 0$ and $\beta > 0$. The exact solution values for all cases are given. The initial and boundary conditions and right hand side function $f(x, y, t)$ can be obtained using the exact solution. The proposed scheme (2.12) is stable.

4. Numerical examples
consider the following hyperbolic equation 
\[
\frac{\partial^2 u}{\partial t^2} + 2(1 + \pi^2) \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 \leq x \leq 1, 0 \leq y \leq 1, t \geq 0,
\]
where the exact solution is given by 
\[
u(x, y, t) = e^{-t} \sin(\pi x) \sin(\pi y).
\]

The initial and boundary conditions are defined so as to agree with the exact solution. We chose time step \(\tau = 0.001\) and space steps \(h = 1/16, 1/32, 1/64, 1/128, 1/256\) for the entire simulation process. The data in Table 1 shows the maximum absolute error between the exact solution and the numerical solution at \(t = 1\) using the difference schemes in [2,7] and in this article. It is confirmed from Table 1 that if \(h\) is reduced by a factor 1/2, then the maximum absolute error indicates that the present method gives fourth-order results.

### Example 4.2
Consider the following hyperbolic equation 
\[
\frac{\partial^2 u}{\partial t^2} + 4\pi \frac{\partial u}{\partial t} + 2\pi^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\pi t \sin \pi(x + y)e^{-(x+y)t}
\]
\[
+ [(x + y - 2\pi)^2 - 2\pi^2] \sin(\pi x) \sin(\pi y)e^{-(x+y)t}, \quad 0 \leq x \leq 1, 0 \leq y \leq 1, t \geq 0.
\]

where the exact solution is given by 
\[
u(x, y, t) = e^{-(x+y)t} \sin(\pi x) \sin(\pi y).
\]

The initial and boundary conditions are defined so as to agree with the exact solution. We chose time step \(\tau = 0.003\) and space steps \(h = 1/12, 1/24, 1/48, 1/96, 1/192\) for the entire simulation process. The data in Table 2 shows the maximum absolute error between the exact solution and the numerical solution at \(t = 1\) using the difference schemes in [2,7] and in this article. It is confirmed from Table 2 that if \(h\) is reduced by a factor 1/2, then the maximum absolute error shows that the present method is of fourth order.

### Example 4.3
Consider the following hyperbolic equation 
\[
\frac{\partial^2 u}{\partial t^2} + 5\pi \frac{\partial u}{\partial t} + 4\pi^4 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + [6\pi^2 y(y-1)^2(x-1)^4(7y^2 - 6y + 1)
\]
\[
+ 6\pi y(x-1)^2(y-1)^4(7x^2 - 6x + 1)]e^{-\pi^2 t}, \quad 0 \leq x \leq 1, 0 \leq y \leq 1, t \geq 0.
\]
where the exact solution is given by

\[ u(x, y, t) = x^3 y^3 (x - 1)^4 (y - 1)^4 e^{-\pi^2 t}. \]

The initial and boundary conditions are defined so as to agree with the exact solution. We chose time step \( \tau = 0.004 \) and space steps \( h = 1/10, 1/20, 1/40, 1/80, 1/160 \) for the entire simulation process. The data in Table 3 shows the maximum absolute error between the exact solution and the numerical solution at \( t = 1 \) using the difference schemes in [2, 7] and in this article. It is confirmed from Table 3 that if \( h \) is reduced by a factor 1/2, then the maximum absolute error indicates that the present method gives fourth-order results.

The computational results presented in Tables 1–3 show that the new difference scheme (2.12) is more accurate than the difference schemes in [2, 7].

5. Concluding remarks

In this work, we get a finite difference scheme for solving two-dimensional second-order non-homogeneous linear hyperbolic equations. The accuracy of the difference scheme reaches to \( O(\tau^4 + h^4) \). Some numerical examples and results are presented to illustrate the efficiency of our proposed difference scheme.

References